State Space Models and MIDAS Regressions*

Jennie Bai† **Eric Ghysels**‡ **Jonathan H. Wright**§

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Abstract

We examine the relationship between MIDAS regressions and the Kalman filter when forecasting with mixed frequency data. In general, state space models involve a system of equations, whereas MIDAS regressions involve a single equation. As a consequence, MIDAS regressions might be less efficient, but could also be less prone to parameter estimation error and/or specification errors. We examine how MIDAS regressions and Kalman filters match up under ideal circumstances, that is in population, and in cases where all the stochastic processes - low and high frequency - are correctly specified. We characterize cases where the MIDAS regression exactly replicates the steady state Kalman filter weights. We compare MIDAS and Kalman filter forecasts in population where the state space model is mis-specified. We also compare MIDAS and Kalman filter forecasts in small samples. The paper concludes with an empirical application. Overall we find that the MIDAS and Kalman filter methods give similar forecasts. In most cases, the Kalman filter is a bit more accurate, but it is also computationally much more demanding.

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†Economist, Capital Markets Function, Federal Reserve Bank of New York, 33 Liberty Street New York, NY 10045, e-mail: *jennie.bai@ny.frb.org*.

‡Department of Finance - Kenan-Flagler Business School and Department of Economics, University of North Carolina, McColl Building, Chapel Hill, NC 27599. e-mail: *eghysels@unc.edu*.

§Department of Economics, Mergenthaler Hall 457, Johns Hopkins University, 3400 N. Charles Street Baltimore, MD 21218, e-mail: *wrightj@jhu.edu*.

1 Introduction

Not all economic data are sampled at the same frequency. Financial data are readily available on a (intra-)daily basis, whereas most macroeconomic data are sampled weekly, monthly, quarterly or even annually. The literature has mainly addressed mixed frequency data in the context of state space models by Harvey and Pierse (1984), Harvey (1989), Bernanke, Gertler, and Watson (1997), Zadrozny (1990), Mariano and Murasawa (2003), Mittnik and Zadrozny (2005), Aruoba, Diebold, and Scotti (2009), Ghysels and Wright (2009), Camacho and Perez-Quiros (2010) and Kuzin, Marcellino, and Schumacher (2011), among others. State space models consist of a system of two equations, a measurement equation which links observed series to a latent state process, and a state equation which describes the state process dynamics. The setup treats the low-frequency data as *"missing data"* and the Kalman filter is a convenient computational device to extract the missing data. The approach has many benefits, but also some drawbacks. State space models can be quite involved, as one must explicitly specify a linear dynamic model for all the series involved: low-frequency data series, latent low-frequency series treated as missing and the high-frequency observed processes. The system of equations therefore can require a lot of parameters, for the measurement equation, the state dynamics and their error processes.

Recently, some alternative approached have been proposed. One approach, considered by Hamilton (2008) is to run an unrestricted regression of the low-frequency data on past low- and high-frequency observations. Another approach to dealing with data sampled at different frequencies has emerged in work by Ghysels, Santa-Clara, and Valkanov (2002), Ghysels, Santa-Clara, and Valkanov (2006) and Andreou, Ghysels, and Kourtellos (2009) using so called MIDAS, meaning Mi(xed) Da(ta) S(ampling), regressions.[1] Recent work has used the regressions in the context of constructing quarterly macro forecasts using monthly data (see e.g. (see e.g. Armesto, Hernández-Murillo, Owyang, and Piger (2009), Clements and Galvão (2008), Clements and Galvão (2009), Galvão (2006), Monteforte and Moretti (2009), Schumacher and Breitung (2008), Tay (2007)), or improving quarterly and monthly macroeconomic predictions with daily financial data (see e.g. Andreou, Ghysels, and Kourtellos (2008), Ghysels and Wright (2009), Hamilton (2008), Tay (2006)).[2]

[1] The original work on MIDAS focused on volatility predictions, see e.g. Alper, Fendoglu, and Saltoglu (2008), Chen and Ghysels (2011), Engle, Ghysels, and Sohn (2008), Forsberg and Ghysels (2006), Ghysels, Santa-Clara, and Valkanov (2005), Ghysels, Santa-Clara, and Valkanov (2006), and Léon, Nave, and Rubio (2007), among others.

[2] A user-friendly introduction to MIDAS regression models appears in Armesto et al. (2010).

Our objective in this paper is to compare MIDAS regressions and the Kalman filter as means of forecasting the low-frequency time series. The MIDAS regression relates the low-frequency time series that we wish to predict to observables at high and low frequencies. The steady state Kalman gain, yields a similar linear projection rule. The purpose of this paper is to examine the relationship between these two filters. The basic Kalman filter applies to linear Gaussian systems and yields an optimal filter in population, provided that the parameters are known and the state space model is correctly specified. There are extensions of the Kalman filter for non-linear and non-Gaussian state space models (Durbin and Koopman (2001)). These are again optimal, but only if the parameters are known and the model is correctly specified. On the other hand, MIDAS regressions provide linear projections given the (high- and low-frequency) regressors without specifying their data generating process. MIDAS regressions cannot hope to do better than the Kalman filter if the state space model is correctly specified and the parameters are known, but otherwise they could give better forecasts of the low-frequency time series. As parameter uncertainty is important and model mis-specification is inevitable, it is not clear whether MIDAS or state space models will forecast the low-frequency data better in practice.

The Kalman filter also has the useful spinoff of providing high-frequency estimates of the latent state variable. The focus of this paper is however more limited. We focus on using information at mixed frequencies to predict the low-frequency time series. This prediction can either be for the current period (nowcasting) or for future periods.

The first objective of this paper is to examine how MIDAS regressions and Kalman filters match up under ideal circumstances, that is in population, and in cases where all the stochastic processes — low- and high-frequency — are correctly specified by a linear state space model. We give conditions under which the equivalence between the steady state Kalman filter and MIDAS regression is *exact*, in population. With mixed sampling frequencies, the steady state Kalman filter has a periodic structure and under certain conditions this maps *exactly* into a multiplicative MIDAS regression model considered by Chen and Ghysels (2011) and Andreou, Ghysels, and Kourtellos (2008). This multiplicative MIDAS regression consists of a parameter-driven aggregation of the high-frequency data, combined with the low-frequency observations using an autoregressive distributed lag (ADL) model. We show that the multiplicative scheme exactly matches the steady state Kalman gain that drives the state space model filter. Next, we compare the MIDAS regressions and Kalman filter in population where the state space model is correctly specified, but the

conditions for the MIDAS regression and Kalman filter to be equivalent do not apply. For those cases, we compute the approximation error, either in terms of forecast mean square errors or in terms of differences in filter weights, and we find that the approximation errors, regardless of the metric chosen, are very small. We also examine how MIDAS regressions perform in comparison to the Kalman filter in population, when the state space model is mis-specified. We find that the population mean square prediction error using the two forecasting methods is very similar in many cases, but in other cases, the Kalman filter model does better despite being mis-specified.

In a small-sample Monte-Carlo simulation, we compare the out-of-sample forecasting properties of predictions from MIDAS regressions with those from the Kalman filter, both in cases where the state-space model is correctly specified, and where it is mis-specified. The two forecasting methods generally behave similarly. The Kalman filter typically gives more accurate forecasts, but there are also cases in which MIDAS regressions give better small-sample predictions.

Finally, the paper concludes with an empirical study similar to that of Kuzin, Marcellino,and Schumacher (2011), except that we compare the MIDAS forecasts to Kalman filter forecasts, rather than to forecasts from the mixed-frequency VAR of Zadrozny (1990) (the latter is a VAR in high-frequency data using temporally disaggregated counterparts of the low-frequency time series). We find the discrepancies between MIDAS and Kalman filter forecasts to be small—with MIDAS performing a bit better in some cases, and a bit worse in others.

Our overall finding is that MIDAS and the Kalman filter generally deliver comparable forecasts. This is true in term of the population filter weights, in small-sample Monte-Carlo simulations, and in our empirical study. In most cases, the Kalman filter gives forecasts that are a bit more accurate, but there are also cases in which MIDAS regressions give better predictions. In guiding the choice of researchers as to which to use, it is however important to note that estimating Kalman filter specifications with large number of series can be numerically quite involved, whereas MIDAS estimation involves only a nonlinear least squares regression and can be computationally a good bit easier. For example, Aruoba, Diebold and Scotti (2009) construct a useful and widely-followed Business Conditions Index published in real time by the Federal Reserve Bank of Philadelphia. The index uses the Kalman filter with data sampled at mixed frequencies, but is limited to six series (weekly initial jobless claims; monthly payroll employment, industrial production, personal income less transfer payments, manufacturing and trade sales; and quarterly real GDP). In contrast,

Andreou, Ghysels, and Kourtellos (2008) compute macroeconomic forecasts with MIDAS regressions using close to a hundred daily financial series which they combine via forecast combinations. This is a fairly straightforward exercise with MIDAS regressions, but would be computationally challenging with a fully specified state space model.

The remainder of this paper is organized as follows. In section 2, we introduce the state space model of Nunes (2005) and derive its relationship with MIDAS regressions. In this section we characterize cases where the MIDAS regression is an exact reduced form representation of the steady state Kalman filter. Section 3 computes measures of the discrepancy between the Kalman filter and MIDAS regressions in cases where the state space model is correctly specified and the MIDAS regression is only an approximation to the Kalman filter. This section also considers cases in which the state space model is mis-specified, and reports some small-sample simulation results. Section 4 contains the empirical work, and section 5 concludes.

2 State space models and MIDAS regressions

We consider a dynamic factor model:

$$\mathbf{F}_{t+j/m} = \sum_{l=1}^{p} \mathbf{\Phi}_l \mathbf{F}_{t+(j-l)/m} + \boldsymbol{\eta}_{t+j/m} \qquad \forall t = 1, \ldots, T, \qquad j = 1, \ldots, m \tag{2.1}$$

where $\mathbf{F}_t$ is a $n_f \times 1$ dimensional vector process and the matrices $\mathbf{\Phi}_l$ are $n_f \times n_f$, with $\boldsymbol{\eta}_t$ being an i.i.d. zero mean Gaussian error process with diagonal covariance matrix $\mathbf{\Sigma}_\eta = \text{diag}(\sigma^2_{i,\eta}, i = 1, \ldots, n_f)$. Apart from the time scale and distributional assumption for the errors, the above equation is a typical multi-factor model used for instance by Stock and Watson (1989), Forni, Hallin, Lippi, and Reichlin (2000), Stock and Watson (2002), Bai and Ng (2004), among others. In anticipation of the mixed frequency sampling scheme, we adopt a time scale expressed in a form that easily accommodates such mixtures. For example, with $m = 3$ we will have monthly data sampled every quarter, or with $m = 22$ we will have daily data sampled every month (approximating the number of business days in a month as 22).[3]

[3] It should be noted that for the purpose of simplicity we select a fixed number of days each month. In practice this is of course not the case. Both the Kalman filter and MIDAS regressions can handle calendar variation in the number of days (m periods). In the context of MIDAS regressions, an example is Francis, Ghysels and Owyang (2011) who measure the impact of FOMC policy shocks on low frequency macro

The monthly/quarterly combination will be most relevant for the empirical application and simulations in later sections, but for the purpose of generality we start with a generic setup.

We have two types of data: (1) time series sampled at a low frequency — every t, and (2) time series sampled at high-frequency — every $t + j/m$ $j = 1, \ldots, m$. We will make two convenient simplifications that depart from generality. First, we assume that there is only *one* low-frequency process and call it y_t. This is without loss of generality — if we had multiple low-frequency time series, then the theory given below would apply to a vector-valued y_t process, though the notation would be a bit more cumbersome. Moreover, focusing on a single low-frequency series is the most common situation involving macroeconomic forecasting of say quarterly GDP growth, or of inflation, etc., using a collection of higher frequency (monthly/weekly/daily) series. Second, we consider the combination of only two sampling frequencies. For example, we do not consider the combination of daily, monthly and quarterly data.

The high-frequency data, denoted $x_{i,t+j/m}$ for $i = 2, \ldots, n$, relates to the factors as follows:

$$x_{i,t+j/m} = \boldsymbol{\gamma}_i' \mathbf{F}_{t+j/m} + u_{i,t+j/m} \qquad i = 2, \ldots, n \qquad \forall t \qquad j = 1, \ldots, m \tag{2.2}$$

where $\{\boldsymbol{\gamma}_i\}$ are $n_f \times 1$ vectors and:

$$d_i(L^{1/m}) u_{i,t+j/m} = \varepsilon_{i,t+j/m} \qquad d_i(L^{1/m}) \equiv 1 - d_{1i} L^{1/m} - \ldots - d_{ki} L^{k/m} \qquad \forall i \tag{2.3}$$

where the lag operator $L^{1/m}$ applies to high-frequency data, i.e. $L^{1/m} u_{i,t} \equiv u_{i,t-1/m}$, and the εs are i.i.d. normal with mean zero and variance σ_ε^2 and are mutually independent. If the low-frequency process were observed at high-frequency, it would similarly relate to the factors as follows:

$$y^*_{t+j/m} = \boldsymbol{\gamma}_1' \mathbf{F}_{t+j/m} + u_{1,t+j/m} \qquad \forall t \qquad j = 1, \ldots, m \tag{2.4}$$

with $u_{1,t+j/m}$ having an AR(k) representation as in (2.3), denoting y^* as the process which is not directly observed. The observed low-frequency process y relates to the y^* via a linear aggregation scheme:

$$y^c_{t+j/m} = \Psi_j y^c_{t+(j-1)/m} + \theta_j y^*_{t+j/m} \tag{2.5}$$

where y_t is equal to y^c_t for integer t, and is not observed otherwise. The above scheme, also

variables - where the FOMC meetings occur on different days of the month.

used by Harvey (1989) and Nunes (2005), covers both stock and flow aggregation, and y_t^c is a cumulator variable. We henceforth consider the case of stock variable only (setting $\Psi_j = 1$ for $j \neq 0$, m, $2m$, ... and zero otherwise and $\theta_j = 1$ for $j = 0$, m, $2m$, ... and zero otherwise). However, if we were instead to pick $\Psi_j = 1$ $j \neq 0$, m, $2m$, ... and zero otherwise with $\theta_j = 1/m$ $\forall$ j, then this would correspond to a flow variable.

