On-Line Appendix to

Valuing the Treasury's Capital Assistance Program

In this appendix, we expand the discussion of the example in Figure 1, we detail the calibration of the model with a jump to default, we extend the model to include stochastic interest rates, and we show that our recursive game formulation is equivalent to an extensive-form version.

A.1 Example

This section continues and expands the discussion of the example in Section 5.2.

The top panel in Figure 6 repeats Figure 1. Following conversion or warrant exercise, the game is governed by the lower figure. The nodes with squares in the lower figure show the warrant exercise region for the UST *after* the QFI has converted (the exercise region for W^c). Similarly, the circles and crosses in the lower figure show the redemption and conversion regions the QFI would use if the UST had already exercised its warrants (the regions for C^e). Beyond the mandatory conversion date, the QFI has no remaining options, so we allow the UST to exercise at every time step between the seventh and tenth years.

One set of decisions remains, not covered by the figures: If the QFI redeems before the UST exercises its warrants, the UST is then left holding a standard warrant (W^r) on undiluted shares. The calculation of the exercise region for a standard warrant is straightforward, so we have not included a separate figure for that case.

In Figure 7, we illustrate how the exercise regions change as we vary the dividend rate on the CAP. The figures correspond to the top half of Figure 6, in the sense that they show exercise regions for C and W, before either party has exercised. In the top left panel, the dividend rate is 2% and thus equal to the risk-free rate; this makes early redemption less attractive to the QFI, creating the possibility that the UST will exercise its warrants before the QFI converts or redeems. In the top right panel, we further reduce the CAP's dividend below the risk-free rate to 1%. Here, the benefit of terminating the dividend payments is reduced relative to the cost of conversion, and a gap opens in the QFI's exercise regions, allowing the CAP to continue beyond the initial two years. In the bottom panel, the CAP pays no dividend and the QFI never chooses early conversion. The QFI still has an incentive to redeem the CAP at the end of the initial two years if the stock price has increased in order to avoid mandatory conversion at the end of seven years at an unattractive conversion price. Indeed, the mandatory conversion feature creates a strong incentive for the QFI to redeem if the stock price increases.

