Appendix of "The Advantage of Flexible Targeting Rules" (NOT FOR PUBLICATION)

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December 19, 2011

Abstract

This appendix shows that the log-linearized equations in "The Advantage of Flexible Targeting Rules" represent a first order log-linear approximation of a model with distortionary taxes on either income or sales. The only difference between these two models concerns the weights of the utility-based welfare objective. The appendix presents the welfare costs of different targeting rules in one particular case. Finally, the appendix also shows that distortionary taxes on consumption can make strict debt and inflation targeting compatible with equilibrium determinacy.

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1 The Baseline Economy with Income Taxes

The model corresponds to Linnemann (2006) except for the absence of cash (Woodford, 2003). Household $j \in (0,1)$ values a consumption index C_t^j and dislikes hours worked ℓ_t^j according to the utility function

$$u_0^j \equiv \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[\frac{(C_t^j)^{1-\tilde{\sigma}^{-1}}}{1-\tilde{\sigma}^{-1}} - \frac{(\ell_t^j)^{1+\omega}}{1+\omega} \right] \right\}, \tag{1}$$

where $\beta \in (0,1)$ is the discount rate, $\tilde{\sigma}^{-1} > 0$ is the coefficient of relative risk aversion and $\omega > 0$ is the inverse Frisch elasticity of labor supply. The consumption index is

$$C_t^j \equiv \left[\int_0^1 c_t^j(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}, \tag{2}$$

where $c_t^j(i)$ is household j's consumption of variety i, whose price is $p_t(i)$, and $\theta > 1$ is the elasticity of substitution among varieties. The flow budget constraint is

$$\int_0^1 p_t(i)\, c_t^j(i)\, di + \mathbb{E}_t(Q_{t,t+1} D_{t+1}^j) = (1-\tau_t)\left(w_t^j \ell_t^j + \Upsilon_t^j\right) + D_t^j - \varsigma_t^j, \tag{3}$$

where D_{t+1}^j denotes the payoff of a portfolio of state-contingent securities purchased by household j at time t whose price is $Q_{t,t+1}$. The nominal wage w_t^j is household-specific because of the assumption of labor market segmentation. The variable Υ_t^j stands for nominal dividends from ownership of the firms accruing to household j. The government levies a distortionary tax τ_t on labor and dividend income and lump-sum taxes ς_t^j.

Firm $i \in (0,1)$ hires labor and produces according to the linear technology

$$y_t(i) = a_t \ell_t(i), \tag{4}$$

where a_t is an economy-wide technology shock. Firms set prices on a staggered basis (Calvo, 1983), taking wages as given. In each period, firm i adjusts its price with probability $1-\alpha$, where $\alpha \in (0,1)$ is independent of previous history. In case of adjustment, the firm chooses its price to maximize the present discounted value of profits conditional on no further price adjustment in the future. The time T profit function of firm i is

$$\Upsilon_{t,T}(i) = (1+s)\, p_t(i)\, y_{t,T}(i) - w_T(i)\, \ell_T(i), \tag{5}$$

where $t \leq T$ is the period of the last price adjustment, s is a constant subsidy that eliminates the steady state goods market distortions and $y_{t,T}(i)$ is the demand of good i at time T conditional on the price of good i not being changed since period t.

The flow government budget constraint is

$$B_t = (1 + i_{t-1}) B_{t-1} - \int_0^1 [\tau_t(w_t^j \ell_t^j + \Upsilon_t^j) + \varsigma_t^j] dj + \int_0^1 p_t(i) [g_t(i) + s y_t(i)] di, \tag{6}$$

where B_t represents government debt and $g_t(i)$ is the amount of government spending on the generic variety i. The government chooses optimally how to allocate a given (exogenous) amount of total expenditure G_t among existing varieties of goods produced in the economy. The aggregator for public spending has the same functional form as the consumption index (2). The monetary policy instrument is the nominal interest rate i_t.

1.1 Private Sector Optimality Conditions

The allocation of consumption among different varieties solves the expenditure minimization problem

$$P_t C_t = \min_{c_t(i)} \int_0^1 p_t(i) c_t^j(i) di$$

subject to

$$\left[\int_0^1 c_t^j(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}} \geq \bar{C},$$

where $\bar{C}$ is a given level of consumption. The optimal allocation of consumption for variety i is a negative function of its relative price and a positive function of total consumption

$$c_t^j(i) = \left[\frac{p_t(i)}{P_t} \right]^{-\theta} C_t^j, \tag{7}$$

where the aggregate price index P_t is

$$P_t = \left[\int_0^1 p_t(i)^{1-\theta} di \right]^{\frac{1}{1-\theta}}. \tag{8}$$

For given composition of expenditure, households choose consumption, savings and hours to maximize utility (1) subject to their budget constraint (3). The optimality condition for

state-contingent securities is

$$Q_{t,t+1} = \beta \left(\frac{C_{t+1}^{j}}{C_{t}^{j}} \right)^{-\tilde{\sigma}^{-1}} \frac{1}{1+\pi_{t+1}}, \tag{9}$$

where $\pi_t \equiv P_t/P_{t-1} - 1$ is the inflation rate between period $t-1$ and t. Expression (9) holds for any state of the world at period $t+1$ given the state of the world at t. No arbitrage requires the safe return on government bonds to equal the return on a portfolio of state-contingent securities that pays one unit of currency with certainty in the next period

$$1 + i_t = \left(\mathbb{E}_t Q_{t,t+1} \right)^{-1} \tag{10}$$

The optimality condition for hours is

$$(1-\tau_t) \frac{w_t^j}{P_t} = \mu_t^w \frac{(\ell_t^j)^{\omega}}{(C_t^j)^{-\tilde{\sigma}^{-1}}}, \tag{11}$$

where $\mu_t^w > 1$ is an exogenous wedge that perturbs the equality between the after-tax real wage and the marginal rate of substitution between consumption and hours.[1]

Total demand for variety i is the sum of private and public demand. Market clearing requires that

$$y_t(i) = \int_0^1 c_t^j(i)\, dj + g_t(i),$$

where (7) is private demand by household j and government spending on good i obeys

$$g_t(i) = \left[\frac{p_t(i)}{P_t} \right]^{-\theta} G_t. \tag{12}$$

In the event of a price change, firm i chooses its price $p_t(i)$ to maximize

$$\mathbb{E}_t \left[\sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \Upsilon_{t,T}(i) \right]$$

subject to (4) and

$$y_{t,T}(i) = \left[\frac{p_t(i)}{P_T} \right]^{-\theta} Y_T, \tag{13}$$

[1] A time-varying elasticity of substitution among differentiated labor inputs would give rise to a markup shock like μ_t^w.

where

$$Y_t = \int_0^1 C_t^j dj + G_t.$$

The first order condition for firm i problem is

$$\mathbb{E}_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left[(1+s) - \frac{\theta}{\theta - 1} \frac{w_T(i)}{a_T p_t(i)} \right] y_{t,T}(i) \right\} = 0. \tag{14}$$

1.2 Equilibrium

Complete markets ensure that consumption is the same across households ($C_t^j = C_t \ \forall j \in (0,1)$). Combining expressions (9) and (10) yields the standard Euler equation

$$1 = \beta (1 + i_t) \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\tilde{\sigma}^{-1}} \frac{1}{1 + \pi_{t+1}} \right]. \tag{15}$$

In equilibrium, all firms within industry j resetting their price at time t charge the same price $p_t(i) = p_t^j$. Further, substituting the labor supply equation (11), combined with the production function (4) and total demand (13), into the optimal pricing condition (14) shows that the optimal price is only function of aggregate variables ($p_t^j = p_t^*$). After some manipulations, the optimal relative price is a function of the ratio between the present discounted value of marginal revenues and marginal costs

$$\frac{p_t^*}{P_t} = \left(\frac{K_t}{F_t} \right)^{\frac{1}{1+\theta\omega}},$$

where

$$K_t \equiv \mathbb{E}_t \left\{ \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\left(\frac{\theta}{\theta - 1} \right) \left(\frac{\mu_T^w}{1 - \tau_T} \right) \left(\frac{P_T}{P_t} \right)^{\theta(1+\omega)} \left(\frac{Y_T}{a_T} \right)^{1+\omega} \right] \right\},$$

and

$$F_t \equiv \mathbb{E}_t \left\{ \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[(1+s) \left(\frac{P_T}{P_t} \right)^{\theta-1} Y_T C_T^{-\tilde{\sigma}^{-1}} \right] \right\}.$$