2.1 Periodic Data Structure and Steady State Predictions

The purpose of this subsection is to derive a steady state Kalman filtering formula that will be used in the next subsections for comparisons with MIDAS regressions. The material in this section is general and uses some derivations that appear in Assimakis and Adam (2009).

The above equations yield a periodic state space model with measurement equation:

$$\mathbf{Y}_t^j = \mathbf{Z}_j \boldsymbol{\alpha}_{t+j/m} \qquad \begin{cases} \mathbf{Y}_t^j = (y_t, x_{2,t}, \ldots, x_{n,t})' & j = m \\ \\ \mathbf{Y}_t^j = (x_{2,t+j/m}, \ldots, x_{n,t+j/m})' & 1 \leq j \leq m-1 \end{cases} \tag{2.6}$$

where

$$\mathbf{Z}_m = \begin{bmatrix} \boldsymbol{\gamma}_1' & & & \\ \boldsymbol{\gamma}_2' & \mathbf{O}_{n \times n_f(p-1)} & \mathbf{I}_n & \mathbf{O}_{n \times n(k-1)} \\ \vdots & & & \\ \boldsymbol{\gamma}_n' & & & \end{bmatrix}$$

$$\mathbf{Z}_j = \begin{bmatrix} \boldsymbol{\gamma}_2' & & & \\ \vdots & \mathbf{O}_{(n-1) \times n_f(p-1)} & \tilde{\mathbf{I}}_{n-1} & \mathbf{O}_{(n-1) \times n(k-1)} \\ \boldsymbol{\gamma}_n' & & & \end{bmatrix}$$

for $1 \leq j \leq m$ - 1 and state vector

$$\boldsymbol{\alpha}_{t+j/m} = \left(\mathbf{F}'_{t+j/m}, \ldots, \mathbf{F}'_{t+(j-p+1)/m}, \boldsymbol{u}'_{t+j/m}, \ldots, \boldsymbol{u}'_{t+(j-k+1)/m}\right)'$$

where $\mathbf{u}_{t+j/m} = (u_{1,t+j/m}, \ldots, u_{n,t+j/m})'$, and $\tilde{\mathbf{I}}_{n-1}$ is a matrix of size $(n-1) \times n$, that corresponds to the identity matrix $\mathbf{I}_n$ with the top row removed.

The transition equation is:

$$\boldsymbol{\alpha}_{t+j/m} = \mathbf{G}\boldsymbol{\alpha}_{t+(j-1)/m} + \mathbf{R}\boldsymbol{\zeta}_{t+j/m} \tag{2.7}$$

where

$$\mathbf{G} = \begin{bmatrix} \boldsymbol{\Phi}_1 \dots \boldsymbol{\Phi}_{p-1} & \boldsymbol{\Phi}_p & \mathbf{O}_{n_f \times (k-1)n} & \mathbf{O}_{n_f \times n} \\ \mathbf{I}_{(p-1)n_f} & \mathbf{O}_{(p-1)n_f \times n_f} & \mathbf{O}_{(p-1)n_f \times (k-1)n} & \mathbf{O}_{(p-1)n_f \times n} \\ \mathbf{O}_{n \times (p-1)n_f} & \mathbf{O}_{n \times n_f} & \mathbf{D}_1 \dots \mathbf{D}_{k-1} & \mathbf{D}_k \\ \mathbf{O}_{(k-1)n \times (p-1)n_f} & \mathbf{O}_{(k-1)n \times n_f} & \mathbf{I}_{(k-1)n} & \mathbf{O}_{(k-1)n \times n} \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} \mathbf{I}_{n_f} & \mathbf{O}_{n_f \times n} \\ \mathbf{O}_{(p-1)n_f \times n_f} & \mathbf{O}_{(p-1)n_f \times n} \\ \mathbf{O}_{n \times n_f} & \mathbf{I}_n \\ \mathbf{O}_{n(k-1) \times n_f} & \mathbf{O}_{n(k-1) \times n} \end{bmatrix}$$

$\mathbf{D}_i = \text{diag}(d_{l,i}, l = 1, \dots, n)$ and $\boldsymbol{\zeta}_{t+j/m} = (\eta'_{t+j/m}, \varepsilon_{1,t+j/m}, \dots \varepsilon_{n,t+j/m})'$. Let $\boldsymbol{\Sigma}_\zeta$ denote the variance-covariance matrix of $\boldsymbol{\zeta}_{t+j/m}$.

The above state space model is periodic as it cycles to the data release pattern that repeats itself every m periods. Such systems have a (periodic) steady state (see e.g. Assimakis and Adam (2009)). If we let $\mathbf{P}_{j|j-1}$ denote the steady state covariance matrix of $\boldsymbol{\alpha}_{t+j/m|t+(j-1)/m}$ then the equations:

$$\begin{aligned} \mathbf{P}_{j+1|j} &= \mathbf{R}\boldsymbol{\Sigma}_\zeta\mathbf{R}' + \mathbf{G}\mathbf{P}_{j|j-1}\mathbf{G}' - \mathbf{G}\mathbf{P}_{j|j-1}\mathbf{Z}'_j[\mathbf{Z}_j\mathbf{P}_{j|j-1}\mathbf{Z}'_j]^{-1}\mathbf{Z}_j\mathbf{P}_{j|j-1}\mathbf{G}' \qquad j = 1, \dots, m-1 \\ \mathbf{P}_{1|m} &= \mathbf{R}\boldsymbol{\Sigma}_\zeta\mathbf{R}' + \mathbf{G}\mathbf{P}_{m|m-1}\mathbf{G}' - \mathbf{G}\mathbf{P}_{m|m-1}\mathbf{Z}'_m[\mathbf{Z}_m\mathbf{P}_{m|m-1}\mathbf{Z}'_m]^{-1}\mathbf{Z}_m\mathbf{P}_{m|m-1}\mathbf{G}' \end{aligned} \tag{2.8}$$

must be satisfied and $\mathbf{P}_{j|j-1} = \mathbf{P}_{j+m|j+m-1}$, $\forall\ j$. The periodic steady state Kalman gain is therefore:

$$\mathbf{K}_{j|j-1} = \mathbf{P}_{j|j-1}\mathbf{Z}'_j[\mathbf{Z}_j\mathbf{P}_{j|j-1}\mathbf{Z}'_j]^{-1} \tag{2.9}$$

with $\mathbf{K}_{j|j-1} \equiv \mathbf{K}_{j+m|j-1+m}$, $\forall\ j$. When we define the extraction of the state vector as:

$$\hat{\boldsymbol{\alpha}}_{(t+j/m)|(t+j/m)} = E[\boldsymbol{\alpha}_{t+j/m}|\mathbf{Y}^j_t, \mathbf{Y}^{j-1}_t, ...\mathbf{Y}^1_t, \mathbf{Y}^m_{t-1}, \mathbf{Y}^m_{t-2}, ...]$$

the filtered states are:

$$\hat{\boldsymbol{\alpha}}_{(t+j/m)|(t+j/m)} = \mathbf{A}_{j|j-1}\hat{\boldsymbol{\alpha}}_{(t+(j-1)/m)|(t+(j-1)/m)} + \mathbf{K}_{j|j-1}\mathbf{Y}^j_t \tag{2.10}$$

where $\mathbf{A}_{j|j-1} = \mathbf{G} - \mathbf{K}_{j|j-1}\mathbf{Z}_j\mathbf{G}$ and $\mathbf{Y}_t^m = \mathbf{Y}_{t+1}^0$, with $\mathbf{A}_{j|j-1} = \mathbf{A}_{1|m}$ for $j = 1$.

Suppose we are interested in predicting at low-frequency intervals only, namely $\hat{\boldsymbol{\alpha}}_{(t+k)|t}$, for k integer valued, using all available low and high-frequency data. First we note that:

$$\hat{\boldsymbol{\alpha}}_{(t+k)|(t+k)} = [\tilde{\mathbf{A}}_1^m]^k \hat{\boldsymbol{\alpha}}_{t|t} + \sum_{i=1}^{m}\sum_{j=1}^{k}[\tilde{\mathbf{A}}_1^m]^{k-j}\tilde{\mathbf{A}}_{i+1}^m \mathbf{K}_{i|i-1}\mathbf{Y}_{t+j-1}^i \tag{2.11}$$

where

$$\tilde{\mathbf{A}}_j^i = \begin{cases} \mathbf{A}_{i|i-1}\mathbf{A}_{i-1|i-2}\ldots\mathbf{A}_{j|j-1} & i \geq j \\ \\ \mathbf{I} & i < j \end{cases}$$

Expression (2.11) can be obtained via straightforward algebra — see Assimakis and Adam (2009). If all eigenvalues of $\mathbf{G}$ lie inside the unit circle, then all the eigenvalues of $\mathbf{A}_{j|j-1}$, j = 1, ..., m - 1, are also inside the unit circle, as are the eigenvalues of the product matrices $\{\tilde{\mathbf{A}}_j^i\}$ (see again Assimakis and Adam). This implies that we can iterate (2.11) backwards to give:

$$\begin{aligned} \hat{\boldsymbol{\alpha}}_{t|t} &= \sum_{j=0}^{\infty}\sum_{i=1}^{m}[\tilde{\mathbf{A}}_1^m]^j\tilde{\mathbf{A}}_{i+1}^m\mathbf{K}_{i|i-1}\mathbf{Y}_{t-j}^i = \sum_{j=0}^{\infty}[\tilde{\mathbf{A}}_1^m]^j\mathbf{K}_{m|m-1}\begin{pmatrix} y_{t-j} \\ x_{2,t-j} \\ \vdots \\ x_{n,t-j} \end{pmatrix} \\ &+ \sum_{j=0}^{\infty}\sum_{i=1}^{m-1}[\tilde{\mathbf{A}}_1^m]^j\tilde{\mathbf{A}}_{i+1}^m\mathbf{K}_{i|i-1}\begin{pmatrix} x_{2,t-1-j+i/m} \\ \vdots \\ x_{n,t-1-j+i/m} \end{pmatrix} \end{aligned} \tag{2.12}$$

from which forecasts can easily be constructed as $E_t[y_{t+h}] = \mathbf{Z}_{m,1}\mathbf{G}^{mh}\hat{\boldsymbol{\alpha}}_{t|t}$, where $\mathbf{Z}_{m,1}$ denotes the first row of the matrix $\mathbf{Z}_m$.

2.2 Using only High-Frequency Data and the DL-MIDAS Regression Model

Suppose for the moment that we discard the observations of low-frequency data and only consider projections on high-frequency data. The purpose of this subsection is to show that this yields a linear projection linked to a standard steady state (aperiodic) Kalman gain

and that this projection has a reduced form representation that maps into what Andreou, Ghysels, and Kourtellos (2008) called a Distributed Lag MIDAS (DL-MIDAS) regression.

Unlike the previous subsection, we will first start with a simple example to illustrate the main finding and then we will cover the general case. In particular, let us consider a single factor AR(1) model, instead of the general case in equation (2.1), namely:

$$f_{t+j/m} = \rho f_{t+(j-1)/m} + \eta_{t+j/m} \qquad \forall t = 1, \ldots, T, \qquad j = 1, \ldots, m \tag{2.13}$$

where η is white noise with variance σ_η^2 and there is only a single high-frequency series related to the latent factor:

$$x_{t+j/m} = f_{t+j/m} + u_{2,t+j/m} \qquad \forall t \qquad j = 1, \ldots, m \tag{2.14}$$

instead of equation (2.2), and we also set the slope coefficient equal to one and assume that $u_{2,t+j/m}$ in the above equation is white noise with variance σ_x^2.

While it is still the case that:

$$y_t = f_t + u_{1,t} \qquad \forall t \tag{2.15}$$

where $u_{1,t}$ is white noise with variance σ_y^2, we assume in this subsection that this measurement is not taken into account. Hence, we compute:

$$E\left[y_{t+h} | I_t^{HF}\right] = \rho^{mh} \hat{f}_{t|t} \tag{2.16}$$

where I_t^{HF} is the high-frequency data set of past xs available at time t and $\hat{f}_{t|t}$ is the filtered estimate of the factor conditional on that information set. Let κ be the steady state Kalman gain so that $\hat{f}_{t|t} = (\rho - \rho\kappa)\hat{f}_{t-1/m|t-1/m} + \kappa x_t$. This implies that:

$$E\left[y_{t+h} | I_t^{HF}\right] = \rho^{mh} \kappa \sum_{j=0}^{\infty} (\rho - \rho\kappa)^j x_{t-j/m} \tag{2.17}$$

Note that κ is a function of all the underlying state space parameters. We have deliberately reduced those parameters to a small number by assuming slopes equal to one and assuming that all measurement noise is uncorrelated. What is left are two variances: σ_η^2 and σ_x^2.