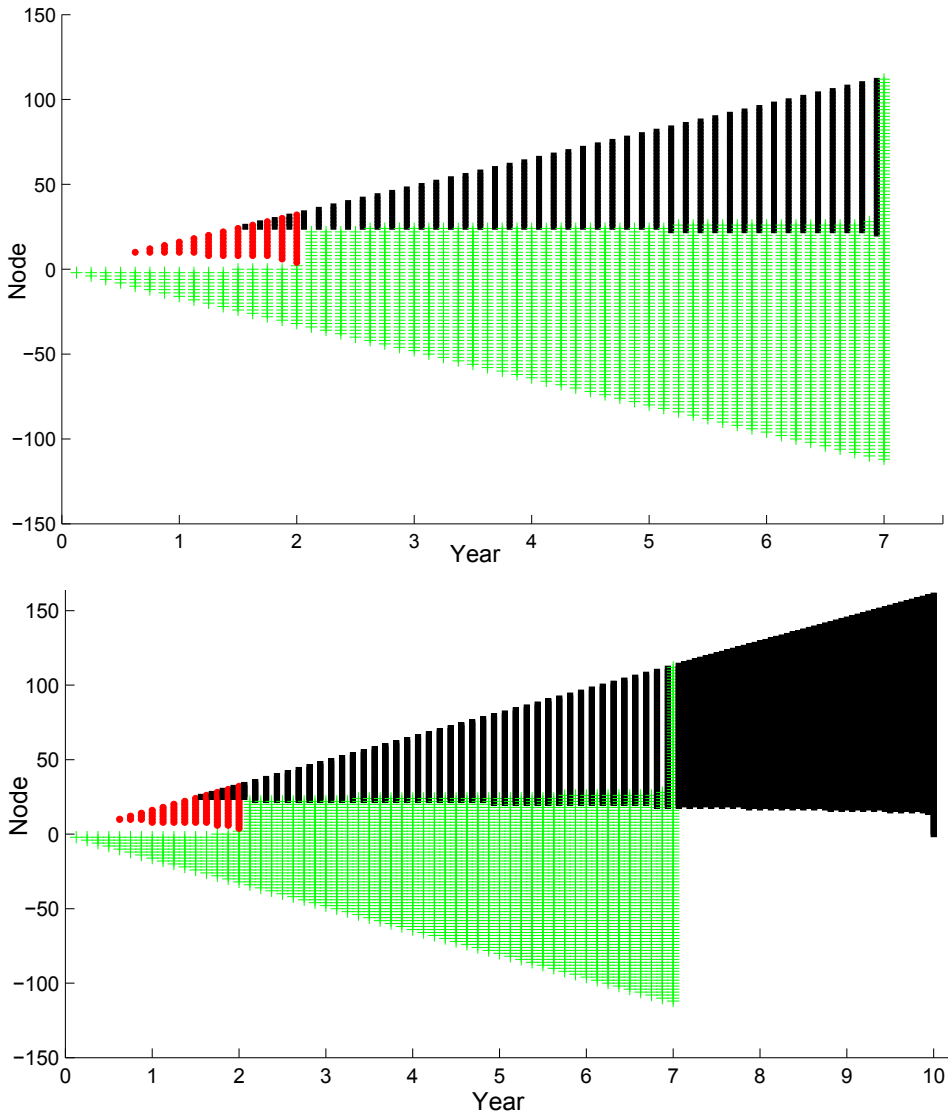

Figure 6: Exercise regions for the CAP: The cross marks show the region in which the QFI exercises its conversion option, the circles show where redemption is optimal, and the squares show the exercise region for the UST's warrants. The top figure applies before either party has exercised an option; once the QFI converts or the UST exercises, the other party's decisions are governed by the lower figure. After year seven, the UST is the only remaining player.

