1 Notes on endogenous capital

These notes extend the model to include endogenous capital. I only consider the most simple case without habits, and compare the solution to the fixed capital stock model treated in the paper and focus on the implications for the contractionary phase under zero inflation targeting (corresponding to the Hoover regime in the model with fixed capital stock). I find that the main conclusions do not change as long as we consider intertemporal disturbances that affect the consumption and investment Euler equations in the same way. This assumption is consistent with the criteria in the paper (P1-P2) for the efficient rate of interest and also necessary to fit the key features of the data (i.e. an investment collapse). The basic finding is in line with recent results in the literature, such as Woodford (2005), that argues that the fixed capital stock model provides a reasonable approximation to the variable capital model in DSGE models with nominal frictions.

2 Model

The household maximization problem is the same as in the paper. For the firms I assume a convex cost of investment as in Christiano (2004) and Woodford (2003,2005). To increase the capital stock to $K_{t+1}(i)$ in the next period from $K_t(i)$ the firm needs to buy

$$I_t(i) = \phi(\frac{K_{t+1}(i)}{K_t(i)}, \xi_t)K_t(i) \tag{1}$$

of the consumption good. Let us denote $I_t^N(i) \equiv \frac{K_{t+1}(i)}{K_t(i)}$ as the net increase in the capital stock in each period. The function ϕ satisfies $\phi(1,\bar{\xi}) = \lambda$, $\phi^I(1,\bar{\xi}) = 1$, $\phi^{II} \geq 0$, $\phi^{\xi}(1,\bar{\xi}) = 0$, $\phi^{I\xi}(1,\bar{\xi}) \neq 0$. Observe that at time t the capital stock is predetermined. I allow for the vector of fundamental shocks to appear in the cost of adjustment function. This is important to generate the same kind of shocks as considered in the paper (namely variations in the efficient rate of interest) and is the key difference relative to Christiano (2004).

Here $I_t(i)$ represents to purchases of firm i of the composite good, defined over all the Dixit-Stiglitz good varieties so that we can write

$$y_t(i) = Y_t(\frac{p_t(i)}{P_t})^{-\theta}$$

Output is produced with the Cobb Douglas function

$$y_t(i) = AK_t(i)^{\alpha}h_t(i)^{1-\alpha}$$

Firms maximize profits facing the resource cost $d(\frac{p_t(i)}{p_{t-1}(i)})$ of changing prices. The firm then solves the Lagrangian

$$\begin{aligned}
\mathcal{L}_t &= \sum_{t=0}^{\infty} Q_{0,t}[y_t(i)p_t(i) - P_t n_t h_t(i) - P_t d(\frac{p_t(i)}{p_{t-1}(i)}) - P_t\phi(\frac{K_{t+1}(i)}{K_t(i)})K_t(i)z_t] = \\
&= \sum Q_{0,t}[(1+s)Y_t P_t^{\theta} p_t(i)^{1-\theta} - n_t P_t[\frac{Y_t p_t(i)^{-\theta} P_t^{\theta}}{A_t K_t(i)^{\alpha}}]^{\frac{1}{1-\alpha}} - P_t d(\frac{p_t(i)}{p_{t-1}(i)}) - P_t\phi(\frac{K_{t+1}(i)}{K_t(i)})K_t(i)z_t]
\end{aligned}$$