The above equation compares directly with a DL-MIDAS regression (again ignoring

intercepts):

$$y_{t+h} = \beta \sum_{j=0}^{\bar{K}} w_j x_{t-j/m} + \varepsilon_t \qquad \forall t \tag{2.18}$$

where the weighting scheme adopted in Ghysels, Santa-Clara and Valkanov (2006) and Andreou, Ghysels, and Kourtellos (2008), among others, is a two-parameter exponential Almon lag polynomial:

$$w_j(\theta_1, \theta_2) = \frac{\exp\{\theta_1 j + \theta_2 j^2\}}{\sum_{j=1}^{\bar{K}} \exp\{\theta_1 j + \theta_2 j^2\}} \tag{2.19}$$

Note that the weights are governed by two parameters and scaled such that they add up to one, hence the presence of a slope parameter β. In the special case of $\theta_2 = 0$ and $\theta_1 = \ln(\rho - \rho\kappa)$ (assuming $\rho > \rho\kappa$), the DL-MIDAS regression involves a weighting scheme identical to that appearing in the conditional mean projection of the Kalman filter appearing in equation (2.17), except truncated at lag $\bar{K}$. Thus the DL-MIDAS regression provides an *exact* fit for the linear projection emerging from the steady state Kalman filter for sufficiently large lag-length $\bar{K}$ (assuming the remaining weights to be negligible).

Consider now the general case of the model with n variables and n_f factors given by equations (2.1) - (2.5) with only the high-frequency data being used for forecasting. Let $\mathbf{K}$ denote the steady state Kalman gain and:

$$\mathbf{Z} = \mathbf{Z}_1 = \begin{bmatrix} \boldsymbol{\gamma}_2' & & & \\ \vdots & \mathbf{O}_{(n-1)\times n_f(p-1)} & \tilde{\mathbf{I}}_{n-1} & \mathbf{O}_{(n-1)\times n(k-1)} \\ \boldsymbol{\gamma}_n' & & & \end{bmatrix}$$

Then (2.12) reduces to

$$\hat{\boldsymbol{\alpha}}_{t|t} = \rho^{mh} \sum_{j=0}^{\infty} (\mathbf{G} - \mathbf{KZG})^j \mathbf{K} \begin{pmatrix} x_{2,t-j/m} \\ \dots \\ x_{n,t-j/m} \end{pmatrix} \tag{2.20}$$

and $E_t[y_{t+h}] = \mathbf{Z}_{m,1}\mathbf{G}^{mh}\hat{\boldsymbol{\alpha}}_{t|t}$, where $\mathbf{Z}_{m,1}$ denotes the first row of the matrix $\mathbf{Z}_m$. This is not exactly a MIDAS regression, but may be well approximated by one — a possibility to which we will return in section 3.

Turning back to the single factor model considered in this subsection, as in equation (2.15), but now assuming many high-frequency series all with uncorrelated measurement noise,

equation (2.20) yields the following interesting result:

$$\hat{y}_{t+h|t} = \rho^{mh} \sum_{j=0}^{\infty} (\rho - \varphi\rho)^j \mathbf{K} \begin{pmatrix} x_{2,t-j/m} \\ \dots \\ x_{n,t-j/m} \end{pmatrix} \tag{2.21}$$

where φ is a scalar and the (1,1) element of the product $\mathbf{KZ}$. We can write this more explicitly as a forecast combination:

$$\hat{y}_{t+h|t} = \sum_{i=2}^{n} (\rho^{mh} \kappa_i) \sum_{j=0}^{\infty} (\rho - \varphi\rho)^j x_{i,t-j/m} \tag{2.22}$$

where $\mathbf{K} = (\kappa_2, \dots, \kappa_n)$.

The above Kalman filter-based prediction can be thought of (in population) as a forecast combination specification, in which the forecast using the ith predictor is given weight $\rho^{mh}\kappa_i$. This is interesting as typically large cross-sections of (financial) high-frequency data are available. The use of forecast combinations generated by MIDAS regressions is in fact advocated by Andreou, Ghysels, and Kourtellos (2008), as one way to handle large cross-sections of daily financial variables. It is interesting to note that (1) the weights relate to the Kalman filter gains and (2) the MIDAS regression polynomials across individual series are constrained to have the same decay profile determined by $\rho - \varphi\rho$. Hence, here again, the DL-MIDAS involving exponential Almon lags provides an exact mapping with $\theta_2 = 0$ and $\theta_1 = \ln(\rho - \rho\varphi)$. The common decay across high-frequency series is of course not imposed in a forecast combination setting—which therefore results in estimation efficiency losses since the DL-MIDAS regressions are estimated with each individual high-frequency series separately. The forecast combination scheme in equation (2.22) is reminiscent of the seminal work by Bates and Granger (1969) who advocated forecast combination method based on variance/covariance properties of forecast errors.[4] It is also worth noting that the above result no longer holds when the individual series involve autocorrelated measurement noise, as in equation (2.3). Here again, DL-MIDAS will provide only an approximation.

[4]There is a substantial literature on forecast combinations - see Timmermann (2006) for an excellent recent survey of forecast combination methods.

2.3 Using Both Low- and High-Frequency Data and the ADL-MIDAS Regression Model

We will start again with the simple example appearing in the previous subsection, yet this time we also take into account past low-frequency measurements of y. For the sake of simplicity we consider the quarterly/monthly data combination ($m = 3$). Hence, we are interested in for instance $E\left[y_{t+h}|I_t^M\right]$, where I_t^M is the mixed data set of past low (quarterly) and high (monthly) frequency data, instead of the linear projection only involving high-frequency data as in equation (2.17). In the latter case we obtained a standard (aperiodic) steady state equation driving the linear projection. Here, however, we deal with a periodic Kalman filter as in subsection 2.1 applied to the model consisting of equations (2.13), (2.14) and (2.15). Then the periodic Kalman gain matrices are:

$$\mathbf{K}_{2|1} = \begin{pmatrix} \kappa_1 \\ * \\ * \end{pmatrix}, \mathbf{K}_{3|2} = \begin{pmatrix} \kappa_2 \\ * \\ * \end{pmatrix} \text{ and } \mathbf{K}_{1|3} = \begin{pmatrix} \kappa_{3,1} & \kappa_{3,2} \\ * & * \\ * & * \end{pmatrix},$$

where " $*$ " denotes some element that does not need to be explicitly named. In addition, let us write $\kappa_3 = \kappa_{3,1} + \kappa_{3,2}$. The state vector is $\boldsymbol{\alpha}_{t+j/m} = (f_{t+j/m}, u_{1,t+j/m}, u_{2,t+j/m})'$, and we have

$$\mathbf{G} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the first rows of the matrices $\tilde{\mathbf{A}}_1^3$, $\tilde{\mathbf{A}}_2^3$ and $\tilde{\mathbf{A}}_3^3$ are $((\rho - \rho\kappa_1)(\rho - \rho\kappa_2)(\rho - \rho\kappa_3), 0, ...0)$, $((\rho - \rho\kappa_2)(\rho - \rho\kappa_3), 0, ...0)$ and $(\rho - \rho\kappa_3, 0, ...0)$, respectively. From equation (2.12) it then follows that:

$$E\left[y_{t+h}|I_t^M\right] = \rho^{3h} f_{t|t} = \rho^{3h}\kappa_{3,1}\sum_{j=0}^{\infty}\vartheta^j y_{t-j} + \rho^{3h}\sum_{j=0}^{\infty}\vartheta^j x(\theta_x)_{t-j} \tag{2.23}$$

where $\vartheta = [(\rho - \rho\kappa_1)(\rho - \rho\kappa_2)(\rho - \rho\kappa_3)]$, and

$$x(\theta_x)_t \equiv [\kappa_{3,2} + (\rho - \rho\kappa_3)\kappa_2 L^{1/3} + (\rho - \rho\kappa_3)(\rho - \rho\kappa_2)\kappa_1 L^{2/3}]x_t \tag{2.24}$$

which is a parameter-driven low-frequency process composed of high-frequency data aggregated at the quarterly level.

The above equation relates to the multiplicative MIDAS regression models considered by Chen and Ghysels (2011) and Andreou, Ghysels, and Kourtellos (2008). In particular consider the following ADL-MIDAS regression:

$$y_{t+h} = \beta_y \sum_{j=0}^{K_y} w_j(\theta_y) y_{t-j} + \beta_x \sum_{j=0}^{K_x} w_j(\theta_x^1) x(\theta_x^2)_{t-j} + \varepsilon_{t+h} \tag{2.25}$$

where $w_j(\theta_y)$, $w_j(\theta_x^1)$ follow an exponential Almon scheme and

$$x(\theta_x^2)_t \equiv \sum_{k=0}^{m-1} w_k(\theta_x^2) L^{k/m} x_{t-k/m} \tag{2.26}$$

also follows an exponential Almon scheme. Provided that $\rho > 0$, equations (2.23) and (2.24) are a special case of this model with $K_y = K_x = \infty$, $w_j(\theta_y) \propto \exp(\log(\vartheta)j)$, $w_j(\theta_x^1) \propto \exp(\log(\vartheta)j)$ and $w_k(\theta_x^2) \propto \exp(\theta_{x,1}^2 k + \theta_{x,2}^2 k^2)$ where $\theta_{x,1}^2$ and $\theta_{x,2}^2$ are parameters that solve the equations

$$\log\{(\rho - \rho\kappa_3)\kappa_2/\kappa_{3,2}\} = \theta_{x,1}^2 + \theta_{x,2}^2$$

$$\log\{(\rho - \rho\kappa_3)(\rho - \rho\kappa_2)\kappa_1/\kappa_{3,2}\} = 2\theta_{x,1}^2 + 4\theta_{x,2}^2$$

This constructed low-frequency regressor is estimated jointly with the other (MIDAS) regression parameters. Hence, one can view $x(\theta_x^2)_t$ as the best *aggregator* that yields the best *prediction*. This ADL-MIDAS regression involves more parameters than the usual specification involving only one polynomial. The multiplicative specification was originally suggested in Chen and Ghysels (2011) to handle seasonal patterns (in their case the intra-daily seasonal of volatility patterns). Comparing equations (2.23) and (2.25) again yields an exact mapping, if $\rho > 0$.

Similar to the previous subsection, let us also consider the case of multiple high-frequency series. Then the periodic structure of the Kalman gain becomes:

$$\mathbf{K}_{2|1} = \begin{pmatrix} \kappa_{1,2} & \cdots & \kappa_{1,n} \\ * & \cdots & * \\ * & \cdots & * \end{pmatrix}, \mathbf{K}_{3|2} = \begin{pmatrix} \kappa_{2,2} & \cdots & \kappa_{2,n} \\ * & \cdots & * \\ * & \cdots & * \end{pmatrix} \text{ and } \mathbf{K}_{1|3} = \begin{pmatrix} \kappa_{3,1} & \cdots & \kappa_{3,n} \\ * & \cdots & * \\ * & \cdots & * \end{pmatrix},$$

where again "$*$" denotes some element that does not need to be explicitly named. Moreover, we also denote $\kappa_i = \sum_{j=2}^{n} \kappa_{i,j}$ for $i = 1, 2$ and $\kappa_3 = \sum_{j=1}^{n} \kappa_{3,j}$. Algebraic derivations similar to the single high-frequency series case yield:

$$E\left[y_{t+h}|I_t^M\right] = \rho^{3h} f_{t|t} = \rho^{3h}\kappa_{3,1}\sum_{j=0}^{\infty}\vartheta^j y_{t-j} + \rho^{3h}\sum_{i=2}^{n}\sum_{j=0}^{\infty}\vartheta^j x(\theta_{i,x})_{i,t-j} \tag{2.27}$$

with similar expressions for ϑ and $x(\theta_{i,x})_{i,t}$. As in the previous case this is again reminiscent of forecasting combinations — involving ADL-MIDAS regressions. In fact, the empirical applications appearing in Andreou, Ghysels, and Kourtellos (2008) actually involves such regression models — rather than the DL-MIDAS discussed before.[5] Note again that the low-frequency decay patterns are identical across the different within-period-aggregated high-frequency series $x(\theta_{i,x})_{i,t}$. This means that estimating ADL-MIDAS regressions one at a time — as is typical in forecast combination settings — involves efficiency losses compared to the systems-based Kalman filter.

Finally, it should also be noted that the appearance of an aggregator series $x(\theta_x)_{i,t}$ is not restricted to cases where $m = 3$. Indeed, it is straightforward to show that the within-period aggregation scheme applies to any sampling frequency combination.

3 Approximation and Specification Errors

From the previous section we know that the mapping between the Kalman filter and MIDAS regressions can be exact. We now analyze cases where the MIDAS regression is instead only an approximation. The purpose of this section is to assess the accuracy of a population approximation to the Kalman filter obtained from a MIDAS regression.

We will focus on two cases where MIDAS regressions do not yield an exact mapping with the Kalman filter. A subsection is devoted to each case. The first is a one-factor state space model with measurement errors that are serially correlated over time. The second is a two-factor state space model. The final subsection covers specification errors.

[5] The appearance is perhaps not so direct — recall however that in the ADL-MIDAS we force the weights of $x(\theta_{i,x})_{i,t}$, to add up to one for the purpose of identification. This means that a weight is attached in front of the MIDAS polynomial proper to each individual series. These weights can be viewed as forecast combination weights - yet they do not relate in any straightforward manner to the Bates and Granger scheme discussed earlier.