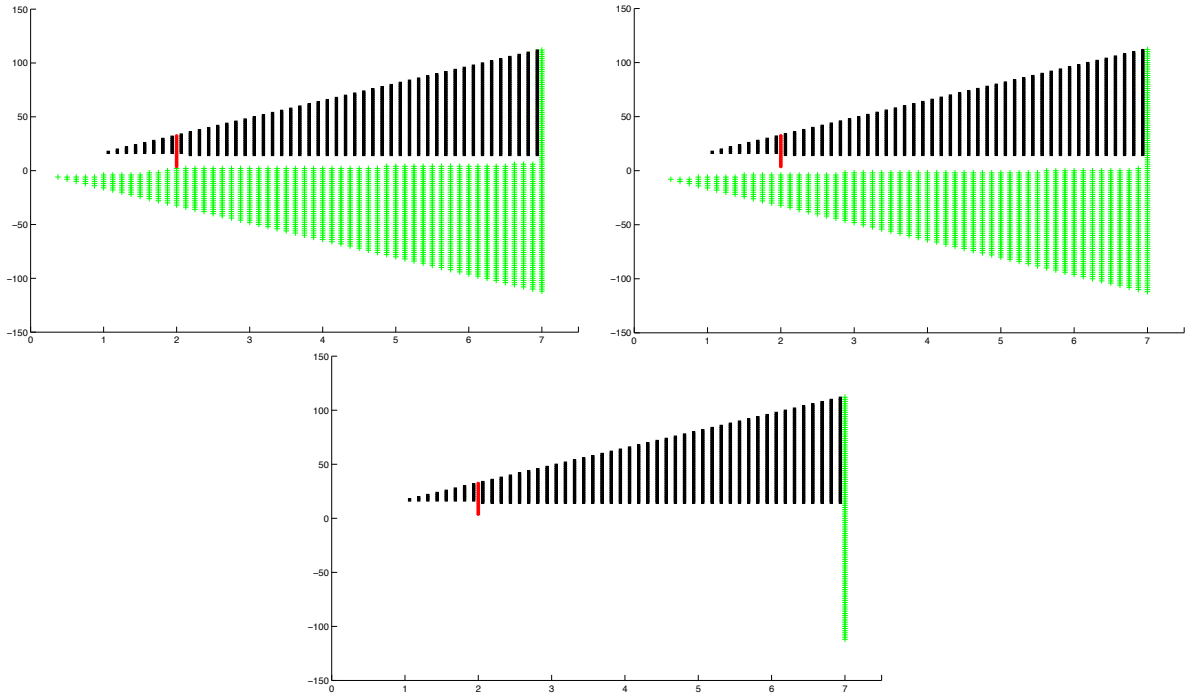

Figure 7: Exercise regions for the CAP with a dividend of 2% (top left), 1% (top right), and 0% (bottom).

One way to gauge the importance of the interaction between the two players is to compare the value of the CAP with the values of two separate securities granting options separately to the two players. In the example of Figure 6, we get a CAP value of $C_0 = \$25.67$ million, or 25.67% of the initial capital G. If we value the warrants in isolation within the same model, we get a value to the Treasury of \$13.85 million, and if we value the CAP stripped of warrants (i.e., with $m = 0$), we get a value to the QFI of \$34.14 million. Ignoring the interaction between the two contracts would yield a net value of $\$34.14 - \$13.85 = \$20.29$ million to the QFI. But the QFI captures more value (25.67 rather than 20.29) when the embedded options are combined. This is a consequence of the conversion option which lowers the value of the warrants through dilution and by shifting part of the cost of subsequent dilution from warrant exercise to the UST.

A.2 Default Calibration

For the extension of the model that includes a potential jump to default, we calibrate parameters for each firm using its credit default swap (CDS) spreads. We use CDS spreads from Markit

at maturities of six months and 1, 2, 3, 4, 5, 7 and 10 years.[21] For consistency with our stock price model, we use spreads for the holding company rather than the bank in cases (like Bank of America) where both are available. We also use CDS recovery rates from Markit, which are 40% in all cases reported, except BAC for which it is 44.86%. We use 40% for BK and RF, for which no Markit recovery rates were available.

For six firms (BBT, BK, FITB, PNC, RF, and STT) some or all spreads were missing for the period of interest (February 2009). For BBT, BK, PNC, and STT just a few maturities were missing, so we imputed values through regression, using as regressors the average spread over all banks for each maturity. To impute, say, a missing 1-year spread at BBT, we regress BBT's 1-year spread against the averages for all maturities, estimating the model with daily data from January 2006 through May 2009. The imputed value on a given day is then the value fit by the model using the estimated coefficients and the average spreads on that day. For FITB and RF, we had too much missing data across maturities and days to apply this procedure, so we used only the 5-year contract (typically the most liquid), for which we had data for both banks, and simply assumed a constant spread across all maturities equal to the 5-year spread.

Using the CDS spreads, we calibrate an adjusted volatility and default intensity for each firm; the results are summarized in Table 6. Recall from the discussion in Section 5.3 that the volatility adjustment offsets the volatility increase that results from adding a jump to zero.[22] We calculate the volatility adjustment as follows. First, we calculate a constant default intensity ξ_{const} calibrated to the 1-year CDS spread for each firm. Then, we solve numerically for the value of the adjusted volatility that yields the same price for a 1-year at-the-money call option in a binomial lattice with a jump intensity of ξ_{const} as would be obtained with no jump to zero and the original volatility in Table 2. The choice of calibration to a 1-year horizon is a compromise between the greater liquidity of options at shorter maturities and the greater liquidity of CDS at somewhat longer maturities.

Once we have an adjusted volatility, we can calibrate the parameters a_0–a_3 of the default intensity (19). For now, we fix a_1 at zero and reintroduce this parameter when we include stochastic interest rates. The calibration proceeds iteratively. We start from initial values for the parameters[23] and calculate CDS spreads in the jump-to-default binomial lattice, as in

[21] We use spreads for senior unsecured debt under document clause XR, which takes the most restrictive definition of a credit event, excluding restructuring.

