

Fiscal and Monetary Rules for a Currency Union

TECHNICAL APPENDIX

(NOT FOR PUBLICATION - PLEASE DO NOT QUOTE)

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March 2, 2007

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1 Equilibrium

This section presents the results of the optimization problems faced by households and firms and the optimal policy problem for a benevolent authority that chooses fiscal and monetary policy for the currency union as a whole.

1.1 Demand

The allocation of total consumption among different varieties can be derived by solving the standard expenditure minimization problems. The implied GDP deflators are

$$P_{H,t} = \left[\frac{1}{n} \int_0^n p_t(h)^{1-\sigma} dh \right]^{\frac{1}{1-\sigma}}, \quad P_{F,t} = \left[\frac{1}{1-n} \int_n^1 p_t(f)^{1-\sigma} df \right]^{\frac{1}{1-\sigma}}. \quad (1)$$

Also from expenditure minimization it follows that the consumption price index (CPI) is

$$P_t = \left[n P_{H,t}^{1-\theta} + (1-n) P_{F,t}^{1-\theta} \right]^{\frac{1}{1-\theta}}. \quad (2)$$

Expression [2] can be rewritten as to highlight the relation between the relative price of country i ($p_{i,t} \equiv P_{i,t}/P_t$) and the terms of trade ($T_t \equiv P_{F,t}/P_{H,t}$)

$$p_{H,t}^{\theta-1} = n + (1-n) T_t^{1-\theta}, \quad p_{F,t}^{\theta-1} = n T_t^{\theta-1} + (1-n). \quad (3)$$

The absence of the nominal exchange rate acting as an automatic stabilizer establishes a one to one link between GDP inflation rate differentials and variations of the terms of trade

$$\frac{T_t}{T_{t-1}} = \frac{\Pi_{F,t}}{\Pi_{H,t}}. \quad (4)$$

where $\Pi_{i,t} = P_{i,t}/P_{i,t-1}$ defines GDP inflation. Given the appropriate CES output aggregators, the total demand for country i goods is

$$Y_{i,t} = p_{i,t}^{-\theta} C_{W,t} + G_{i,t}, \quad (5)$$

where $C_{W,t} \equiv \int_0^1 C_t^j dj$ is total consumption in the currency union and $G_{i,t}$ represents the exogenous amount of public spending in country i .

The consumption-saving and labor-leisure decisions can be derived by maximizing the utility function subject to the sequence of flow budget constraints. The optimality condition for the allocation of wealth among state-contingent securities at the individual level delivers the standard expression for the stochastic discount factor

$$Q_{t,t+1} = \beta \left(\frac{P_t}{P_{t+1}} \right) \left(\frac{C_{t+1}^j}{C_t^j} \right)^{-\rho}. \quad (6)$$

Idiosyncratic risk is fully shared among households, both within and across countries. For each couple of households in the currency union, consumption levels are proportional, and the proportionality term depends on the initial distribution of wealth among households. Assuming that the initial state-contingent distribution of wealth is such that the life-time budget constraints of all households are identical, the consumption levels are actually equalized across individuals, that is, $C_t^j = C_t, \forall j \in (0, n)$, $C_t^j = C_t^*, \forall j \in (n, 1)$ and $C_t = C_t^* = C_{W,t}$. Given the expression of the stochastic discount factor

[6], no arbitrage implies that the gross return on a one-period risk-free bond R_t satisfies the Euler equation

$$1 = \beta R_t E_t \left\{ \frac{1}{\Pi_{t+1}} \left(\frac{C_{W,t+1}}{C_{W,t}} \right)^{-\rho} \right\}, \quad (7)$$

where $\Pi_t \equiv P_t/P_{t-1}$ defines CPI inflation. From [2], CPI inflation can be written as a function of GDP inflation rates and lagged relative prices

$$\Pi_t^{1-\theta} = n (\Pi_{H,t} p_{H,t-1})^{1-\theta} + (1-n) (\Pi_{F,t} p_{F,t-1})^{1-\theta}. \quad (8)$$

An exogenous country-specific wage markup ($\mu_{i,t}^w > 1$) introduces a wedge between the real wage and the marginal rate of substitution between labor and leisure¹

$$\frac{w_t^j}{P_t} = \mu_{i,t}^w \frac{(\ell_t^j)^\eta}{C_{W,t}^{-\rho}}. \quad (9)$$

The last first order necessary condition for household's optimization is the intertemporal budget constraint which corresponds to the flow budget constraint coupled with the appropriate transversality condition

$$\lim_{T \rightarrow \infty} E_t \{ Q_{t,T} D_T^j \} = 0. \quad (10)$$

1.2 Supply

The program for firm k consists of choosing the price $p_t(k)$ that maximizes the present discounted value of profits subject to the technology constraint and the demand for its own product

$$y_{t,T}(k) = \left[\frac{p_t(k)}{P_{i,T}} \right]^{-\sigma} Y_{i,T}, \quad \forall T \geq t. \quad (11)$$

The first order condition of the firm's problem can be arranged as to express the optimal relative price $\tilde{p}_t(k)/P_{i,t}$ as a function of aggregate variables only

$$\frac{\tilde{p}_t(k)}{P_{i,t}} = \left(\frac{K_{i,t}}{F_{i,t}} \right)^{\frac{1}{1+\sigma\eta}}.$$

The determinants of the optimal relative price for firm k can be written recursively as

$$K_{i,t} = \left(\frac{\sigma}{\sigma-1} \right) \mu_{i,t}^w \left(\frac{Y_{i,t}}{a_{i,t}} \right)^{1+\eta} + \alpha_i \beta E_t \{ \Pi_{i,t+1}^{\sigma(1+\eta)} K_{i,t+1} \} \quad (12)$$

and

$$F_{i,t} = (1 - \tau_{i,t}) C_{W,t}^{-\rho} p_{i,t} Y_{i,t} + \alpha_i \beta E_t \{ \Pi_{i,t+1}^{\sigma-1} F_{i,t+1} \}. \quad (13)$$

Expression [12] is the present discounted value of a time-varying gross markup over current and future (total) marginal costs. The gross markup can be decomposed in a combination of the price markup (the constant term) and of the distortions in the labor market due to the exogenous wage markup $\mu_{i,t}^w$. Expression [13] is the present discounted value of current and future (total) revenues net of taxation.

¹This wedge can be thought of as capturing either the monopolistic distortions in input supply or inefficient contracting.

Differently from the closed economy case, total revenues display a terms of trade correction due to foreign sales (the term $p_{i,t}$).

All firms that reset their price in a given period choose the same optimal figure. By the law of large numbers, the relevant price for all the firms that do not adjust is the GDP deflator of the previous period. From the definition of the price index [1], the resulting Phillips curve for country i is

$$\left(\frac{1 - \alpha_i \Pi_{i,t}^{\sigma-1}}{1 - \alpha_i} \right)^{\frac{1+\sigma\eta}{\sigma-1}} = \frac{F_{i,t}}{K_{i,t}}. \quad (14)$$

The presence of nominal rigidities creates price dispersion across firms which is the source of the inefficient level of output in the model. The index $\Delta_{i,t}$ measures the level of price dispersion for country i

$$\Delta_{H,t} \equiv \frac{1}{n} \int_0^n \left[\frac{p_t(h)}{P_{H,t}} \right]^{-\sigma(1+\eta)} dh, \quad \Delta_{F,t} \equiv \frac{1}{1-n} \int_n^1 \left[\frac{p_t(f)}{P_{F,t}} \right]^{-\sigma(1+\eta)} df. \quad (15)$$

The evolution of the index of price dispersion $\Delta_{i,t}$ can be derived from the definition of the price index [1]

$$\Delta_{i,t} = \alpha_i \Delta_{i,t-1} \Pi_{i,t}^{\sigma(1+\eta)} + (1 - \alpha_i) \left(\frac{1 - \alpha_i \Pi_{i,t}^{\sigma-1}}{1 - \alpha_i} \right)^{\frac{\sigma(1+\eta)}{\sigma-1}}. \quad (16)$$

1.3 Government Budget Constraint

The flow government budget constraint for country i can be rewritten more compactly using the demand for the k^{th} variety [11] with $T = t$ and the definition of the GDP deflator [1]

$$B_{i,t} = R_{t-1} B_{i,t-1} - P_t s_{i,t}, \quad (17)$$

where real per-capita surplus is defined as

$$s_{i,t} \equiv p_{i,t} (\tau_{i,t} Y_{i,t} - G_{i,t}) - \varsigma_{i,t}. \quad (18)$$

In what follows, it will be useful to work with an alternative expression for government liabilities. In particular, I define the real value of debt at maturity as

$$b_{i,t} \equiv \frac{R_t B_{i,t}}{P_t} \quad (19)$$

Exploiting the definition of the stochastic discount factor, expression [17] can be rewritten as

$$\frac{C_{W,t}^{-\rho} b_{i,t-1}}{\Pi_t} = C_{W,t}^{-\rho} p_{i,t} (\tau_{i,t} Y_{i,t} - G_{i,t} - \varsigma_{i,t}) + \beta E_t \left\{ \frac{C_{W,t+1}^{-\rho} b_{i,t}}{\Pi_{t+1}} \right\}. \quad (20)$$

In an open economy, the household's transversality condition does not necessarily imply a correspondent restriction on the value of debt issued by each national government. Indeed, the only constraint brought about by the transversality conditions of the private sector is a correspondent transversality condition on the sum of the asset positions of the two governments (i.e., on consolidated debt)

$$\lim_{T \rightarrow \infty} E_t \left\{ \beta^{T-t} \frac{C_{W,T}^{-\rho}}{\Pi_T} [n b_{H,T-1} + (1-n) b_{F,T-1}] \right\} = 0. \quad (21)$$

The resulting consolidated intertemporal budget constraint is then

$$\frac{C_{W,t}^{-\rho}}{\Pi_t} [nb_{H,t-1} + (1-n)b_{F,t-1}] = E_t \left\{ \sum_{T=t}^{\infty} \beta^{T-t} C_{W,T}^{-\rho} [ns_{H,T} + (1-n)s_{F,T}] \right\}. \quad (22)$$

2 The Steady State

This section shows that there exists an optimal policy plan that supports a symmetric steady state with zero inflation and constant values for all variables.

The proof is by construction. I need to find a system of prices and quantities that a benevolent policymaker would implement to maximize aggregate welfare at the union level for a given constant vector of exogenous disturbances, equal across countries ($\bar{a}_{i,t} = \bar{a}$, $\bar{\mu}_{i,t}^w = \bar{\mu}^w > 1$, $\bar{G}_{i,t} = \bar{G} \geq 0$ and $\bar{\varsigma}_{i,t} = \bar{\varsigma}$).

The steady state is indexed by per-capita debt, the terms of trade and price dispersion. I focus on a steady state with positive debt ($\bar{b}_{i,t} = \bar{b} > 0$), terms of trade equal to one ($T_{-1} = 1$) and no price dispersion ($\Delta_{i,-1} = 1$). While the value of \bar{b} is arbitrary, I will discuss below how it is nonetheless subject to an upper bound.

Given a set of initial commitments X_0 that make policy optimal from a timeless perspective, I wish to find a solution that involves a constant policy plus a constant set of commitments, debt levels, terms of trade and price dispersions equal to the initial ones.

The centralized policymaker chooses the sequence

$$\{Y_{i,t}, \Delta_{i,t}, K_{i,t}, F_{i,t}, \Pi_{i,t}, p_{i,t}, b_{i,t}, \tau_{i,t}, C_{W,t}, T_t, \Pi_t\}_{t=0}^{\infty}, \quad i = \{H, F\}$$

that maximizes

$$u_{W,0} = \sum_{t=0}^{\infty} \beta^t \left[\frac{C_{W,t}^{1-\rho}}{1-\rho} - n\Delta_{H,t} \frac{(Y_{H,t}/a_{H,t})^{1+\eta}}{1+\eta} - (1-n)\Delta_{F,t} \frac{(Y_{F,t}/a_{F,t})^{1+\eta}}{1+\eta} \right],$$

subject to

$$\begin{aligned} K_{i,t} \left(\frac{1 - \alpha_i \Pi_{i,t}^{\sigma-1}}{1 - \alpha_i} \right)^{\frac{1+\sigma\eta}{\sigma-1}} &= F_{i,t}, \\ K_{i,t} &= \left(\frac{\sigma}{\sigma-1} \right) \mu_{i,t}^w \left(\frac{Y_{i,t}}{a_{i,t}} \right)^{1+\eta} + \alpha_i \beta \Pi_{i,t+1}^{\sigma(1+\eta)} K_{i,t+1}, \\ F_{i,t} &= (1 - \tau_{i,t}) C_{W,t}^{-\rho} p_{i,t} Y_{i,t} + \alpha_i \beta E_t \Pi_{i,t+1}^{\sigma-1} F_{i,t+1}, \\ \frac{C_{W,t}^{-\rho} b_{i,t-1}}{\Pi_t} &= C_{W,t}^{-\rho} p_{i,t} (\tau_{i,t} Y_{i,t} - \bar{G} - \bar{\varsigma}) + \beta \frac{C_{W,t+1}^{-\rho} b_{i,t}}{\Pi_{t+1}}, \\ \Delta_{i,t} &= \alpha_i \Delta_{i,t-1} \Pi_{i,t}^{\sigma(1+\eta)} + (1 - \alpha_i) \left(\frac{1 - \alpha_i \Pi_{i,t}^{\sigma-1}}{1 - \alpha_i} \right)^{\frac{\sigma(1+\eta)}{\sigma-1}}, \\ Y_{i,t} &= p_{i,t}^{-\theta} C_{W,t} + \bar{G}, \end{aligned}$$

$$\begin{aligned}
p_{H,t}^{\theta-1} &= n + (1-n)T_t^{1-\theta}, & p_{F,t}^{\theta-1} &= nT_t^{\theta-1} + (1-n), \\
\frac{T_t}{T_{t-1}} &= \frac{\Pi_{F,t}}{\Pi_{H,t}}, \\
\Pi_t^{1-\theta} &= n\Pi_{H,t}^{1-\theta}p_{H,t-1}^{1-\theta} + (1-n)\Pi_{F,t}^{1-\theta}p_{F,t-1}^{1-\theta},
\end{aligned}$$

given the initial conditions X_0 , $\Delta_{i,-1}$, $b_{i,-1}$ and T_{-1} . Notice that the relative price for country i has been defined as $p_{i,t} \equiv P_{i,t}/P_t$.

I attach Lagrange multipliers $\phi_{1,t}^i$ through $\phi_{7,t}^i$ to the constraints of the country i ,² and multipliers $\phi_{8,t}^W$ and $\phi_{9,t}^W$ to the last two union-wide constraints. In order to impose constant commitments $X_0 = \bar{X}$, the initial conditions X_0 require additional Lagrange multipliers. These multipliers are normalized in such a way that the first order conditions for $t = 0$ look the same as the first order conditions at a generic period $t > 0$. Since the stationarity constraints for fiscal policy can only be imposed on the consolidated real value of debt at maturity, this additional multiplier is denoted by $\phi_{4,-1}^W$ and in principle differs from $\phi_{4,t}^H$ and $\phi_{4,t}^F$.

Before formally proving the existence of the steady state characterized so far, I take a step back and discuss the upper bound implicit on steady state debt. The government budget constraint is

$$(1 - \beta)\bar{b} = \bar{\tau}\bar{Y} - (\bar{G} + \bar{\zeta}).$$

Assuming that lump-sum transfers alone do not compensate for debt (assumed to be positive) and spending (assumed to be non-negative), the steady state tax rate must be positive. Substituting the resource constraint into the pricing equation gives

$$(1 - \bar{\tau}) \left(1 - \frac{\bar{G}}{\bar{Y}}\right)^{-\rho} = \left(\frac{\sigma}{\sigma - 1}\right) \bar{\mu}^w \frac{\bar{Y}^{\rho+\eta}}{\bar{a}^{1+\eta}},$$

which determines steady state output as a negative function of the steady state tax rate and the steady state markup and a positive function of steady state government purchases and productivity, where the fraction of steady state spending to output is taken as exogenous from the benchmark calibration. In any solution, the steady state tax rate $\bar{\tau}$ must then be less than unity, otherwise output would be zero and so would be revenues for the government. Hence, over the $(0, 1)$ interval, revenues are bounded above (as in a standard Laffer curve argument) and so must be \bar{b} for a steady state to exist consistent.