where the discount factor is given by

$$Q_{t,T} = \beta^{T-t} \frac{u_{c,t+1}}{u_{c,t}} \frac{P_t}{P_{t+1}}$$

and the term $1+s$ is a production subsidy. This maximization problem gives rise rise to the first order conditions.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial p_t(i)} &= (1+sub)(1-\theta) Y_t P_t^{\theta} p_t(i)^{-\theta} + \frac{\theta}{1-\alpha} n_t(i) \left[\frac{Y_t p_t(i)^{-\theta} P_t^{\theta}}{A_t K_t(i)^{\alpha}} \right]^{\frac{1}{1-\alpha}-1} \frac{Y_t P_t^{\theta+1} p_t(i)^{-\theta-1}}{A_t K_t(i)^{\alpha}} \\
&\quad -P_t d'() \frac{1}{p_{t-1}(i)} + E_t Q_{t,t+1} P_{t+1} d'() \frac{p_{t+1}(i)}{p_t(i)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial K_{t+1}(i)} &= -P_t [\phi'(\frac{K_{t+1}(i)}{K_t(i)}) z_t] + E_t Q_{t+1} P_{t+1} [\phi'(\frac{K_{t+2}(i)}{K_{t+1}(i)}) \frac{K_{t+2}(i) z_{t+1}}{K_{t+1}(i)} + \phi(\frac{K_{t+2}(i)}{K_{t+1}(i)}) z_{t+1}] \\
&\quad + \frac{\alpha}{1-\alpha} E_t Q_{t+1} n_{t+1} P_{t+1} \left[\frac{Y_{t+1} p_{t+1}(i)^{-\theta} P_{t+1}^{\theta}}{A K_{t+1}(i)^{\alpha}} \right]^{\frac{1}{1-\alpha}-1} \frac{Y_{t+1} p_{t+1}(i)^{-\theta} P_{t+1}^{\theta}}{A_{t+1} K_{t+1}(i)^{\alpha+1}}
\end{aligned}$$

Assuming a symmetric equilibrium so that $p_t(i) = P_t$, $y_t(i) = Y_t$, $I_t(i)^N = I_t^N$ and $K_t(i) = K_t$ and that $1+s = \frac{\theta}{\theta-1}$ we obtain from these two first order conditions

$$\theta Y_t [\frac{1}{1-\alpha} \frac{v_{L,t}}{u_{c,t}} \frac{L_t}{Y_t} - 1] - d'(\Pi_t)\Pi_t + \beta E_t \frac{u_{c,t+1}}{u_{c,t}} d'(\Pi_{t+1})\Pi_{t+1} \tag{2}$$

and

$$\begin{aligned}
&-\phi^I(I_t^N, \xi_t) + E_t Q_{t+1} \Pi_{t+1} [\phi^I(I_{t+1}^N, \xi_{t+1}) I_{t+1}^N - \phi(I_{t+1}^N, \xi_{t+1})] \\
&+E_t Q_{t+1} \Pi_{t+1} \rho_{t+1}
\end{aligned} \tag{3}$$

where

$$\rho_t \equiv \frac{\alpha}{1-\alpha} \frac{v_L}{u_c} \frac{L_t}{K_t} \tag{4}$$

Observe that the characterization of the model is much simpler when we have convex cost of price adjustment than in the Calvo case. The key simplification is that all firms are identical which turns out to simplify the analytics considerably. We will see, however, that the two models are identical to the first order so that we can (as in the paper) use information on the frequency of price adjustment under Calvo prices to calibrate the model.

3 Linearization

Let us now linearize the model. Equation (2) is:

$$\pi_t = \kappa \hat{Y}_t - \kappa_K \hat{K}_t - \kappa_I \hat{I}_t + \beta E_t \pi_{t+1} \tag{5}$$

where $\kappa \equiv [\frac{\alpha}{1-\alpha} + \frac{\omega}{1-\alpha} + \sigma^{-1}\delta_c^{-1}] \frac{\theta}{d''}$, $\kappa_K \equiv (1+\omega) \frac{\alpha}{1-\alpha} \frac{\theta}{d''}$, $\kappa_I \equiv \sigma^{-1}\delta_c^{-1} \frac{\theta}{d''}$, $\pi_t \equiv \log \Pi_t, \omega \equiv \frac{v_{LL} L}{v_L}, \sigma \equiv -\frac{u_c}{u_{cc} C}$, $\delta_c \equiv \frac{C}{Y}$. In log-linearizing this equation I have used the aggregate resource constraint $Y_t = C_t + I_t$