[22] Whether such an adjustment is needed depends on whether one interprets volatility in the binomial tree as measuring market implied volatility, including the possibility of default, or as measuring stock price fluctuations in the absence of default. In our calibration to market data, we adopt the first interpretation.

[23] We used $a_0 = 0$, $a_2 = 1$, and $a_3 = .1$ as initial values, except in the case of USB for which we needed to start at $a_0 = 5$ to get a satisfactory fit. We compared results from multiple starting points.

Ticker	Spread	ξ_{const}	Adj. Vol.	a_0	a_2	a_3	$\lambda_{\max}$
AXP	394.71	0.0673	0.7627	1.5584	0.181	-0.1407	0.0500
BAC	285.06	0.0512	1.1643	-2.8121	0.0078	-0.0653	0.0500
BBT	152.53	0.0256	0.8182	14.508	0.8152	-0.1971	0.0500
BK	74.57	0.0125	0.7936	9.9998	0.5943	-0.1229	0.1063
C	523.17	0.0897	1.152	-1.5994	0.0357	-0.1159	0.0100
COF	410.06	0.0705	0.9988	1.3649	0.1773	-0.2331	0.0719
FITB	70.81	0.0119	1.8758	-2.9019	0.0739	-0.1257	0.1492
GS	340.65	0.0577	0.6927	0.7806	0.1479	-0.1247	0.0469
JPM	106.84	0.0179	0.803	11.503	0.6204	-0.165	0.0500
KEY	298.13	0.0512	1.0169	1.4622	0.2117	0.1458	0.0080
MET	659.18	0.1122	0.8454	-2.1008	0.0035	-0.0406	0.0724
MS	499.94	0.0865	0.8351	0.7394	0.1331	-0.1726	0.0500
PNC	242.90	0.0416	0.8355	0.429	0.1553	-0.0321	0.0750
RF	509.47	0.0865	1.2972	-2.453	0.0018	0.0926	0.0450
STI	192.20	0.0320	1.0937	1.16	0.2092	-0.0288	0.0069
STT	127.71	0.0214	0.9024	24.5275	1.2986	0.077	0.0500
USB	93.75	0.0157	0.8459	9.919	0.5888	-0.0608	0.0100
WFC	200.48	0.0352	0.9047	-0.6811	0.1104	-0.0675	0.0427

Table 6: Parameters for jump-to-default model. The spread column reports 1-year CDS spreads; these determine the value of λ_{const}, which we use to calculate the adjusted volatility. The last four columns report the parameters of the default intensity obtained through calibration to CDS spreads of maturities from six months to ten years.

equations (12)–(15) in Das and Sundaram [7]. We compare the model spreads with our observed (or imputed) spreads and adjust a_0, a_2 and a_3 to minimize the mean square error between the two. At sufficiently large values of the default intensity, the up or down probabilities in the binomial lattice may become negative. To avoid this problem, we cap the default probability at $\lambda_{\max}$ and include $\lambda_{\max}$ in the calibration process itself, starting at 0.075. The final values are included in Table 6.

For reference, Table 3 also includes the average CDS spread for each QFI, averaged over maturities from six months to ten years and averaged over the ten days ending on February 25. In general, the larger the spread the greater the difference in CAP values with and without default; see Figure 8. The correlation between the price differences and the average spread is 0.85. Although not included in the table, we have also calculated prices for the CAP warrants *in isolation* with and without a jump to default. The results are in most cases identical, and the difference is never more than half a percentage point. Recall that we calibrated the default model to keep the one-year implied volatility unchanged when we add jumps, so the agreement in warrant values between the two models is consistent with our objective in the calibration.