In order to verify the existence of the proposed steady state, I need to check that the first order conditions of the optimal policy problem above are satisfied for time-invariant Lagrange multipliers.

The first order condition for price dispersion determines the solution for ϕ_5^i

$$\phi_5^H = \frac{n(\bar{Y}/\bar{a})^{1+\eta}}{(1 - \alpha_H\beta)}, \quad \phi_5^F = \frac{(1-n)(\bar{Y}/\bar{a})^{1+\eta}}{(1 - \alpha_F\beta)}.$$

The optimality conditions for debt give $\phi_{4,t}^i = \phi_{4,t-1}^i \forall t$. Here, it is important to stress that the above condition holds at time zero as well, so that $\phi_4^i = \phi_4^W$. This result combined with the first order condition for CPI inflation then implies $\phi_9^W = 0$.

²The Lagrange multiplier on each country's government budget constraint is normalized by the size of its country, which is equivalent to write the government budget constraint in terms of total rather than per-capita levels.

The first order conditions for the tax rates give $\phi_3^H = n\phi_4^W$ and $\phi_3^F = (1-n)\phi_4^W$. The first order conditions for K_i and F_i yield

$$\phi_1^i = -(1-\alpha_i)\phi_2^i$$

and

$$\phi_1^i = (1-\alpha_i)\phi_3^i,$$

which clearly imply $\phi_2^i = -\phi_3^i$. These results, together with the previous finding for ϕ_9^W , can be substituted into one of the first order conditions for GDP inflation to show that $\phi_8^W = 0$.

The set of first order conditions sufficient to determine the steady state value of remaining Lagrange multipliers (ϕ_4^W plus ϕ_6^i, ϕ_7^i) finally consists of (i) the first order condition for consumption

$$-\rho\bar{C}^{-\rho}\phi_4^W + (\phi_6^H + \phi_6^F) = \bar{C}^{-\rho}; \quad (23)$$

(ii) The two first order conditions for output

$$-n\bar{C}^{-\rho}\bar{\tau}\phi_4^W + \phi_6^H = \frac{n\bar{Y}^\eta}{\bar{a}^{1+\eta}} \quad (24)$$

and

$$-(1-n)\bar{C}^{-\rho}\bar{\tau}\phi_4^W + \phi_6^F = \frac{(1-n)\bar{Y}^\eta}{\bar{a}^{1+\eta}}; \quad (25)$$

(iii) The two first order conditions for relative prices

$$-n\bar{C}^{1-\rho}\phi_4^W + \theta\bar{C}\phi_6^H + (\theta-1)\phi_7^H = 0 \quad (26)$$

and

$$-(1-n)\bar{C}^{1-\rho}\phi_4^W + \theta\bar{C}\phi_6^F + (\theta-1)\phi_7^F = 0; \quad (27)$$

(iv) The first order condition for the terms of trade

$$(1-n)\phi_7^H = n\phi_7^F. \quad (28)$$

Hence, there are six conditions left to pin down five Lagrange multipliers. I will show that one condition is redundant so that the system is in fact uniquely determined. To this extent, I add [24] and [25] to obtain

$$-\bar{C}^{-\rho}\bar{\tau}\phi_4^W + (\phi_6^H + \phi_6^F) = \frac{\bar{Y}^\eta}{\bar{a}^{1+\eta}}.$$

I solve for $\phi_6^H + \phi_6^F$ and substitute into [23] to derive the value of ϕ_4^W

$$\phi_4^W = \frac{1-\bar{\mu}^{-1}}{\bar{\tau}-\rho},$$

where

$$\bar{\mu} \equiv \left(\frac{\sigma}{\sigma-1}\right) \left(\frac{\bar{\mu}^w}{1-\bar{\tau}}\right) = \frac{\bar{Y}^\eta/\bar{a}^{1+\eta}}{\bar{C}^{-\rho}}.$$

It then follows that ϕ_6^H and ϕ_6^F can be expressed as a function of the solution for ϕ_4^W and, most importantly, that the solution for their values satisfies

$$(1-n)\phi_6^H = n\phi_6^F. \quad (29)$$

Now, I multiply [26] by $(1 - n)$ and [27] by n . I subtract one from the other and obtain

$$\theta \bar{C} \left[(1 - n) \phi_6^H - n \phi_6^F \right] + (\theta - 1) \left[(1 - n) \phi_7^H - n \phi_7^F \right] = 0.$$

Since the first term of the last equation is zero because of [29], it follows that [28] is indeed redundant. This argument completes the proof of the existence of a well-defined symmetric steady state with zero inflation and positive debt.

The solution for the remaining multipliers is

$$\begin{aligned} \phi_1^H &= n(1 - \alpha_H) \left(\frac{1 - \bar{\mu}^{-1}}{\bar{\tau} - \rho} \right), & \phi_1^F &= (1 - n)(1 - \alpha_F) \left(\frac{1 - \bar{\mu}^{-1}}{\bar{\tau} - \rho} \right) \\ \phi_2^H &= n \left(\frac{\bar{\mu}^{-1} - 1}{\bar{\tau} - \rho} \right), & \phi_2^F &= (1 - n) \left(\frac{\bar{\mu}^{-1} - 1}{\bar{\tau} - \rho} \right) \\ \phi_3^H &= n \left(\frac{1 - \bar{\mu}^{-1}}{\bar{\tau} - \rho} \right), & \phi_3^F &= (1 - n) \left(\frac{1 - \bar{\mu}^{-1}}{\bar{\tau} - \rho} \right) \\ \phi_6^H &= n \bar{C}^{-\rho} \left(\frac{\bar{\tau} - \rho \bar{\mu}^{-1}}{\bar{\tau} - \rho} \right), & \phi_6^F &= (1 - n) \bar{C}^{-\rho} \left(\frac{\bar{\tau} - \rho \bar{\mu}^{-1}}{\bar{\tau} - \rho} \right). \\ \phi_7^H &= \frac{n \bar{C}^{1-\rho}}{\theta - 1} \left[\frac{(1 - \bar{\tau})(1 - \bar{\mu}^{-1})}{\bar{\tau} - \rho} - \theta \bar{\mu}^{-1} \right], & \phi_7^F &= \frac{(1 - n) \bar{C}^{1-\rho}}{\theta - 1} \left[\frac{(1 - \bar{\tau})(1 - \bar{\mu}^{-1})}{\bar{\tau} - \rho} - \theta \bar{\mu}^{-1} \right]. \end{aligned}$$

3 Second Order Approximation of the Utility Function

The second order approximation of the utility function is fairly standard and follows along the lines of Woodford (2003). Up to the second order, the total welfare for country i is

$$\begin{aligned} u_{i,t} &= \bar{C}^{1-\rho} E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[\hat{C}_{W,s} + \frac{1}{2} (1 - \rho) \hat{C}_{W,s}^2 \right. \right. \\ &\quad \left. \left. - (s_c \bar{\mu})^{-1} \left(\hat{Y}_{i,s} + \frac{1}{2} (1 + \eta) \hat{Y}_{i,s}^2 - (1 + \eta) \hat{a}_{i,t} \hat{Y}_{i,s} + \frac{1}{2} \sigma \kappa_i^{-1} \pi_{i,s}^2 \right) \right] \right\} + t.i.p. + \mathcal{O} \left(\left\| \hat{\xi}_{i,t} \right\|^3 \right), \end{aligned} \quad (30)$$

where

$$\kappa_i \equiv \frac{(1 - \alpha_i)(1 - \alpha_i \beta)}{\alpha_i(1 + \sigma \eta)}$$

and $s_c \equiv \bar{C}/\bar{Y}$.

Although consumption is perfectly insured across countries, along the derivation of the approximate policy problem, it will be useful to exploit the symmetry of the model and distinguish consumption in country H ($\hat{C}_t = \hat{C}_{W,t}$) from consumption in country F ($\hat{C}_t^* = \hat{C}_{W,t}$). The vector of endogenous variables in deviations from the steady state can then be defined as

$$x'_t \equiv \left[\hat{Y}_{H,t} \quad \hat{C}_t \quad \hat{p}_{H,t} \quad \hat{\tau}_{H,t} \quad \hat{Y}_{F,t} \quad \hat{C}_t^* \quad \hat{p}_{F,t} \quad \hat{\tau}_{F,t} \quad \hat{T}_t \right]'$$

The single vector of exogenous shocks can be denoted by $\xi_t = \left[\hat{\xi}_{H,t} \quad \hat{\xi}_{F,t} \right]$. The six entries are

$$\xi'_t \equiv \left[\hat{a}_{H,t} \quad \hat{\mu}_{H,t}^w \quad \hat{G}_{H,t} \quad \hat{\varsigma}_{H,t} \quad \hat{a}_{F,t} \quad \hat{\mu}_{F,t}^w \quad \hat{G}_{F,t} \quad \hat{\varsigma}_{F,t} \right]'$$

In matrix notation, welfare for country H is

$$u_{H,t} = \bar{C}^{1-\rho} E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[z'_{H,x} x_s - \frac{1}{2} x'_s Z_{H,x} x_s - x'_s Z_{H,\xi} \xi_s - \frac{1}{2} z_{\pi_H} \pi_{H,s}^2 \right] \right\} + t.i.p. + \mathcal{O}(\|\xi_t\|^3),$$

where

$$z'_{H,x} \equiv [-(s_c \bar{\mu})^{-1} \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],$$

$$Z_{H,x} \equiv \begin{bmatrix} (s_c \bar{\mu})^{-1} (1 + \eta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(1 - \rho) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Z_{H,\xi} \equiv \begin{bmatrix} -(s_c \bar{\mu})^{-1} (1 + \eta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$z_{\pi_H} \equiv \sigma (s_c \bar{\mu} \kappa_H)^{-1}.$$

The Foreign counterpart is

$$u_{F,t} = \bar{C}^{1-\rho} E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[z'_{F,x} x_s - \frac{1}{2} x'_s Z_{F,x} x_s - x'_s Z_{F,\xi} \xi_s - \frac{1}{2} z_{\pi_F} \pi_{F,s}^2 \right] \right\} + t.i.p. + \mathcal{O}(\|\xi_t\|^3),$$

where

$$z'_{F,x} \equiv [0 \quad 0 \quad 0 \quad 0 \quad -(s_c \bar{\mu})^{-1} \quad 1 \quad 0 \quad 0 \quad 0],$$

$$Z_{F,x} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (s_c \bar{\mu})^{-1} (1 + \eta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(1 - \rho) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Z_{F,\xi} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(s_c \bar{\mu})^{-1} (1 + \eta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$z_{\pi_F} \equiv \sigma (s_c \bar{\mu} \kappa_F)^{-1}.$$

3.1 Second Order Approximation of the Equilibrium Relations

The second order approximation of the utility function [30] displays non-zero linear terms in consumption and output. As discussed by Woodford (2003) and Benigno and Woodford (2003), in this case a log-linear approximation of the equilibrium conditions might not deliver the correct approximation of the optimal plan and might lead to an incorrect welfare ranking of alternative policies. The idea is that a linear approximation to the equilibrium conditions implies errors of second order which matter for welfare if linear terms in the objective function are non-zero. In the absence of alternative corrections that eliminate the linear terms from the objective functions,³ Benigno and Woodford (2006b) propose a methodology which (i) derives a second order approximation of the equilibrium conditions and (ii) exploits the relation between those conditions and the coefficients of the linear terms in the second order approximation of the welfare criterion from the deterministic non-linear optimal policy problem. This appendix applies the methodology of Benigno and Woodford (2006b) to the case of a currency union. For simplicity, I will show the derivations only for the Home country.

3.1.1 Optimal Price Setting

The non-linear Phillips curve admits an exact log-linear approximation

$$\left(\frac{1 + \sigma \eta}{\sigma - 1} \right) \hat{V}_{H,t} = \hat{F}_{H,t} - \hat{K}_{H,t}, \quad (31)$$

where

$$V_{H,t} \equiv \frac{1 - \alpha_H \Pi_{H,t}^{\sigma-1}}{1 - \alpha_H}.$$

A second order approximation of $V_{H,t}$ yields

$$\hat{V}_{H,t} = - \left(\frac{\alpha_H}{1 - \alpha_H} \right) (\sigma - 1) \left[\pi_{H,t} + \frac{1}{2} \left(\frac{\sigma - 1}{1 - \alpha_H} \right) \pi_{H,t}^2 \right] + \mathcal{O}(\|\xi_t\|^3). \quad (32)$$

³In the current example, the coefficients on linear terms depend on the steady state total markup. Differently from the previous literature (see Rotemberg and Woodford, 1997), a steady state subsidy that offsets the monopolistic distortions is not a feasible device to reduce the welfare objective to a purely quadratic criterion. The reason is that the presence of either positive debt or public spending requires the steady state tax rate to be positive to satisfy the government budget constraint, provided lump-sum taxes do not cover entirely the fiscal deficit.

A second order approximation of $F_{H,t}$ and $K_{H,t}$ gives

$$\hat{F}_{H,t} + \frac{1}{2}\hat{F}_{H,t}^2 = (1 - \alpha_H\beta) E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H\beta)^{s-t} \left(\hat{f}_{H,s} + \frac{1}{2}\hat{f}_{H,s}^2 \right) \right\} + \mathcal{O}(\|\xi_t\|^3)$$

and

$$\hat{K}_{H,t} + \frac{1}{2}\hat{K}_{H,t}^2 = (1 - \alpha_H\beta) E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H\beta)^{s-t} \left(\hat{k}_{H,s} + \frac{1}{2}\hat{k}_{H,s}^2 \right) \right\} + \mathcal{O}(\|\xi_t\|^3),$$

where $f_{H,s}$ and $k_{H,s}$ are the forcing terms of $F_{H,t}$ and $K_{H,t}$ whose difference and sum give

$$\hat{f}_{H,s} - \hat{k}_{H,s} = -\hat{\mu}_{H,s}^w - \eta\hat{Y}_{H,s} + (1 + \eta)\hat{a}_{H,s} - (1 + \sigma\eta)\pi_{H,s}^t + \hat{p}_{H,s} + \hat{S}_{H,s} - \rho\hat{C}_s$$

and

$$\hat{f}_{H,s} + \hat{k}_{H,s} = \hat{\mu}_{H,s}^w + (2 + \eta)\hat{Y}_{H,s} - (1 + \eta)\hat{a}_{H,s} + [\sigma(1 + \eta) + (\sigma - 1)]\pi_{H,s}^t + \hat{p}_{H,s} + \hat{S}_{H,s} - \rho\hat{C}_s.$$

In the previous expressions, the GDP inflation rate between time t and time s is defined as

$$\pi_{H,s}^t \equiv \ln \left(\frac{P_{H,s}}{P_{H,t}} \right),$$

whereas

$$\hat{S}_{H,t} \equiv \ln \left(\frac{1 - \tau_{H,t}}{1 - \bar{\tau}} \right). \quad (33)$$

Substituting the approximations of $F_{H,t}$ and $K_{H,t}$ into [31] gives

$$\begin{aligned} \left(\frac{1 + \sigma\eta}{\sigma - 1} \right) \hat{V}_{H,t} = & (1 - \alpha_H\beta) E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H\beta)^{s-t} \left[\left(\hat{f}_{H,s} - \hat{k}_{H,s} \right) + \frac{1}{2} \left(\hat{f}_{H,s}^2 - \hat{k}_{H,s}^2 \right) \right] \right\} + \\ & - \frac{1}{2} \left(\hat{F}_{H,t}^2 - \hat{K}_{H,t}^2 \right) + \mathcal{O}(\|\xi_t\|^3). \end{aligned}$$

I further define

$$\hat{X}_{H,t} \equiv (2 + \eta)\hat{Y}_{H,t} - (1 + \eta)\hat{a}_{H,t} - \rho\hat{C}_t + \hat{p}_{H,t} + \hat{\mu}_{H,t}^w + \hat{S}_{H,t},$$

which implies

$$\hat{f}_{H,s} + \hat{k}_{H,s} = \hat{X}_{H,s} + [\sigma(1 + \eta) + (\sigma - 1)]\pi_{H,s}^t.$$