3.1 One-Factor State Space Model versus MIDAS

We start again with the example of a single factor AR(1) model in equation (2.13) with a single high-frequency series appearing in Section 2.2. But now we allow for persistence in the measurement errors, and use both high- and low-frequency data for forecasting. We again consider the quarterly-monthly data combination ($m = 3$), without loss of generality. This yields ($\forall$ t and $j = 1, \ldots, m$):

$$\begin{aligned} f_{t+j/m} &= \rho f_{t+(j-1)/m} + \eta_{t+j/m} \\ y^*_{t+j/m} &= \gamma_1 f_{t+j/m} + u_{1,t+j/m} \\ x_{t+j/m} &= \gamma_2 f_{t+j/m} + u_{2,t+j/m} \end{aligned} \tag{3.1}$$

where

$$u_{i,t+j/m} - d_i u_{i,t+(j-1)/m} = \epsilon_{i,t+j/m} \qquad i = 1, 2. \tag{3.2}$$

Then the periodic Kalman gain matrices are:

$$\mathbf{K}_{2|1} = \begin{pmatrix} \kappa^1_1 \\ \kappa^1_2 \\ \kappa^1_3 \end{pmatrix}, \mathbf{K}_{3|2} = \begin{pmatrix} \kappa^2_1 \\ \kappa^2_2 \\ \kappa^2_3 \end{pmatrix} \text{ and } \mathbf{K}_{1|3} = \begin{pmatrix} \kappa^3_{1,1} & \kappa^3_{1,2} \\ \kappa^3_{2,1} & \kappa^3_{2,2} \\ \kappa^3_{3,1} & \kappa^3_{3,2} \end{pmatrix}.$$

The state vector is $\boldsymbol{\alpha}_{t+j/m} = (f_{t+j/m}, u_{1,t+j/m}, u_{2,t+j/m})'$ and we have

$$\mathbf{G} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_2 \end{pmatrix},$$

$$\mathbf{Z}_j = \begin{pmatrix} \gamma_2 & 0 & 1 \end{pmatrix} \qquad 1 \leq j \leq m-1$$

$$\mathbf{Z}_m = \begin{pmatrix} \gamma_1 & 1 & 0 \\ \gamma_2 & 0 & 1 \end{pmatrix}$$

Correspondingly, since $\mathbf{A}_{j|j-1} = \mathbf{G} - \mathbf{K}_{j|j-1}\mathbf{Z}_j\mathbf{G}$, we can compute $\mathbf{A}_{2|1}$, $\mathbf{A}_{3|2}$ and $\mathbf{A}_{1|3}$ appearing respectively in equations (A.1) through (A.3) in Appendix A. Using these matrices we can compute the Kalman filter equation for h-quarter-ahead prediction, a long expression appearing in equation (A.4) also in Appendix A. To simplify notation, write the Kalman

filter prediction as:

$$E_{KF}(y_{t+h}|I_t^M) = \sum_{j=0}^{\infty} w_{y,j}^{KF} y_{t-j} + \sum_{j=0}^{\infty} w_{x,j}^{KF} x_{t-j/m} \tag{3.3}$$

and the corresponding MIDAS regression as:

$$E_{Mds}(y_{t+h}|I_t^M) = \sum_{j=0}^{\bar{K}} w_{y,j}^{Mds} y_{t-j} + \sum_{j=0}^{3\bar{K}} w_{x,j}^{Mds} x_{t-j/m} \tag{3.4}$$

We will consider two types of MIDAS regression specifications, both relate to the above regression as follows: a multiplicative scheme referring to the ADL-MIDAS regression appearing in equation (2.25) with $K_y = K_x = \bar{K}$, and a 'regular' MIDAS regression which does not involve the aggregator scheme, but instead has a single polynomial specification for the high-frequency data, namely:

$$y_{t+h} = \beta_y \sum_{j=0}^{\bar{K}} w_j(\theta_y) y_{t-j} + \beta_x \sum_{j=0}^{3\bar{K}} w_j(\theta_x) x_{t-j/m} + \varepsilon_{t+h} \tag{3.5}$$

where $w_j(\theta_y)$ and $w_j(\theta_x)$ are both distributed lags of the form of equation (2.19). We will compare the models using two criteria. The first is the prediction error minimization. Assuming that the Kalman Filter weights are negligible beyond lag length $\bar{K}$, let $\mathbf{\Sigma}_{xy}$ denote the variance-covariance matrix of $\boldsymbol{\xi}_t = (x_{t+h}, y^*_{t+h}, x_{t+h-1/m}, y^*_{t+h-1/m}, ..., x_{t-\bar{K}}, y^*_{t-\bar{K}})'$, the elements of which are as follows:

$$Cov(y^*_{t-i/m}, y^*_{t-j/m}) = \gamma_1^2 \frac{\rho^{|i-j|}\sigma_\eta^2}{1-\rho^2} + \frac{d_1^{|i-j|}\sigma_y^2}{1-d_1^2}$$

$$Cov(x_{t-i/m}, x_{t-j/m}) = \gamma_2^2 \frac{\rho^{|i-j|}\sigma_\eta^2}{1-\rho^2} + \frac{d_2^{|i-j|}\sigma_x^2}{1-d_2^2}$$

$$Cov(x_{t-i/m}, y^*_{t-j/m}) = \gamma_1\gamma_2 \frac{\rho^{|i-j|}\sigma_\eta^2}{1-\rho^2}$$

for $i, j = -3h, -3h+1, -3h+2, \ldots, 3\bar{K}$, where $\sigma_\eta^2 = Var(\eta_t)$, $\sigma_y^2 = Var(\varepsilon_{1,t})$ and $\sigma_x^2 = Var(\varepsilon_{2,t})$. Then, the h-quarter-ahead Kalman Filter prediction error can be written as $\boldsymbol{w}'_{KF}\boldsymbol{\xi}_t$ where the vector of weights $\boldsymbol{w}_{KF}$ is shown at the end of Appendix A and the variance of the prediction error is $\boldsymbol{w}'_{KF}\mathbf{\Sigma}_{xy}\boldsymbol{w}_{KF}$. Similarly, the corresponding MIDAS prediction error

variance is $\boldsymbol{w}'_{Mds}\boldsymbol{\Sigma}_{xy}\boldsymbol{w}_{Mds}$, with $\boldsymbol{w}_{Mds}$ also at the end of the aforementioned Appendix.

We choose the MIDAS parameters to minimize the variance of prediction errors, that is:

$$min\ \boldsymbol{w}'_{Mds}\boldsymbol{\Sigma}_{xy}\boldsymbol{w}_{Mds} \tag{3.6}$$

It will be convenient to report the results in relative terms, namely the ratio of prediction error variances (we will refer to as PE distance):

$$\frac{\boldsymbol{w}'_{KF}\boldsymbol{\Sigma}_{xy}\boldsymbol{w}_{KF}}{\boldsymbol{w}'_{Mds}\boldsymbol{\Sigma}_{xy}\boldsymbol{w}_{Mds}} \tag{3.7}$$

with the MIDAS parameters minimizing (3.6) and the Kalman filter weights maximizing the likelihood function. An alternative measure that we also consider is the minimized value of an $\mathbb{L}^2$ distance between the weights:

$$\mathbb{L}^2 \equiv \sum_{j=0}^{3\bar{K}}(w_{x,j}^{KF} - w_{x,j}^{Mds})^2 + \sum_{j=0}^{\bar{K}}(w_{y,j}^{KF} - w_{y,j}^{Mds})^2 \tag{3.8}$$

Table 1 shows the minimized values of $\mathbb{L}^2$ comparing Kalman Filter and MIDAS regressions (regular and multiplicative), with $d = d_1 = d_2$, $\gamma_1 = \gamma_2 = 1$ and $\sigma_\eta^2 = \sigma_y^2 = \sigma_x^2 = 1$. Results are shown for combinations of d and ρ, and the forecast horizons $h = 1$ (Panel A) and 4 (Panel B) quarters ahead. Both panels cover the monthly/quarterly sampling mix, i.e. $m = 3$. Panels C and D cover the quarterly/weekly mix with $m = 13$.[6] We do not actually report the results for the prediction error distances as they are easy to summarize — for all combinations of d and ρ, forecast horizon and sampling frequency combinations the MIDAS and Kalman filter-based predictions are for all practical purposes identical, i.e. the value of the PE distance is numerically extremely close to one.

For $d = 0$ and $\rho > 0$, the multiplicative MIDAS provides, by construction, a perfect fit to the Kalman Filter, and so both distance measures are equal to zero. In contrast to the multiplicative MIDAS, we do not expect the fit with the regular specification to be exact. Yet the results in Table 1 show that the differences between the regular MIDAS and Kalman filter weights are also negligible. For other combinations of d and ρ we occasionally observe some significant differences. However, they are concentrated around the extreme values for either

[6]The comment in footnote 3 about fixed m applies here too, i.e. both Kalman filtering and MIDAS regressions can handle time-varying weekly/quarterly calendar effects.

d or ρ (-0.9 or 0.95). For all other entries to Table 1 the differences between MIDAS weights and the Kalman filter ones are small. The multiplicative MIDAS specification generally yields smaller errors than regular MIDAS. This is somewhat expected since the former provides an exact match for some parameter combinations. It is also worth noting that the impact of forecast horizon appears to be small, judging by the differences between $h = 1$ and 4 in Table 1. In contrast, Panels C and D show that increasing m from 3 to 13 uniformly reduces the $\mathbb{L}^2$ distances.

In Table 2, we turn to the forecast combination issue. Namely, consider the following system:

$$\begin{aligned} f_{t+j/m} &= \rho f_{t+(j-1)/m} + \eta_{t+j/m} \\ y^*_{t+j/m} &= \gamma_1 f_{t+j/m} + u_{1,t+j/m} \\ x_{1,t+j/m} &= \gamma_2 f_{t+j/m} + u_{2,t+j/m} \\ x_{2,t+j/m} &= \gamma_2 f_{t+j/m} + u_{3,t+j/m} \end{aligned} \tag{3.9}$$

Hence, we have two high-frequency series and we examine cases where $var(u_{2,t+j/m}) = var(u_{3,t+j/m})$ and cases where $var(u_{2,t+j/m}) = var(u_{3,t+j/m})/10$, which we call respectively equal and unequal noise variance cases in the Table. We also vary m, namely Panels A and B pertain to $m = 3$, while C and D cover $m = 13$. All four assume the forecast horizon $h = 1$. The results in Table 2 indicate that forecast combinations with MIDAS regressions work well and achieve the same weighting as the Kalman filter. We report again only the $\mathbb{L}^2$ distance measure results as the PE distances are almost equal to one. Comparisons between Tables 1 and 2 also allow us to appraise the effect of increasing the number of high-frequency series. We note a slight deterioration of the $\mathbb{L}^2$ distance as we add another high-frequency series. Typically that effect seems negligible, though. Moreover, moving from $m = 3$ to 13 also improves the fit.

3.2 Two-Factor State Space Model versus MIDAS

We also consider cases where the MIDAS regression is only an approximation. To do so, we specify a two-factor state space model:

$$\begin{aligned}
\mathbf{F}_{t+j/m} &= \begin{pmatrix} f_{1,t+j/m} \\ f_{2,t+j/m} \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} f_{1,t+(j-1)/m} \\ f_{2,t+(j-1)/m} \end{pmatrix} + \begin{pmatrix} \eta_{1,t+j/m} \\ \eta_{2,t+j/m} \end{pmatrix} \\
y^*_{t+j/m} &= \gamma_{1,1} f_{1,t+j/m} + \gamma_{1,2} f_{2,t+j/m} + u_{1,t+j/m} = \boldsymbol{\gamma}_1' \mathbf{F}_{t+j/m} + u_{1,t+j/m} \\
x_{2,t+j/m} &= \gamma_{2,1} f_{1,t+j/m} + \gamma_{2,2} f_{2,t+j/m} + u_{2,t+j/m} = \boldsymbol{\gamma}_2' \mathbf{F}_{t+j/m} + u_{2,t+j/m} \qquad (3.10)
\end{aligned}$$

where

$$u_{i,t+j/m} - d u_{i,t+(j-1)/m} = \epsilon_{i,t+j/m} \qquad i = 1, 2 \qquad (3.11)$$

Then the periodic Kalman gain matrices are:

$$\mathbf{K}_{1|0} = \begin{pmatrix} \kappa_1^1 \\ \kappa_2^1 \\ \kappa_3^1 \\ \kappa_4^1 \end{pmatrix}, \mathbf{K}_{2|1} = \begin{pmatrix} \kappa_1^2 \\ \kappa_2^2 \\ \kappa_3^2 \\ \kappa_4^2 \end{pmatrix} \text{ and } \mathbf{K}_{3|1} = \begin{pmatrix} \kappa_{1,1}^3 & \kappa_{1,2}^3 \\ \kappa_{2,1}^3 & \kappa_{2,2}^3 \\ \kappa_{3,1}^3 & \kappa_{3,2}^3 \\ \kappa_{4,1}^3 & \kappa_{4,2}^3 \end{pmatrix},$$

The state vector is $\boldsymbol{\alpha}_{t+j/m} = (f_{1,t+j/m}, f_{2,t+j/m}, u_{1,t+j/m}, u_{2,t+j/m})'$ and we have

$$\mathbf{G} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{pmatrix},$$

$$\mathbf{Z}_j = \begin{pmatrix} \gamma_{2,1} & \gamma_{2,2} & 0 & 1 \end{pmatrix} 1 \leq j \leq m$$

$$\mathbf{Z}_m = \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} & 1 & 0 \\ \gamma_{2,1} & \gamma_{2,2} & 0 & 1 \end{pmatrix} j = 0.$$

Correspondingly, since $\mathbf{A}_{j|j-1} = \mathbf{G} - \mathbf{K}_{j|j-1} \mathbf{Z}_j \mathbf{G}$, we can compute again $\mathbf{A}_{1|0}$, $\mathbf{A}_{2|1}$ and $\mathbf{A}_{3|2}$ appearing respectively in equations (B.1) through (B.3) in Appendix B.