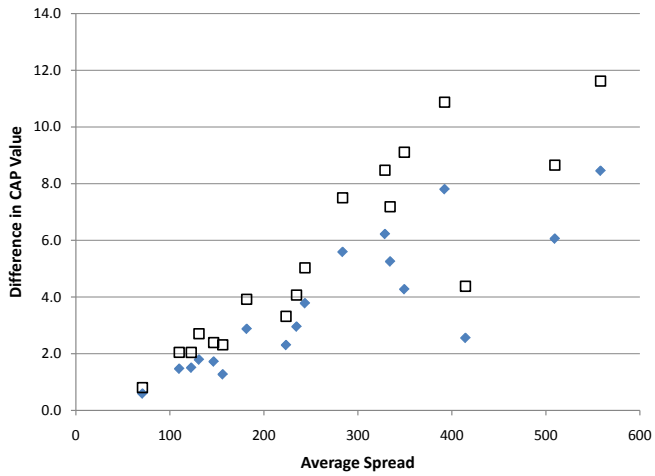

Figure 8: The figure shows differences in CAP values (with and without default) plotted against average CDS spreads. The diamonds mark differences with fixed interest rates, and the squares mark differences with stochastic rates. The correlations are 0.85 and 0.88, respectively.

A.3 Stochastic Interest Rates

We now further extend our valuation model to allow for a stochastic term structure of interest rates. For this, we apply the implementation of the Heath-Jarrow-Morton (HJM) [13] framework developed by Das and Sundaram [7]. Their model allows for consistent valuation of equity and credit derivatives in a stochastic interest rate setting.

The central modeling element in the HJM setting is the forward rate curve. In a discrete-time setting, it is convenient to use the same time step h both for calendar time and maturity. Thus, $f(t,T)$ denotes the forward rate fixed at time t for borrowing or lending over the interval from T to $T+h$, and the HJM framework describes the evolution of all such rates from time t to time $t+h$. In the notation of Das and Sundaram [7], we have

$$f(t+h,T) = f(t,T) + \alpha(t,T)h + \sigma(t,T)X_f(t)\sqrt{h}, \tag{20}$$

where α is the drift coefficient, σ is the diffusion coefficient, and $X_f(t)$ is a random variable taking the values ± 1. This is a discrete-time approximation to a diffusion model, with $X_f(t)\sqrt{h}$ approximating a Brownian increment. It is implicit in Das-Sundaram [7] that $\sigma(t,T)$ depends only on T; given σ, the form of α is determined by the absence of arbitrage through the key result of HJM. The short rate at time t is given by $r(t) = f(t,t)$.

In a binomial model of the evolution of the stock price S and the forward curve f, each node records a pair (S,f). The stock price may take a step up or down, and the forward curve may

receive a positive or negative shock X_f, so there are four possible transitions out of each node corresponding to the four possible combinations of up and down moves. The special feature of the Das and Sundaram [7] model is that it recombines in the sense that a move up followed by a move down reaches the same node as a move down followed by a move up. This makes the model computationally tractable compared to a tree in which the number of nodes grows exponentially. It is, however, computationally much more demanding than a binomial model of the stock alone. Whereas a single-asset binomial model produces a triangular lattice of $O(T^2)$ nodes after T steps, the bivariate model produces a pyramid of $O(T^3)$ nodes.[24]

The equations in Section 4 require some modification to apply to the current setting. The difficulty we face in a direct application of the previous equations is that when rates are stochastic, the value of cash becomes path-dependent. Suppose, for example, that the capital G is invested at the risk-free rate at time zero. At some future node (S, f), we would like to determine the liquidation value of the CAP contract, but the value of the initial investment of G is not uniquely determined by the state information (S, f). Different paths from the root node to (S, f) would produce different paths of the short rate $r(t)$ and thus different values of the investment at the same ending node.