I can then replace in the expression for $\hat{V}_{H,t}$ to obtain

$$\begin{aligned} \left(\frac{1 + \sigma\eta}{\sigma - 1} \right) \hat{V}_{H,t} = & (1 - \alpha_H\beta) E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H\beta)^{s-t} \left(\hat{f}_{H,s} - \hat{k}_{H,s} \right) \right\} - \frac{1}{2} \left(\hat{F}_{H,t}^2 - \hat{K}_{H,t}^2 \right) \\ + \left(\frac{1 - \alpha_H\beta}{2} \right) E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H\beta)^{s-t} \left[-\hat{\mu}_{H,s}^w + \hat{S}_{H,s} - \eta\hat{Y}_{H,s} + (1 + \eta)\hat{a}_{H,s} - (1 + \sigma\eta)\pi_{H,s}^t + \hat{p}_{H,s} - \rho\hat{C}_s \right] \right. \\ & \left. \cdot \left[\hat{X}_{H,s} + (2\sigma + \sigma\eta - 1)\pi_{H,s}^t \right] \right\} + \mathcal{O}(\|\xi_t\|^3). \end{aligned}$$

From the approximation of $F_{H,t}$ and $K_{H,t}$, one can then observe that

$$\begin{aligned} \hat{F}_{H,t} + \hat{K}_{H,t} &= (1 - \alpha_H \beta) E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H \beta)^{s-t} \left[(\hat{f}_{H,s} + \hat{k}_{H,s}) + \frac{1}{2} (\hat{f}_{H,s}^2 + \hat{k}_{H,s}^2) \right] \right\} \\ &\quad - \frac{1}{2} (\hat{F}_{H,t}^2 + \hat{K}_{H,t}^2) + \mathcal{O}(\|\xi_t\|^3). \end{aligned}$$

I multiply the last expression by [31] and I have that

$$\hat{F}_{H,t}^2 - \hat{K}_{H,t}^2 = \left(\frac{1 + \sigma \eta}{\sigma - 1} \right) \hat{V}_{H,t} \cdot E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H \beta)^{s-t} (\hat{f}_{H,s} + \hat{k}_{H,s}) \right\} + \mathcal{O}(\|\xi_t\|^3).$$

Substituting back into the expression for $\hat{V}_{H,t}$ above, I get

$$\begin{aligned} \left(\frac{1 + \sigma \eta}{\sigma - 1} \right) \hat{V}_{H,t} &= (1 - \alpha_H \beta) E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H \beta)^{s-t} (\hat{f}_{H,s} - \hat{k}_{H,s}) \right\} \\ &\quad + \left(\frac{1 - \alpha_H \beta}{2} \right) E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H \beta)^{s-t} (\hat{f}_{H,s} - \hat{k}_{H,s}) \left[\hat{X}_{H,s} + (2\sigma + \sigma \eta - 1) \pi_{H,s}^t \right] \right\} \\ &\quad - \frac{1}{2} (1 - \alpha_H \beta) \left(\frac{1 + \sigma \eta}{\sigma - 1} \right) \hat{V}_{H,t} \cdot E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H \beta)^{s-t} \left[\hat{X}_{H,s} + (2\sigma + \sigma \eta - 1) \pi_{H,s}^t \right] \right\} + \mathcal{O}(\|\xi_t\|^3). \end{aligned}$$

I define

$$Z_{H,t} \equiv E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H \beta)^{s-t} \left[\hat{X}_{H,s} + (2\sigma + \sigma \eta - 1) \pi_{H,s}^t \right] \right\}. \quad (34)$$

I can then rewrite

$$\begin{aligned} \frac{(1 + \sigma \eta)}{(1 - \alpha_H \beta)(\sigma - 1)} \hat{V}_{H,t} &= -\frac{1}{2} \left(\frac{1 + \sigma \eta}{\sigma - 1} \right) \hat{V}_{H,t} Z_{H,t} + E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H \beta)^{s-t} (\hat{f}_{H,s} - \hat{k}_{H,s}) \right\} \\ &\quad + \frac{1}{2} E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H \beta)^{s-t} (\hat{f}_{H,s} - \hat{k}_{H,s}) \left[\hat{X}_{H,s} + (2\sigma + \sigma \eta - 1) \pi_{H,s}^t \right] \right\} + \mathcal{O}(\|\xi_t\|^3). \end{aligned} \quad (35)$$

I update [35] one period, multiply by $\alpha_H \beta$ and take expectations at time t . I subtract the resulting expression from the time t version as to obtain

$$\begin{aligned} \frac{(1 + \sigma \eta)}{(1 - \alpha_H \beta)(\sigma - 1)} \left[\hat{V}_{H,t} - (\alpha_H \beta) E_t \hat{V}_{H,t+1} \right] &= -\frac{1}{2} \left(\frac{1 + \sigma \eta}{\sigma - 1} \right) \left[\hat{V}_{H,t} Z_{H,t} - (\alpha_H \beta) E_t \hat{V}_{H,t+1} Z_{H,t+1} \right] \\ &\quad + E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H \beta)^{s-t} (\hat{f}_{H,s} - \hat{k}_{H,s}) \right\} - (\alpha_H \beta) E_t \left\{ \sum_{s=t+1}^{\infty} (\alpha_H \beta)^{s-t-1} (\hat{f}_{H,s} - \hat{k}_{H,s}) \right\} \\ &\quad + \frac{1}{2} E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H \beta)^{s-t} (\hat{f}_{H,s} - \hat{k}_{H,s}) \left[\hat{X}_{H,s} + (2\sigma + \sigma \eta - 1) \pi_{H,s}^t \right] \right\} \\ &\quad - \frac{1}{2} (\alpha_H \beta) E_t \left\{ \sum_{s=t+1}^{\infty} (\alpha_H \beta)^{s-t-1} (\hat{f}_{H,s} - \hat{k}_{H,s}) \left[\hat{X}_{H,s} + (2\sigma + \sigma \eta - 1) \pi_{H,s}^{t+1} \right] \right\} + \mathcal{O}(\|\xi_t\|^3). \end{aligned}$$

I define

$$z_{H,s} \equiv \hat{f}_{H,s} - \hat{k}_{H,s} + (1 + \sigma\eta) \pi_{H,s}^t = -\hat{\mu}_{H,s}^w + \hat{S}_{H,s} - \eta \hat{Y}_{H,s} + (1 + \eta) \hat{a}_{H,s} + \hat{p}_{H,s} - \rho \hat{C}_s. \quad (36)$$

I can then rewrite the previous expression as

$$\begin{aligned} & \frac{(1 + \sigma\eta)}{(1 - \alpha_H\beta)(\sigma - 1)} \left[\hat{V}_{H,t} - (\alpha_H\beta) E_t \hat{V}_{H,t+1} \right] = -\frac{1}{2} \left(\frac{1 + \sigma\eta}{\sigma - 1} \right) \left[\hat{V}_{H,t} Z_{H,t} - (\alpha_H\beta) E_t \hat{V}_{H,t+1} Z_{H,t+1} \right] \\ & + E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H\beta)^{s-t} [z_{H,s} - (1 + \sigma\eta) \pi_{H,s}^t] \right\} - (\alpha_H\beta) E_t \left\{ \sum_{s=t+1}^{\infty} (\alpha_H\beta)^{s-t-1} [z_{H,s} - (1 + \sigma\eta) \pi_{H,s}^{t+1}] \right\} \\ & \quad + \frac{1}{2} E_t \left\{ \sum_{s=t}^{\infty} (\alpha_H\beta)^{s-t} [z_{H,s} - (1 + \sigma\eta) \pi_{H,s}^t] \left[\hat{X}_{H,s} + (2\sigma + \sigma\eta - 1) \pi_{H,s}^t \right] \right\} \\ & - \frac{1}{2} (\alpha_H\beta) E_t \left\{ \sum_{s=t+1}^{\infty} (\alpha_H\beta)^{s-t-1} [z_{H,s} - (1 + \sigma\eta) \pi_{H,s}^{t+1}] \left[\hat{X}_{H,s} + (2\sigma + \sigma\eta - 1) \pi_{H,s}^{t+1} \right] \right\} + \mathcal{O}(\|\xi_t\|^3). \end{aligned}$$

I use the discount factors in the sums and the fact that $\pi_{H,t}^t = 0$ and that, $\forall s, \pi_{H,s}^t - \pi_{H,s}^{t+1} = \pi_{H,t+1}^t = \pi_{H,t+1}$ to rewrite

$$\begin{aligned} & \frac{(1 + \sigma\eta)}{(1 - \alpha_H\beta)(\sigma - 1)} \left[\hat{V}_{H,t} - (\alpha_H\beta) E_t \hat{V}_{H,t+1} \right] = -\frac{1}{2} \left(\frac{1 + \sigma\eta}{\sigma - 1} \right) \left[\hat{V}_{H,t} Z_{H,t} - (\alpha_H\beta) E_t \hat{V}_{H,t+1} Z_{H,t+1} \right] \\ & \quad + z_{H,t} + \frac{1}{2} z_{H,t} X_{H,t} - \frac{(\alpha_H\beta)(1 + \sigma\eta)}{(1 - \alpha_H\beta)} E_t \pi_{H,t+1} \\ & \quad + \left(\frac{\alpha_H\beta}{2} \right) E_t \left\{ \sum_{s=t+1}^{\infty} (\alpha_H\beta)^{s-t-1} \left[(2\sigma + \sigma\eta - 1) \pi_{H,t+1} z_{H,s} - (1 + \sigma\eta) \pi_{H,t+1} \hat{X}_{H,s} \right. \right. \\ & \quad \left. \left. - (1 + \sigma\eta)(2\sigma + \sigma\eta - 1) \left(\pi_{H,t+1}^2 + 2\pi_{H,t+1} \pi_{H,s}^{t+1} \right) \right] \right\} + \mathcal{O}(\|\xi_t\|^3). \end{aligned}$$

Observe that $\pi_{H,t+1} + \pi_{H,s}^{t+1} = \pi_{H,s}^t$. Using the previous definition of $Z_{H,t}$, I can readjust to get

$$\begin{aligned} & \frac{(1 + \sigma\eta)}{(1 - \alpha_H\beta)(\sigma - 1)} \left[\hat{V}_{H,t} - (\alpha_H\beta) E_t \hat{V}_{H,t+1} \right] = -\frac{1}{2} \left(\frac{1 + \sigma\eta}{\sigma - 1} \right) \left[\hat{V}_{H,t} Z_{H,t} - (\alpha_H\beta) E_t \hat{V}_{H,t+1} Z_{H,t+1} \right] \\ & \quad + z_{H,t} + \frac{1}{2} z_{H,t} X_{H,t} - \frac{(\alpha_H\beta)(1 + \sigma\eta)}{(1 - \alpha_H\beta)} E_t \pi_{H,t+1} - \frac{(\alpha_H\beta)(1 + \sigma\eta)(2\sigma + \sigma\eta - 1)}{(1 - \alpha_H\beta)} E_t \pi_{H,t+1}^2 \\ & \quad + \left(\frac{\alpha_H\beta}{2} \right) E_t \left\{ \pi_{H,t+1} \sum_{s=t+1}^{\infty} (\alpha_H\beta)^{s-t-1} (2\sigma + \sigma\eta - 1) [z_{H,s} - (1 + \sigma\eta) \pi_{H,s}^{t+1}] \right\} \\ & \quad - \left(\frac{\alpha_H\beta}{2} \right) (1 + \sigma\eta) E_t (\pi_{H,t+1} Z_{H,t+1}). \end{aligned}$$

From the definition of $z_{H,t}$ [36] and [35] it follows that

$$E_t \left\{ \pi_{H,t+1} \sum_{s=t+1}^{\infty} (\alpha_H\beta)^{s-t-1} [z_{H,s} - (1 + \sigma\eta) \pi_{H,s}^{t+1}] \right\} = \frac{(1 + \sigma\eta)}{(1 - \alpha_H\beta)(\sigma - 1)} E_t \left(\hat{V}_{H,t+1} \pi_{H,t+1} \right) + \mathcal{O}(\|\xi_t\|^3).$$

I substitute back to obtain

$$\begin{aligned} \frac{(1+\sigma\eta)}{(1-\alpha_H\beta)(\sigma-1)} \left[\hat{V}_{H,t} - (\alpha_H\beta) E_t \hat{V}_{H,t+1} \right] &= -\frac{1}{2} \left(\frac{1+\sigma\eta}{\sigma-1} \right) \left[\hat{V}_{H,t} Z_{H,t} - (\alpha_H\beta) E_t \hat{V}_{H,t+1} Z_{H,t+1} \right] \\ &+ z_{H,t} + \frac{1}{2} z_{H,t} X_{H,t} - \frac{(\alpha_H\beta)(1+\sigma\eta)}{(1-\alpha_H\beta)} E_t \pi_{H,t+1} - \frac{(\alpha_H\beta)(1+\sigma\eta)(2\sigma+\sigma\eta-1)}{(1-\alpha_H\beta)} E_t \pi_{H,t+1}^2 \\ &+ \frac{(\alpha_H\beta)(1+\sigma\eta)}{2(1-\alpha_H\beta)(\sigma-1)} E_t \left(\hat{V}_{H,t+1} \pi_{H,t+1} \right) - \left(\frac{\alpha_H\beta}{2} \right) (1+\sigma\eta) E_t (\pi_{H,t+1} Z_{H,t+1}). \end{aligned}$$

Next, I use the expression for $\hat{V}_{H,t}$ in [32] together with the definition of κ_H

$$\begin{aligned} \pi_{H,t} + \left[\frac{(\sigma-1)}{2(1-\alpha_H)} \right] \pi_{H,t}^2 + \left(\frac{1-\alpha_H\beta}{2} \right) \pi_{H,t} Z_{H,t} &= \beta E_t \pi_{H,t+1} + \left[\frac{\beta(1-\alpha_H\beta)}{2} \right] E_t (\pi_{H,t+1} Z_{H,t+1}) \\ -\kappa_H \left(z_{H,t} + \frac{1}{2} z_{H,t} X_{H,t} \right) + \left[\frac{(\alpha_H\beta)(\sigma-1)}{2(1-\alpha_H)} \right] E_t \pi_{H,t+1}^2 &+ \left[\frac{\beta(2\sigma+\sigma\eta-1)}{2} \right] E_t \pi_{H,t+1}^2 + \mathcal{O}(\|\xi_t\|^3). \end{aligned}$$

I define

$$\tilde{V}_{H,t} \equiv \pi_{H,t} + \left[\frac{(\sigma-1)}{2(1-\alpha_H)} \right] \pi_{H,t}^2 + \left(\frac{1-\alpha_H\beta}{2} \right) \pi_{H,t} Z_{H,t} + \left[\frac{\sigma(1+\eta)}{2} \right] \pi_{H,t}^2.$$

Notice that

$$\frac{\alpha_H(\sigma-1)}{1-\alpha_H} + (2\sigma+\sigma\eta-1) = \frac{\sigma-1}{1-\alpha_H} + \sigma(1+\eta).$$

I further define the parameters

$$v_{\pi_H} \equiv \sigma(1+\eta) + \frac{\sigma-1}{1-\alpha_H}, \quad v_{z_H} \equiv \frac{1-\alpha_H\beta}{2}.$$

Then, it is straightforward to see that

$$\tilde{V}_{H,t} = \pi_{H,t} + \frac{1}{2} v_{\pi_H} \pi_{H,t}^2 + v_{z_H} \pi_{H,t} Z_{H,t}.$$

It then follows that

$$\begin{aligned} \tilde{V}_{H,t} &= \beta \left\{ E_t \pi_{H,t+1} + \left(\frac{1-\alpha_H\beta}{2} \right) E_t (\pi_{H,t+1} Z_{H,t+1}) + \frac{1}{2} \left[\frac{\alpha_H(\sigma-1)}{2(1-\alpha_H)} + \frac{(2\sigma+\sigma\eta-1)}{2} \right] E_t \pi_{H,t+1}^2 \right\} \\ &- \kappa_H \left(z_{H,t} + \frac{1}{2} z_{H,t} X_{H,t} \right) + \left[\frac{\sigma(1+\eta)}{2} \right] \pi_{H,t}^2 + \mathcal{O}(\|\xi_t\|^3). \end{aligned}$$