The last two expressions in recursive form read as

$$K_t = \left(\frac{\theta}{\theta - 1} \right) \left(\frac{\mu_t^w}{1 - \tau_t} \right) \left(\frac{Y_t}{a_t} \right)^{1+\omega} + \alpha\beta \mathbb{E}_t [(1 + \pi_{t+1})^{\theta(1+\omega)} K_{t+1}] \tag{16}$$

and

$$F_t = (1+s) C_t^{-\tilde{\sigma}^{-1}} Y_t + \alpha\beta \mathbb{E}_t [(1 + \pi_{t+1})^{\theta-1} F_{t+1}]. \tag{17}$$

Given the law of large numbers, the aggregate price index (8) becomes

$$P_t = [\alpha P_{t-1}^{1-\theta} + (1-\alpha)\,(p_t^*)^{1-\theta}]^{\frac{1}{1-\theta}}.$$

Combining the last equation with the expression for the optimal relative price p_t^*/P_t yields a non-linear Phillips curve of the form

$$\left[\frac{1-\alpha\,(1+\pi_t)^{\theta-1}}{1-\alpha}\right]^{\frac{1+\theta\omega}{\theta-1}} = \frac{F_t}{K_t}. \tag{18}$$

The government finances the subsidy to firms with lump-sum taxes. Using the demand functions (12) and (13) as well as the expression for profits (5), the government budget constraint in real terms becomes

$$\frac{B_t}{P_t} = \left(\frac{1+i_{t-1}}{1+\pi_t}\right)\frac{B_{t-1}}{P_{t-1}} - (\tau_t Y_t - G_t)\,,$$

where the relative price simplifies given the definition of the price index (8). Define the real value of debt at maturity as

$$b_t \equiv \frac{(1+i_t)\,B_t}{P_t}.$$

The previous version of the government budget constraint becomes

$$\frac{b_t}{1+i_t} = \frac{b_{t-1}}{1+\pi_t} - (\tau_t Y_t - G_t)\,.$$

Substituting out the nominal interest rate from the Euler equation (15) and rearranging yields

$$\frac{C_t^{-\tilde{\sigma}^{-1}} b_{t-1}}{1+\pi_t} = C_t^{-\tilde{\sigma}^{-1}}\,(\tau_t Y_t - G_t) + \beta \mathbb{E}_t \left(\frac{C_{t+1}^{-\tilde{\sigma}^{-1}} b_t}{1+\pi_{t+1}}\right). \tag{19}$$

Equation (19) clarifies why this forward looking formulation of the government budget constraint is independent of the nominal interest rate.

Finally, the resource constraint is

$$Y_t = C_t + G_t. \tag{20}$$

For a given specification of monetary and fiscal policy (simple rules or optimal policy), an equilibrium for this economy is a sequence $\{C_t, Y_t, F_t, K_t, \pi_t, b_t, i_t, \tau_t\}_{t=0}^{\infty}$ that satisfies equations (15) – (20).

1.3 The Policy Problem

One possible specification of monetary and fiscal policy would consist of writing simple instrument rules for i_t (interest rate rule for the monetary authority) and τ_t (tax rate rule for the fiscal authority). The paper follows an alternative strategy and specifies targeting rules directly on inflation π_t and debt b_t. In this case, the equilibrium endogenously determines the nominal interest rate and the tax rate. In particular, the nominal interest rate is residual (or block-exogenous) unless explicitly included in one of the targeting rules.

This appendix studies the optimal monetary and fiscal policy plan which results from maximizing aggregate households' utility subject to the constraint of the allocation being an equilibrium for the private sector and compares the outcome with the case of optimized flexible targeting rules. In this case, a benevolent planner maximizes

$$u_0 \equiv \int_0^1 u_0^j dj = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\tilde{\sigma}^{-1}}}{1-\tilde{\sigma}^{-1}} - \frac{(Y_t/a_t)^{1+\omega}}{1+\omega} \Delta_t \right] \right\}, \tag{21}$$

where Δ_t is an index that captures the output costs associated with price dispersion

$$\Delta_t \equiv \int_0^1 \left[\frac{p_t(i)}{P_t} \right]^{-\theta(1+\omega)} di.$$

This first term of the welfare objective (21) features aggregate consumption because of the assumption of complete markets. Substituting for labor from the production function (4) and using the demand equation (13) leads to expressing the second term of (21) as a function of aggregate output and price dispersion only. Given its definition, the price dispersion index evolves according to

$$\Delta_t = \alpha \Delta_{t-1} (1+\pi_t)^{\theta(1+\omega)} + (1-\alpha) \left[\frac{1-\alpha(1+\pi_t)^{\theta-1}}{1-\alpha} \right]^{\frac{\theta(1+\omega)}{\theta-1}}. \tag{22}$$

The constraints for the optimal policy problem are equations (15) through (20) that characterize the equilibrium discussed above, plus expression (22).

1.4 Steady State

The analytical derivations and the quantitative results in the paper rely on approximated dynamics around a non-stochastic steady state with zero inflation and no price dispersion.

In such a steady state, the government sets the subsidy s to eliminate goods market

distortions

$$s = \left(\frac{\theta}{\theta - 1}\right)\left(\frac{\mu^w}{1-\tau}\right) - 1 \Rightarrow C^{-\tilde{\sigma}^{-1}} = \frac{Y_t^{\omega}}{a_t^{1+\omega}}.$$

Using the resource constraint, the last equation gives a solution for steady state output as a function of productivity and the ratio of government spending to GDP.

Under the assumption that lump-sum taxes finance the subsidy to firms, the government budget constraint pins down the steady state tax rate as a function of the ratios of debt and spending to GDP

$$\tau = (1-\beta)\frac{b}{Y} + \frac{G}{Y}.$$

1.5 Second Order Approximation of the Welfare Objective

The welfare calculations below are derived in the context of a linear-quadratic framework. To this end, a second order approximation of the utility from consumption after replacing for the resource constraint is

$$\frac{(Y_t - G_t)^{1-\tilde{\sigma}^{-1}}}{1-\tilde{\sigma}^{-1}} = C^{-\tilde{\sigma}^{-1}} Y \left[\hat{Y}_t + \frac{1}{2}\left(1-\sigma^{-1}\right)\hat{Y}_t^2 + \sigma^{-1}\hat{Y}_t\hat{G}_t\right] + t.i.p. \tag{23}$$

where $\sigma \equiv \tilde{\sigma} s_C$, $s_C \equiv C/Y$, $\hat{G}_t \equiv (G_t - G)/Y$ and $t.i.p.$ stands for "terms independent of policy" (in this case the steady state value of utility from consumption and a quadratic term in $\hat{G}_t$).[2]

A second order approximation of expression (22) shows that the index of price dispersion is of order two[3]

$$\hat{\Delta}_t = \alpha\hat{\Delta}_{t-1} + \frac{\alpha\theta(1+\omega)(1+\theta\omega)}{2(1-\alpha)}\pi_t^2.$$

Therefore, up to second order, the disutility from labor is

$$\frac{(Y_t/a_t)^{1+\omega}}{1+\omega}\Delta_t = \left(\frac{Y}{a}\right)^{1+\omega}\left[\hat{Y}_t + \frac{1}{2}(1+\omega)\hat{Y}_t^2 - (1+\omega)\hat{Y}_t\hat{a}_t + \left(\frac{1}{1+\omega}\right)\hat{\Delta}_t\right] + t.i.p. \tag{24}$$

Combining the approximations of the two components of the welfare objective and using the steady state optimal pricing condition yields

$$u_0 = -(C^{-\tilde{\sigma}^{-1}}Y)\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left\{\frac{1}{2}\left(\sigma^{-1}+\omega\right)\hat{Y}_t^2 - \left[(1+\omega)\hat{a}_t + \sigma^{-1}\hat{G}_t\right]\hat{Y}_t + \left(\frac{1}{1+\omega}\right)\hat{\Delta}_t\right\} + t.i.p.$$

[2] The standard notation for the log-linear approximation of a generic variable X_t, whose steady state value is X, is $\hat{X}_t \equiv \ln(X_t/X) \simeq (X_t - X)/X$.

[3] The index of price dispersion is a measure of the variance of relative prices across firms.

The second order approximation of household utility only features quadratic terms thanks to the presence of the steady state subsidy that eliminates goods market distortions.