We circumvent this difficulty by working with bonds rather than money market accounts. At time zero, part of the capital G is used to buy a strip of zero-coupon bonds whose cashflows match the CAP's dividend payments all the way to the mandatory conversion date T_c. This strip "defeases" the promised dividend payments. Let B_t denote the value of the remaining cashflows from this strip as of time t; this value can be calculated at any node (S, f) because the remaining cashflows can be discounted off the forward curve.

Next we suppose that, at time zero, the remaining cash $G - B_0$ (which could be negative) is invested in zero-coupon bonds maturing at T_c. If we let $P_t(T_c)$ denote the time-t price of such a zero-coupon bond with a face value of 1, then the number of bonds N satisfies

$$G - B_0 = N \cdot P_0(T_c).$$

If at some future time t we liquidate the CAP (due to conversion or redemption, for example), the UST returns the bond strip B_t to the QFI to stop the remaining dividend payments, and the QFI keeps the N zero-coupon bonds, now worth $P_t(T_c)$ each. (In the case of redemption, the QFI also pays the par value G to the UST.) Thus, the net effect of these transaction is that

[24] This limits our numerical calculations to a time step of 1/8 of a year, compared with 1/32 for our basic binomial model.

everywhere in equations (8)–(15) that we previously had $G_t - D_t$, we now have

$$B_t + N \cdot P_t(T_c), \quad N = \frac{G - B_0}{P_0(T_c)}.$$

The new expression has the advantage that the prices needed at each node can be computed from the forward curve at that node.

A simple calculation verifies that the expression we have derived here actually reduces to the previous expression in the case of a constant interest rate r. This follows from the fact that, with a fixed rate,

$$B_t = e^{-r(T-t)h} D_T - D_t \quad \text{and} \quad P_t(T_c) = e^{-rT_c h}$$

which yields $B_t + N \cdot P_t(T_c) = G_t - D_t$ after some algebraic simplification.

Details of the construction of the Das-Sundaram [7] model are given in their paper, so we do not repeat them here but simply note the parameter values we use. We fix the parameter γ in their paper at 1, meaning that we continue to model the stock price process as a binomial approximation to a geometric Brownian motion. Their framework also allows correlation between the increments of the stock price and forward curve process; we fix this parameter at 0.3; however, we have found our results to be relatively insensitive to this parameter.

We extract a forward curve from overnight index swap (OIS) rates on February 25, 2009. OIS maturities range from one week to five years, extending the overnight Fed funds rates. We build zero-coupon bond prices from the OIS rates, interpolate the bonds, and then extract forward rates for any maturity time step h. Beyond five years, we assume the forward curve remains flat. For the diffusion coefficients σ, we start with implied volatilities for swaptions from Bloomberg on February 25. These are "Black volatilities" in the sense that they are based on the Black formula for swaptions, as in Hull [15], p.660. We use interpolation to get volatilities for forward rates at arbitrary maturities. As these are proportional or lognormal volatilities, to convert to the diffusion coefficients used in the Das-Sundaram [7] model, we multiply each lognormal volatility by the corresponding rate. In other words, we set $\sigma(0,T) = \sigma_L(0,T) f(0,T)$, where $\sigma_L(0,T)$ is a lognormal volatility; we then take $\sigma(t,T) = \sigma(0,T)$ for all t.

To include a possible jump to default, we need to recalibrate the default intensity to account for the addition of stochastic interest rates. The results are displayed in Table 7. CAP values with and without default in the presence of stochastic interest rates are recorded in Table 3. These results are in line with those we found with fixed rates, adding to the robustness of the results.