$$\begin{aligned}
E(y_{t+h}|I_t^M) &= E(\gamma_{1,1} f_{1,t+h} + \gamma_{1,2} f_{2,t+h} + u_{1,t+h}|I_t^M) \\
&= \gamma_{1,1} \rho_1^{3h} E(f_{1,t}|I_t^M) + \gamma_{1,2} \rho_2^{3h} E(f_{2,t}|I_t^M) + d_1^{3h} E(u_{1,t}|I_t^M),
\end{aligned}$$

and so we have:

$$E(y_{t+h}|I_t^M) = \begin{pmatrix} \gamma_{1,1}\rho_1^{3h} & \gamma_{1,2}\rho_2^{3h} & d_1^{3h} & 0 \end{pmatrix} \hat{\alpha}_{t|t}$$

This gives a Kalman filter prediction that can be written as

$$E_{KF}(y_{t+h}|I_t^M) = \sum_{j=0}^{\infty} w_{y,j}^{KF} y_{t-j} + \sum_{j=0}^{\infty} w_{x,j}^{KF} x_{t-j/m}$$

As in the previous subsection, we can find the regular or multiplicative MIDAS parameters using the objective functions given in equations (3.6) and (3.8). In the two-factor model that we consider, the elements of $\mathbf{\Sigma}_{xy}$, the variance-covariance matrix of $(x_{t+h}, y^*_{t+h}, x_{t+h-1/m}, y^*_{t+h-1/m}, \ldots, x_{t-\bar{K}}, y^*_{t-\bar{K}})'$, are as follows:

$$Cov(y^*_{t-i/m}, y^*_{t-j/m}) = \gamma_{1,1}^2 \frac{\rho_1^{|i-j|}\sigma_{\eta,1}^2}{1-\rho_1^2} + \gamma_{1,2}^2 \frac{\rho_2^{|i-j|}\sigma_{\eta,2}^2}{1-\rho_2^2} + \frac{d_1^{|i-j|}\sigma_y^2}{1-d_1^2}$$

$$Cov(x_{t-i/m}, x_{t-j/m}) = \gamma_{2,1}^2 \frac{\rho_1^{|i-j|}\sigma_{\eta,1}^2}{1-\rho_1^2} + \gamma_{2,2}^2 \frac{\rho_2^{|i-j|}\sigma_{\eta,2}^2}{1-\rho_2^2} + \frac{d_2^{|i-j|}\sigma_x^2}{1-d_2^2}$$

$$Cov(x_{t-i/m}, y^*_{t-j/m}) = 2\gamma_{1,1}\gamma_{2,1} \frac{\rho_1^{|i-j|}\sigma_{\eta,1}^2}{1-\rho_1^2} + 2\gamma_{1,2}\gamma_{2,2} \frac{\rho_2^{|i-j|}\sigma_{\eta,2}^2}{1-\rho_2^2}$$

for $i, j = -3h, -3h+1, -3h+2, \ldots, 3\bar{K}$, where $\sigma_{\eta,1}^2 = Var(\eta_{1,t})$, $\sigma_{\eta,2}^2 = Var(\eta_{2,t})$, $\sigma_y^2 = Var(\varepsilon_{1,t})$ and $\sigma_x^2 = Var(\varepsilon_{2,t})$.

To save space, we do not report the results in a table, as they are quite similar to those reported in Table 1. There are however a few differences with the results for the one-factor case. First, for $d = 0$ and $\rho > 0$, multiplicative MIDAS is no longer a perfect fit to the Kalman Filter. Yet, we find again that the fit is for all practical purposes identical as in the one-factor case. This also applies to the regular MIDAS specification. Second, differences between the multiplicative and regular specifications for the extremes in the parameter space with regards to persistence in the factors and/or measurement errors, are smaller than in the one-factor case considered in Table 1.

3.3 Specification Errors

All the models considered so far are correctly specified, and so the MIDAS regression cannot hope to do better than the Kalman filter, in population at least. However, this is not true any more if the state space model is mis-specified. Accordingly in this subsection, we consider the case in which the Kalman filter weights are computed assuming that the data are generated by a one-factor model, whereas in fact the data are generated by a two-factor model. The MIDAS regressions are selected so as to approximate the data generating process minimizing the objective functions (3.6) or (3.8) with respect to a two-factor model.

More specifically, the mis-specified state space model in this case is that appearing in equations (3.1) and (3.2). Hence, we let the six parameters ρ, γ_1, γ_2, and the three error variances appearing in those equations determine how close a fit a one-factor model is to the correctly specified two-factor model. We pick the MIDAS parameters also according to either one of the objectives: (3.6) or (3.8).

Table 3 compares the MIDAS regression with parameters chosen in (3.6) and the Kalman Filter in terms of PE distances (equation (3.7)). The structure of the table is similar to that of Table 1, except that we report only results for the regular MIDAS (the results for the multiplicative case are similar). The results tell us that mis-specified state space models and MIDAS regressions generally perform roughly similarly, as many PE distances are close to one. For extremes in the parameter space, either of the persistence of the factor (ρ), or of the persistence of the measurement error (d), the Kalman filter performs better than the MIDAS regression—despite being mis-specified (the PE distance being smaller than one).

In the next and last subsection we turn our attention to simulation results, in which we study finite sample behavior via Monte-Carlo experiments. These will also be useful to examine to what extent the findings we have reported so far also apply in a small-sample setting.

3.4 Monte Carlo Simulations

We consider three Monte-Carlo designs. The first specifies that the true data generating process is a one-factor model given by equations (3.1) and (3.2) with $\gamma_1 = \gamma_2 = 1$, $d_1 = d_2 = d$, and where the errors $\{\eta_{t+j/m}\}$ and $\{\varepsilon_{i,t+j/m}\}$ are all independent standard normal random variables. In each Monte-Carlo simulation, we generate T draws of the low-frequency series $\{y_t\}$ and Tm draws of the high-frequency series $x_{t+j/m}$. We then consider forecasting y_{t+h}

(h periods ahead, measured in the units of time of the low-frequency series) using three different methods: least-squares estimation of the regular MIDAS regression, least-squares estimation of the multiplicative MIDAS regression, and maximum-likelihood estimation of the one-factor Kalman filter (a model that is correctly specified in this design). The sample size is $T = 40$ and we consider values of m equal to 3 (which we think of as quarterly-monthly mixes).

Table 4, Panels A and B, reports the simulated root-mean-square prediction error (RMSPE) from the Kalman filter, relative to the RMSPE from the two MIDAS regressions. Results are shown for different values of d (persistence of the measurement error) and ρ (persistence of the factor). All entries in Table 4 are a little below 1, indicating that the Kalman filter gives slightly more accurate predictions than either MIDAS regression, uniformly in the parameter space. As the Kalman filter is correctly specified in this design, it is not surprising that maximum-likelihood estimation of this model gives the best forecasts. The magnitude of the improvement from the Kalman filter is up to about 20 percent.

In the second Monte-Carlo design, the data generating process is a two-factor model given by equations (3.10) and (3.11) where the errors $\{\eta_{i,t+j/m}\}$ and $\{\varepsilon_{i,t+j/m}\}$ are all independent standard normal random variables, with $\gamma_{1,1} = 0.9$, $\gamma_{1,2} = 0.1$, $\gamma_{2,1} = 0.1$ and $\gamma_{2,2} = 0.9$. As before, we consider forecasting y_{t+h} using three different methods: the regular MIDAS regression, the multiplicative MIDAS regression, and maximum-likelihood estimation of the one-factor Kalman filter. But notice that the one-factor state space model is now mis-specified. The setup is therefore the small-sample analog of the asymptotic results for the mis-specified case considered in Table 3 above.

Table 4, Panels C and D, reports the simulated RMSPE from the Kalman filter relative to the RMSPE from the two MIDAS regressions for $T = 40$, $m = 3$ and for different values of d and ρ. Most entries in panels C and D are a little below 1, indicating that the Kalman filter again gives slightly more accurate predictions than either MIDAS regression. But for $h = 1$, when d and ρ are of large absolute magnitude but opposite sign, the ratios of RMSPEs are actually above 1, meaning that the MIDAS regressions (either regular or multiplicative) are more accurate than the mis-specified one-factor Kalman filter. For example, with $h = 1$, $d = -0.5$ and $\rho = 0.95$, the regular MIDAS gives predictions that are 15 percent more accurate (in RMSPE terms) than the Kalman filter. Thus the combination of a small sample with mis-specification of the state space model can cause the MIDAS regression to give better forecasts than the Kalman filter.

In the third and final Monte-Carlo design, the data generating process is again the one-factor model given by equations (3.1) and (3.2) with $\gamma_1 = \gamma_2 = 1$, $d_1 = d_2 = d$, and where the errors $\{\eta_{t+j/m}\}$ and $\{\varepsilon_{i,t+j/m}\}$ are all independent standard normal random variables. The MIDAS regressions are considered as above, but the Kalman filter is now applied to either a one- or two-factor state space model, depending on which gives the higher value of the Akaike Information Criterion (AIC) or Bayes Information Criterion (BIC). The simulation design thus leaves open the possibility of the state space model being over-specified, which may affect its performance in finite samples, although not in population.

Table 5 reports the simulated RMSPE from the Kalman filter relative to the RMSPE from the two MIDAS regressions for $T = 40$, $m = 3$ and for different values of d and ρ. Panels A and C show the results using AIC (at one- and four-quarter forecasting horizons, respectively). Because AIC has a tendency to overfit, it often selects the over-specified two-factor state space model. This hurts the finite-sample forecasting performance of the state space model, quite considerably in some cases. In the most extreme case, the state space model using AIC gives a RMSPE that is 62 percent higher than the RMSPE from the regular MIDAS regression. On the other hand, the BIC is more parsimonious, and nearly always correctly picks the single-factor model. As a result, the forecasting performance of the Kalman filter with BIC is a bit better than the predictive accuracy from either MIDAS regression, in almost all the simulations considered here.

4 Empirical Study

As an illustration of the theoretical results in sections 2 and 3, we present an empirical application to forecasting of U.S. GDP growth. In a first subsection we describe the data. The results are discussed in a second subsection.

4.1 The Data

We use a dataset with mixed frequencies: monthly and quarterly. The quarterly variable to be predicted is the growth rate of real GDP from 1959Q1 to 2009Q1. The explanatory variables include nine monthly indicators until May 2009. In particular, we consider the term spread (TERM), stock market returns (SP500), industrial production (IP), employment

(Emply), consumer expectations (Exptn), personal income (PI), the leading index (LEI), manufacturing (Manu), and oil prices (Oil). They are transformed to induce stationarity and to ensure that the transformed variables correspond to the real GDP growth observed at the end of the quarter. See Table 6 for more details on the definition and data transformations.[7] It should also be noted that we focus exclusively on one-factor state space models.

Each model uses just one out of nine monthly indicators. The forecasts are in all cases made using monthly data up to and including the second month of the quarter. We evaluate the state space and MIDAS forecasts in a standard recursive prediction exercise. The first estimation window is from 1959:Q1 to 1978:Q4, and is recursively expanded over time. For example, for MIDAS, a one-step-ahead forecast of 1979:Q1 is generated from regressing GDP growth up to 1978:Q4 on its own lags and the monthly predictor up to 1978:11 (November). Then the values of GDP growth through 1978:Q4 and of the monthly predictor up to 1979:02 (February) are used with the estimated coefficients to predict the 1979:Q1 GDP growth rate. We also do two- to eight-quarter-ahead forecasting in a similar fashion. The evaluation sample is from 1979:Q1 to 2009Q1. Some monthly predictors are available only for more recent subsamples (e.g. crude oil price and manufacturing). In these cases, we use the first 40 quarters as the estimation sample and the remaining period until 2009Q1 as the evaluation sample. We should also note that—as usually is done in the context of state space models, all series are normalized by the (full sample) mean and variance.

In line with Kuzin, Marcellino, and Schumacher (2011), we specify the lag order in the mixed-frequency state space model by applying the Bayesian information criterion (BIC) with a maximum lag order of $p = 4$ months. We also find that the chosen lag lengths are usually small with only one or two lags in most cases. In both the regular and multiplicative MIDAS model, we set the maximum number of lags as $K_y = 1$ and $K_x = 6$ quarters and choose the lag length by the minimum in-sample fitting error criterion. Finally, we use the root mean squared forecasting error (RMSE) to evaluate each model's forecasting accuracy:

$$RMSE(h) = \sqrt{\frac{1}{T_2 - T_1 - h + 1} \sum_{t=T_1}^{T_2-h} (\widehat{Y}_{t+h} - Y_{t+h})^2},$$

where the model is estimated for the period of $t = [1, T_1]$, and the forecasting period is given

[7] Note that, because real-time vintages for all the series in the panel are not available, we did not perform a pure real-time forecasting exercise. Authors such as Bernanke and Boivin (2003) and Schumacher and Breitung (2008) find that data revisions have limited impact on forecasting accuracy for economic activity.

by $t = [T_1 + h, T_2]$.

4.2 Forecasting Results

Table 7 compares the forecasting performance between the regular MIDAS, multiplicative MIDAS and state space models. We consider horizons from one quarter up to two years. Recall that all the series are normalized by the (full sample) mean and variance, including real GDP growth. So the root mean squared forecasting errors reported in Table 7 are in standard deviation units. We report the level of root mean squared forecasting errors for state space models (denoted m0), and for regular MIDAS (denoted m1) and multiplicative MIDAS (denoted m2). In addition, we also report the ratios (m0/m1) and (m0/m2). When we see entries for ratios of say 0.80, we can interpret this as gains equivalent to 20 % of the full sample standard deviation of GDP growth. The ratios above one imply that MIDAS regressions produce better forecasts. Conversely, ratios below one imply that the Kalman filter produces better forecasts.