Note that the present discounted value of the index of price dispersion is

$$\sum_{t=0}^{\infty} \beta^t \hat{\Delta}_t = \frac{\alpha\theta\left(1+\omega\right)\left(1+\theta\omega\right)}{2\left(1-\alpha\right)\left(1-\alpha\beta\right)} \sum_{t=0}^{\infty} \beta^t \pi_t^2 + t.i.p.$$

Replacing the last expression in the previous approximation of the welfare objective gives

$$u_0 = -\left(\frac{C^{-\tilde{\sigma}^{-1}}Y}{2}\right) \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \{\left(\sigma^{-1}+\omega\right)\hat{Y}_t^2 - 2\left[\left(1+\omega\right)\hat{a}_t + \sigma^{-1}\hat{G}_t\right]\hat{Y}_t + \theta\xi^{-1}\pi_t^2\} + t.i.p., \tag{25}$$

where

$$\xi \equiv \frac{\left(1-\alpha\right)\left(1-\alpha\beta\right)}{\alpha\left(1+\theta\omega\right)}.$$

In an efficient equilibrium (i.e. absent distortionary taxes and markup shocks), up to a first order approximation, output would be equal to

$$\hat{Y}_t^* \equiv \left(\frac{1+\omega}{\sigma^{-1}+\omega}\right)\hat{a}_t + \left(\frac{\sigma^{-1}}{\sigma^{-1}+\omega}\right)\hat{G}_t. \tag{26}$$

Using the previous expression in (25), the welfare approximation becomes

$$u_0 = -\frac{\Omega}{2}\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\pi_t^2 + q_y y_t^2\right), \tag{27}$$

where the efficient output gap is

$$y_t \equiv \hat{Y}_t - \hat{Y}_t^*$$

and the coefficients in the welfare objective are

$$\Omega \equiv \theta\xi^{-1}C^{-\tilde{\sigma}^{-1}}Y$$

and

$$q_y \equiv \theta^{-1}\xi\left(\sigma^{-1}+\omega\right). \tag{28}$$

1.6 Linear Approximation of the Equilibrium Conditions

A first order approximation of equation (15) yields

$$\hat{C}_t = -\tilde{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1}\right) + \mathbb{E}_t \hat{C}_{t+1}. \tag{29}$$

A first order approximation to the non-linear Phillips curve (18) yields

$$\hat{F}_t - \hat{K}_t = -\left(\frac{1+\theta\omega}{\theta-1}\right)\left(\frac{\alpha\theta}{1-\alpha}\right)\pi_t,$$

where the two approximated left-hand side terms are

$$\hat{F}_t = (1-\alpha\beta)\left(\hat{Y}_t - \tilde{\sigma}^{-1}\hat{C}_t\right) + \alpha\beta\mathbb{E}_t[(\theta-1)\,\pi_{t+1} + \hat{F}_{t+1}]$$

and

$$\hat{K}_t = (1-\alpha\beta)\left[(1+\omega)\left(\hat{Y}_t - \hat{a}_t\right) + \omega_\tau \hat{\tau}_t + \hat{\mu}_t^w\right] + \alpha\beta\mathbb{E}_t[\theta\,(1+\omega)\,\pi_{t+1} + \hat{K}_{t+1}],$$

and where $\omega_\tau \equiv \tau/(1-\tau)$. Combining the last three expressions and rearranging yields a standard linearized Phillips curve

$$\pi_t = \xi[\omega\hat{Y}_t + \tilde{\sigma}^{-1}\hat{C}_t + \omega_\tau\hat{\tau}_t - (1+\omega)\,\hat{a}_t + \hat{\mu}_t^w] + \beta\mathbb{E}_t\pi_{t+1}. \tag{30}$$

A first order approximation of the government budget constraint (19) is

$$\begin{aligned} -\tilde{\sigma}^{-1}\hat{C}_t + \hat{b}_{t-1} - \pi_t = (1-\beta)\left[-\tilde{\sigma}^{-1}\hat{C}_t + b_\tau(\hat{\tau}_t + \hat{Y}_t) - s_d^{-1}\hat{G}_t\right] \\ + \beta\mathbb{E}_t(-\tilde{\sigma}^{-1}\hat{C}_{t+1} + \hat{b}_t - \pi_{t+1}), \end{aligned} \tag{31}$$

where $s_d \equiv \tau - G/Y$ and $b_\tau \equiv \tau s_d^{-1}$.

Finally, a first order approximation of the resource constraints is

$$\hat{Y}_t = s_C\hat{C}_t + \hat{G}_t. \tag{32}$$

Replacing equation (32) into the linearized Euler equation (29), Phillips curve (30) and government budget constraint (31) allows for substituting out consumption as a function of output and exogenous government spending

$$\hat{Y}_t = -\sigma(i_t - \mathbb{E}_t\pi_{t+1}) + \mathbb{E}_t\hat{Y}_{t+1} + \hat{G}_t - \mathbb{E}_t\hat{G}_{t+1}.$$

$$\pi_t = \xi[(\omega + \sigma^{-1})\hat{Y}_t + \omega_\tau \hat{\tau}_t - (1+\omega)\hat{a}_t - \sigma^{-1}\hat{G}_t + \hat{\mu}_t^w] + \beta \mathbb{E}_t \pi_{t+1},$$

$$\begin{aligned}\hat{b}_{t-1} - \sigma^{-1}\hat{Y}_t - \pi_t + \sigma^{-1}\hat{G}_t = (1-\beta)\,[b_y \hat{Y}_t + b_\tau \hat{\tau}_t + (\sigma^{-1} - s_d^{-1})\hat{G}_t] \\ + \beta \mathbb{E}_t(\hat{b}_t - \sigma^{-1}\hat{Y}_{t+1} - \pi_{t+1} + \sigma^{-1}\hat{G}_{t+1}),\end{aligned}$$

where $b_y \equiv b_\tau - \sigma^{-1}$.

The expression for the efficient level of output (26) leads to rewriting the Phillips curve as

$$\pi_t = \xi[(\omega + \sigma^{-1})y_t + \omega_\tau \hat{\tau}_t + \hat{\mu}_t^w] + \beta \mathbb{E}_t \pi_{t+1},$$

or, as in the text,

$$\pi_t = \kappa[y_t + \psi(\hat{\tau}_t - \hat{\tau}_t^*)] + \beta \mathbb{E}_t \pi_{t+1},$$

where the slope of the Phillips curve with respect to the output gap is $\kappa \equiv \xi(\omega + \sigma^{-1})$. The coefficient in front of the tax gap is $\psi \equiv \omega_\tau/(\omega + \sigma^{-1})$ and $\hat{\tau}_t^* \equiv -\omega_\tau^{-1}\hat{\mu}_t^w$ (a linear transformation of the markup) is the tax rate that satisfies the Phillips curve when both inflation and the output gap are at their desired (welfare-maximizing) levels.

The Euler equation in terms of the output gap becomes

$$\hat{Y}_t - \hat{Y}_t^* = -\sigma(i_t - \mathbb{E}_t \pi_{t+1}) + \mathbb{E}_t(\hat{Y}_{t+1} - \hat{Y}_{t+1}^*) + \mathbb{E}_t \hat{Y}_{t+1}^* - \hat{Y}_t^* + \hat{G}_t - \mathbb{E}_t \hat{G}_{t+1}$$

or, more compactly,

$$y_t = -\sigma(i_t - \mathbb{E}_t \pi_{t+1} - r_t^*) + \mathbb{E}_t y_{t+1},$$

where the efficient real interest rate is

$$r_t^* \equiv \sigma^{-1}[\mathbb{E}_t(\hat{Y}_{t+1}^* - \hat{Y}_t^*) - \mathbb{E}_t(\hat{G}_{t+1} - \hat{G}_t)].$$

Finally, the intertemporal government budget constraint in terms of output gap and tax gap is

$$\begin{aligned}&\hat{b}_{t-1} - \sigma^{-1}(\hat{Y}_t - \hat{Y}_t^*) - \pi_t - \sigma^{-1}(\hat{Y}_t^* - \hat{G}_t) \\ &\quad = (1-\beta)\,\mathbb{E}_t \sum_{s=t}^{\infty} \beta^{s-t}[b_y(\hat{Y}_s - \hat{Y}_s^*) + b_\tau(\hat{\tau}_s - \hat{\tau}_s^*) + b_y \hat{Y}_s^* + b_\tau \hat{\tau}_s^* + (\sigma^{-1} - s_d^{-1})\hat{G}_s].\end{aligned}$$