Ticker	ξ_{const}	Adj. Vol.	a_0	a_1	a_2	a_3	$\lambda_{\max}$
AXP	0.0675	0.7013	1.5011	-8.0150	0.1790	-0.0241	0.1500
BAC	0.0530	1.0896	-2.1610	-29.7816	0.0324	0.1421	0.0700
BBT	0.0256	0.7923	14.4938	5.0000	0.8166	-0.1392	0.1200
BK	0.0128	0.7798	10.8446	7.1339	0.6325	-0.1142	0.1000
C	0.0901	1.0277	-1.5604	-18.7977	0.0370	0.0693	0.2000
COF	0.0707	0.9152	1.8990	-14.4515	0.2015	-0.0778	0.1500
FITB	0.0120	1.8432	-2.9026	-2.6515	0.0743	-0.0670	0.1400
GS	0.0578	0.6447	0.7011	-27.1054	0.1432	0.0990	0.0498
JPM	0.0176	0.7856	11.3953	-1.0196	0.6172	-0.0926	0.0500
KEY	0.0514	0.9553	-2.1085	-3.5350	0.0041	0.0422	0.0724
MET	0.1136	0.7239	-2.1245	-1.0055	0.0029	0.0515	0.0500
MS	0.0853	0.7513	0.6824	-21.6812	0.1303	0.0392	0.0224
PNC	0.0401	0.7949	0.5101	3.7111	0.1613	0.0044	0.0500
RF	0.0869	1.1584	-2.3995	1.3801	0.0043	0.1095	0.0500
STI	0.0353	1.0451	0.6813	-3.3801	0.1886	0.0688	0.0700
STT	0.0216	0.8781	22.6242	-4.3263	1.2148	0.2656	0.1200
USB	0.0157	0.8292	10.4310	1.6605	0.6125	-0.0105	0.0700
WFC	0.0337	0.8701	0.2264	-9.4252	0.1469	0.0210	0.0343

Table 7: Parameters for jump-to-default model with stochastic interest rates.

A.4 From Normal Form to Extensive Form

In this appendix, we formulate the game between the QFI and the UST created by the CAP's options more precisely, and we justify the extensive form of the game that underlies the recursive equations in Section 4.

We start with a more generic formulation of the type used in Ohtsubo [25] and references therein. Recall our key modeling assumption that the exercise of the CAP's embedded options does not affect total firm value, only how value is allocated between the two players. Thus, any payoff to one party comes at the expense of the other. We focus (arbitrarily) on the payoff to the QFI and thus designate the QFI to be the maximizing player and the UST to be the minimizing player.

The strategies available to the two parties are stopping times with respect to the history of the underlying state process. Initially, we allow these stopping times (the admissible strategies) to take values in the set $\{0, 1, \dots\}$. We introduce payoffs X_t and Y_t, for $t = 0, 1, \dots$. If the QFI exercises at t, it receives X_t; if the UST exercises at t, the QFI receives Y_t. In either case, the game stops when either party exercises. Thus, if the QFI and UST choose strategies τ_Q and

τ_U, respectively, the payoff to the QFI is

$$\Pi(\tau_Q, \tau_U) = X_{\tau_Q} I_{\{\tau_Q \leq \tau_U\}} + Y_{\tau_U} I_{\{\tau_U < \tau_Q\}}.$$

We have arbitrarily chosen X_t as the payoff if both players exercise at t, though we rule out this possibility shortly. The upper and lower values of the game are given by

$$v^+ = \inf_{\tau_U} \sup_{\tau_Q} E[e^{-r \min(\tau_Q, \tau_U)h} \Pi(\tau_Q, \tau_U)], \quad v^- = \sup_{\tau_Q} \inf_{\tau_U} E[e^{-r \min(\tau_Q, \tau_U)h} \Pi(\tau_Q, \tau_U)],$$

with obvious modifications in the case of stochastic interest rates. We will impose conditions ensuring the existence of the expectations in these expressions and thus ensuring that the values are well-defined. We will also establish that for the game created by the CAP's options, the upper and lower values are equal and can be calculated through backward induction.

We first need to fit the CAP game into this generic formulation. To have the players alternate turns, we can restrict the set of admissible strategies, limiting one player to exercise at even times and the other to exercise at odd times. We can similarly force exercise by a specific date (conversion at year seven, warrant exercise at year ten) by restricting the admissible strategies. In a discrete-time, discrete-state model (like our extended binomial model), we can alternatively impose these restrictions by adding sufficiently large penalties to payoffs at stopping times that violate the restrictions.