and the production function to substitute out for C_t and L_t. Equation (3) is

$$\hat{I}_t^N = \beta E_t \hat{I}_{t+1}^N - \frac{1}{\phi^{II}}(i_t - E_t\pi_{t+1} - r_t^I) + \frac{1}{\phi^{II}}\beta\bar{\rho}E_t\hat{\rho}_{t+1} \tag{6}$$

where $r_t^I \equiv \log\beta^{-1} - \phi^{I\xi}\xi_t + \beta\phi^{I\xi}E_t\xi_{t+1}$ and $\hat{\rho}_t = \log\frac{\rho_t}{\bar{\rho}}$, $\hat{I}_t^N = \log I_t^N$.

Observe that this IS equation takes essentially the same form as the consumption Euler equations and this is the reason for why the extension yields very similar results once it is assumed that r_t^I – the shock to the investment Euler equation – parallels the shock to the consumption Euler equation (more on this below).

Linearizing the definition of ρ_t yields

$$\hat{\rho}_t = (1+\omega)\hat{L}_t + \sigma^{-1}\delta_c^{-1}\hat{C}_t - \hat{K}_t$$

where $\hat{C}_t \equiv \log\frac{C_t}{Y}$, $\hat{L}_t \equiv \log\frac{L_t}{L}$ and $\hat{K}_t \equiv \log\frac{K_t}{K}$. Linearizing the definition of I_t^N yields

$$\hat{I}_t^N \equiv \hat{K}_{t+1} - \hat{K}_t$$

Linearizing (1) yields

$$\hat{I}_t = \delta_K\hat{I}_t^N + \lambda\delta_K\hat{K}_t$$

Linearizing the resource constraint $Y_t = C_t + I_t$ yields

$$\hat{Y}_t = \hat{C}_t + \hat{I}_t = \hat{C}_t + \delta_K\hat{I}_t^N + \zeta\delta_K\hat{K}_t$$

Linearizing the consumption Euler equation yields

$$\hat{C}_t = E_t\hat{C}_{t+1} - \sigma\delta_c(i_t - E_t\pi_{t+1} - r_t^c) \tag{7}$$

where $r_t^e \equiv \log\beta^{-1} + \frac{\bar{u}_{c\xi}}{\bar{u}_c}\xi_t - \frac{\bar{u}_{c\xi}}{\bar{u}_c}E_t\xi_{t+1}$. The production function is

$$\hat{Y}_t = \alpha\hat{K}_t + (1-\alpha)\hat{L}_t$$

where I have assumed no productivity shocks.

4 The efficient rate of interest

Observe that in the current model we have two IS equations – (6) and (7) – the first relating investment to current and expected future short-term real interest rates and the second consumption to current and expected future real short term interest rates. Recall that our definition of the shocks in the paper was that it was they correspond to intertemporal disturbances that only change the efficient rate of interest, leaving the efficient level of output and consumption constant. It is easy to see that in the model with endogenous capital, the disturbance that satisfies this criteria is one in which $r_t^c = r_t^I = r_t^e$. This disturbance leads

to a decline in the efficient rate of interest, leaving the efficient level of output, capital, investment and consumption constant.

This shock has different properties than the one studied in Christiano (2004). In his model the shock he considers is only a shock to the discount factor in the household utility. This does not satisfy the criteria in the paper, because a shock that only affects the consumption Euler equation will then lead to an increase in investment that offsets this shock, having a much smaller effect on the efficient interest rate. The fact that the shock only appears in the consumption Euler equation also has the implication that it perturbs the efficient allocation for investment, capital, output and consumption. This kind of shock is less appealing for my purposes because it implies that the Great Depression was associated with an investment boom. Instead investment collapsed together with output and consumption during the Great Depression. More generally if one thinks of the intertemporal disturbance as a reduced form representation of financial frictions it makes sense to assume that it affected the cost of lending by both consumers and firms in the same way.