Using again the definition of $\tilde{V}_{H,t}$ and the identity of the coefficients above, a recursive representation of the previous expression is

$$\tilde{V}_{H,t} = -\kappa_H \left(z_{H,t} + \frac{1}{2} z_{H,t} X_{H,t} \right) + \left[\frac{\sigma(1+\eta)}{2} \right] \pi_{H,t}^2 + \beta E_t \tilde{V}_{H,t+1} + \mathcal{O}(\|\xi_t\|^3).$$

Integrating forward the last equation gives

$$\tilde{V}_{H,t} = E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[-\kappa_H \left(z_{H,s} + \frac{1}{2} z_{H,s} X_{H,s} \right) + \frac{\sigma(1+\eta)}{2} \pi_{H,s}^2 \right] \right\} + \mathcal{O}(\|\xi_t\|^3).$$

For simplicity, I redefine $V_{H,t} = \kappa_H^{-1} \tilde{V}_{H,t}$. The previous expression becomes

$$V_{H,t} = E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[- \left(z_{H,s} + \frac{1}{2} z_{H,s} X_{H,s} \right) + \frac{\sigma(1+\eta)}{2\kappa_H} \pi_{H,s}^2 \right] \right\} + \mathcal{O} \left(\|\xi_t\|^3 \right).$$

From definition [33], I take a second order approximation as to obtain

$$\hat{S}_{H,t} = -\omega_\tau \hat{\tau}_{H,t} - \frac{1}{2} \left(\frac{\omega_\tau}{1-\bar{\tau}} \right) \hat{\tau}_{H,t}^2 + \mathcal{O} \left(\|\xi_t\|^3 \right),$$

where

$$\omega_\tau \equiv \frac{\bar{\tau}}{1-\bar{\tau}}.$$

I can then substitute into the expression for $V_{H,t}$ using also the definitions of $\hat{X}_{H,t}$ and $z_{H,t}$. The result is

$$\begin{aligned} V_{H,t} = E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[\eta \hat{Y}_{H,s} - (1+\eta) \hat{a}_{H,s} + \rho \hat{C}_s - \hat{p}_{H,s} + \omega_\tau \hat{\tau}_{H,s} + \frac{1}{2} \omega_\tau (1+\omega_\tau) \hat{\tau}_{H,s}^2 + \hat{\mu}_{H,s}^w \right] \right. \\ \left. + \frac{1}{2} \left[\eta \hat{Y}_{H,s} - (1+\eta) \hat{a}_{H,s} + \rho \hat{C}_s - \hat{p}_{H,s} + \omega_\tau \hat{\tau}_{H,s} + \hat{\mu}_{H,s}^w \right] \right. \\ \left. \left[(2+\eta) \hat{Y}_{H,s} - (1+\eta) \hat{a}_{H,s} - \rho \hat{C}_s + \hat{p}_{H,s} + \hat{\mu}_{H,s}^w - \omega_\tau \hat{\tau}_{H,s} \right] + \frac{\sigma(1+\eta)}{2\kappa_H} \pi_{H,s}^2 \right\} + s.o.t.i.p. + \mathcal{O} \left(\|\xi_t\|^3 \right), \end{aligned}$$

where *s.o.t.i.p.* stands for “second order terms independent of policy”. I can rewrite the aggregate supply relation in matrix form using the vectors defined in the previous section as

$$V_{H,t} = E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[a'_{H,x} x_s + a'_{H,\xi} \xi_s + \frac{1}{2} x'_s A_{H,x} x_s + x'_s A_{H,\xi} \xi_s + \frac{1}{2} a_{\pi_H} \pi_{H,s}^2 \right] \right\} + s.o.t.i.p. + \mathcal{O} \left(\|\xi_t\|^3 \right),$$

where

$$a'_{H,x} \equiv [\eta \quad \rho \quad -1 \quad \omega_\tau \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],$$

$$a'_{H,\xi} \equiv [-(1+\eta) \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],$$

$$A_{H,x} \equiv \begin{bmatrix} \eta(2+\eta) & \rho & -1 & \omega_\tau & 0 & 0 & 0 & 0 & 0 \\ \rho & -\rho^2 & \rho & -\rho\omega_\tau & 0 & 0 & 0 & 0 & 0 \\ -1 & \rho & -1 & \omega_\tau & 0 & 0 & 0 & 0 & 0 \\ \omega_\tau & -\rho\omega_\tau & \omega_\tau & \omega_\tau & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{H,\xi} \equiv \begin{bmatrix} -(1+\eta)^2 & 1+\eta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$a_{\pi_H} \equiv \sigma(1+\eta)\kappa_H^{-1}.$$

Analogously, the Foreign aggregate supply equation in matrix notation is

$$V_{F,t} = E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[a'_{F,x} x_s + a'_{F,\xi} \xi_s + \frac{1}{2} x'_s A_{F,x} x_s + x'_s A_{F,\xi} \xi_s + \frac{1}{2} a_{\pi_F} \pi_{F,s}^2 \right] \right\} + s.o.t.i.p. + \mathcal{O}(\|\xi_t\|^3),$$

where

$$a'_{F,x} \equiv [0 \ 0 \ 0 \ 0 \ \eta \ \rho \ -1 \ \omega_\tau \ 0],$$

$$a'_{F,\xi} \equiv [0 \ 0 \ 0 \ 0 \ -(1+\eta) \ 1 \ 0 \ 0],$$

$$A_{F,x} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta(2+\eta) & \rho & -1 & \omega_\tau & 0 \\ 0 & 0 & 0 & 0 & \rho & -\rho^2 & \rho & -\rho\omega_\tau & 0 \\ 0 & 0 & 0 & 0 & -1 & \rho & -1 & \omega_\tau & 0 \\ 0 & 0 & 0 & 0 & \omega_\tau & -\rho\omega_\tau & \omega_\tau & \omega_\tau & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{F,\xi} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(1+\eta)^2 & 1+\eta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$a_{\pi_F} \equiv \sigma(1+\eta)\kappa_F^{-1}.$$

3.1.2 Intertemporal Government Budget Constraint

In the global problem, the national government budget constraints can be written recursively but not necessarily in intertemporal form. The reason is that in an open economy the only restriction implied by the households' transversality condition concerns the sum of debt issued by the two governments, not the stock issued by the single fiscal authorities. Nonetheless, the existence of a symmetric steady state with finite debt and the equilibrium concept adopted in this paper rule out equilibria in which (i) one fiscal authority issues an infinite amount of liabilities, (ii) one fiscal authority indefinitely accumulates government securities issued by the other country, (iii) the transversality condition is never violated. Hence, the focus will be on the case in which the intertemporal government budget constraint actually holds at the country level, at least locally. This section derives a second order approximation of the intertemporal government budget constraint for country H .

The flow government budget constraint can be cast as

$$R_t^{-1}b_{H,t} = \frac{b_{H,t-1}}{\Pi_t} - s_{H,t}.$$

From the definition of the nominal interest rate and the fact that b_t belongs to the information set at time t , the previous expression can be written alternatively as

$$W_{H,t} = C_t^{-\rho} s_{H,t} + \beta E_t W_{H,t+1}, \quad (37)$$

where, $W_{H,t} \equiv C_t^{-\rho} b_{H,t-1} / \Pi_t$. A second order approximation of [37] gives

$$\begin{aligned} \frac{W_{H,t} - \bar{W}}{\bar{W}} &= (1 - \beta) \left[-\rho \hat{C}_t + \frac{1}{2} \rho^2 \hat{C}_t^2 + \left(\frac{s_{H,t} - \bar{s}}{\bar{s}} \right) - \rho \hat{C}_t \left(\frac{s_{H,t} - \bar{s}}{\bar{s}} \right) \right] \\ &+ \beta E_t \left\{ \frac{W_{H,t} - \bar{W}}{\bar{W}} \right\} + \mathcal{O}(\|\xi_t\|^3). \end{aligned} \quad (38)$$

The definition of per-capita surplus is

$$s_{H,t} = p_{H,t} (\tau_{H,t} Y_{H,t} - G_{H,t}) - \varsigma_{H,t}.$$

Therefore, a second order approximation can be derived as

$$\begin{aligned} \frac{s_{H,t} - \bar{s}}{\bar{s}} &= (1 + \omega_g) \hat{Y}_{H,t} + \hat{p}_{H,t} + (1 + \omega_g) \hat{\tau}_{H,t} - s_d \left(\hat{G}_{H,t} + \hat{\varsigma}_{H,t} \right) \\ &+ \frac{1}{2} \left[(1 + \omega_g) \hat{Y}_{H,t}^2 + \hat{p}_{H,t}^2 + (1 + \omega_g) \hat{\tau}_{H,t}^2 \right] \\ &+ (1 + \omega_g) \hat{Y}_{H,t} \hat{p}_{H,t} + (1 + \omega_g) \hat{Y}_{H,t} \hat{\tau}_{H,t} \\ &+ (1 + \omega_g) \hat{p}_{H,t} \hat{\tau}_{H,t} - s_d \hat{p}_{H,t} \left(\hat{G}_{H,t} + \hat{\varsigma}_{H,t} \right) + s.o.t.i.p. + \mathcal{O}(\|\xi_t\|^3), \end{aligned} \quad (39)$$

where, in steady state, $\bar{s} = \bar{\tau} \bar{Y} - (\bar{G} + \bar{\varsigma})$ and I have defined $s_d \equiv \bar{Y} / \bar{s}$ and $\omega_g \equiv s_g s_d$. Let $\tilde{W}_{H,t} \equiv (W_{H,t} - \bar{W}) / \bar{W}$. Up to a second order approximation, it follows that

$$\tilde{W}_{H,t} = \hat{W}_{H,t} + \frac{1}{2} \hat{W}_{H,t}^2 + \mathcal{O}(\|\xi_t\|^3),$$

where $\hat{W}_{H,t}$ stands for the percentage deviations of $W_{H,t}$ from the steady state value in per-capita terms \bar{W} . It then follows that

$$\tilde{W}_{H,t} = \hat{b}_{H,t-1} - \rho \hat{C}_t - \pi_t + \frac{1}{2} \left(\hat{b}_{H,t-1} - \rho \hat{C}_t - \pi_t \right)^2 + \mathcal{O}(\|\xi_t\|^3).$$

Combining [39] and [38] gives

$$\begin{aligned}
\tilde{W}_{H,t} &= (1 - \beta) \left\{ (1 + \omega_g) \hat{Y}_{H,t} - \rho \hat{C}_t + \hat{p}_{H,t} + (1 + \omega_g) \hat{\tau}_{H,t} \right. \\
&\quad + \frac{1}{2} \left[(1 + \omega_g) \hat{Y}_{H,t}^2 + \rho^2 \hat{C}_t^2 + \hat{p}_{H,t}^2 + (1 + \omega_g) \hat{\tau}_{H,t}^2 \right] \\
&\quad - \rho (1 + \omega_g) \hat{Y}_{H,t} \hat{C}_t + (1 + \omega_g) \hat{Y}_{H,t} \hat{p}_{H,t} + (1 + \omega_g) \hat{Y}_{H,t} \hat{\tau}_{H,t} \\
&\quad - \rho \hat{C}_t \hat{p}_{H,t} - \rho (1 + \omega_g) \hat{C}_t \hat{\tau}_{H,t} + (1 + \omega_g) \hat{p}_{H,t} \hat{\tau}_{H,t} \\
&\quad \left. + s_d \left(\rho \hat{C}_t - \hat{p}_{H,t} - 1 \right) \left(\hat{G}_{H,t} + \hat{\varsigma}_{H,t} \right) \right\} + \beta E_t \tilde{W}_{H,t+1} + s.o.t.i.p. + \mathcal{O} \left(\|\xi_t\|^3 \right), \tag{40}
\end{aligned}$$

Expression [40] is now “local”, in the sense that it involves small deviations of variables from their steady state values. Since it has been proved that such a steady state is not characterized by explosive paths for the real value of debt at maturity, I can safely integrate the previous expression forward to get

$$\begin{aligned}
\tilde{W}_{H,t} &= (1 - \beta) E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[(1 + \omega_g) \hat{Y}_{H,s} - \rho \hat{C}_s + \hat{p}_{H,s} + (1 + \omega_g) \hat{\tau}_{H,s} \right. \right. \\
&\quad + \frac{1}{2} \left((1 + \omega_g) \hat{Y}_{H,s}^2 + \rho^2 \hat{C}_s^2 + \hat{p}_{H,s}^2 + (1 + \omega_g) \hat{\tau}_{H,s}^2 \right) \\
&\quad - \rho (1 + \omega_g) \hat{Y}_{H,s} \hat{C}_s + (1 + \omega_g) \hat{Y}_{H,s} \hat{p}_{H,s} + (1 + \omega_g) \hat{Y}_{H,s} \hat{\tau}_{H,s} \\
&\quad - \rho \hat{C}_s \hat{p}_{H,s} - \rho (1 + \omega_g) \hat{C}_s \hat{\tau}_{H,s} + (1 + \omega_g) \hat{p}_{H,s} \hat{\tau}_{H,s} \\
&\quad \left. \left. + s_d \left(\rho \hat{C}_s - \hat{p}_{H,s} - 1 \right) \left(\hat{G}_{H,s} + \hat{\varsigma}_{H,s} \right) \right] \right\} + s.o.t.i.p. + \mathcal{O} \left(\|\xi_t\|^3 \right). \tag{41}
\end{aligned}$$

I rewrite the previous expression in matrix notation as

$$\tilde{W}_{H,t} = (1 - \beta) E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[\omega'_{H,x} x_s + \omega'_{H,\xi} \xi_s + \frac{1}{2} x'_s W_{H,x} x_s + x'_s W_{H,\xi} \xi_s \right] \right\} + s.o.t.i.p. + \mathcal{O} \left(\|\xi_t\|^3 \right),$$

where I have defined

$$\omega'_{H,x} \equiv [1 + \omega_g \quad -\rho \quad 1 \quad 1 + \omega_g \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],$$

$$\omega'_{H,\xi} \equiv [0 \quad 0 \quad -s_d \quad -s_d \quad 0 \quad 0 \quad 0 \quad 0],$$

$$W_{H,x} \equiv \begin{bmatrix} 1 + \omega_g & -\rho(1 + \omega_g) & 1 + \omega_g & 1 + \omega_g & 0 & 0 & 0 & 0 & 0 \\ -\rho(1 + \omega_g) & \rho^2 & -\rho & -\rho(1 + \omega_g) & 0 & 0 & 0 & 0 & 0 \\ 1 + \omega_g & -\rho & 1 & 1 + \omega_g & 0 & 0 & 0 & 0 & 0 \\ 1 + \omega_g & -\rho(1 + \omega_g) & 1 + \omega_g & 1 + \omega_g & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$W_{H,\xi} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho s_d & \rho s_d & 0 & 0 & 0 & 0 \\ 0 & 0 & -s_d & -s_d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Foreign equation is

$$\tilde{W}_{F,t} = (1 - \beta) E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[\omega'_{F,x} x_s + \omega'_{F,\xi} \xi_s + \frac{1}{2} x'_s W_{F,x} x_s + x'_s W_{F,\xi} \xi_s \right] \right\} + s.o.t.i.p. + \mathcal{O}(\|\xi_t\|^3),$$

where

$$\omega'_{F,x} \equiv [0 \ 0 \ 0 \ 0 \ 1 + \omega_g \ -\rho \ 1 \ 1 + \omega_g \ 0],$$

$$\omega'_{F,\xi} \equiv [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ -s_d \ -s_d],$$

$$W_{F,x} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 + \omega_g & -\rho(1 + \omega_g) & 1 + \omega_g & 1 + \omega_g \\ 0 & 0 & 0 & 0 & -\rho(1 + \omega_g) & \rho^2 & -\rho & -\rho(1 + \omega_g) \\ 0 & 0 & 0 & 0 & 1 + \omega_g & -\rho & 1 & 1 + \omega_g \\ 0 & 0 & 0 & 0 & 1 + \omega_g & -\rho(1 + \omega_g) & 1 + \omega_g & 1 + \omega_g \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$W_{F,\xi} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho s_d & \rho s_d \\ 0 & 0 & 0 & 0 & 0 & 0 & -s_d & -s_d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3.1.3 Aggregate Demand