When we consider the various series reported in Table 7, we see that MIDAS gives better forecasts when the term spread and consumer expectations are used as predictors. On the other hand, for the personal income and manufacturing series, the Kalman filter dominates at all horizons. For the other series the results are mixed, with ratios generally slightly above or below one. The results also differ across horizons, without a clear pattern. At the longest horizon ($h = 8$), except for term spread and consumer expectations, we note a slight preference for the Kalman filter—although the ratios are typically within a 5 to 10 % range.

Overall, the results support the theoretical results obtained earlier. There is little difference between the MIDAS and state space forecasts. In some cases the MIDAS forecasts do a little better, while in others, the state space forecasts appear to be slightly more accurate.

To conclude it is worth summarizing Table 7 across all series — and by doing so, we observe the best predictor with the regular/multiplicative MIDAS and state space models is the crude oil price, except at the longest horizons.

	h (Quarter)	1	2	3	4	5	6	7	8
Best Predictor	State Space	Oil	Oil	Oil	Oil	Oil	Oil	LEI	LEI
	Regular MIDAS	Oil	Oil	Oil	Oil	Oil	LEI	Emply	Emply
	Multiplicative MIDAS	Oil	Oil	Oil	Oil	Oil	Term	Emply	IP
RMSE	State Space	0.69	0.65	0.68	0.67	0.70	0.70	0.74	0.76
	Regular MIDAS	0.65	0.76	0.70	0.74	0.72	0.78	0.80	0.79
	Multiplicative MIDAS	0.65	0.77	0.72	0.76	0.70	0.78	0.80	0.79

When we look at the best performance series in the above table we find evidence similar to Kuzin, Marcellino, and Schumacher (2011)—they find gains at short horizons from using MIDAS and the reverse for longer horizons (two years, as in our application). For intermediate horizons we find the Kalman filter to be best. The differences are however small.

5 Conclusion

The canonical approach in the literature to dealing with mixed frequency data is to use a state space model estimated via the Kalman filter. State space models consist of a system of two equations, a measurement equation which links observed series to a latent state process, and a state equation which describes the state process dynamics. The system of equations can require a lot of parameters, for the measurement equation, the state dynamics and their error processes. In contrast, recent work by Ghysels, Santa-Clara, and Valkanov (2002), Ghysels, Santa-Clara, and Valkanov (2006) and Andreou, Ghysels, and Kourtellos (2009) using MIDAS regressions handles mixed sample frequencies in a simple single-equation setting that is easy to estimate. This paper has examined the relationship between MIDAS regressions and the Kalman filter applied to mixed frequency data. We showed that, in population, MIDAS regressions and the correctly-specified Kalman steady state linear filter can be identical—and if they are not, they are very close in terms of prediction behavior. In small samples, MIDAS regressions and the Kalman filter also generally behave similarly. The Kalman filter typically gives more accurate forecasts, but there are also cases in which

MIDAS regressions give better small-sample predictions. Finally, it is important to note that estimating Kalman filter specifications is numerically much more involved. In contrast, all MIDAS estimations are in comparison computationally simple. This is relevant as the computational complexity limits the applicability of the Kalman filter to a small set of series.

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Technical Appendices

A One-Factor State Space Model with Correlated Measurement Errors

We start from the state space model appearing in subsection 3.1 repeated here for convenience:

$$\begin{aligned} f_{t+j/m} &= \rho f_{t+(j-1)/m} + \eta_{t+j/m} \qquad \forall t \qquad j = 1, \ldots, m \\ y^*_{t+j/m} &= \gamma_1 f_{t+j/m} + u_{1,t+j/m} \qquad \forall t \qquad j = 1, \ldots, m \\ x_{t+j/m} &= \gamma_2 f_{t+j/m} + u_{2,t+j/m} \qquad \forall t \qquad j = 1, \ldots, m \end{aligned}$$

where

$$u_{i,t+j/m} - d_i u_{i,t+(j-1)/m} = \epsilon_{i,t+j/m} \qquad i = 1, 2.$$

with periodic Kalman gain matrices:

$$\mathbf{K}_{2|1} = \begin{pmatrix} \kappa_1^1 \\ \kappa_2^1 \\ \kappa_3^1 \end{pmatrix}, \mathbf{K}_{3|2} = \begin{pmatrix} \kappa_1^2 \\ \kappa_2^2 \\ \kappa_3^2 \end{pmatrix} \text{ and } \mathbf{K}_{1|3} = \begin{pmatrix} \kappa_{1,1}^3 & \kappa_{1,2}^3 \\ \kappa_{2,1}^3 & \kappa_{2,2}^3 \\ \kappa_{3,1}^3 & \kappa_{3,2}^3 \end{pmatrix}.$$

As noted in section 3.1, the state vector is $\boldsymbol{\alpha}_{t+j/m} = (f_{t+j/m}, u_{1,t+j/m}, u_{2,t+j/m})'$ and we have

$$\mathbf{G} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_2 \end{pmatrix},$$

$\mathbf{Z}_j = \begin{pmatrix} \gamma_2 & 0 & 1 \end{pmatrix}$ for $1 \leq j \leq m-1$ and $\mathbf{Z}_m = \begin{pmatrix} \gamma_1 & 1 & 0 \\ \gamma_2 & 0 & 1 \end{pmatrix}$ Using the formula $\mathbf{A}_{j|j-1}$ $=\mathbf{G} - \mathbf{K}_{j|j-1}\mathbf{Z}_j\mathbf{G}$, we can write

$$\mathbf{A}_{2|1} = \begin{pmatrix} \rho - \rho\gamma_2\kappa_1^1 & 0 & -\kappa_1^1 d_2 \\ -\rho\gamma_2\kappa_2^1 & d_1 & -\kappa_2^1 d_2 \\ -\rho\gamma_2\kappa_3^1 & 0 & d_2 - \kappa_3^1 d_2 \end{pmatrix} \tag{A.1}$$

$$\mathbf{A}_{3|2} = \begin{pmatrix} \rho - \rho\gamma_2\kappa_1^2 & 0 & -\kappa_1^2 d_2 \\ -\rho\gamma_2\kappa_2^2 & d_1 & -\kappa_2^2 d_2 \\ -\rho\gamma_2\kappa_3^2 & 0 & d_2 - \kappa_3^2 d_2 \end{pmatrix} \tag{A.2}$$

and

$$\mathbf{A}_{1|3} = \begin{pmatrix} \rho - \rho(\gamma_1\kappa_{1,1}^3 + \gamma_2\kappa_{1,2}^3) & -\kappa_{1,1}^3 d_1 & -\kappa_{1,2}^3 d_2 \\ -\rho(\gamma_1\kappa_{2,1}^3 + \gamma_2\kappa_{2,2}^3) & d_1 - \kappa_{2,1}^3 d_1 & -\kappa_{2,2}^3 d_2 \\ -\rho(\gamma_1\kappa_{3,1}^3 + \gamma_2\kappa_{3,2}^3) & -\kappa_{3,1}^3 d_1 & d_2 - \kappa_{3,2}^3 d_2 \end{pmatrix} \tag{A.3}$$

Letting $\tilde{\mathbf{A}}_3^3 = \mathbf{A}_{1|3}$, $\tilde{\mathbf{A}}_2^3 = \mathbf{A}_{1|3}\mathbf{A}_{3|2}$ and $\tilde{\mathbf{A}}_1^3 = \mathbf{A}_{1|3}\mathbf{A}_{3|2}\mathbf{A}_{1|3}$ as before, and adopting the notation that $[\mathbf{A}]_{ij}$ refers to the ijth element of the matrix $\mathbf{A}$, from equation (2.12), the Kalman filter implies the following equation for h-quarter-ahead prediction:

$$\begin{aligned}
E_{KF}[y_{t+h}|I_t^M] &= E(\gamma_1 f_{t+h} + u_{1,t+h}|I_t^M) = \gamma_1\rho^{3h}E(f_t|I_t^M) + d_1^{3h}E(u_{1,t}|I_t^M) \\
&= \gamma_1\rho^{3h}\sum_{j=0}^{\infty}\{[(\tilde{\mathbf{A}}_1^3)^j]_{11}\kappa_{1,1}^3 + [(\tilde{\mathbf{A}}_1^3)^j]_{12}\kappa_{2,1}^3 + [(\tilde{\mathbf{A}}_1^3)^j]_{13}\kappa_{3,1}^3\}y_{t-j} \\
&\quad +\gamma_1\rho^{3h}\sum_{j=0}^{\infty}\{[(\tilde{\mathbf{A}}_1^3)^j]_{11}\kappa_{1,2}^3 + [(\tilde{\mathbf{A}}_1^3)^j]_{12}\kappa_{2,2}^3 + [(\tilde{\mathbf{A}}_1^3)^j]_{13}\kappa_{3,2}^3\}x_{t-j} \\
&\quad +\gamma_1\rho^{3h}\sum_{j=0}^{\infty}\{[(\tilde{\mathbf{A}}_1^3)^j\tilde{\mathbf{A}}_2^3]_{11}\kappa_1^2 + [(\tilde{\mathbf{A}}_1^3)^j\tilde{\mathbf{A}}_2^3]_{12}\kappa_2^2 + [(\tilde{\mathbf{A}}_1^3)^j\tilde{\mathbf{A}}_2^3]_{13}\kappa_3^2\}x_{t-j-1/3} \\
&\quad +\gamma_1\rho^{3h}\sum_{j=0}^{\infty}\{[(\tilde{\mathbf{A}}_1^3)^j\tilde{\mathbf{A}}_3^3]_{11}\kappa_1^1 + [(\tilde{\mathbf{A}}_1^3)^j\tilde{\mathbf{A}}_3^3]_{12}\kappa_2^1 + [(\tilde{\mathbf{A}}_1^3)^j\tilde{\mathbf{A}}_3^3]_{13}\kappa_3^1\}x_{t-j-2/3} \\
&\quad +d_1^{3h}\sum_{j=0}^{\infty}\{[(\tilde{\mathbf{A}}_1^3)^j]_{21}\kappa_{1,1}^3 + [(\tilde{\mathbf{A}}_1^3)^j]_{22}\kappa_{2,1}^3 + [(\tilde{\mathbf{A}}_1^3)^j]_{23}\kappa_{3,1}^3\}y_{t-j} \\
&\quad +d_1^{3h}\sum_{j=0}^{\infty}\{[(\tilde{\mathbf{A}}_1^3)^j]_{21}\kappa_{1,2}^3 + [(\tilde{\mathbf{A}}_1^3)^j]_{22}\kappa_{2,2}^3 + [(\tilde{\mathbf{A}}_1^3)^j]_{23}\kappa_{3,2}^3\}x_{t-j} \\
&\quad +d_1^{3h}\sum_{j=0}^{\infty}\{[(\tilde{\mathbf{A}}_1^3)^j\tilde{\mathbf{A}}_2^3]_{21}\kappa_1^2 + [(\tilde{\mathbf{A}}_1^3)^j\tilde{\mathbf{A}}_2^3]_{22}\kappa_2^2 + [(\tilde{\mathbf{A}}_1^3)^j\tilde{\mathbf{A}}_2^3]_{23}\kappa_3^2\}x_{t-j-1/3} \\
&\quad +d_1^{3h}\sum_{j=0}^{\infty}\{[(\tilde{\mathbf{A}}_1^3)^j\tilde{\mathbf{A}}_3^3]_{21}\kappa_1^1 + [(\tilde{\mathbf{A}}_1^3)^j\tilde{\mathbf{A}}_3^3]_{22}\kappa_2^1 + [(\tilde{\mathbf{A}}_1^3)^j\tilde{\mathbf{A}}_3^3]_{23}\kappa_3^1\}x_{t-j-2/3}
\end{aligned} \tag{A.4}$$

Hence, as stated in section 3.1, the h-quarter-ahead Kalman Filter forecast error is $\boldsymbol{w}'_{KF}\boldsymbol{\xi}_t$ and forecast error variance is $\boldsymbol{w}'_{KF}\boldsymbol{\Sigma}_{xy}\boldsymbol{w}_{KF}$ where

$$\begin{aligned}
\boldsymbol{w}_{KF} &= (1, 0_{1\times mh-1}, -w_{x,0}^{KF}, -w_{y,0}^{KF}, -w_{x,1}^{KF}, 0, -w_{x,2}^{KF}, 0, \\
&\quad -w_{x,3}^{KF}, -w_{y,1}^{KF}, -w_{x,4}^{KF}, 0, -w_{x,5}^{KF}, 0, ... - w_{x,3\bar{K}-2}^{KF}, 0, -w_{x,3\bar{K}-1}^{KF}, 0, -w_{x,3\bar{K}}^{KF}, -w_{y,\bar{K}}^{KF})'
\end{aligned}$$

Similarly, the variance of the MIDAS forecast error is $\boldsymbol{w}'_{Mds}\boldsymbol{\Sigma}_{xy}\boldsymbol{w}_{Mds}$ where

$$\begin{aligned}\boldsymbol{w}_{Mds} &= (1, 0_{1\times mh-1}, -w^{Mds}_{x,0}, -w^{Mds}_{y,0}, -w^{Mds}_{x,1}, 0, -w^{Mds}_{x,2}, 0, \\ &\quad -w^{Mds}_{x,3}, -w^{Mds}_{y,1}, -w^{Mds}_{x,4}, 0, -w^{Mds}_{x,5}, 0, ... - w^{Mds}_{x,3\bar{K}-2}, 0, -w^{Mds}_{x,3\bar{K}-1}, 0, -w^{Mds}_{x,3\bar{K}}, -w^{Mds}_{y,\bar{K}})'\end{aligned}$$