5 Calibration

To calibrate the model we use the same parameters as in the paper, i.e. $\omega = \sigma = 1$, $\theta = 11$, $\beta = 0.96$. Of the new parameters I set $\alpha = 0.25$ and $\lambda = 0.05$ which is the depreciation rate. In the steady state we have $\delta_K = 2.72$. To calibrate the cost function ϕ^{II} I follow Christiano and Davis (2006) by assuming that $\phi^{II} = \lambda^{-1}$ (see further discussion in that paper).

In order to calibrate the cost of price adjustment parameter, I use the same approach as in the paper. The first order dynamics of the current model are identical to those of the model with a Calvo adjustment shown in Woodford (2005). I use Proposition 1 of Woodford (2005) to map the value of κ in the equation at the bottom of page 25 in that paper to κ in equation (5). I use the same parameters as above, but assume as in the paper that the value of the Calvo parameter is 0.66. To obtain the value of κ in annual frequencies I multiply by 4 yielding $\kappa = 0.1637$.

6 Results

I consider here the consequence of a zero inflation targeting regime of the following kind

$$\pi_t = 0 \text{ for } t \geq \tau$$

$$i_t = 0 \text{ for } t \leq \tau$$

This corresponds to the Hoover regime in the model without the capital stock. It may not map exactly into the MPE in the model with endogenous capital, because now the capital stock is an endogenous state

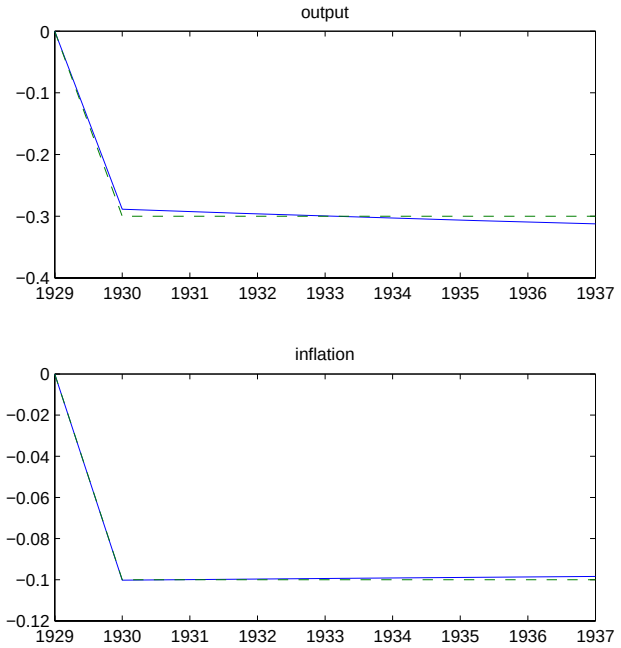

Figure 1: The evolution of ouput and inflation is similar in the fixed capital stock model (dashed line) as in the variable capital model (solid line).

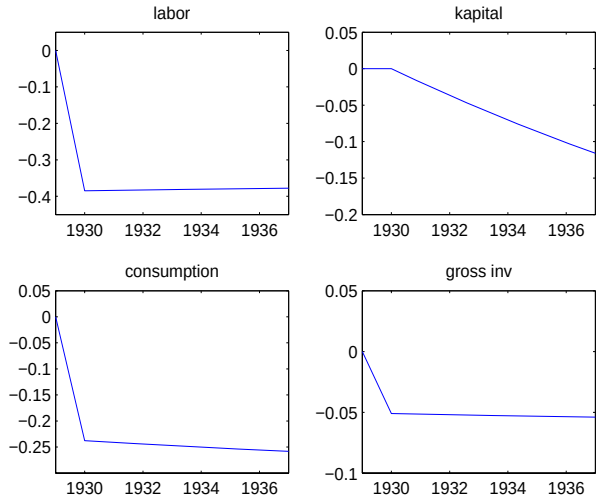

Figure 2: The figure shows the evolution of labor, capital, consumption and gross investment in the contractionary phase.