The equilibrium in the goods market for country H is

$$Y_{H,t} = p_{H,t}^{-\theta} C_{W,t} + G_{H,t}.$$

I can write total (average) consumption as

$$C_{W,t} = nC_t + (1-n)C_t^*.$$

Combining the previous two expressions and taking a second order approximation gives

$$\begin{aligned} \hat{Y}_{H,t} = & ns_c \hat{C}_t - \theta s_c \hat{p}_{H,t} + (1-n) s_c \hat{C}_t^* + \hat{G}_{H,t} + \\ & + \frac{1}{2} \left[ns_c (1 - ns_c) \hat{C}_t^2 + \theta^2 s_c (1 - s_c) \hat{p}_{H,t}^2 + (1-n) s_c (1 - (1-n) s_c) \left(\hat{C}_t^* \right)^2 \right] \\ & - n (1-n) s_c^2 \hat{C}_t \hat{C}_t^* - \theta ns_c (1 - s_c) \hat{C}_t \hat{p}_{H,t} - \theta (1-n) s_c (1 - s_c) \hat{p}_{H,t} \hat{C}_t^* \\ & - ns_c \hat{C}_t \hat{G}_{H,t} + \theta s_c \hat{p}_{H,t} \hat{G}_{H,t} - (1-n) s_c \hat{C}_t^* \hat{G}_{H,t} + \mathcal{O} \left(\|\xi_t\|^3 \right). \end{aligned}$$

In vector notation, the present discounted value of the previous expression is

$$0 = \sum_{s=t}^{\infty} \beta^{s-t} \left[d'_{H,x} x_s + d'_{H,\xi} \xi_s + \frac{1}{2} x'_s D_{H,x} x_s + x'_s D_{H,\xi} \xi_s \right] + \mathcal{O} \left(\|\xi_t\|^3 \right),$$

where

$$d'_{H,x} \equiv \left[-1 \quad ns_c \quad -\theta s_c \quad 0 \quad 0 \quad (1-n) s_c \quad 0 \quad 0 \quad 0 \right],$$

$$d'_{H,\xi} \equiv \left[0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \right],$$

$$D_{H,x} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & ns_c (1 - ns_c) & -n\theta s_c (1 - s_c) & 0 & 0 & -n(1-n) s_c^2 & 0 & 0 & 0 \\ 0 & -n\theta s_c (1 - s_c) & \theta^2 s_c (1 - s_c) & 0 & 0 & -(1-n) \theta s_c (1 - s_c) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -n(1-n) s_c^2 & -(1-n) \theta s_c (1 - s_c) & 0 & 0 & (1-n) s_c (1 - (1-n) s_c) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$D_{H,\xi} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -ns_c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta s_c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(1-n) s_c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Foreign aggregate demand can be represented as

$$0 = \sum_{s=t}^{\infty} \beta^{s-t} \left[d'_{F,x} x_s + d'_{F,\xi} \xi_s + \frac{1}{2} x'_s D_{F,x} x_s + x'_s D_{F,\xi} \xi_s \right] + \mathcal{O} \left(\|\xi_t\|^3 \right),$$

where

$$d'_{F,x} \equiv [0 \quad ns_c \quad 0 \quad 0 \quad -1 \quad (1-n)s_c \quad -\theta s_c \quad 0 \quad 0],$$

$$d'_{F,\xi} \equiv [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0],$$

$$D_{F,x} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & ns_c(1-ns_c) & 0 & 0 & 0 & -n(1-n)s_c^2 & -n\theta s_c(1-s_c) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -n(1-n)s_c^2 & 0 & 0 & 0 & (1-n)s_c(1-(1-n)s_c) & -(1-n)\theta s_c(1-s_c) & 0 & 0 \\ 0 & -n\theta s_c(1-s_c) & 0 & 0 & 0 & -(1-n)\theta s_c(1-s_c) & \theta^2 s_c(1-s_c) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$D_{F,\xi} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -ns_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(1-n)s_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \theta s_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3.1.4 The Terms of Trade

The relation between the terms of trade and the Home relative price is

$$p_{H,t}^{\theta-1} = n + (1-n)T_t^{1-\theta}.$$

A second order approximation of this equation yields

$$\hat{p}_{H,t} = -(1-n)\hat{T}_t - \frac{1}{2}n(1-n)(1-\theta)\hat{T}_t^2 + \mathcal{O}(\|\xi_t\|^3).$$

The matrix form present discounted value is then

$$0 = \sum_{s=t}^{\infty} \beta^{s-t} \left[h'_{H,x} x_s + \frac{1}{2} x'_s H_{H,x} x_s \right] + \mathcal{O}(\|\xi_t\|^3),$$

where

$$h'_{H,x} \equiv [0 \quad 0 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -(1-n)],$$

and

$$H_{H,x} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -n(1-n)(1-\theta) \end{bmatrix}.$$

The Foreign counterpart is

$$0 = \sum_{s=t}^{\infty} \beta^{s-t} \left[h'_{F,x} x_s + \frac{1}{2} x'_s H_{F,x} x_s \right] + \mathcal{O}(\|\xi_t\|^3),$$

where

$$h'_{F,x} \equiv [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1 \quad 0 \quad n],$$

and

$$H_{F,x} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -n(1-n)(1-\theta) \end{bmatrix}.$$

3.1.5 Risk Sharing

The risk sharing equation gives

$$\hat{C}_t - \hat{C}_t^* = 0,$$

which can be written as

$$\sum_{s=t}^{\infty} \beta^{s-t} (m'_{W,x} x_s) = 0,$$

where

$$m'_{W,x} \equiv [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad -1 \quad 0 \quad 0 \quad 0].$$

3.2 Elimination of First Order Terms and Aggregate Welfare

This section shows how to eliminate the first order terms in the second order approximation of the aggregate welfare function [30]. The procedure consists of two steps. First, I need to find the steady state Lagrange multipliers of the optimal policy problem. These multipliers relate the coefficients in front of the first order terms of the objective function with the coefficients in front of the first order terms of the equilibrium relations. Second, I need to express the first order terms of the equilibrium relations as a function of the second order terms. In this second step, the weights in front of the linear terms coincide with the weights in front of the second order terms.

To this extent, I construct the (9×9) matrix Γ defined as

$$\Gamma \equiv \begin{bmatrix} a_{H,x} & a_{F,x} & \omega_{H,x} & \omega_{F,x} & d_{H,x} & d_{F,x} & h_{H,x} & h_{F,x} & m_{W,x} \end{bmatrix}.$$

The matrix Γ collects the vectors that multiply the linear terms in the endogenous variables of the second order approximation of the structural equilibrium conditions

$$\Gamma = \begin{bmatrix} \eta & 0 & 1 + \omega_g & 0 & -1 & 0 & 0 & 0 & 0 \\ \rho & 0 & -\rho & 0 & ns_c & ns_c & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -\theta s_c & 0 & -1 & 0 & 0 \\ \omega_\tau & 0 & 1 + \omega_g & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 1 + \omega_g & 0 & -1 & 0 & 0 & 0 \\ 0 & \rho & 0 & -\rho & (1-n)s_c & (1-n)s_c & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & -\theta s_c & 0 & -1 & 0 \\ 0 & \omega_\tau & 0 & 1 + \omega_g & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(1-n) & n & 0 \end{bmatrix}.$$

The steady state Lagrange multipliers that are sought are the solution of the linear system

$$\Gamma \zeta_i = z_{i,x},$$

where ζ_i is a (9×1) vectors of weights for country i .

The resulting quadratic criterion for country i becomes

$$u_{i,0} = -\frac{1}{2} \bar{C}^{1-\rho} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [x_t' Q_{i,x} x_t + 2x_t' Q_{i,\xi} \xi_t + q_{i,\pi_H} \pi_{H,t}^2 + q_{i,\pi_F} \pi_{F,t}^2] \right\} + J_{i,0} + t.i.p. + \mathcal{O}(\|\xi_t\|^3),$$

where

$$\begin{aligned} Q_{i,x} &\equiv Z_{i,x} + \zeta_{i,1} A_{H,x} + \zeta_{i,2} A_{F,x} + \zeta_{i,3} W_{H,x} + \zeta_{i,4} W_{F,x} + \zeta_{i,5} D_{H,x} + \zeta_{i,6} D_{F,x} + \zeta_{i,7} H_{H,x} + \zeta_{i,8} H_{F,x} \\ Q_{i,\xi} &\equiv Z_{i,\xi} + \zeta_{i,1} A_{H,\xi} + \zeta_{i,2} A_{F,\xi} + \zeta_{i,3} W_{H,\xi} + \zeta_{i,4} W_{F,\xi} + \zeta_{i,5} D_{H,\xi} + \zeta_{i,6} D_{F,\xi}, \end{aligned}$$

$$\begin{aligned} q_{H,\pi_H} &\equiv z_{\pi_H} + \zeta_{H,1} a_{\pi_H}, & q_{H,\pi_F} &\equiv \zeta_{H,2} a_{\pi_F} \\ q_{F,\pi_H} &\equiv \zeta_{F,1} a_{\pi_H}, & q_{F,\pi_F} &\equiv z_{\pi_F} + \zeta_{F,2} a_{\pi_F} \end{aligned}$$

$$J_{i,0} \equiv \bar{C}^{1-\rho} \left[\zeta_{i,1} V_{H,0} + \zeta_{i,2} V_{F,0} + \left(\frac{1}{1-\beta} \right) (\zeta_{i,3} \tilde{W}_{H,0} + \zeta_{i,4} \tilde{W}_{F,0}) \right].$$

3.2.1 Further Simplifications

The dimension of the vector of endogenous variables x_t can actually be reduced by appropriately using the first order approximation of some equilibrium relations. The objective functions $u_{i,0}$ can then be expressed in terms of quadratic deviations of the remaining control variables from their targets, to be defined as linear combinations of the exogenous shocks.

Up to a first order approximation, the aggregate demand equations in the two countries can be written as

$$\hat{Y}_{H,t} = s_c \hat{C}_{W,t} - \theta s_c \hat{p}_{H,t} + \hat{G}_{H,t} + \mathcal{O}(\|\xi_t\|^2),$$

$$\hat{Y}_{F,t} = s_c \hat{C}_{W,t} - \theta s_c \hat{p}_{F,t} + \hat{G}_{F,t} + \mathcal{O}(\|\xi_t\|^2).$$

The relation between relative prices and the terms of trade is

$$\hat{p}_{H,t} = -(1-n)\hat{T}_t + \mathcal{O}\left(\|\xi_t\|^2\right), \quad \hat{p}_{F,t} = n\hat{T}_t + \mathcal{O}\left(\|\xi_t\|^2\right). \quad (42)$$

Combining the previous two relationships gives

$$\hat{Y}_{H,t} = s_c \hat{C}_{W,t} + (1-n)\theta s_c \hat{T}_t + \hat{G}_{H,t} + \mathcal{O}\left(\|\xi_t\|^2\right),$$

$$\hat{Y}_{F,t} = s_c \hat{C}_{W,t} - n\theta s_c \hat{T}_t + \hat{G}_{F,t} + \mathcal{O}\left(\|\xi_t\|^2\right).$$

I define average output as the population-weighted linear combination of national GDP

$$\hat{Y}_{W,t} \equiv n\hat{Y}_{H,t} + (1-n)\hat{Y}_{F,t}.$$

The equilibrium in the goods market at the currency union level in log-linear form requires that

$$\hat{Y}_{W,t} = s_c \hat{C}_{W,t} + n\hat{G}_{H,t} + (1-n)\hat{G}_{F,t} + \mathcal{O}\left(\|\xi_t\|^2\right). \quad (43)$$

I can then replace average consumption by average output and get

$$\hat{Y}_{H,t} = \hat{Y}_{W,t} + (1-n)\theta s_c \hat{T}_t + (1-n)\hat{G}_{H,t} - (1-n)\hat{G}_{F,t} + \mathcal{O}\left(\|\xi_t\|^2\right),$$

$$\hat{Y}_{F,t} = \hat{Y}_{W,t} - n\theta s_c \hat{T}_t - n\hat{G}_{H,t} + n\hat{G}_{F,t} + \mathcal{O}\left(\|\xi_t\|^2\right). \quad (44)$$

I define a reduced vector of endogenous variables as

$$\hat{y}'_t \equiv \left[\hat{Y}_{W,t} \quad \hat{\tau}_H \quad \hat{\tau}_F \quad \hat{T}_t \right].$$

The relation between the previous vector of endogenous variables x_t and the newly defined reduced vector is described by the two relations in [42], equation [43] applied to consumption in each country (using the risk sharing condition $\hat{C}_t = \hat{C}_t^* = \hat{C}_{W,t}$) and the two relations in [44], plus three identities (one for the terms of trade and one for each of the two tax rates). In vector notation, the previous system of equation can be written compactly as

$$x_t = N_x \hat{y}_t + N_\xi \xi_t + \mathcal{O}\left(\|\xi_t\|^2\right),$$

where

$$N_x \equiv \begin{bmatrix} 1 & 0 & 0 & (1-n)\theta s_c \\ s_c^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-n) \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -n\theta s_c \\ s_c^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$N_\xi = \begin{bmatrix} 0 & 0 & 1-n & 0 & 0 & 0 & -(1-n) & 0 \\ 0 & 0 & -ns_c^{-1} & 0 & 0 & 0 & -(1-n)s_c^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -n & 0 & 0 & 0 & n & 0 \\ 0 & 0 & -ns_c^{-1} & 0 & 0 & 0 & -(1-n)s_c^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The relation between the reduced and the original vector can be substituted back into the welfare criterion to obtain

$$u_{i,0} = -\frac{1}{2}\bar{C}^{1-\rho}E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[\hat{y}'_t \tilde{Q}_{i,x} \hat{y}_t + 2\hat{y}'_t \tilde{Q}_{i,\xi} \xi_t + q_{i,\pi_H} \pi_{H,t}^2 + q_{i,\pi_F} \pi_{F,t}^2 \right] \right\} + J_{i,0} + t.i.p. + \mathcal{O}(\|\xi_t\|^3),$$

where

$$\tilde{Q}_{i,x} \equiv N'_x Q_{i,x} N_x, \quad \tilde{Q}_{i,\xi} \equiv N'_x Q_{i,x} N_\xi + N'_x Q_{i,\xi}.$$

Since this paper is concerned with the optimal policy from a centralized perspective, the welfare criterion for the currency union as a whole is the population weighted average of the national welfare criteria

$$u_{W,0} \equiv nu_{H,0} + (1-n)u_{F,0}.$$

From the results derived above, this corresponds to

$$u_{W,0} = -\frac{1}{2}\bar{C}^{1-\rho}E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[\hat{y}'_t \tilde{Q}_{W,x} \hat{y}_t + 2\hat{y}'_t \tilde{Q}_{W,\xi} \xi_t + q_{W,\pi_H} \pi_{H,t}^2 + q_{W,\pi_F} \pi_{F,t}^2 \right] \right\} + J_{W,0} + t.i.p. + \mathcal{O}(\|\xi_t\|^3), \quad (45)$$

where

$$\begin{aligned} \tilde{Q}_{W,x} &\equiv n\tilde{Q}_{H,x} + (1-n)\tilde{Q}_{F,x} = N'_x Q_{W,x} N_x \\ \tilde{Q}_{W,\xi} &\equiv n\tilde{Q}_{H,\xi} + (1-n)\tilde{Q}_{F,\xi} = N'_x Q_{W,x} N_\xi + N'_x Q_{W,\xi} \\ q_{W,\pi_H} &\equiv nq_{H,\pi_H} + (1-n)q_{F,\pi_H} \\ q_{W,\pi_F} &\equiv nq_{H,\pi_F} + (1-n)q_{F,\pi_F} \end{aligned}$$

and

$$J_{W,0} \equiv \bar{C}^{1-\rho} \left[\zeta_{W,1} V_{H,0} + \zeta_{W,2} V_{F,0} + \left(\frac{1}{1-\beta} \right) \left(\zeta_{W,3} \tilde{W}_{H,0} + \zeta_{W,4} \tilde{W}_{F,0} \right) \right].$$

It is actually convenient to compute the relevant objects for the currency union welfare objective without deriving the national welfare objectives. In other words, I will compute only the relevant matrices for expression [45]. To this extent, it suffices deriving the three objects $Z_{W,x} = nZ_{H,x} + (1-n)Z_{F,x}$, $Z_{W,\xi} = nZ_{H,\xi} + (1-n)Z_{F,\xi}$ and $\zeta_W = n\zeta_H + (1-n)\zeta_F$. The first two matrices are

easy to calculate from the quadratic approximation of utility

$$Z_{W,x} \equiv \begin{bmatrix} n(s_c \bar{\mu})^{-1}(1+\eta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -n(1-\rho) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-n)(s_c \bar{\mu})^{-1}(1+\eta) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(1-n)(1-\rho) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$Z_{W,\xi} = \begin{bmatrix} -n(s_c \bar{\mu})^{-1}(1+\eta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(1-n)(s_c \bar{\mu})^{-1}(1+\eta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The third vector is the solution of the linear system

$$\Gamma \zeta_W = z_{W,x},$$

where $z_{W,x} \equiv nz_{H,x} + (1-n)z_{F,x}$.