Define

$$v_t \equiv (1-\beta)\,\mathbb{E}_t \sum_{s=t}^{\infty} \beta^{s-t}[b_y \hat{Y}_s^* + b_\tau \hat{\tau}_s^* + (\sigma^{-1} - s_d^{-1})\hat{G}_s].$$

The previous expression becomes

$$\hat{b}_{t-1} - \sigma^{-1} y_t - \pi_t + f_t = (1-\beta)\, \mathbb{E}_t \sum_{s=t}^{\infty} \beta^{s-t} [b_y y_s + b_\tau (\hat{\tau}_s - \hat{\tau}_s^*)],$$

or, in flow form

$$\hat{b}_{t-1} - \sigma^{-1} y_t - \pi_t + f_t = (1-\beta)\, [b_y y_t + b_\tau (\hat{\tau}_t - \hat{\tau}_t^*)] + \beta \mathbb{E}_t (\hat{b}_t - \sigma^{-1} y_{t+1} - \pi_{t+1} + f_{t+1}),$$

where

$$f_t \equiv -\sigma^{-1} (\hat{Y}_t^* - \hat{G}_t) - v_t.$$

Benigno and Woodford (2003) call f_t the "fiscal stress,"— a measure of departure from contemporaneous stabilization of inflation and output gap. In this model, the policymaker can use the tax rate as an additional stabilization instrument to eliminate the tradeoff between inflation and output gap arising from the Phillips curve. If the fiscal stress were zero in every period, the fiscal authority could keep debt constant at its steady state level ($\hat{b}_t = 0$) and achieve an equilibrium with zero inflation and output always equal to its efficient level. Therefore, the fiscal stress measures the extent to which fiscal considerations prevent this allocation to occur.

The linear-quadratic optimal policy problem discussed in the text represents a correct approximation, up to the second order, to the non-linear optimal policy problem presented in this section.

1.7 The Optimal Policy Plan

The optimal policy plan under commitment is a sequence $\left\{\pi_t, y_t, \hat{\tau}_t, \hat{b}_t\right\}_{t=0}^{\infty}$ which solves

$$\min -\frac{\Omega}{2} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\pi_t^2 + q_y y_t^2\right)$$

subject to

$$\pi_t = \kappa [y_t + \psi (\hat{\tau}_t - \hat{\tau}_t^*)] + \beta \mathbb{E}_t \pi_{t+1}$$

and

$$\hat{b}_{t-1} - \sigma^{-1} y_t - \pi_t + f_t = (1-\beta)\, [b_y y_t + b_\tau (\hat{\tau}_t - \hat{\tau}_t^*)] + \beta \mathbb{E}_t (\hat{b}_t - \sigma^{-1} y_{t+1} - \pi_{t+1} + f_{t+1}),$$

plus initial commitments that make policy optimal from a timeless perspective (Woodford, 1999).

The first order condition for π_t is

$$\pi_t = \kappa^{-1}\left(\varphi_{1t} - \varphi_{1t-1}\right) + \left(\varphi_{2t} - \varphi_{2t-1}\right),$$

where φ_{1t} is the lagrange multiplier on the aggregate supply and φ_{2t} is the lagrange multiplier on the government budget constraint.

The first order condition for y_t is

$$q_y y_t = -\varphi_{1t} + \left[(1-\beta)\, b_y + \sigma^{-1}\right]\varphi_{2t} - \sigma^{-1}\varphi_{2t-1}.$$

The first order condition for $\hat{\tau}_t$ is

$$\psi\varphi_{1t} = (1-\beta)\, b_\tau \varphi_{2t}.$$

Finally, the first order condition for $\hat{b}_t$ is

$$\varphi_{2t} = \mathbb{E}_t \varphi_{2t+1}. \tag{33}$$

Replacing the first order condition for $\hat{\tau}_t$ into the first two yields

$$\pi_t = -\omega_\varphi\left(\varphi_{2t} - \varphi_{2t-1}\right) \tag{34}$$

and

$$y_t = m_\varphi \varphi_{2t} + n_{\varphi 2t-1}, \tag{35}$$

where

$$\begin{aligned}
\omega_\varphi &\equiv -\left[1 + (1-\beta)\, b_\tau\, (\kappa\psi)^{-1}\right] \\
m_\varphi &\equiv \left[(1-\beta)\left(b_y - b_\tau \psi^{-1}\right) + \sigma^{-1}\right] q_y^{-1} \\
n_\varphi &\equiv -\left(\sigma q_y\right)^{-1}.
\end{aligned}$$

Solving the first condition for φ_{2t-1} and the second for φ_{2t} and combining yields the first optimal targeting rule

$$\pi_t + \frac{n_\varphi}{m_\varphi}\pi_{t-1} + \frac{\omega_\varphi}{m_\varphi}\left(y_t - y_{t-1}\right) = 0. \tag{36}$$

Expression (36) resembles the optimal policy rule in the baseline New Keynesian model, ex-

cept for the term in lagged inflation which captures the additional degree of inertia introduced by the government budget constraint.

The second optimal targeting rule follows from combining the first order condition for $\hat{b}_t$ and the expression above for π_t as a function of φ_{2t} and φ_{2t-1}

$$\mathbb{E}_t \pi_{t+1} = 0. \tag{37}$$

The optimal fiscal rule (37) commands the government to set taxes without creating inflationary expectations. The interpretation of (37) as a targeting rule for fiscal policy is perhaps not completely obvious. However, replacing (37) into the Phillips curve yields a more intuitive fiscal policy rule in terms of taxes, output gap and inflation

$$(\hat{\tau}_t - \hat{\tau}_t^*) + m_\tau y_t + n_\tau \pi_t = 0, \tag{38}$$

where $m_\tau \equiv \psi^{-1}$ and $n_\tau \equiv -(\kappa\psi)^{-1}$. Equation (38) indicates that the government sets taxes trying to balance movements in the output gap and the inflation rate.

Benigno and Woodford (2003) prove (see their appendix) that the equilibrium under optimal targeting rules is determinate. Furthermore, the policy plan has the additional advantage of being robust to different assumptions about the nature of the stochastic process assumed for the exogenous disturbances.

Economically, one important feature of the optimal policy plan is the unit root in the process of debt. This result can be proved analytically in case of i.i.d. fiscal stress shocks. The first step consists of replacing (38) into the present discounted value version of the government budget constraint. Then, equations (34) and (35) can be used to substitute for π_t and y_t as a function of the Lagrange multiplier φ_{2t}. The resulting expression is

$$\varphi_{2t} = \frac{m_b}{m_b + n_b}\varphi_{2t-1} - \frac{1}{m_b + n_b}(f_t + \hat{b}_{t-1}),$$

where

$$\begin{aligned} m_b &\equiv \omega_\varphi + n_\varphi \sigma^{-1} + (1-\beta)\left[b_y n_\varphi - \left(1 - \omega_\varphi \kappa^{-1}\right) b_\tau \psi^{-1}\right] \\ n_b &\equiv b_\tau \left(\psi^{-1} - 1\right)(m_\varphi + n_\varphi). \end{aligned}$$

This equation implies that debt evolves according to

$$\hat{b}_t = -\left(\mathbb{E}_t f_{t+1} + n_b \varphi_{2t}\right).$$

If the fiscal is stress is i.i.d., $\mathbb{E}_t f_{t+1} = 0$, the first order condition for debt (33) then implies that

$$\hat{b}_t = \mathbb{E}_t \hat{b}_{t+1}.$$

1.8 Optimal Flexible Targeting Rules

This section compares the welfare properties of the optimized version of the flexible targeting rules for monetary policy

$$\pi_t + \gamma y_t = 0 \tag{39}$$

and fiscal policy

$$\hat{b}_t + \lambda y_t = 0 \tag{40}$$

with the optimal policy plan derived in the previous section.

The welfare metric for the comparison is

$$d_p \equiv -\left(\frac{1-\beta}{2C^{1-\tilde{\sigma}^{-1}}}\right)\left[\mathbb{E}(u_0^p) - \mathbb{E}(u_0^{opt})\right], \tag{41}$$

where u_0^p represents welfare under an arbitrary suboptimal policy plan p consistent with timeless perspective commitments (in this case, the simple targeting rules) and u_0^{opt} stands for welfare under the optimal policy plan. The operator $\mathbb{E}(\cdot)$ defines the expectation over the distribution of shocks at time zero. The welfare measure d_p is conditional on the system being in a steady state before time 0 and corresponds to a correct second order approximation of the consumption equivalent of the two policies (Lucas, 2003).[4] In other words, d_p measures the fraction of consumption under the optimal policy regime that the average household in the economy would be willing to sacrifice in each period to avoid switching to the suboptimal regime p.