B Two-Factor State Space Model with Correlated Measurement Errors

We consider a two-factor state space model appearing in section 3.2 repeated here for convenience:

$$\begin{aligned}\mathbf{F}_{t+j/m} &= \begin{pmatrix} f_{1,t+j/m} \\ f_{2,t+j/m}\end{pmatrix} = \begin{pmatrix}\rho_1 & 0 \\ 0 & \rho_2\end{pmatrix}\begin{pmatrix} f_{1,t+(j-1)/m} \\ f_{2,t+(j-1)/m}\end{pmatrix} + \begin{pmatrix}\eta_{1,t+j/m} \\ \eta_{2,t+j/m}\end{pmatrix} \qquad j = 1,\ldots,m \\ y^*_{t+j/m} &= \boldsymbol{\gamma}'_1\mathbf{F}_{t+j/m} + u_{1,t+j/m} \qquad \forall t \qquad j = 1,\ldots,m \\ x_{2,t+j/m} &= \boldsymbol{\gamma}'_2\mathbf{F}_{t+j/m} + u_{2,t+j/m} \qquad \forall t \qquad j = 1,\ldots,m\end{aligned}$$

where

$$u_{i,t+j/m} - d_i u_{i,t+(j-1)/m} = \epsilon_{i,t+j/m} \qquad i = 1, 2.$$

Then the periodic Kalman gain matrices are:

$$\mathbf{K}_{2|1} = \begin{pmatrix}\kappa^1_1 \\ \kappa^1_2 \\ \kappa^1_3 \\ \kappa^1_4\end{pmatrix}, \mathbf{K}_{3|2} = \begin{pmatrix}\kappa^2_1 \\ \kappa^2_2 \\ \kappa^2_3 \\ \kappa^2_4\end{pmatrix} \text{ and } \mathbf{K}_{1|3} = \begin{pmatrix}\kappa^3_{1,1} & \kappa^3_{1,2} \\ \kappa^3_{2,1} & \kappa^3_{2,2} \\ \kappa^3_{3,1} & \kappa^3_{3,2} \\ \kappa^3_{4,1} & \kappa^3_{4,2}\end{pmatrix},$$

The state vector is $\boldsymbol{\alpha}_{t+j/m} = (f_{1,t+j/m}, f_{2,t+j/m}, u_{1,t+j/m}, u_{2,t+j/m})'$ and we have $\mathbf{G} = \begin{pmatrix}\rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & d_1 & 0 \\ 0 & 0 & 0 & d_2\end{pmatrix}$, $\mathbf{Z}_j = \begin{pmatrix}\gamma_{2,1} & \gamma_{2,2} & 0 & 1\end{pmatrix}$ and $\mathbf{Z}_m = \begin{pmatrix}\gamma_{1,1} & \gamma_{1,2} & 1 & 0 \\ \gamma_{2,1} & \gamma_{2,2} & 0 & 1\end{pmatrix}$. Hence, since

$\mathbf{A}_{j|j-1} = \mathbf{G}$ - $\mathbf{K}_{j|j-1}\mathbf{Z}_j\mathbf{G}$, we can write

$$\mathbf{A}_{2|1} = \begin{pmatrix} \rho_1 - \rho_1\gamma_{2,1}\kappa_1^1 & -\rho_2\gamma_{2,2}\kappa_1^1 & 0 & -\kappa_1^1 d_2 \\ -\rho_1\gamma_{2,1}\kappa_2^1 & \rho_2 - \rho_2\gamma_{2,2}\kappa_2^1 & 0 & -\kappa_2^1 d_2 \\ -\rho_1\gamma_{2,1}\kappa_3^1 & -\rho_2\gamma_{2,2}\kappa_3^1 & d_1 & -\kappa_3^1 d_2 \\ -\rho_1\gamma_{2,1}\kappa_4^1 & -\rho_2\gamma_{2,2}\kappa_4^1 & 0 & d_2 - \kappa_4^1 d_2 \end{pmatrix} \tag{B.1}$$

$$\mathbf{A}_{3|2} = \begin{pmatrix} \rho_1 - \rho_1\gamma_{2,1}\kappa_1^1 & -\rho_2\gamma_{2,2}\kappa_1^1 & 0 & -\kappa_1^1 d_2 \\ -\rho_1\gamma_{2,1}\kappa_2^1 & \rho_2 - \rho_2\gamma_{2,2}\kappa_2^1 & 0 & -\kappa_2^1 d_2 \\ -\rho_1\gamma_{2,1}\kappa_3^1 & -\rho_2\gamma_{2,2}\kappa_3^1 & d_1 & -\kappa_3^1 d_2 \\ -\rho_1\gamma_{2,1}\kappa_4^1 & -\rho_2\gamma_{2,2}\kappa_4^1 & 0 & d_2 - \kappa_4^1 d_2 \end{pmatrix} \tag{B.2}$$

and

$$\mathbf{A}_{1|3} = \begin{pmatrix} \rho_1 - \rho_1(\gamma_{1,1}\kappa_{1,1}^3 + \gamma_{2,1}\kappa_{1,2}^3) & -\rho_2(\gamma_{1,2}\kappa_{1,1}^3 + \gamma_{2,2}\kappa_{1,2}^3) & -\kappa_{1,1}^3 d_1 & -\kappa_{1,1}^3 d_2 \\ -\rho_1(\gamma_{1,1}\kappa_{2,1}^3 + \gamma_{2,1}\kappa_{2,2}^3) & \rho_2 - \rho_2(\gamma_{1,2}\kappa_{2,1}^3 + \gamma_{2,2}\kappa_{2,2}^3) & -\kappa_{2,1}^3 d_1 & -\kappa_{2,1}^3 d_2 \\ -\rho_1(\gamma_{1,1}\kappa_{3,1}^3 + \gamma_{2,1}\kappa_{3,2}^3) & -\rho_2(\gamma_{1,2}\kappa_{3,1}^3 + \gamma_{2,2}\kappa_{3,2}^3) & d_1 - \kappa_{3,1}^3 d_1 & -\kappa_{3,1}^3 d_2 \\ -\rho_1(\gamma_{1,1}\kappa_{4,1}^3 + \gamma_{2,1}\kappa_{4,2}^3) & -\rho_2(\gamma_{1,2}\kappa_{4,1}^3 + \gamma_{2,2}\kappa_{4,2}^3) & -\kappa_{4,1}^3 d_1 & d_2 - \kappa_{4,1}^3 d_2 \end{pmatrix} \tag{B.3}$$

Table 1: Approximation Results between MIDAS and *One-Factor* State Space Model

Entries are the values of the L^2 measure appearing in equation (3.8) comparing the Kalman Filter and MIDAS specifications for the one-factor model appearing in equations (3.1) and (3.2), with $d = d_1 = d_2$, $\gamma_1 = \gamma_2 = 1$ and $\sigma_\eta^2 = \sigma_y^2 = \sigma_x^2 = 1$. The multiplicative MIDAS scheme refers to the ADL-MIDAS regression appearing in equation (2.25), whereas the regular one refers to equation (3.5). Panels A through D pertain to different configurations of relative sampling frequency for high- and low-frequency data, m, and forecast horizon, h.

	Regular MIDAS					Multiplicative MIDAS					Regular MIDAS					Multiplicative MIDAS				
$d \setminus \rho$	-0.9	-0.5	0	0.5	0.95	-0.9	-0.5	0	0.5	0.95	-0.9	-0.5	0	0.5	0.95	-0.9	-0.5	0	0.5	0.95
$\mathbb{L}^2$ distance - Sampling frequency m = 3																				
	Panel A: One Quarter Horizon (h=1)										Panel B: Four Quarters Horizon (h=4)									
$-.9$	.000	.004	.013	.075	.102	.000	.004	.049	.075	.139	.000	.006	.007	.008	.000	.000	.001	.007	.008	.006
$-.5$	.052	.000	.002	.000	.109	.001	.000	.002	.006	.109	.000	.000	.000	.000	.000	.012	.000	.000	.000	.032
.0	.011	.000	.000	.000	.000	.011	.002	.000	.000	.000	.000	.000	.000	.000	.000	.011	.000	.000	.000	.000
.5	.023	.000	.002	.000	.000	.082	.002	.002	.000	.080	.009	.000	.000	.000	.000	.009	.000	.000	.000	.043
.95	.020	.006	.005	.003	.000	.012	.001	.052	.035	.000	.001	.002	.002	.002	.000	.000	.023	.001	.005	.000
$\mathbb{L}^2$ distance - Sampling frequency m = 13																				
	Panel C: One Quarter Horizon (h=1)										Panel D: Four Quarters Horizon (h=4)									
$-.9$	.000	.005	.005	.005	.004	.000	.005	.005	.005	.003	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
$-.5$	.018	.000	.000	.000	.003	.007	.000	.000	.000	.001	.000	.000	.000	.000	.001	.000	.000	.000	.000	.001
.0	.009	.000	.000	.000	.000	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
.5	.010	.000	.000	.000	.002	.007	.000	.000	.000	.005	.000	.000	.000	.000	.001	.000	.000	.000	.000	.001
.95	.048	.017	.011	.007	.000	.001	.003	.003	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000

Table 2: Multiple High Frequency Series: MIDAS Forecast Combinations and *One-Factor* State Space Model

The entries are similar to those in Table 1. Entries are the values of the L^2 measure appearing in equation (3.8) comparing the Kalman Filter and MIDAS specifications for the one-factor model appearing in equations (3.1) and (3.2), with $d = d_1 = d_2$, $\gamma_1 = \gamma_2 = 1$ and $\sigma_\eta^2 = \sigma_y^2 = \sigma_x^2 = 1$. The multiplicative MIDAS scheme refers to the ADL-MIDAS regression appearing in equation (2.25), whereas the regular one refers to equation (3.5). The data generating process is described by the system of equations appearing in (3.9). Hence, we have two high frequency series and we examine cases where $var(u_{2,t+j/m}) = var(u_{3,t+j/m})$ and cases where $var(u_{2,t+j/m}) = var(u_{3,t+j/m})/10$, which we call respectively equal and unequal noise variance cases in the Table. We vary m, namely Panels A and B pertain to $m = 3$, while C and D cover $m = 13$. All four assume that the forecast horizon is $h = 1$.

	Regular MIDAS					Multiplicative MIDAS					Regular MIDAS					Multiplicative MIDAS				
$d \setminus \rho$	-0.9	-0.5	0	0.5	0.95	-0.9	-0.5	0	0.5	0.95	-0.9	-0.5	0	0.5	0.95	-0.9	-0.5	0	0.5	0.95
$\mathbb{L}^2$ distance - Sampling frequency m = 3 - Horizon: One Quarter (h=1)																				
	Panel A: Two high frequency series equal noise variance										Panel B: Two high frequency series unequal noise variance									
$-.9$	.034	.018	.010	.044	.221	.000	.033	.030	.044	.322	.004	.000	.000	.003	.013	.000	.000	.000	.000	.002
$-.5$	.018	.000	.001	.004	.056	.027	.000	.002	.008	.069	.001	.000	.000	.000	.012	.001	.000	.000	.000	.008
.0	.038	.002	.000	.001	.000	.037	.002	.000	.000	.000	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000
.5	.047	.007	.002	.000	.082	.049	.004	.002	.000	.041	.001	.000	.000	.000	.005	.000	.000	.000	.000	.011
.95	.052	.044	.033	.023	.000	.029	.043	.059	.026	.000	.001	.005	.000	.003	.000	.002	.003	.005	.005	.000
$\mathbb{L}^2$ distance - Sampling frequency m = 13 - Horizon: One Quarter (h=1)																				
	Panel C: Two high frequency series equal noise variance										Panel D: Two high frequency series unequal noise variance									
$-.9$	.055	.003	.005	.006	.063	.000	.006	.006	.003	.006	.055	.117	.060	.002	.013	.000	.001	.005	.003	.005
$-.5$	.014	.000	.000	.000	.035	.005	.000	.000	.000	.015	.061	.000	.000	.000	.014	.053	.000	.000	.000	.001
.0	.013	.000	.000	.000	.000	.005	.000	.000	.000	.000	.002	.000	.000	.000	.000	.002	.000	.000	.000	.000
.5	.004	.000	.000	.000	.000	.007	.000	.000	.000	.040	.060	.000	.000	.000	.002	.060	.000	.000	.000	.008
.95	.001	.011	.021	.003	.000	.031	.022	.021	.020	.000	.017	.225	.224	.001	.000	.034	.022	.016	.012	.000

Table 3: Prediction Error Distances: MIDAS/One-Factor State Space Model

Entries are the prediction error ratios in equation (3.7), with MIDAS parameters chosen so as to miminize equation (3.6). The distance ratio is computed for a one-factor Kalman Filter (denoted SS1) and regular MIDAS (see equation (3.5)). The data generating process is a two-factor model appearing in section 3.2 with $\rho_1 = \rho_2 = \rho$, $d_1 = d_2 = d$. Entries pertain to two prediction horizons $h = 1$ and 4.