In the CAP game, the QFI holds two options (conversion and redemption) in the first two years. For purposes of valuation, we may reduce the QFI's three choices — convert, redeem, continue — to two choices — stop or continue — where the payoff upon stopping is the greater of the payoff from conversion and (when permissible) redemption. Also, under the terms of the CAP the game continues after one party exercises an option, though our generic formulation has the game terminate at that point. We accommodate this feature by defining the payoffs X_t and Y_t appropriately.

In more detail, if the QFI redeems at time t (assumed to be the QFI's turn), its payoff is

$$G_t - D_t - G - W_t^r(S_t),$$

where

$$W_t^r(S) = \sup_{\tau \in \mathcal{T}_t} \frac{m}{m+n} E[e^{-r(\tau - t)h} \max(S_\tau - K, 0) | S_t = S],$$

and $\mathcal{T}_t$ is the set of stopping times taking values in $\{t, t+1, \ldots, T\}$. Here, $W_t^r(S_t)$ is just the value at time t of a standard warrant for m shares with strike price K and expiration T. In this formulation, when the QFI redeems the game stops and the cash value of the warrants

is deducted from the payoff to the QFI. Similarly, as explained following (14), in the case of conversion the payoff is

$$-\frac{qn}{n+q}S_t + G_t - D_t - W_t^c(\frac{nS_t}{n+q}),$$

in which the last term is the value of a warrant on m shares with $n+q$ shares outstanding at a stock price of $nS_t/(n+q)$. Thus, we take

$$X_t = \max\{-\frac{qn}{n+q}S_t + G_t - D_t - W_t^c(\frac{nS_t}{n+q}), G_t - D_t - G - W_t^r(S_t)\},$$

omitting the second term inside the max once the redemption option expires. In the case of exercise by the UST, the payoff to the QFI is

$$Y_t = -\frac{mn}{m+n}(S_t - K) + C_t^e\left(\frac{nS_t + mK}{n+m}\right);$$

see the discussion following (18). The key point is that C_t^e (like the warrant prices in the previous cases) is the value of a security in which only one player (in this case, the QFI) holds an option. In our formulation, the value of this security is paid to the QFI when the UST exercises and the game stops.

We now have the following:

Proposition. Suppose that $E[S_t] < \infty$ for $t = 1, 2, \ldots, T$. Then the upper and lower values of the game v^+ and v^- are equal and given by $C_0(S_0)$, the value found recursively through (7)–(18).

Proof. This is a special case of Proposition 2.1 in Ohtsubo [25], except for some minor differences. Ohtsubo's [25] setting is infinite horizon, but his result applies to our simpler setting if we restrict the admissible strategies so that the players exercise their options in accordance with the CAP rules. Ohtsubo's [25] payoffs are not discounted; if we redefine our payoffs by setting $\tilde{X}_t = \exp(-rth)X_t$ and $\tilde{Y}_t = \exp(-rth)Y_t$ (in other words, if we denominate our payoffs in time-0 dollars) we put our problem in his setting. The discounted case is treated explicitly in Kifer [18]. Our equations (14) and (18) can be expressed as

$$C_t(S_t) = \max\{X_t, e^{-rh}E[C_{t+1}(S_{t+1})|S_t]\} \qquad \text{(QFI turn)} \tag{21}$$

$$C_t(S_t) = \min\{Y_t, e^{-rh}E[C_{t+1}(S_{t+1})|S_t]\} \qquad \text{(UST turn)} \tag{22}$$

because $W_t = -C_t$. Ohtsubo [25] requires $X_t \le Y_t$ for all t and concludes (in our notation) that $C_t(S_t)$ is the median of the three values X_t, Y_t, and $e^{-rh}E[C_{t+1}(S_{t+1})|S_t]$. In our game, the players alternate turns; we do not need $X_t \le Y_t$, nor do (21)–(22) require or imply the median property. □