In order to compute the optimal coefficients in the policy rules and the consumption equivalents, the persistence of the fiscal stress is assumed to be equal to 0.9 and the innovations to its process to be i.i.d. standard normal. Table 1 summarizes the main results of the welfare analysis. The first two columns report the value of the optimized coefficient and the consumption equivalent when only one policy authority at a time follows a flexible targeting rule. The third column reports the optimal coefficients and the consumption equivalent when both policy authorities follow flexible targeting rules.

The highlight of table 1 is the large welfare gain of moving from a flexible inflation targeting to a flexible debt targeting regime. There is no additional gain in granting flexibility to

[4] See the extended version of Schmitt-Grohé and Uribe (2007) for the derivation.

Table 1: Optimized targeting rules coefficients (γ, λ) and consumption equivalents (d_p).

		Flexible Targeting Rule		
		Monetary	Fiscal	Both
Optimized	γ	0.02	0	0.01
Coefficients	λ	0	12.2	12.2
Consumption Equivalent	d_p	10.45%	0.36%	0.36%

the monetary authority when the fiscal authority already follows an optimized flexible debt targeting rule. The optimized flexible debt targeting rule implements an aggressive response of debt to variations of the output gap. This feature attempts to mimic the permanent changes of debt in response to a fiscal stress shock typical of the optimal policy plan.

A simple back-of-the-envelope calculation can shed some light on the magnitudes associated with the optimized value of the coefficient λ. An adverse shock that pushes output 5% below potential over the course of a year implies a 61% increase in debt over the same horizon. Starting from a 40% steady state (roughly the value of net debt as a fraction of GDP in the U.S. at the end of 2008), debt would increase to 64.4%, not too far from the official estimates for 2010 (63.6%, according to the Office of Management and Budget).

If only the monetary authority follows a flexible targeting rule, the weight on the output gap is negligible.[5] When both policy authorities follow flexible targeting rules, the weight on the output gap in the inflation targeting rule decreases. This result is not very surprising. In this model, taxes can offset the tradeoff between inflation and output gap stabilization arising from markup shocks. The fundamental costs of business cycles are instead associated with the fiscal stress. Fiscal stabilizations are the most appropriate tool in response to this type of disturbances, while strict inflation targeting is nearly optimal.

The general lesson of the welfare analysis is that flexibility is clearly a more desirable characteristic of fiscal targeting rules. This finding partially collides with the robustness results in the text, hence creating a welfare-versus-robustness tradeoff.

[5] If the flexible inflation targeting rule allows inflation to respond to variations in the growth rate of the output gap Δy_t, optimality would require a much higher weight (of the order of 0.3). This alternative formulation is interesting because it corresponds to the optimal targeting rule in the baseline New Keynesian model.

Table 2: Optimized targeting rules coefficients (γ, ϕ) and consumption equivalents (d_p).

		Flexible Targeting Rule		
		Monetary	Fiscal	Both
Optimized	γ	0.06	0	0
Coefficients	ϕ	0	12.6	12.6
Consumption Equivalent	d_p	37.47%	0.39%	0.39%

1.9 Balanced Budget Rules and Welfare

The paper shows that the rational expectations equilibrium is determinate under a balanced budget requirement if the fiscal rule is the strict version of

$$\hat{b}_t - i_t + \phi y_t = 0. \tag{42}$$

Also in this case, the optimized version of (42) is the result of the optimal choice of the coefficient ϕ to minimize the welfare costs of departing from the fully optimal plan. Obviously, the answer also bears some consequences for the optimal degree of flexibility in monetary policy. Table 2 reports the quantitative results in the same format as in the previous section.

One preliminary observation is in order. The consumption equivalent numbers in tables 1 and 2 are not directly comparable. The solution of the model under the modified flexible debt targeting rule (42) requires the specification of a stochastic process for the equilibrium real interest rate r_t^*. For simplicity, this variable is assumed to follow a first order autoregressive process with persistence equal to 0.9 and i.i.d. standard normal innovations. This assumption adds an independent source of exogenous variability relative to the fiscal stress and increases the overall costs of business cycle fluctuations.[6] Since the emphasis here is on the gains from fiscal relative to monetary flexibility, the simplification of making the two processes independent of each other does not affect the main result.

The main messages of the previous section are unchanged. The gains of granting flexibility to the fiscal authority rather than to the monetary authority remain substantial. Moreover, in this case, strict inflation targeting is exactly the optimal specification of the flexible inflation targeting rule (39) when the fiscal authority follows a flexible debt targeting rule. These results stress even further the superior welfare properties of flexible fiscal rules. In all simulations, optimality pushes the flexibility coefficient ϕ to its determinacy threshold. As in section 1.8, the optimal flexible fiscal targeting rule tries to approximate the unit root in

[6] In principle, r_t^* and f_t are functions of the same fundamental shocks and hence correlated.

debt typical of the optimal policy plan.

2 An Alternative Economy with Sales Taxes

The model in this section corresponds to Benigno and Woodford (2003). The only difference with the previous section is that the distortionary tax instrument is a sales tax as opposed to an income tax.

The utility function and consumption index are the same as in the previous section. The flow budget constraint does not display the income tax

$$\int_0^1 p_t(i)\, c_t^j(i)\, di + \mathbb{E}_t(Q_{t,t+1} D_{t+1}^j) = w_t^j \ell_t^j + \Upsilon_t^j + D_t^j - \varsigma_t^j. \tag{43}$$

Conversely, the tax rate now appears in the per-period profit function

$$\Upsilon_{t,T}(i) = (1-\tau_T)\, p_t(i)\, y_{t,T}(i) - w_T(i)\, \ell_T(i). \tag{44}$$

Furthermore, the firm does not receive the subsidy s any longer. The modified flow government budget constraint is

$$B_t = (1+i_{t-1})\, B_{t-1} - \int_0^1 p_t(i)\left[\tau_t y_t(i) - g_t(i)\right] di - \int_0^1 \varsigma_t^j dj. \tag{45}$$

In this specification, lump-sum transfers represent a purely exogenous fiscal shock. The rest of the model coincides with the previous section.

2.1 Private Sector Optimality Conditions

The only differences involve the optimality condition for hours, which is no longer affected by taxes

$$\frac{w_t^j}{P_t} = \mu_t^w \frac{(\ell_t^j)^\omega}{(C_t^j)^{-\tilde{\sigma}^{-1}}}, \tag{46}$$

and the first order condition for firm i which features the sales taxes influencing the revenue term

$$\mathbb{E}_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left[(1-\tau_T) - \frac{\theta}{\theta-1} \frac{w_T(i)}{a_T p_t(i)} \right] y_{t,T}(i) \right\} = 0. \tag{47}$$

2.2 Equilibrium

As a consequence of the previous modifications, the expression that characterize the optimal price setting becomes

$$K_t \equiv \mathbb{E}_t \left\{ \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\left(\frac{\theta}{\theta - 1} \right) \mu_T^w \left(\frac{P_T}{P_t} \right)^{\theta(1+\omega)} \left(\frac{Y_T}{a_T} \right)^{1+\omega} \right] \right\},$$

and

$$F_t \equiv \mathbb{E}_t \left\{ \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[(1 - \tau_T) \left(\frac{P_T}{P_t} \right)^{\theta-1} Y_T C_T^{-\tilde{\sigma}^{-1}} \right] \right\},$$

which in recursive form read as

$$K_t = \left(\frac{\theta}{\theta - 1} \right) \mu_t^w \left(\frac{Y_t}{a_t} \right)^{1+\omega} + \alpha\beta \mathbb{E}_t [(1 + \pi_{t+1})^{\theta(1+\omega)} K_{t+1}] \tag{48}$$

and

$$F_t = (1 - \tau_t) C_t^{-\tilde{\sigma}^{-1}} Y_t + \alpha\beta \mathbb{E}_t [(1 + \pi_{t+1})^{\theta-1} F_{t+1}]. \tag{49}$$

Under the assumption of no lump-sum fiscal shocks (i.e. $\varsigma_t = 0 \ \forall t$), the government budget constraint coincides with the previous section.