Given these three objects, I can then solve for $Q_{W,x}$, $Q_{W,\xi}$, q_{W,π_H} and q_{W,π_F} by observing that

$$\begin{aligned} Q_{W,x} &= Z_{W,x} + \zeta_{W,1}A_{H,x} + \zeta_{W,2}A_{F,x} + \zeta_{W,3}W_{H,x} + \zeta_{W,4}W_{F,x} + \zeta_{W,5}D_{H,x} + \zeta_{W,6}D_{F,x} + \zeta_{W,7}H_{H,x} + \zeta_{W,8}H_{F,x} \\ Q_{W,\xi} &= Z_{W,\xi} + \zeta_{W,1}A_{H,\xi} + \zeta_{W,2}A_{F,\xi} + \zeta_{W,3}W_{H,\xi} + \zeta_{W,4}W_{F,\xi} + \zeta_{W,5}D_{H,\xi} + \zeta_{W,6}D_{H,\xi} \\ q_{W,\pi_H} &= nz_{\pi_H} + \zeta_{W,1}a_{\pi_H} \\ q_{W,\pi_F} &= (1-n)z_{\pi_F} + \zeta_{W,2}a_{\pi_F}. \end{aligned}$$

It is possible to show that the quadratic terms in the tax rates are equal to zero, that is

$$\left(\tilde{Q}_{W,x}\right)_{2i} = \left(\tilde{Q}_{W,x}\right)_{3i} = \left(\tilde{Q}_{W,x}\right)_{j2} = \left(\tilde{Q}_{W,x}\right)_{j3} = 0,$$

$\forall i, j = 1, \dots, 4$. Moreover, the matrix $\tilde{Q}_{W,x}$ is diagonal

$$\left(\tilde{Q}_{W,x}\right)_{14} = \left(\tilde{Q}_{W,x}\right)_{41} = 0$$

and the coefficients of the cross terms that involve taxes and the exogenous shocks in the matrix $\tilde{Q}_{W,\xi}$ are also all equal to zero, that is

$$\left(\tilde{Q}_{W,\xi}\right)_{i2} = \left(\tilde{Q}_{W,\xi}\right)_{i3} = 0.$$

It follows that the tax rates do not enter directly the average welfare objective for the currency union. While distortionary taxation does imply welfare costs to the extent that it contributes to create a wedge between actual and desired output levels, tax rates do not appear as an explicit term in the welfare objective⁴.

⁴The results about tax rates are consistent with the correspondent closed economy case in Benigno and Woodford (2003).

4 Coefficients and Targets in the Objective Function

The elimination of first order terms in the second order approximation of total welfare requires the computation of the steady state Lagrange multipliers of the optimal policy problem, which are $\zeta_{W,i}$, $i = \{1, \dots, 8\}$

$$\begin{aligned}\zeta_{W,1} &\equiv n(\bar{\mu} - 1)(1 + \omega_g)(\bar{\mu}d_\zeta)^{-1} & \zeta_{W,2} &\equiv n^{-1}(1 - n)\zeta_{W,1} \\ \zeta_{W,3} &\equiv -\omega_\tau(\bar{\mu} - 1)(\bar{\mu}d_\zeta)^{-1} & \zeta_{W,4} &\equiv n^{-1}(1 - n)\zeta_{W,3} \\ \zeta_{W,5} &\equiv \varphi(s_c\bar{\mu}d_\zeta)^{-1} & \zeta_{W,6} &\equiv n^{-1}(1 - n)\zeta_{W,5} \\ \zeta_{W,7} &\equiv -\vartheta(\bar{\mu}d_\zeta)^{-1} & \zeta_{W,8} &\equiv n^{-1}(1 - n)\zeta_{W,7}\end{aligned}$$

where

$$\begin{aligned}\bar{\mu} &\equiv \sigma\bar{\mu}^w / [(\sigma - 1)(1 - \bar{\tau})] \\ \omega_g &\equiv s_g / (\bar{\tau} - s_g) \\ s_g &\equiv (\bar{G} + \bar{\zeta}) / \bar{Y} \\ \omega_\tau &\equiv (1 - \bar{\tau}) / \bar{\tau} \\ s_c &\equiv \bar{C} / \bar{Y} \\ d_\zeta &\equiv s_c(1 + \omega_g)(\rho s_c^{-1} + \eta - \omega_\tau) + \rho\omega_\tau \\ \varphi &\equiv \bar{\mu}s_c(1 + \omega_g)(\eta - \omega_\tau) + \rho(1 + \omega_g + \omega_\tau) \\ \vartheta &\equiv (\bar{\mu} - 1)(1 + \omega_g + \omega_\tau) + \theta\varphi.\end{aligned}$$

The weights in the purely quadratic second order approximation of the welfare function are combinations of these Lagrange multipliers and of other structural parameters.

The weight on the output gap is

$$\lambda_y \equiv \left(\tilde{Q}_{W,x}\right)_{11},$$

where

$$\left(\tilde{Q}_{W,x}\right)_{11} \equiv Q_{w,x}^1 + s_c^{-2}Q_{w,x}^2 + 2s_c^{-1}Q_{w,x}^3$$

and

$$\begin{aligned}Q_{w,x}^1 &\equiv (s_c\bar{\mu})^{-1}(1 + \eta) + \eta(2 + \eta)(\zeta_{W,1} + \zeta_{W,2}) + (1 + \omega_g)(\zeta_{W,3} + \zeta_{W,4}) \\ Q_{w,x}^2 &\equiv -(1 - \rho) - \rho^2 [(\zeta_{W,1} + \zeta_{W,2}) - (\zeta_{W,3} + \zeta_{W,4})] + s_c(1 - s_c)(\zeta_{W,5} + \zeta_{W,6}) \\ Q_{w,x}^3 &\equiv \rho [(\zeta_{W,1} + \zeta_{W,2}) - (1 + \omega_g)(\zeta_{W,3} + \zeta_{W,4})].\end{aligned}$$

The weight on the terms of trade gap is

$$\lambda_q \equiv n^{-1}(1 - n)^{-1} \left(\tilde{Q}_{W,x}\right)_{44},$$

where

$$\left(\tilde{Q}_{W,x}\right)_{44} \equiv (\theta s_c)^2 Q_{w,x}^4 + Q_{w,x}^5 + Q_{w,x}^6 - 2\theta s_c Q_{w,x}^7$$

and

$$\begin{aligned}Q_{w,x}^4 &\equiv n(1 - n)Q_{w,x}^1 \\ Q_{w,x}^5 &\equiv n(1 - n) [-(\zeta_{W,1} + \zeta_{W,2}) + (\zeta_{W,3} + \zeta_{W,4}) + \theta^2 s_c(1 - s_c)(\zeta_{W,5} + \zeta_{W,6})] \\ Q_{w,x}^6 &\equiv n(1 - n)(\theta - 1)(\zeta_{W,7} + \zeta_{W,8}) \\ Q_{w,x}^7 &\equiv -n(1 - n)\rho^{-1}Q_{w,x}^3.\end{aligned}$$

The weight on the GDP inflation rate for country i is

$$\lambda_{\pi_i} \equiv (\kappa_i \bar{\mu})^{-1} \sigma \left[s_c^{-1} + (\bar{\mu} - 1)(1 + \omega_g)(1 + \eta) d_\zeta^{-1} \right],$$

where

$$\kappa_i \equiv \frac{(1 - \alpha_i)(1 - \alpha_i \beta)}{\alpha_i(1 + \sigma \eta)}.$$

It is easy to check that the weights on GDP inflation rates are always positive. On the other hand, it is generally not obvious to find conditions that guarantee the weights on the output and terms of trade gap to be positive too. The example presented in section 3.3 in the paper satisfies the conditions for strict concavity for any calibration. In the quantitative experiments of section 4, I check concavity numerically for the baseline calibration and find that λ_y and λ_q are both strictly positive.

In order to allow for government spending and lump-sum fiscal transfers to be zero in steady state, the deviations of those variables from their steady state values are defined in percentage of output, that is, up to the first order, $\hat{G}_{i,t} \equiv (G_{i,t} - \bar{G})/\bar{Y}$ and $\hat{\varsigma}_{i,t} \equiv (\varsigma_{i,t} - \bar{\varsigma})/\bar{Y}$. The vector of exogenous shocks ξ_t collects Home and Foreign disturbances

$$\xi_t = \begin{bmatrix} \hat{\xi}_{H,t} & \hat{\xi}_{F,t} \end{bmatrix},$$

where

$$\hat{\xi}'_{i,t} \equiv \begin{bmatrix} \hat{a}_{i,t} & \hat{\mu}_{i,t}^w & \hat{G}_{i,t} & \hat{\varsigma}_{i,t} \end{bmatrix}.$$

In what follows, average and relative disturbances will be referred to as

$$\xi_{W,t} \equiv n\xi_{H,t} + (1 - n)\xi_{F,t}, \quad \xi_{R,t} \equiv \xi_{F,t} - \xi_{H,t}.$$

The welfare-relevant target level for the output gap is

$$\tilde{Y}_{W,t} \equiv -(n\lambda_y)^{-1} \left[\left(\tilde{Q}_{W,\xi} \right)_{11} \hat{a}_{W,t} + \left(\tilde{Q}_{W,\xi} \right)_{12} \hat{\mu}_{W,t}^w + \left(\tilde{Q}_{W,\xi} \right)_{13} \hat{G}_{W,t} + \left(\tilde{Q}_{W,\xi} \right)_{14} \hat{\varsigma}_{W,t} \right],$$

where

$$\begin{aligned} \left(\tilde{Q}_{W,\xi} \right)_{11} &\equiv -n(s_c \bar{\mu})^{-1}(1 + \eta) - (1 + \eta)^2 \zeta_{W,1} \\ \left(\tilde{Q}_{W,\xi} \right)_{12} &\equiv (1 + \eta) \zeta_{W,1} \\ \left(\tilde{Q}_{W,\xi} \right)_{13} &\equiv s_c^{-1} Q_{w,\xi}^1 - n s_c^{-2} Q_{w,x}^2 - 2n s_c^{-1} Q_{w,x}^3 \\ \left(\tilde{Q}_{W,\xi} \right)_{14} &\equiv \rho s_c^{-1} s_d \zeta_{W,3} \end{aligned}$$

and

$$\begin{aligned} Q_{w,\xi}^1 &\equiv \rho s_d \zeta_{W,3} - s_c \zeta_{W,5} \\ s_d &\equiv \omega_g / s_g. \end{aligned}$$

The welfare-relevant target for the terms of trade gap is

$$\tilde{T}_t \equiv (\lambda_q)^{-1} \left[\left(\tilde{Q}_{W,\xi} \right)_{41} \hat{a}_{R,t} + \left(\tilde{Q}_{W,\xi} \right)_{42} \hat{\mu}_{R,t}^w + \left(\tilde{Q}_{W,\xi} \right)_{43} \hat{G}_{R,t} + \left(\tilde{Q}_{W,\xi} \right)_{44} \hat{\varsigma}_{R,t} \right],$$

where

$$\begin{aligned}
\left(\tilde{Q}_{W,\xi}\right)_{41} &\equiv (1-n)\theta s_c \left(\tilde{Q}_{W,\xi}\right)_{11} \\
\left(\tilde{Q}_{W,\xi}\right)_{42} &\equiv (1-n)\theta s_c \left(\tilde{Q}_{W,\xi}\right)_{12} \\
\left(\tilde{Q}_{W,\xi}\right)_{43} &\equiv -(1-n)Q_{w,\xi}^2 + \theta s_c Q_{w,x}^1 - Q_{w,x}^7 \\
\left(\tilde{Q}_{W,\xi}\right)_{44} &\equiv (1-n)\rho^{-1}s_c \left(\tilde{Q}_{W,\xi}\right)_{14}
\end{aligned}$$

and

$$Q_{w,\xi}^2 \equiv -s_d \zeta_{W,3} + \theta s_c \zeta_{W,5}.$$

Given the definition of the target levels for average output and the terms of trade, it is then possible to define the target levels also for Home and Foreign output as

$$\tilde{Y}_{H,t} \equiv \tilde{Y}_{W,t} + (1-n)\theta s_c \tilde{T}_t + (1-n)\hat{G}_{R,t}$$

and

$$\tilde{Y}_{F,t} \equiv \tilde{Y}_{W,t} - n\theta s_c \tilde{T}_t - n\hat{G}_{R,t}.$$

Subtracting the target levels from their actual counterparts, the Home and Foreign output gaps are

$$y_{H,t} = y_t + (1-n)\theta s_c q_t$$

and

$$y_{F,t} = y_t - n\theta s_c q_t,$$

where $y_{i,t} \equiv \hat{Y}_{i,t} - \tilde{Y}_{i,t}$. Substituting the Home and Foreign output gap into the per-period loss function provides a rationale for the definition of the welfare-relevant targets for national output

$$L_{W,t} = n\lambda_y y_{H,t}^2 + (1-n)\lambda_y y_{F,t}^2 + n(1-n)\tilde{\lambda}_q q_t^2 + n\lambda_{\pi_H}\pi_{H,t}^2 + (1-n)\lambda_{\pi_F}\pi_{F,t}^2,$$

where $\tilde{\lambda}_q \equiv \lambda_q - (\theta s_c)^2 \lambda_y$.

The target for the tax rates are

$$\tilde{\tau}_{H,t} \equiv -\omega_\tau^{-1} \left[\delta_y \tilde{Y}_{W,t} + (1-n)\delta_q \tilde{T}_t - (1+\eta)\hat{a}_{H,t} + \hat{\mu}_{H,t}^w - (1-n)\eta\hat{G}_{R,t} - \rho s_c^{-1}\hat{G}_{W,t} \right]$$

and

$$\tilde{\tau}_{F,t} \equiv -\omega_\tau^{-1} \left[\delta_y \tilde{Y}_{W,t} - n\delta_q \tilde{T}_t - (1+\eta)\hat{a}_{F,t} + \hat{\mu}_{F,t}^w + n\eta\hat{G}_{R,t} - \rho s_c^{-1}\hat{G}_{W,t} \right],$$

where $\delta_y \equiv \eta + \rho s_c^{-1}$ and $\delta_q \equiv 1 + \eta\theta s_c$.

The fiscal stress terms for country i is

$$\psi_{i,t} \equiv -\rho s_c^{-1} \left(\tilde{Y}_{W,t} - \hat{G}_{W,t} \right) - (1-\beta) E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} v_{i,s} \right\},$$

where

$$v_{H,t} \equiv b_y \tilde{Y}_{W,t} + (1-n)b_q \tilde{T}_t + (1+\omega_g)\tilde{\tau}_{H,t} - s_d \left(\hat{G}_{H,t} + \hat{\varsigma}_{H,t} \right) + \rho s_c^{-1}\hat{G}_{W,t} - (1-n)(1+\omega_g)\hat{G}_{R,t},$$

$$v_{F,t} \equiv b_y \tilde{Y}_{W,t} - nb_q \tilde{T}_t + (1+\omega_g)\tilde{\tau}_{F,t} - s_d \left(\hat{G}_{F,t} + \hat{\varsigma}_{F,t} \right) + \rho s_c^{-1}\hat{G}_{W,t} + n(1+\omega_g)\hat{G}_{R,t}$$

and $b_y \equiv (1+\omega_g) - \rho s_c^{-1}$ and $b_q \equiv (1+\omega_g)\theta s_c - 1$.

5 Determinacy

This section studies analytically the determinacy properties of the optimal policy under the assumption that the degree of price rigidity is the same across countries ($\alpha_H = \alpha_F$) and the additional simplifications on the parameters such that $\rho = \theta = 1$ and $\bar{G} = \bar{\zeta} = 0$. I begin by proving Lemma 3 and Proposition 5 in the text.