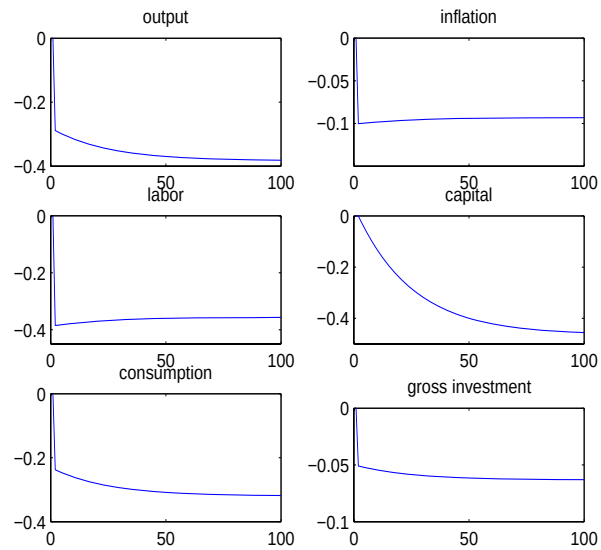

Figure 3: This figure shows the evolution of the endogeous variables over a very long horizon under the assumption that the shock stayes in the "low" state the whole time.

variable which complicates the game significantly. The shock takes the form

$$r_t^e = r_t^I = r_t^C = 1/\beta - 1 \text{ for } t \geq \tau$$

$$r_t^e = r_t^I = r_t^C = r_L^e < 0 \text{ for } t \geq \tau$$

Figure 1 and 2 show the results for the variable capital stock case for inflation and output. The dashed line denotes the fixed capital stock, the solid line the variable capital stock. The criteria for the shock was to pick them so as to generate a 10 percent deflation in 1933 and output contraction of 30 percent. This is essentially the same criteria as in the paper because inflation was approximately 10 percent in 1931-33 and the output contraction about 30 percent.

The figure reveals that the two model generate approximately the same outcome. The second figures shows how the decline in output is distributed between a decline in gross investment and consumption. About 20 percent of the contraction in output is caused by the decline in investment which suggests that investment collapsed by a 50 percent because investment accounts for only 13 percent of spending in steady state in the calibration. Figure 3 shows the long run evolution of the model if it stays in the r_L^e state for a very long time.

In the variable capital model the shocks that give rise to this equilibrium are $r_L^e = -0.0505$ and $\alpha = 0.4120$ but for the fixed capital model we have $r_L^e = -0.0767$ and $\alpha = 0.39$ suggesting that the shock is smaller both in terms absolute level and persistence in the model with endogeous capital. This is because

investment is more sensitive to a decline in the efficient rate of interest than consumption (which explains why it declines more in percentage terms than consumption).

7 Some remarks

The key difference between the result here and the one in Christiano (2004) is the way in which the shock is introduced. Once it satisfies the criteria in the paper for the efficient rate of interest, the results are very similar across the two models. The shock that satisfies the criterion of the paper is slightly smaller in the variable capital model both in terms absolute level and persistence, although I do not know if this will hold for all parameterization (my conjecture is that this depends on the calibrated value of ϕ^{II} among other things). There appears to be little reason to believe that the extension to a MPE with external habits would yield substantially different results from those already in the paper, althout this remains to be confirmed. Those results would be considerable more complicated to derive, however, because the game would have many more state variables so I would not be able to produce any closed form solution (although a numerical characterization seems computationally feasible). Not only would the K_t be a state, but also L_{t-1} and Y_{t-1}. The key simplification in the model with fixed capital stock was the specification of the habit which meant that the state Y_{t-1} dropped out and the model could be written in terms of quasi growth rate of output. This appears no longer possible in the variable capital model because the production function is not linear in labor.

8 References

Christiano, Lawrence, "The Zero-Bound, Low Inflation, and Output Collapse," mimeo, Northwestern.

Christiano, Lawrence and Davis, Joshua "Two Flaws in Business Cycle Accounting," mimeo, Northwestern.

Woodford, Michael (2005), "Firm-Specific Capital and the New-Keynesian Phillips Curve," International Journal of Central Banking.