$d \setminus \rho$	-0.9	-0.5	0	0.5	0.95	-0.9	-0.5	0	0.5	0.95
	PE distance - Sampling frequency m = 3									
	Panel A: One Quarter Horizon (h=1)					Panel B: Four Quarters Horizon (h=4)				
$-.9$	.888	.980	.980	.714	.730	.743	.855	.854	.870	.772
$-.5$	.635	.999	1.00	.999	.753	.880	1.00	1.00	1.00	.761
.0	.624	1.00	1.00	1.00	.746	.874	1.00	1.00	1.00	.753
.5	.666	.999	1.00	.999	.766	.889	1.00	1.00	1.00	.772
.95	.722	.734	.725	.735	.775	.751	.750	.732	.741	.679
	PE distance - Sampling frequency m = 13									
	Panel C: One Quarter Horizon (h=1)					Panel D: Four Quarters Horizon (h=4)				
$-.9$	.984	.990	.990	.991	.870	1.00	1.00	1.00	1.00	.991
$-.5$	.992	1.00	1.00	1.00	.847	1.00	1.00	1.00	1.00	.989
.0	.991	1.00	1.00	1.00	.838	1.00	1.00	1.00	1.00	.987
.5	.992	1.00	1.00	1.00	.842	1.00	1.00	1.00	1.00	.988
.95	.856	.837	.824	.824	.770	.991	.987	.986	.986	.981

Table 4: Simulated Out-of-Sample RMSPE for One-Factor Kalman filter relative to MIDAS

Panels A and B report the out-of-sample root mean square prediction error (RMSPE) from the one-factor state space model implemented via the Kalman filter relative to that of the MIDAS model (regular or multiplicative) in the Monte-Carlo simulation described in the text. The multiplicative MIDAS scheme refers to the ADL-MIDAS regression appearing in equation (2.25), whereas the regular one refers to equation (3.5). In this design, the true data generating process (DGP) is a one-factor model, and the sample consists of T=*40* low-frequency observations and mT high-frequency observations where $m = 3$. The objective is to forecast the low-frequency series at a horizon of h quarters ahead (thinking of the low-frequency data as measured in quarters). Entries less than 1 mean that the MIDAS predictions are less accurate than those from the Kalman filter. In Panels C and D the true data generating process is instead a two-factor model given by equations (3.10) and (3.11) where the errors are all independent standard normal random variables. The Kalman filter is however still based on a one-factor model, that is consequently mis-specified.

	Regular MIDAS				Multiplicative MIDAS				Regular MIDAS				Multiplicative MIDAS			
$d \setminus \rho$	-0.9	-0.5	0.5	0.95	-0.9	-0.5	0.5	0.95	-0.9	-0.5	0.5	0.95	-0.9	-0.5	0.5	0.95
	Panel A: One-Factor DGP: h=1								Panel B: One-Factor DGP: h=4							
$-.9$	0.93	0.91	0.94	0.92	0.95	0.93	0.93	0.92	0.92	0.86	0.92	0.91	0.96	0.86	0.90	0.88
$-.5$	0.94	0.95	0.95	0.96	0.92	0.96	0.95	0.93	0.93	0.90	0.91	0.88	0.94	0.91	0.88	0.85
0	0.92	0.92	0.96	0.95	0.92	0.91	0.95	0.95	0.94	0.91	0.91	0.88	0.93	0.90	0.88	0.86
0.5	0.92	0.92	0.97	0.94	0.91	0.92	0.95	0.92	0.92	0.93	0.91	0.87	0.93	0.91	0.89	0.84
0.95	0.79	0.89	0.97	0.97	0.79	0.88	0.97	0.97	0.91	0.84	0.86	0.84	0.90	0.82	0.82	0.87
	Panel C: Two-Factor DGP: h=1								Panel D: Two-Factor DGP: h=4							
$-.9$	0.96	0.95	0.98	1.09	0.93	0.91	0.95	1.08	0.91	0.94	0.93	0.97	0.88	0.92	0.92	0.95
$-.5$	0.93	0.94	0.94	1.15	0.94	0.91	0.93	1.13	0.93	0.91	0.96	0.91	0.89	0.89	0.94	0.89
0	1.05	0.91	0.93	1.09	1.04	0.90	0.91	1.09	0.92	0.93	0.95	0.89	0.91	0.91	0.93	0.86
0.5	1.04	0.93	0.95	0.94	1.03	0.93	0.93	0.93	0.94	0.94	0.94	0.87	0.93	0.92	0.93	0.86
0.95	1.08	0.93	0.96	0.95	1.07	0.92	0.96	0.94	0.98	0.86	0.87	0.82	0.95	0.84	0.86	0.82

Table 5: Simulated Out-of-Sample RMSPE for Kalman filter with AIC/BIC relative to MIDAS

Panels A and B report the out-of-sample root mean square prediction error (RMSPE) from the state space model implemented via the Kalman filter relative to that of the MIDAS model (regular or multiplicative) in the Monte-Carlo simulation described in the text. The number of factors in the state space model is either one or two, chosen on the basis of the AIC. The multiplicative MIDAS scheme refers to the ADL-MIDAS regression appearing in equation (2.25), whereas the regular one refers to equation (3.5). In this design, the true data generating process is a one-factor model, and the sample consists of $T = 40$, low-frequency observations and mT high-frequency observations, where $m = 3$. The objective is to forecast the low-frequency series at a horizon of h quarters ahead (thinking of the low-frequency data as measured in quarters). Entries less than 1 mean that the MIDAS predictions are less accurate than those from the Kalman filter. In Panels C and D, the number of factors in the state space model is instead chosen on the basis of the BIC.

	Regular MIDAS				Multiplicative MIDAS				Regular MIDAS				Multiplicative MIDAS			
$d \setminus \rho$	-0.9	-0.5	0.5	0.95	-0.9	-0.5	0.5	0.95	-0.9	-0.5	0.5	0.95	-0.9	-0.5	0.5	0.95
	Panel A: AIC (h=1)								Panel B: BIC (h=1)							
-.9	0.93	0.91	1.54	1.14	0.95	0.93	1.52	1.15	0.93	0.91	0.94	0.92	0.95	0.93	0.93	0.92
-.5	1.10	0.98	1.36	1.11	1.08	1.00	1.36	1.07	0.94	0.99	0.95	0.96	0.92	1.01	0.95	0.93
0	1.05	1.35	1.62	1.23	1.05	1.33	1.61	1.22	0.92	0.92	0.96	0.95	0.92	0.91	0.95	0.95
0.5	1.03	1.13	0.96	0.94	1.01	1.13	0.95	0.93	0.92	0.92	0.97	0.94	0.91	0.92	0.95	0.92
0.95	0.93	1.07	0.97	1.03	0.92	1.07	0.97	1.09	0.79	0.90	0.97	0.97	0.79	0.89	0.97	0.92
	Panel C: AIC (h=4)								Panel D: BIC (h=4)							
-.9	0.92	0.86	1.34	1.09	0.96	0.75	1.31	1.06	0.92	0.86	0.92	0.91	0.96	0.86	0.90	0.88
-.5	1.12	0.93	1.41	0.95	1.12	0.94	1.36	0.92	0.93	0.93	0.91	0.88	0.94	0.93	0.88	0.85
0	1.05	1.32	1.44	1.05	1.04	1.31	1.39	1.02	0.94	0.91	0.91	0.88	0.93	0.90	0.88	0.86
0.5	0.99	1.12	0.91	0.87	1.00	1.10	0.89	0.84	0.92	0.93	0.91	0.87	0.93	0.91	0.89	0.84
0.95	0.96	0.89	0.86	0.83	0.95	0.87	0.82	0.86	0.91	0.84	0.86	0.84	0.90	0.82	0.82	0.87

Table 6: Definition of the Regressors

Name	Description	Period	Transformation
TERM	Term spread (10yr T-Bond - 1yr T-bond)	1959:01 - 2009:05	lv
SP500	Stock Price Index: Standard & Poor's 500 Composite	1959:01 - 2009:05	$\Delta \ln$
IP	Industry Production Index (SA)	1959:01 - 2009:05	$\Delta \ln$
Emply	All Employees: Total Nonfarm Payroll(SA, Thous)	1959:01 - 2009:05	$\Delta \ln$
Exptn	Consumer Expectations (Q1-66=100)	1959:01 - 2009:05	ln
PI	Personal Income less Transfer Payments (SAAR)	1959:02 - 2009:05	$\Delta \ln$
LEI	Leading index, percent change from previous month	1959:02 - 2009:05	lv
Manu	Real Manufacturing & Trade Inventories: All Industries (SA)	1967:02 - 2009:05	$\Delta \ln$
Oil	Crude Oil Spot Price: WTI Cushing	1982:01 - 2009:05	$\Delta \ln$

Data Source: Federal Reserve Board and Haver Analytics.

Table 7: RMSE Forecasting Comparison for MIDAS and State Space Model

We use a dataset with mixed frequencies: monthly and quarterly. The forecasting variable is the growth rate of real GDP from 1959Q1 to 2009Q1. Each model consists of real GDP growth and one of the monthly indicators, with the latter as observed up to the second month of each quarter. Table 6 contains the details on the definitions and data transformations. We use the root mean squared forecasting error (RMSE) to evaluate each models forecasting accuracy. Section 4 contains the sample configurations.

h (Quarters)	1	2	3	4	5	6	7	8
Term Spread								
State Space (m0)	1.06	0.97	1.06	1.02	0.99	1.00	0.97	1.00
Regular Midas (m1)	0.82	0.81	0.85	0.80	0.85	0.78	0.81	0.83
Multiple Midas (m2)	0.80	0.84	0.84	0.81	0.85	0.78	0.87	0.86
Ratio (m0/m1)	1.29	1.19	1.25	1.28	1.16	1.29	1.20	1.21
Ratio (m0/m2)	1.33	1.16	1.26	1.26	1.16	1.28	1.12	1.17
S&P 500								
State Space (m0)	0.81	0.77	0.77	0.77	0.78	0.74	0.77	0.80
Regular Midas (m1)	0.79	0.82	0.82	0.84	0.84	0.82	0.84	0.84
Multiple Midas (m2)	0.80	0.87	0.81	0.85	0.85	0.81	0.82	0.82
Ratio (m0/m1)	1.02	0.94	0.94	0.92	0.93	0.90	0.91	0.95
Ratio (m0/m2)	1.01	0.88	0.96	0.91	0.92	0.91	0.93	0.97
Industrial Production								
State Space (m0)	0.75	0.76	0.81	0.81	0.80	0.75	0.77	0.81
Regular Midas (m1)	0.78	0.86	0.83	0.86	0.84	0.79	0.81	0.81
Multiple Midas (m2)	0.79	0.86	0.85	0.87	0.84	0.80	0.82	0.79
Ratio (m0/m1)	0.96	0.88	0.98	0.94	0.95	0.95	0.95	1.00
Ratio (m0/m2)	0.96	0.88	0.96	0.93	0.95	0.94	0.93	1.03

Table continued on next page ...

Table 7 (Cont'd) RMSE Forecasting Comparison for MIDAS and State Space Model

h (Quarters)	1	2	3	4	5	6	7	8
Employment								
State Space (m0)	0.72	0.81	0.84	0.85	0.88	0.79	0.81	0.84
Regular Midas (m1)	0.78	0.83	0.82	0.84	0.81	0.80	0.80	0.79
Multiple Midas (m2)	0.78	0.84	0.83	0.85	0.82	0.81	0.80	0.79
Ratio (m0/m1)	0.93	0.97	1.02	1.01	1.09	1.00	1.01	1.07
Ratio (m0/m2)	0.93	0.96	1.01	1.00	1.07	0.98	1.01	1.08
Consumer Expectations								
State Space (m0)	0.96	1.02	1.07	1.06	1.09	1.07	1.07	1.07
Regular Midas (m1)	0.74	0.82	0.79	0.84	0.84	0.81	0.83	0.83
Multiple Midas (m2)	0.74	0.82	0.80	0.84	0.84	0.79	0.83	0.83
Ratio (m0/m1)	1.30	1.24	1.34	1.26	1.31	1.33	1.28	1.30
Ratio (m0/m2)	1.30	1.24	1.34	1.26	1.31	1.34	1.29	1.30
Personal Income								
State Space (m0)	0.73	0.78	0.81	0.79	0.76	0.74	0.77	0.80
Regular Midas (m1)	0.87	0.98	0.84	0.87	0.82	0.84	0.84	0.81
Multiple Midas (m2)	0.94	1.00	0.92	0.91	0.82	0.85	0.89	0.84
Ratio (m0/m1)	0.84	0.79	0.96	0.91	0.93	0.89	0.93	0.99
Ratio (m0/m2)	0.78	0.77	0.87	0.87	0.93	0.87	0.87	0.95

Table 7 (Cont'd) RMSE Forecasting Comparison for MIDAS and State Space Model

h (Quarters)	1	2	3	4	5	6	7	8
Leading Index (LEI)								
State Space (m0)	0.74	0.72	0.78	0.79	0.74	0.74	0.74	0.77
Regular Midas (m1)	0.66	0.81	0.81	0.79	0.76	0.78	0.83	0.84
Multiple Midas (m2)	0.68	0.82	0.79	0.78	0.77	0.80	0.84	0.81
Ratio (m0/m1)	1.11	0.89	0.96	1.00	0.96	0.94	0.90	0.92
Ratio (m0/m2)	1.09	0.88	1.00	1.01	0.95	0.93	0.88	0.95
Manufacturing								
State Space (m0)	0.90	0.88	0.88	0.86	0.80	0.81	0.84	0.87
Regular Midas (m1)	1.03	1.02	0.96	0.89	0.86	0.87	0.94	0.93
Multiple Midas (m2)	1.08	1.02	0.97	0.94	0.84	0.88	0.96	0.97
Ratio (m0/m1)	0.87	0.87	0.91	0.97	0.92	0.92	0.89	0.94
Ratio (m0/m2)	0.83	0.87	0.90	0.92	0.96	0.91	0.87	0.90
Crude Oil Price								
State Space (m0)	0.69	0.65	0.68	0.67	0.70	0.70	0.80	0.89
Regular Midas (m1)	0.65	0.76	0.70	0.74	0.72	0.80	0.90	0.95
Multiple Midas (m2)	0.65	0.77	0.72	0.76	0.70	0.81	0.92	0.97
Ratio (m0/m1)	1.06	0.86	0.97	0.91	0.97	0.87	0.89	0.94
Ratio (m0/m2)	1.06	0.85	0.95	0.88	0.99	0.86	0.88	0.92