5.1 Proof of Lemma 3

Lemma 3 *If the degree of price rigidity is the same across countries, the currency union can be represented in terms of an average and a relative block. The optimal policy plan for each block can be determined independently.*

Proof. Since the degree of price rigidity is the same, the per-period loss function can be written as

$$L_{W,t} \equiv \lambda y_t^2 + n(1-n)\lambda q_t^2 + \pi_t^2 + n(1-n)\pi_{R,t}^2, \quad (46)$$

where $\lambda \equiv (1+\eta)\kappa\sigma^{-1}$ and $\pi_{R,t} \equiv \pi_{F,t} - \pi_{H,t}$.

Average Block. Taking the average, with weights n and $(1-n)$ respectively, of the country-specific Phillips curves gives

$$\pi_t = \sigma\lambda[\tilde{\omega}(\hat{\tau}_{W,t} - \tilde{\tau}_{W,t}) + y_t] + \beta E_t \pi_{t+1}. \quad (47)$$

Similarly, the average of the two national government budget constraints is

$$\hat{b}_{W,t-1} - y_t - \pi_t + \psi_{W,t} = (1-\beta)(\hat{\tau}_{W,t} - \tilde{\tau}_{W,t}) + \beta E_t(\hat{b}_{W,t} - y_{t+1} - \pi_t + \psi_{W,t+1}). \quad (48)$$

Equations [47] and [48] represent the average block of the currency union. The optimal policy for the average block is the sequence $\{y_t, \pi_t, \hat{\tau}_{W,t}, \hat{b}_{W,t}\}_{t=0}^{\infty}$ that maximizes the present discounted value of [46] subject to [59] and [60] and the additional constraints for the timeless perspective.

Relative Block. Taking the difference between the country-specific aggregate supply equations yields the relative Phillips curve

$$\pi_{R,t} = \sigma\lambda[\tilde{\omega}(\hat{\tau}_{R,t} - \tilde{\tau}_{R,t}) - q_t] + \beta E_t \pi_{R,t+1}. \quad (49)$$

Similarly, the difference between the national government budget constraints gives

$$\hat{b}_{R,t-1} + \psi_{R,t} = (1-\beta)(\hat{\tau}_{R,t} - \tilde{\tau}_{R,t}) + \beta E_t(\hat{b}_{R,t} + \psi_{R,t+1}). \quad (50)$$

The law of motion of the terms of trade is already expressed in terms of relative variables

$$q_t = q_{t-1} + \pi_{R,t} - \Delta \tilde{T}_t. \quad (51)$$

Equations [49], [50] and [51] represent the relative block of the currency union. The optimal policy for the relative block is the sequence $\{q_t, \pi_{R,t}, \hat{\tau}_{R,t}, \hat{b}_{R,t}\}_{t=0}^{\infty}$ that maximizes the present discounted value of [46] subject to [49], [50] and [51] and the additional constraints for the timeless perspective.

The two problems are independent, hence completing the proof of the claim. ■

5.2 Proof of Proposition 5

Proposition 5 *If the degree of price rigidity is the same across countries, the optimal policy for the relative block can be represented by a flexible tax targeting rule adjusted for the terms of trade.*

Proof. The first order conditions for optimal policy in the relative block are given by

$$\lambda q_t = \sigma \lambda \varphi_{1,t}^R + \varphi_{3,t} - \beta E_t \varphi_{3,t+1}, \quad (52)$$

$$\pi_{R,t} = (\varphi_{1,t}^R - \varphi_{1,t-1}^R) - \varphi_{3,t}, \quad (53)$$

$$E_t \varphi_{1,t+1}^R = \varphi_{1,t}^R \quad (54)$$

and by the constraints [49]–[51]. From [53], I solve for $\varphi_{3,t}$ and substitute into [52]. After using [54] and [49], I obtain a solution for $\varphi_{1,t}^R$ as a function of the equilibrium values of the terms of trade gap and of the relative tax gap

$$\varphi_{1,t}^R = \rho_\varphi \varphi_{1,t-1}^R + (1 - \rho_\varphi) \mathcal{T}_t,$$

where ρ_φ and \mathcal{T}_t have been defined in the text. Since [54] implies that $\varphi_{1,t}^R$ must follow a random walk, the previous expression can be rewritten as

$$E_t \mathcal{T}_{t+1} = \rho_\varphi E_{t-1} \mathcal{T}_t + (1 - \rho_\varphi) \mathcal{T}_t,$$

which corresponds to the optimal targeting rule reported in the text. ■

5.3 Determinacy under Optimal Targeting Rules

The following proposition characterizes the sign and magnitude of the eigenvalues of a system of two difference equations using the characteristic equation and will be used extensively below to study the determinacy properties of the model.

Proposition 6 *Let $P(\lambda) \equiv \lambda^2 + A_1 \lambda + A_0 = 0$ and let λ_1 and λ_2 be the roots of $P(\lambda)$. Then, the absolute values of λ_1 and λ_2 split across the unit circle if and only if $P(1) > 0$ and $P(-1) < 0$ or viceversa.*

Proof. First, notice that one can always rewrite the polynomial $P(\lambda)$ as

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \quad (55)$$

1. I first show that $P(1) > 0$ and $P(-1) < 0$ imply that the absolute values of the two roots λ_1 and λ_2 split across the unit circle. From the right hand side of [55], it is easy to see that

$$P(1) = (1 - \lambda_1)(1 - \lambda_2) > 0, \quad (56)$$

and

$$P(-1) = (1 + \lambda_1)(1 + \lambda_2) < 0. \quad (57)$$

If $P(1) > 0$, it means that λ_1 and λ_2 are on the same side of 1. Similarly, if $P(-1) < 0$, it means that λ_1 and λ_2 are on opposite sides of -1 . It then follows that one root must lie inside the unit circle and the other outside. The case $P(1) < 0$ and $P(-1) > 0$ is totally symmetric.

2. Next, I show that if $|\lambda_1|$ and $|\lambda_2|$ lie on opposite sides of 1, it must be the case that $P(1)$ and $P(-1)$ lie on opposite sides of 0. Without loss of generality, suppose $|\lambda_1| > 1$ and $|\lambda_2| < 1$. There are two cases to be considered. First, if $\lambda_1 > 1$, then, one can see from [56] that $P(1) < 0$ and from

[57] that $P(-1) > 0$, which confirms the claim. Second, if $\lambda_1 < -1$, then, again from [56] and [57], $P(1) > 0$ and $P(-1) < 0$. The case $|\lambda_1| < 1$ and $|\lambda_2| > 1$ is symmetric. ■

Proposition 6 complements Proposition C.1 in Woodford (2003) which gives necessary and sufficient conditions for the two roots of $P(\lambda)$ to be both outside the unit circle.

Proposition 7 *The optimal policy plan brings about a determinate equilibrium.*

Proof. Given Lemma 3, I will analyze the determinacy properties of the average and relative block separately. The equilibrium will be determinate if only if both blocks admit a determinate solution.

Average Block. Proposition 4 in the text implies that the determinacy properties of the average block coincide with the closed economy case of Benigno and Woodford (2003) (section 7.11). Hence, as in their model, the optimal policy brings about a determinate equilibrium for the average block.

Relative Block. The dynamics of the relative block under the optimal policy are described by the system

$$ME_t z_{t+1} = N_z z_t + N_\varepsilon \varepsilon_t, \quad (58)$$

where the vector of endogenous variables is

$$z_t \equiv \left[\pi_{R,t} \quad \hat{\tau}_{R,t} - \tilde{\tau}_{R,t} \quad q_{t-1} \quad x_{t-1} \quad \hat{b}_{R,t-1} \right],$$

and $x_t \equiv E_t T_{t+1}$. The matrixes of coefficients relevant for the determinacy properties are

$$M \equiv \begin{bmatrix} \beta & 0 & -\sigma\lambda & 0 & 0 \\ -(\sigma-1)\sigma^{-1} & \tilde{\omega} & -(\sigma-1)\sigma^{-1} & -1 & 0 \\ 0 & 0 & 0 & 0 & \beta \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & (1-\rho_\varphi)(\sigma-1)\sigma^{-1} & 1 & 0 \end{bmatrix}$$

and

$$N_z \equiv \begin{bmatrix} 1 & -\tilde{\omega}\sigma\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -(1-\beta) & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & (1-\rho_\varphi)\tilde{\omega} & 0 & \rho_\varphi & 0 \end{bmatrix}.$$

The vector of shocks ε_t and the matrix of coefficients N_ε are irrelevant for the determinacy of properties of the system [58] and hence not reported here. The determinacy properties of the relative block depend instead on the eigenvalues of the matrix $M^{-1}N_z$. First, notice that the determinant of M is given by $\beta^2\tilde{\omega}$ and hence M is always invertible. Second, it is easy to check that three eigenvalues of $M^{-1}N_z$ are 0, 1 and β^{-1} . The remaining two eigenvalues are the roots v_1 and v_2 of the polynomial

$$P(v) = v^2 - \left[1 + \beta^{-1} + \beta^{-1}\sigma\lambda \left(\frac{\mu-1}{\mu} \right) \right] v + \beta^{-1}.$$

It then follows that

$$P(1) = -\beta^{-1}\sigma\lambda \left(\frac{\mu-1}{\mu} \right) < 0$$

and

$$P(-1) = 2(1 + \beta^{-1}) + \beta^{-1}\sigma\lambda \left(\frac{\mu-1}{\mu} \right) > 0.$$

Since $P(1)$ and $P(-1)$ have opposite sign, Proposition 6 ensures that the absolute value of the two roots of $P(v)$ split across the unit circle. Since the system has three predetermined variables, the solution is determinate although it possesses a unit root. ■

5.4 Determinacy under Simple Rules

This section studies the determinacy properties of the model under simple rules assuming the same degree of price rigidity across countries ($\alpha_H = \alpha_F = \alpha$) but relaxing all the other restrictions on the parameters imposed in the previous section. Therefore, Lemma 3 still applies and the aggregate and relative block can be treated independently. For convenience, I report the equations that characterize the average and relative block.

Average Block. The average Phillips curve is

$$\pi_t = \kappa [\delta_y y_t + \omega_\tau (\hat{\tau}_{W,t} - \tilde{\tau}_{W,t})] + \beta E_t \pi_{t+1}. \quad (59)$$

The average government budget constraint can be written as

$$\hat{b}_{W,t-1} - \rho s_c^{-1} y_t - \pi_t + \psi_{W,t} = (1 - \beta) [(1 + \omega_g) (\hat{\tau}_{W,t} - \tilde{\tau}_{W,t}) + b_y y_t] + \beta E_t (\hat{b}_{W,t} - \rho s_c^{-1} y_{t+1} - \pi_{t+1} + \psi_{W,t+1}). \quad (60)$$

The analysis of simple rules requires also the specification of the log-linear approximation of the Euler equation

$$r_t = \tilde{r}_t + E_t \pi_{t+1} + \rho s_c^{-1} (E_t y_{t+1} - y_t), \quad (61)$$

where

$$\tilde{r}_t \equiv \rho s_c^{-1} \left[\left(E_t \tilde{Y}_{W,t+1} - \tilde{Y}_{W,t} \right) - (E_t \hat{G}_{W,t+1} + \hat{G}_{W,t}) \right].$$

Relative Block. The relative Phillips curve is

$$\pi_{R,t} = \kappa [\omega_\tau (\hat{\tau}_{R,t} - \tilde{\tau}_{R,t}) - \delta_q q_t] + \beta E_t \pi_{R,t+1}. \quad (62)$$

The relative government budget constraint is

$$\hat{b}_{R,t-1} + \psi_{R,t} = (1 - \beta) [(1 + \omega_g) (\hat{\tau}_{R,t} - \tilde{\tau}_{R,t}) - b_q q_t] + \beta E_t (\hat{b}_{R,t} + \psi_{R,t+1}). \quad (63)$$

The terms of trade evolves according to

$$q_t = q_{t-1} + \pi_{R,t} - \Delta \tilde{T}_t. \quad (64)$$

5.4.1 Determinacy under Simple Strict Rules

Average Block. The monetary rule for the average block is

$$\pi_t = 0. \quad (65)$$

The fiscal rule is

$$\hat{b}_{W,t} = r_t. \quad (66)$$

From [59], the tax gap simply becomes a linear function of the output gap

$$\hat{\tau}_{W,t} - \tilde{\tau}_{W,t} = -\frac{\delta_y}{\omega_\tau} y_t.$$

I substitute the result into [60], together with [65] and get

$$\hat{b}_{W,t-1} + u_{W,t}^f = \omega_f (\omega_\tau - \delta_y) y_t + \beta \hat{b}_{W,t} - \beta \rho s_c^{-1} (E_t y_{t+1} - y_t),$$

where I have exploited the definition of b_y to simplify terms and I have defined $\omega_f \equiv (1 - \beta)(1 + \omega_g)\omega_\tau^{-1}$ and $u_{W,t}^f \equiv \psi_{W,t} - \beta E_t \psi_{W,t+1}$. I can then substitute for the innovation in the output gap from [61] to eliminate the last term in the previous expression and finally apply the fiscal policy rule [66]. The result is a closed form solution for the output gap as a function of the existing stock of liabilities and the exogenous shocks of the form

$$y_t = \frac{1}{\omega_f(\omega_\tau - \delta_y)} \left(\hat{b}_{W,t-1} + u_{W,t}^f - \beta \tilde{r}_t \right). \quad (67)$$

From [67] and [61], I can then derive the dynamic evolution of debt using the fiscal rule [66]. The resulting expression is

$$\hat{b}_{W,t} = \delta_b^s \hat{b}_{W,t-1} + \epsilon_{W,t}, \quad (68)$$

where

$$\delta_b^s \equiv \frac{\rho s_c^{-1}}{\rho s_c^{-1} - \omega_f(\omega_\tau - \delta_y)}$$

and $\epsilon_{W,t}$ is a composite shock whose definition is immaterial for the determinacy properties. A determinate solution requires $|\delta_b^s| < 1$. Under the baseline calibration, $\delta_b^s = 0.7724$, thus ensuring determinacy.⁵

Relative Block. The relative block is characterized by the fiscal rule

$$\hat{b}_{R,t} = 0. \quad (69)$$

I can solve for the tax gap from [62] and plug the result into the government budget constraint. I can also eliminate debt using [69] and inflation rate differentials from [64]. The outcome is a second order linear difference equation in the terms of trade given by

$$E_t q_{t+1} - \left(1 + \frac{1}{\beta} + \frac{\kappa \omega_q}{\beta \omega_f} \right) q_t + \frac{1}{\beta} q_{t-1} = \frac{\kappa}{\beta \omega_f} u_{R,t}^\psi + \frac{1}{\beta} u_{1,t}^{\tilde{T}},$$

where $\omega_q \equiv \omega_f \delta_q - \nu_q$, $\nu_q \equiv (1 - \beta) b_q$, $u_{R,t}^\psi \equiv \psi_{R,t} - \beta E_t \psi_{R,t+1}$ and $u_{1,t}^{\tilde{T}} \equiv \Delta \tilde{T}_t - \beta E_t \Delta \tilde{T}_{t+1}$.

The associated characteristic equation is

$$P(\lambda) = \lambda^2 - \left(1 + \frac{1}{\beta} + \frac{\kappa \omega_q}{\beta \omega_f} \right) \lambda + \frac{1}{\beta} = 0.$$

One can easily see that

$$P(1) = -\frac{\kappa \omega_q}{\beta \omega_f}, \quad P(-1) = \frac{1}{\beta} \left[2(1 + \beta) + \frac{\kappa \omega_q}{\omega_f} \right].$$

From the definition of the parameters, it is easy to see that ω_f is always positive. Under the baseline calibration, $\omega_q > 0$, hence, $P(1) < 0$ and $P(-1) > 0$. From Proposition ??, it follows that the relative block is determinate under the baseline calibration.

⁵Robustness exercises to different parameters configurations are available upon request.