2.3 Policy Problem, Steady State and Quadratic Welfare Objective

The welfare objective and the index of price distortion Δ_t coincide with the previous section. The absence of the steady state subsidy s, however, makes the steady state in this section inefficient.[7] The optimal pricing condition becomes

$$C^{-\tilde{\sigma}^{-1}} = \left(\frac{\theta}{\theta - 1} \right) \left(\frac{\mu^w}{1 - \tau} \right) \frac{Y^\omega}{a^{1+\omega}}.$$

The second order approximation of the utility from consumption and of the disutility from labor coincide with (23) and (24) respectively. The absence of the subsidy, however, implies

[7] Benigno and Woodford (2003) show that there exists nevertheless an optimal commitment plan that supports a zero inflation steady state with positive debt even if the real allocation is inefficient.

that after combining the two terms, a linear term remains present

$$u_0 \equiv -(C^{-\tilde{\sigma}^{-1}}Y)\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left[-\Phi \hat{Y}_t + \frac{1}{2} \left[\left(\sigma^{-1} + \omega \right) - \Phi \left(1 + \omega \right) \right] \hat{Y}_t^2 \right] \right.$$
$$\left. - \left[\left(1 - \Phi \right) \left(1 + \omega \right) \hat{a}_t + \sigma^{-1} \hat{G}_t \right] \hat{Y}_t + \left(\frac{1 - \Phi}{1 + \omega} \right) \hat{\Delta}_t \right\}.$$

where

$$\Phi \equiv 1 - \left(\frac{\theta - 1}{\theta} \right) \left(\frac{1 - \tau}{\mu^w} \right) < 1.$$

Since the second order approximation of the index of price distortion Δ_t is unchanged, the welfare objective becomes

$$u_0 \equiv -(C^{-\tilde{\sigma}^{-1}}Y)\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left[-\Phi \hat{Y}_t + \frac{1}{2} \left[\left(\sigma^{-1} + \omega \right) - \Phi \left(1 + \omega \right) \right] \hat{Y}_t^2 \right] \right.$$
$$\left. - \left[\left(1 - \Phi \right) \left(1 + \omega \right) \hat{a}_t + \sigma^{-1} \hat{G}_t \right] \hat{Y}_t + \frac{1}{2} \frac{\theta \left(\sigma^{-1} + \omega \right) \left(1 - \Phi \right)}{\kappa} \pi_t^2 \right\}. \quad (50)$$

Expression (50) contains a linear term in output which is multiplied by a measure of steady state distortions Φ. The previous section assumed that the fiscal authority eliminates the steady state distortions by setting steady state subsidy s so that $\Phi = 0$. In that case, the welfare objective becomes purely quadratic and a first order approximation to the equilibrium conditions is sufficient to correctly characterize optimal policy up to the first order. If the fiscal authority is either not willing or not able to eliminate the steady state monopolistic distortions by resorting to a subsidy, the linear term in (50) can still be eliminated by appropriately using a second order approximation to the equilibrium conditions.[8] The resulting product is again a purely quadratic loss function like (27), which, combined with a linear approximation to the equilibrium conditions, still allows for evaluating optimal policy correctly up to the first order and welfare up to the second order. The expression for the relative weight on the output gap, however, now becomes a different function of the structural parameters relative to (28).

[8] The intuition relies on the relation which exists under optimal policy between the linear terms in the second order approximation of the utility function and the linear terms in the first order approximation of the equilibrium conditions. Because, up to the second order, the linear terms in the approximation of the equilibrium conditions are a function of the quadratic terms, the welfare objective can be written as a sum of second order terms only. See Benigno and Woodford (2003) for details.

2.4 Linear Approximation of the Equilibrium Conditions

Only two equilibrium conditions differ across the two specifications (income vs. sales taxes). These two equations are the expressions for K_t and F_t in (48) and (49), whose first order approximation are

$$\hat{F}_t = (1-\alpha\beta)\left(\hat{Y}_t - \tilde{\sigma}^{-1}\hat{C}_t - \omega_\tau \hat{\tau}_t\right) + \alpha\beta\mathbb{E}_t[(\theta-1)\,\pi_{t+1} + \hat{F}_{t+1}]$$

and

$$\hat{K}_t = (1-\alpha\beta)\left[(1+\omega)\left(\hat{Y}_t - \hat{a}_t\right) + \hat{\mu}_t^w\right] + \alpha\beta\mathbb{E}_t[\theta\,(1+\omega)\,\pi_{t+1} + \hat{K}_{t+1}].$$

The distortionary tax rate now appears in $\hat{F}_t$, as opposed to $\hat{K}_t$ as in the previous section, but with the opposite sign. Therefore, the resulting Phillips curve coincides with (30).

In sum, the linearized model is the same whether the distortionary tax instrument is an income or sales tax rate. A difference arises between the two models if for some reasons the fiscal authority cannot use a steady state subsidy to eliminate the monopolistic distortions. In this case, the steady state differs between the two models. As a consequence, the absolute welfare costs are not comparable. Nevertheless, the second order approximation of the welfare objective is purely quadratic in inflation and the output gap in both models, although the relative weight on the output gap in the welfare objective also differs across specifications.

3 Consumption Tax

The utility function and consumption index are again the same as in the baseline model. In this case, household j's budget constraint is

$$(1+\tau_t)\int_0^1 p_t\,(i)\,c_t^j\,(i)\,di + \mathbb{E}_t(Q_{t,t+1}D_{t+1}^j) = (w_t^j\ell_t^j + \Upsilon_t^j) + D_t^j - \varsigma_t^i, \tag{51}$$

where τ_t is a distortionary consumption tax. The problem of the firm coincides with the baseline model too. The flow government budget constraint becomes

$$B_t = (1+i_{t-1})\,B_{t-1} - \int_0^1\left[\tau_t\int_0^1 p_t\,(i)\,c_t^j\,(i)\,di + \varsigma_t^j\right]dj + \int_0^1 p_t\,(i)\,[g_t\,(i) + sy_t\,(i)]\,di. \tag{52}$$

3.1 Private Sector Optimality Conditions

The presence of a distortionary consumption tax modifies the optimality condition for state-contingent securities

$$Q_{t,t+1} = \beta \left(\frac{1+\tau_t}{1+\tau_{t+1}} \right) \left(\frac{C_{t+1}^j}{C_t^j} \right)^{-\tilde{\sigma}^{-1}} \frac{1}{1+\pi_{t+1}} \tag{53}$$

and the optimality condition for hours is

$$\frac{w_t^j}{P_t} = \mu_t^w \left(1+\tau_t\right) \frac{(\ell_t^j)^{\omega}}{(C_t^j)^{-\tilde{\sigma}^{-1}}}. \tag{54}$$

where $\mu_t^w > 1$ is an exogenous wedge that perturbs the equality between the after-tax real wage and the marginal rate of substitution between consumption and hours.

3.2 Equilibrium

Given the no arbitrage relation (10), the Euler equation becomes

$$1 = \beta \left(1+i_t\right) \mathbb{E}_t \left[\left(\frac{1+\tau_t}{1+\tau_{t+1}} \right) \left(\frac{C_{t+1}}{C_t} \right)^{-\tilde{\sigma}^{-1}} \frac{1}{1+\pi_{t+1}} \right]. \tag{55}$$

The non-linear Phillips curve is once again the same function of F_t and K_t as in the baseline case. The recursive expression for F_t is

$$F_t = (1+s) \frac{C_t^{-\tilde{\sigma}^{-1}} Y_t}{1+\tau_T} + \alpha\beta \mathbb{E}_t[(1+\pi_{t+1})^{\theta-1} F_{t+1}]. \tag{56}$$

The recursive expression for K_t coincides with (48).

The forward looking rendition of the government budget constraint in real terms becomes

$$\frac{C_t^{-\tilde{\sigma}^{-1}} b_{t-1}}{(1+\tau_t)(1+\pi_t)} = \frac{C_t^{-\tilde{\sigma}^{-1}}}{1+\tau_t} (\tau_t C_t - G_t) + \beta \mathbb{E}_t \left[\frac{C_{t+1}^{-\tilde{\sigma}^{-1}} b_t}{(1+\tau_{t+1})(1+\pi_{t+1})} \right]. \tag{57}$$

3.3 Policy Problem, Steady State and Welfare Objective

The presence of the subsidy s ensures that the steady state is efficient. The policy problem, the steady state allocation and the welfare objective coincide with the baseline case. The

only difference is that the steady state subsidy is

$$s = \left(\frac{\theta}{\theta - 1}\right)(1+\tau)\,\mu^w - 1$$

and the steady state tax rate which supports the real allocation is determined by

$$s_C \tau = (1-\beta)\,\frac{b}{Y} + \frac{G}{Y}.$$

3.4 Linear Approximation of the Equilibrium Conditions

A first order approximation of (55) yields

$$\hat{C}_t = -\tilde{\sigma}\left[i_t - \mathbb{E}_t \pi_{t+1} + \omega_\tau\left(\tau_t - \mathbb{E}_t \tau_{t+1}\right)\right] + \mathbb{E}_t \hat{C}_{t+1}, \tag{58}$$

where now $\omega_\tau \equiv \tau/(1+\tau)$.

A first order approximation of (56) is

$$\hat{F}_t = (1-\alpha\beta)\left(\hat{Y}_t - \tilde{\sigma}^{-1}\hat{C}_t - \omega_\tau \hat{\tau}_t\right) + \alpha\beta\mathbb{E}_t[(\theta-1)\,\pi_{t+1} + \hat{F}_{t+1}],$$

which, combined with the first order approximation of (48) derived in the previous section and the first order approximation of the non-linear Phillips curve (18) yields the modified aggregate supply equation

$$\pi_t = \xi[\omega\hat{Y}_t + \tilde{\sigma}^{-1}\hat{C}_t + \omega_\tau\hat{\tau}_t - (1+\omega)\,\hat{a}_t + \hat{\mu}_t^w] + \beta\mathbb{E}_t\pi_{t+1}. \tag{59}$$

A first order approximation of the government budget constraint (57) is

$$\begin{aligned} -\tilde{\sigma}^{-1}\hat{C}_t + \hat{b}_{t-1} - \omega_\tau\hat{\tau}_t - \pi_t = (1-\beta)\left[-\tilde{\sigma}^{-1}\hat{C}_t - \omega_\tau\hat{\tau}_t + \tilde{b}_\tau(\hat{\tau}_t + \hat{C}_t) - \tilde{s}_d^{-1}\hat{G}_t\right] \\ + \beta\mathbb{E}_t(-\tilde{\sigma}^{-1}\hat{C}_{t+1} + \hat{b}_t - \omega_\tau\hat{\tau}_{t+1} - \pi_{t+1}), \end{aligned}$$

where now $s_d \equiv s_C\tau - G/Y$ and $b_\tau \equiv s_C\tau s_d^{-1}$.

In this case, it is convenient to use the resource constraint (32) to substitute out output into the linearized Phillips curve (30)

$$\pi_t = \xi[s_C(\omega + \sigma^{-1})\hat{C}_t + \omega_\tau\hat{\tau}_t - (1+\omega)\hat{a}_t + \omega\hat{G}_t + \hat{\mu}_t^w] + \beta\mathbb{E}_t\pi_{t+1}. \tag{60}$$

Combining the efficient level of output (26) with the resource constraint (32) gives an

expression for the efficient level of consumption

$$\hat{C}_t^* = s_C^{-1}\left[\left(\frac{1+\omega}{\sigma^{-1}+\omega}\right)\hat{a}_t + \left(\frac{\omega}{\sigma^{-1}+\omega}\right)\hat{G}_t\right] \tag{61}$$

Substituting (61) into (60) yields

$$\pi_t = \xi[s_C(\omega+\sigma^{-1})c_t + \omega_\tau\hat{\tau}_t + \hat{\mu}_t^w] + \beta\mathbb{E}_t\pi_{t+1},$$

where now $c_t \equiv \hat{C}_t - \hat{C}_t^*$. Alternatively, the Phillips curve in terms of consumption can also be written as

$$\pi_t = \kappa[c_t + \psi(\hat{\tau}_t - \hat{\tau}_t^*)] + \beta\mathbb{E}_t\pi_{t+1},$$

where now the slope of the Phillips curve with respect to the consumption gap is $\kappa \equiv \xi s_C(\omega+\sigma^{-1})$ and the coefficient in front of the tax gap is $\psi \equiv \omega_\tau s_C^{-1}/(\omega+\sigma^{-1})$.

The Euler equation in terms of consumption and tax gaps becomes

$$\begin{aligned}\hat{C}_t - \hat{C}_t^* = -\tilde{\sigma}\left\{i_t - \mathbb{E}_t\pi_{t+1} + \omega_\tau\left[(\tau_t - \tau_t^*) - \left(\mathbb{E}_t\tau_{t+1} - \tau_{t+1}^*\right)\right]\right\} \\ + \mathbb{E}_t(\hat{C}_{t+1} - \hat{C}_{t+1}^*) + \left(\mathbb{E}_t\hat{C}_{t+1}^* - \hat{C}_t^*\right) - \left(\mathbb{E}_t\tau_{t+1}^* - \tau_t^*\right)\end{aligned}$$

or, more compactly,

$$c_t = -\tilde{\sigma}\left\{i_t - \mathbb{E}_t\pi_{t+1} + \omega_\tau\left[(\tau_t - \tau_t^*) - \left(\mathbb{E}_t\tau_{t+1} - \tau_{t+1}^*\right)\right] - r_t^*\right\} + \mathbb{E}_t c_{t+1},$$

where the efficient real interest rate now is

$$r_t^* \equiv \tilde{\sigma}^{-1}\left[\left(\mathbb{E}_t\hat{C}_{t+1}^* - \hat{C}_t^*\right) - \left(\mathbb{E}_t\tau_{t+1}^* - \tau_t^*\right)\right].$$

Finally, the intertemporal government budget constraint in terms of consumption gap and tax gap is

$$\begin{aligned}\hat{b}_{t-1} - \tilde{\sigma}^{-1}(\hat{C}_t - \hat{C}_t^*) - \omega_\tau(\tau_t - \tau_t^*) - \pi_t - \tilde{\sigma}^{-1}\hat{C}_t^* - \omega_\tau\tau_t^* \\ = (1-\beta)\,\mathbb{E}_t\sum_{s=t}^{\infty}\beta^{s-t}[b_c(\hat{C}_s - \hat{C}_s^*) + \tilde{b}_\tau(\hat{\tau}_s - \hat{\tau}_s^*) + b_c\hat{C}_s^* + \tilde{b}_\tau\hat{\tau}_s^* - s_d^{-1}\hat{G}_s],\end{aligned}$$

where $b_c \equiv b_\tau - \tilde{\sigma}^{-1}$ and $\tilde{b}_\tau \equiv b_\tau - \omega_\tau$. Define

$$v_t \equiv (1-\beta)\,\mathbb{E}_t \sum_{s=t}^{\infty} \beta^{s-t}(b_c \hat{C}_t^* + \tilde{b}_\tau \hat{\tau}_s^* - s_d^{-1}\hat{G}_s).$$

The previous expression becomes

$$\hat{b}_{t-1} - \tilde{\sigma}^{-1}c_t - \omega_\tau\,(\tau_t - \tau_t^*) - \pi_t + f_t = (1-\beta)\,\mathbb{E}_t \sum_{s=t}^{\infty} \beta^{s-t}[b_c c_s + \tilde{b}_\tau(\hat{\tau}_s - \hat{\tau}_s^*)],$$

or, in flow form

$$\begin{aligned}\hat{b}_{t-1} - \tilde{\sigma}^{-1}c_t - \omega_\tau\,(\tau_t - \tau_t^*) - \pi_t + f_t = (1-\beta)\,[b_c c_t + \tilde{b}_\tau(\hat{\tau}_t - \hat{\tau}_t^*)]& \\ + \beta\mathbb{E}_t[\hat{b}_t - \tilde{\sigma}^{-1}c_{t+1} - \omega_\tau\left(\tau_{t+1} - \tau_{t+1}^*\right) - \pi_{t+1} + f_{t+1}]&,\end{aligned}$$

where

$$f_t \equiv -\tilde{\sigma}^{-1}\hat{C}_t^* - \omega_\tau \tau_t^* - v_t.$$

To summarize, the equilibrium conditions are:

1. Aggregate demand

$$c_t = -\tilde{\sigma}\left\{i_t - \mathbb{E}_t\pi_{t+1} + \omega_\tau\left[(\tau_t - \tau_t^*) - \left(\mathbb{E}_t\tau_{t+1} - \tau_{t+1}^*\right)\right] - r_t^*\right\} + \mathbb{E}_t c_{t+1}, \tag{62}$$

2. Aggregate supply

$$\pi_t = \kappa[c_t + \psi(\hat{\tau}_t - \hat{\tau}_t^*)] + \beta\mathbb{E}_t\pi_{t+1}, \tag{63}$$