5.4.2 Determinacy under Simple Flexible Rules

Average Block. The monetary policy rule is

$$\pi_t + \gamma \Delta y_t = 0. \quad (70)$$

The flexible fiscal rule is

$$\hat{b}_{W,t} + \phi y_t = r_t. \quad (71)$$

As in the case of strict simple rules, I eliminate taxes from the average Phillips curve [59]. Since now CPI inflation is different from zero, I use the monetary rule [70] to obtain

$$\hat{\tau}_{W,t} - \tilde{\tau}_{W,t} = -\frac{\gamma}{\kappa \omega_\tau} (\Delta y_t - \beta E_t \Delta y_{t+1}) - \frac{\delta_y}{\omega_\tau} y_t.$$

I replace the result into the average government budget constraint, together with the monetary rule [70] to eliminate CPI inflation

$$\begin{aligned} \hat{b}_{W,t-1} + \gamma \Delta y_t + u_{W,t}^\psi &= \omega_f (\omega_\tau - \delta_y) y_t - \frac{\omega_f \gamma}{\kappa} (\Delta y_t - \beta E_t \Delta y_{t+1}) \\ &\quad + \beta \hat{b}_{W,t} - \beta \rho s_c^{-1} E_t \Delta y_{t+1} + \beta \gamma E_t \Delta y_{t+1}. \end{aligned}$$

I can collect terms and express the last expression as

$$\begin{aligned} \beta \hat{b}_{W,t} + \beta \left[\gamma \left(1 + \frac{\omega_f}{\kappa} \right) - \rho s_c^{-1} \right] E_t \Delta y_{t+1} + \omega_f (\omega_\tau - \delta_y) y_t &= \\ \hat{b}_{W,t-1} + \gamma \left(1 + \frac{\omega_f}{\kappa} \right) \Delta y_t + u_{W,t}^\psi. \end{aligned} \quad (72)$$

I can also substitute the Euler equation [61] and the monetary rule [70] into the fiscal rule [71] to get

$$\hat{b}_{W,t} + (\gamma - \rho s_c^{-1}) E_t \Delta y_{t+1} + \phi y_t = \tilde{r}_t. \quad (73)$$

The determinacy properties of the average block depend upon the system constituted by expressions [73] and [72], together with the definition

$$y_t \equiv \Delta y_t + y_{t-1}. \quad (74)$$

The system can be written in the form

$$A E_t z_{t+1} = B z_t + C \varepsilon_{W,t},$$

where the vector of endogenous variables is

$$z_t \equiv \left[\hat{b}_{W,t-1} \quad \Delta y_t \quad y_{t-1} \right]',$$

the vector of exogenous shocks is

$$\varepsilon_{W,t} \equiv \left[u_{W,t}^\psi \quad \tilde{r}_t \right]$$

and the matrices of the system are

$$A \equiv \begin{bmatrix} 1 & a_{12} & \phi \\ \beta & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}, \quad B \equiv \begin{bmatrix} 0 & 0 & 0 \\ 1 & b_{22} & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad C \equiv \begin{bmatrix} 0 & 0 & 0 \\ 1 & -\omega_f/\kappa & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with

$$\begin{aligned} a_{12} &\equiv \gamma - \rho s_c^{-1} \\ a_{22} &\equiv \beta (b_{22} - \rho s_c^{-1}) \\ a_{23} &\equiv \omega_f (\omega_\tau - \delta_y) \\ b_{22} &\equiv \gamma (1 + \omega_f / \kappa). \end{aligned}$$

The determinant of the matrix A is

$$\det(A) \equiv d = a_{22} - \beta a_{12} = \beta \gamma (\omega_f / \kappa) > 0.$$

It is then possible to show that

$$A^{-1}B = \begin{bmatrix} c_{11} & c_{11}b_{22} + c_{13} & c_{13} \\ c_{21} & c_{21}b_{22} + c_{23} & c_{23} \\ 0 & 1 & 1 \end{bmatrix},$$

where

$$\begin{aligned} c_{11} &= -a_{12}/d \\ c_{13} &= (a_{12}a_{23} - \phi a_{22})/d \\ c_{21} &= 1/d \\ c_{23} &= (\beta\phi - a_{23})/d. \end{aligned}$$

Since at least one eigenvalue of $A^{-1}B$ is equal to zero,⁶ determinacy requires that the two remaining eigenvalues lie on different sides of 1. The other two eigenvalues are the solution of the characteristic equation

$$P(\lambda) = \lambda^2 - (1 + c_{11} + b_{22}c_{11} + c_{23})\lambda + (c_{11} + b_{22}c_{11} + c_{11}c_{23} - c_{21}c_{13}) = 0.$$

When $\phi = 0$ and $\gamma > 0$ (flexibility is granted to monetary policy only), little algebra leads to

$$P(1) = a_{23}/d$$

and

$$P(-1) = 2 \left(1 - \frac{a_{12}}{d} + \frac{b_{22}}{d} \right) - \frac{a_{23}}{d}.$$

From the definitions of the parameters, one can see that under the baseline calibration $P(1) < 0$ and $P(-1) > 0 \forall \gamma > 0$. It follows that in this case the average block has always a determinate solution.

On the other hand, when $\phi > 0$ and $\gamma = 0$ (flexibility is granted to fiscal policy only), it is possible to obtain a closed form solution for output similar to [67] and derive an expression for the evolution of debt analogous to [68]. The autoregressive coefficient that governs the dynamics of debt is

$$\delta_b^f \equiv \frac{\rho s_c^{-1} - \phi}{\rho s_c^{-1} + \beta\phi - \omega_f (\omega_\tau - \delta_y)}.$$

Since $\phi > 0$, $\delta_b^f < \delta_b^s$. Hence, the inequality $\delta_b^f > -1$ gives the relevant restriction on ϕ for this case

$$\phi < \frac{2\rho s_c^{-1} - a_{23}}{1 - \beta} = 950.794. \quad (75)$$

⁶The difference of the second and third column is equal to the first column multiplied by b_{22} . Hence, $A^{-1}B$ is singular and must have at least one null eigenvalue

In the more general situation in which both flexibility parameters are positive, the evaluation of the characteristic equation yields

$$P(1) = -c_{23}(1 - c_{11}) - c_{21}c_{13}$$

and

$$P(-1) = 2(1 + c_{11} + b_{22}c_{21}) + c_{23}(1 + c_{11}) - c_{21}c_{13}.$$

I solve numerically for these two conditions to find the set of values for γ and ϕ that guarantee $P(1) > 0$ and $P(-1) < 0$ or viceversa.

Relative Block. The fiscal rule for the relative block is

$$\hat{b}_{R,t} + \theta s_c \phi q_t = 0. \quad (76)$$

As in the previous section, one can derive a second order difference equation in the terms of trade gap, which in this case reads as

$$E_t q_{t+1} - \left(1 + \frac{1}{\beta} + \frac{\kappa \omega_q}{\beta \omega_f} + \frac{\kappa \theta s_c \phi}{\omega_f}\right) q_t + \frac{1}{\beta} \left(1 + \frac{\kappa \theta s_c \phi}{\omega_f}\right) + q_{t-1} = \frac{\kappa}{\beta \omega_f} u_{R,t}^\psi + \frac{1}{\beta} u_{1,t}^{\tilde{T}}.$$

From the associated characteristic polynomial $P(\lambda)$, it follows that

$$P(1) = \frac{\kappa}{\beta \omega_f} [(1 - \beta) \theta s_c \phi - \omega_q]$$

and

$$P(-1) = \frac{(1 + \beta) \theta s_c \kappa}{\beta \omega_f} + \frac{2(1 + \beta)}{\beta} + \frac{\kappa \omega_q}{\beta \omega_f}.$$

Given that under the baseline calibration ω_q and ω_f are positive, $P(-1) > 0, \forall \phi > 0$. Therefore, a necessary condition for determinacy under flexible rules is given by $P(1) < 0$ which can be solved for ϕ as to yield

$$\phi < \frac{\omega_q}{(1 - \beta) \theta s_c} = 10.4676. \quad (77)$$

Clearly, condition [77] is stronger than [75]. Condition [77] continues to bind also with flexible monetary rules ($\gamma > 0$), thus representing the relevant upper bound for the feedback coefficient in the flexible formulation of simple fiscal rules.

6 Welfare Analysis

This section shows an analytical method to derive the value of welfare under a given policy rule and provides a second order approximation of the consumption equivalent proposed by Lucas (1987).

6.1 The Welfare Objective

The welfare objective is assumed to be

$$L(y_{-1}) \equiv (1 - \beta) E_0 \left\{ \sum_{t=0}^{\infty} \beta^t L_t \right\},$$

where $L_t \equiv y_t' Q y_t$ is the per-period quadratic loss function. The vector y_t has dimension $(n \times 1)$ and contains all the endogenous variables and Q is a given $(n \times n)$ symmetric matrix. The solution of the model is written in vector auto-regression representation

$$y_t = A y_{t-1} + B \varepsilon_t, \quad (78)$$

for initial conditions $y_{-1} = 0$ and *i.i.d.* innovations ε_t with mean zero and variance-covariance matrix Ω .

Substituting the solution [78], the per-period loss function can then be rewritten as

$$\begin{aligned} L_t &= (A y_{t-1} + B \varepsilon_t)' Q (A y_{t-1} + B \varepsilon_t) \\ &= y_{t-1}' A' Q A y_{t-1} + 2 \varepsilon_t' B' Q A y_{t-1} + \varepsilon_t' B' Q B \varepsilon_t. \end{aligned}$$

Hence, the welfare objective becomes

$$L(y_{-1}) = (1 - \beta) E_0 \left\{ \sum_{t=0}^{\infty} \beta^t (y_{t-1}' A' Q A y_{t-1} + \varepsilon_t' B' Q B \varepsilon_t) \right\}, \quad (79)$$

where the expectation of the cross-product simplifies to zero because of the *i.i.d.* property of the shocks.

The evaluation of welfare from an ex-ante perspective requires to integrate over the distribution of shocks at time 0. I denote with $E\{\cdot\}$ the expectation operator over the initial distribution of shocks. I analyze the two components of the welfare objective [79] separately.

For the second element of [79], it is straightforward to see that

$$(1 - \beta) E \left\{ \sum_{t=0}^{\infty} \beta^t (\varepsilon_t' B' Q B \varepsilon_t) \right\} = \text{tr}(B' Q B \Omega).$$

The first component of [79] can again be written as

$$\begin{aligned} (1 - \beta) E \left\{ \sum_{t=0}^{\infty} \beta^t (y_{t-1}' A' Q A y_{t-1}) \right\} &= (1 - \beta) E \left\{ \sum_{t=0}^{\infty} \beta^t [\text{tr}(A' Q A y_{t-1} y_{t-1}')] \right\} \\ &= (1 - \beta) \text{tr} \left\{ \sum_{t=0}^{\infty} \beta^t [A' Q A E(y_{t-1} y_{t-1}')] \right\} \\ &= \text{tr}(A' Q A J), \end{aligned}$$

where

$$J = J(y_{-1}) \equiv E \left\{ (1 - \beta) \sum_{t=0}^{\infty} \beta^t y_{t-1} y_{t-1}' \right\}.$$

The variable J can be rewritten in recursive form using the initial condition $y_{-1} = 0$ as

$$\begin{aligned} J &= (1 - \beta) y_{-1} y_{-1}' + \beta E \left\{ (1 - \beta) \sum_{t=0}^{\infty} \beta^t y_t y_t' \right\} \\ &= \beta E \left\{ (1 - \beta) \sum_{t=0}^{\infty} \beta^t (A y_{t-1} + B \varepsilon_t) (A y_{t-1} + B \varepsilon_t)' \right\} \\ &= \beta (A J A' + B \Omega B'). \end{aligned}$$

The last expression can be solved for the matrix J using a two-step procedure. First, one can solve for $vec(J)$ from the last equality

$$vec(J) = [I_{n^2} - \beta(A \otimes A)]^{-1} vec(\beta B \Omega B').$$

Second, one can retrieve the matrix J by “reshaping” $vec(J)$.⁷

The value of welfare is then given by

$$E\{L(y_{-1})\} = tr(A'QAJ) + tr(B'QB\Omega).$$

Expression $E\{L(y_{-1})\}$ constitute the basis for the welfare analysis conducted in the text.

6.2 The Consumption Equivalent

The objective is to compute the number λ such that

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [u((1-\lambda)C_t^{opt}) - v(\ell_t^{opt})] \right\} = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [u(C_t^{alt}) - v(\ell_t^{alt})] \right\}, \quad (80)$$

where

$$u(C_t) = \frac{C_t^{1-\rho}}{1-\rho}.$$

The present discounted value of welfare under the optimal policy is

$$V_0^{opt} \equiv E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [u(C_t^{opt}) - v(\ell_t^{opt})] \right\},$$

whereas the present discounted value of welfare under an alternative policy is

$$V_0^{alt} \equiv E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [u(C_t^{alt}) - v(\ell_t^{alt})] \right\}.$$

It follows that [80] can be rewritten as

$$[(1-\lambda)^{1-\rho} - 1] W_0^{opt} = V_0^{alt} - V_0^{opt}, \quad (81)$$

where

$$W_0^{opt} \equiv E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(C_t^{opt}) \right\}. \quad (82)$$

In equilibrium, W_0^{opt} , V_0^{alt} and V_0^{opt} are a function of the initial state vector x_0 and of the parameter σ_ε scaling the standard deviation of the exogenous shocks. Therefore, in what follows, I adopt the following notation

$$W_0^{opt} = W^{opt}(x_0, \sigma_\varepsilon), \quad V_0^{alt} = V^{alt}(x_0, \sigma_\varepsilon), \quad V_0^{opt} = V^{opt}(x_0, \sigma_\varepsilon).$$

This implies that also λ will be a function of x_0 and σ_ε

$$\lambda = \Lambda(x_0, \sigma_\varepsilon).$$

⁷This operation can be performed, for example, using the command “`reshape`” in Matlab.

I want to consider a second order approximation of Λ around the vector $(x_0, \sigma_\varepsilon) = (x, 0)$ where x represents the optimal deterministic steady state. Since the approximation is characterized for the initial state being the deterministic steady state, only the first and second order derivatives with respect to σ_ε shall be considered

$$\lambda = \Lambda(x, 0) + \Lambda_{\sigma_\varepsilon}(x, 0) \sigma_\varepsilon + \frac{1}{2} \Lambda_{\sigma_\varepsilon \sigma_\varepsilon}(x, 0) \sigma_\varepsilon^2 \quad (83)$$

Because all policies are assumed to deliver the same level of welfare in steady state, it follows that

$$\Lambda(x, 0) = 0.$$

Furthermore, because the first derivatives of the policy functions with respect to σ_ε , evaluated $(x_0, \sigma_\varepsilon) = (x, 0)$, are zero (see Schmitt-Grohé and Uribe, 2004), also the first derivatives of the value functions evaluated at that point are zero ($V^{alt}(x, 0) = 0$ and $V^{opt}(x, 0) = 0$). From a first order approximation of [81], it follows that

$$\Lambda_{\sigma_\varepsilon}(x, 0) = 0.$$

Hence, a second order approximation of [81] gives

$$\Lambda_{\sigma_\varepsilon \sigma_\varepsilon}(x, 0) = - \left[\frac{V_{\sigma_\varepsilon \sigma_\varepsilon}^{alt}(x, 0) - V_{\sigma_\varepsilon \sigma_\varepsilon}^{opt}(x, 0)}{(1 - \rho) W^{opt}(x, 0)} \right]$$

Substituting into [83] yields

$$\lambda = - \left[\frac{V_{\sigma_\varepsilon \sigma_\varepsilon}^{alt}(x, 0) - V_{\sigma_\varepsilon \sigma_\varepsilon}^{opt}(x, 0)}{(1 - \rho) W^{opt}(x, 0)} \right] \frac{\sigma_\varepsilon^2}{2}.$$

From the definition [82] it follows

$$(1 - \rho) W^{opt}(x, 0) = \frac{\bar{C}^{1-\rho}}{1 - \beta},$$

where \bar{C} is the steady state level of consumption. Plugging back into the expression for λ gives the second order approximation of the consumption equivalent

$$\lambda = - \frac{(1 - \beta) \sigma_\varepsilon^2}{2 \bar{C}^{1-\rho}} [V_{\sigma_\varepsilon \sigma_\varepsilon}^{alt}(x, 0) - V_{\sigma_\varepsilon \sigma_\varepsilon}^{opt}(x, 0)].$$

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