3. Government budget constraint

$$\begin{aligned}\hat{b}_{t-1} - \tilde{\sigma}^{-1}c_t - \omega_\tau\,(\tau_t - \tau_t^*) - \pi_t + f_t = (1-\beta)\,[b_c c_t + \tilde{b}_\tau(\hat{\tau}_t - \hat{\tau}_t^*)]& \\ + \beta\mathbb{E}_t[\hat{b}_t - \tilde{\sigma}^{-1}c_{t+1} - \omega_\tau\left(\tau_{t+1} - \tau_{t+1}^*\right) - \pi_{t+1} + f_{t+1}]&. \end{aligned} \tag{64}$$

3.5 Determinacy under Strict Targeting Rules

Consumption taxes partly change the determinacy analysis under the strict targeting rules

$$\pi_t = 0$$

and

$$\hat{b}_t = 0.$$

As in the paper, given that the targeting rules have no reference to the nominal interest rate, the aggregate demand equation (62) residually determines i_t. Replacing the targeting rules into the aggregate supply equation (63) and (64) yields

$$0 = c_t + \psi(\hat{\tau}_t - \hat{\tau}_t^*)$$

and

$$\tilde{\sigma}^{-1} c_t - \omega_\tau \left(\tau_t - \tau_t^*\right) = (1-\beta)\left[b_c c_t + \tilde{b}_\tau(\hat{\tau}_t - \hat{\tau}_t^*)\right] \\ + \beta \mathbb{E}_t[-\tilde{\sigma}^{-1} c_{t+1} - \omega_\tau \left(\tau_{t+1} - \tau_{t+1}^*\right)] + \varepsilon_t^f.$$

Replacing for the tax gap from the simplified aggregate demand equation gives

$$\beta\left(\tilde{\sigma}^{-1} - \omega_\tau \psi^{-1}\right) \mathbb{E}_t c_{t+1} = \left[(1-\beta)\, b_\tau \left(1 - \psi^{-1}\right) + \beta\left(\tilde{\sigma}^{-1} - \omega_\tau \psi^{-1}\right)\right] c_t + \varepsilon_t^f.$$

The equilibrium is determinate if and only if $|\rho_c| > 1$, where

$$\rho_c \equiv 1 + \frac{(1-\beta)\, b_\tau \left(1 - \psi^{-1}\right)}{\beta\left(\tilde{\sigma}^{-1} - \omega_\tau \psi^{-1}\right)}.$$

The condition for determinacy is satisfied for values of $\tau > 20\%$ (see figure 1). Therefore, strictly speaking, the equilibrium is indeterminate under the baseline calibration. However, this is a knife-hedge result. The threshold for determinacy is decreasing in σ^{-1} (remember that ψ is also a function of σ^{-1}). If $\sigma^{-1} = 1$, the equilibrium is determinate with $\tau = 20\%$. Conversely, determinacy is harder to achieve if the ratio of government spending to GDP is higher. For example, if $G/Y = 20\%$, determinacy occurs for $\tau \geq 25\%$.

4 Roots of a Second Order Difference Equation

Proposition 1 *Let $P(\lambda) \equiv \lambda^2 + A_1\lambda + A_0 = 0$ and let λ_1 and λ_2 be the roots of $P(\lambda)$. Then, the absolute values of λ_1 and λ_2 split across the unit circle if and only if $P(1) > 0$ and $P(-1) < 0$ or vice versa.*

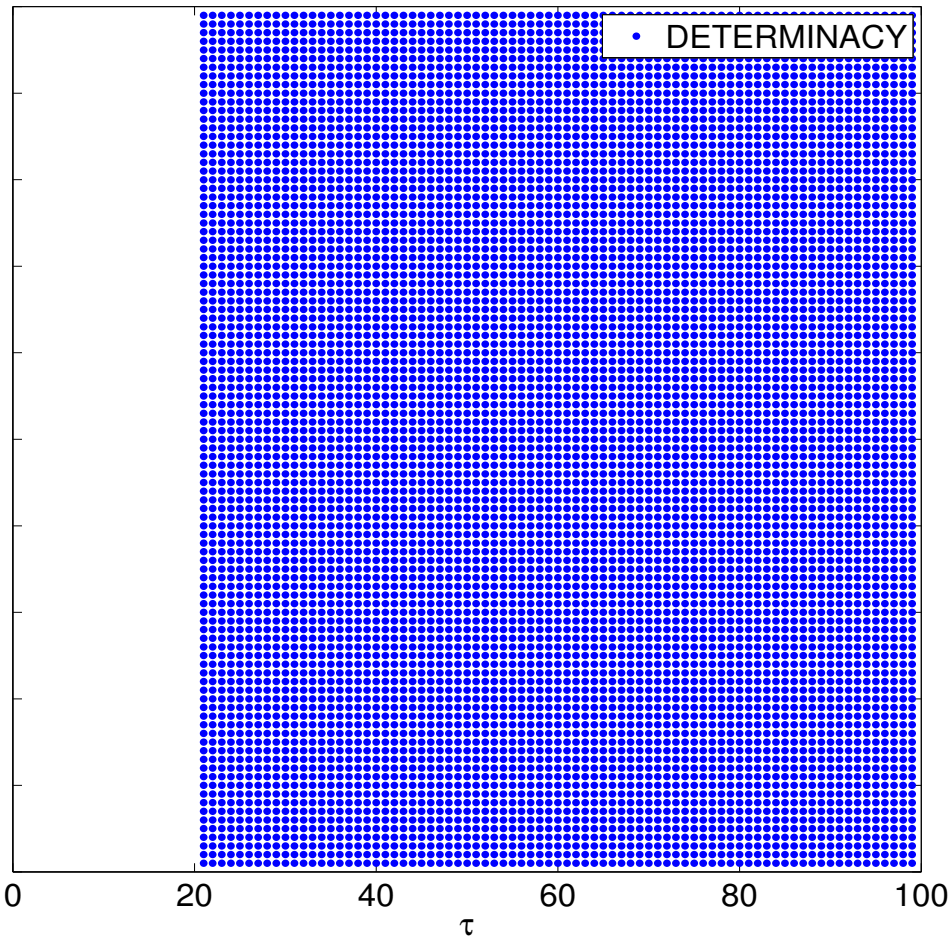

Figure 1: Determinacy regions as a function of the steady state consumption tax rate τ under strict targeting rules.

Proof. Notice that one can always rewrite the polynomial $P(\lambda)$ as

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \tag{65}$$

1. First, show that $P(1) > 0$ and $P(-1) < 0$ imply that the absolute values of the two roots λ_1 and λ_2 split across the unit circle.

   From the right hand side of [65], it is easy to see that

   $$P(1) = (1 - \lambda_1)(1 - \lambda_2) > 0, \tag{66}$$

   and

   $$P(-1) = (1 + \lambda_1)(1 + \lambda_2) < 0. \tag{67}$$

   If $P(1) > 0$, it means that λ_1 and λ_2 are on the same side of 1. Similarly, if $P(-1) < 0$, it means that λ_1 and λ_2 are on opposite sides of -1. It then follows that one root must lie inside the unit circle and the other outside. The case $P(1) < 0$ and $P(-1) > 0$ is totally symmetric.

2. Next, show that if $|\lambda_1|$ and $|\lambda_2|$ lie on opposite sides of 1, it must be the case that $P(1)$ and $P(-1)$ lie on opposite sides of 0.

   Without loss of generality, suppose $|\lambda_1| > 1$ and $|\lambda_2| < 1$. There are two cases two be considered. First, if $\lambda_1 > 1$, then, one can see from [66] that $P(1) < 0$ and from [67] that $P(-1) > 0$, which confirms the claim. Second, if $\lambda_1 < -1$, then, again from [66] and [67], $P(1) > 0$ and $P(-1) < 0$. The case $|\lambda_1| < 1$ and $|\lambda_2| > 1$ is symmetric.

■

Proposition 1 complements Proposition $C.1$ in Woodford (2003) which gives necessary and sufficient conditions for the two roots of $P(\lambda)$ to be both outside the unit circle.

References

[1] Lucas, R. (2003), "Macroeconomic Priorities," *American Economic Review* 93 (2003), pp. 1-14.

[2] Woodford, M. (1999), "Commentary: How Should Monetary Policy Be Conducted in an Era of Price Stability?" in *New Challenges for Monetary Policy*, Federal Reserve Bank of Kansas City.