The Pricing and Hedging of Index Amortizing Rate Swaps

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Index amortizing rate (IAR) swaps have been popular yield enhancement instruments over the past few years.[1] The enhanced yields associated with these instruments result from premiums earned on options embedded in the swaps. Because these options depend on the path of interest rates, the pricing of IAR swaps requires a model of interest rate movements.[2]

This article presents a simple example of an interest rate model, outlines IAR swap pricing derived from the model, and develops a hedging strategy to offset the uncertain cash flows from the swap. Finally, the article discusses the complications that arise in more realistic pricing and hedging situations.

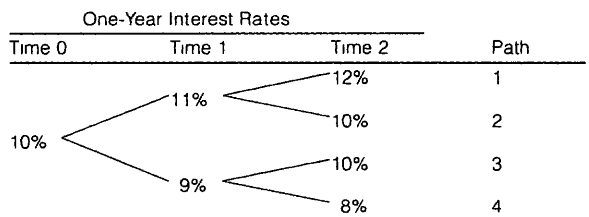
Interest rate model

In this example, we assume that one-year interest rates are well represented by a model with the binomial tree structure illustrated in the figure.[3] The tree is consistent with initial two- and three-year interest rates of 9.995 percent and 9.988 percent, respectively, if the probabilities of rates rising or falling equal one-half.[4]

Description of the swap

Although the interest rate tree has only two periods of uncertainty, the IAR swap in our example has three cash flow payments. If we assume an IAR swap with a one-year lockout period, the first cash flow at time 0 is based on an original notional amount of $100 and the current one-year rate. The two subsequent payments depend on the realization of the one-year rates at time 1 and time 2 and on the amortization schedule in Table 1.

Figure: Binomial Distribution of One-Year Interest Rates



[1] See Lisa Galaif, "Index Amortizing Rate Swaps," in this issue of the *Quarterly Review*.

[2] Models used to value path-dependent interest rate options must be free from arbitrage in the sense that they price fixed-income instruments consistently with the current term structure of interest rates. The models can be represented by interest rate trees or lattices that give possible outcomes of future short-term interest rates. These representations are used to calculate both the initial price of the IAR swap and the dynamic hedges that swap dealers would enter over time.

[3] Our example assumes that future short-term rates are determined by one factor. The example is consistent with one-year rates that are normally distributed with a constant annual volatility of 1.0 percentage point.

[4] The price of a two-period zero coupon bond with an interest rate of 9.995 percent equals the price of a two-year zero coupon bond derived from the tree.

$$\frac{1}{1.10} * \frac{1}{2} * \left(\frac{1}{1.09} + \frac{1}{1.11}\right) = .827$$

In the pricing and hedging of IAR swaps, the relevant probabilities are those that make the binomial tree consistent with the current term structure of interest rates.

Because the swap's notional principal amortizes on the basis of the short rate, the swap cash flows at each period depend not only on the rate that period but also on the path of previous rates. Table 2 shows the four possible cash flow paths (from the perspective of the fixed rate payer) that arise from our interest rate model In this example, F is the fixed rate paid on the IAR swap.

Pricing

As with any swap, the fixed rate on a IAR swap is determined such that the initial present value of the swap's cash flows is zero The present value of the cash flows from an IAR swap is more difficult to calculate than the corresponding value for a plain vanilla swap, however, and depends on the assumed arbitrage-free interest rate model. In pricing our IAR swap, we find the fixed rate consistent with the predetermined amortization schedule, the assumed distribution of one-year interest rates, and our binomial representation of the model. The cash flows are functions of the fixed rate F, the current rate, and the path of previous rates. Because we have only four possible cash flow paths, we can solve explicitly for the fixed rate, F, that makes the average present value over these possible cash flow paths equal to zero In this way, we obtain a fixed rate of 10.26 percent [5]

In this example, with its virtually flat 10 percent term structure, the fixed rate on a plain vanilla swap is approximately 10 percent. The 26 basis point premium in the IAR swap fixed rate is the value of the embedded options that the fixed rate payer implicitly purchases.

Table 3 shows the fixed rate payer's cash flows over the four paths and the three time steps, given the 10.26 percent fixed rate Notice that when the interest rate is 10 percent at time 2 (paths 2 and 3), the cash flows depend on the interest rate at time 1 This difference illustrates the path-dependent nature of the IAR swap.

[5] Let $R_{p,t}$ be the one-year interest rates and let $CF_{p,t}$ be the cash flows for the four possible paths, p, and the three time periods, t We solve for the fixed rate that sets the present value of the cash flows, or

$$\frac{1}{4} * \sum_{p=1}^{4} \sum_{t=0}^{2} \frac{CF_{p,t}}{\prod_{q=0}^{t} (1+R_{p,q})}$$

equal to zero

Table 1

Amortization Schedule

Interest Rate (Percent)	Notional Amortization (Percent)
12	0
11	10
10	20
9	50
8	100

Hedging

Fixed rate payers (usually swap dealers) may wish to hedge their highly variable payments For example, if rates rise in the first period, dealers receive $.663, but if rates fall, the dealer pays $ 631 In the second period, dealers face a similarly variable outcome that depends on the path of interest rates. We show that if fixed rate payers hedge the uncertain cash flows every period, they will earn exactly the additional 26 basis points that they pay as option premium.

Although there are many ways to implement hedges, all methods involve calculating changes in the swap's value given small changes in the underlying interest rates. Because our interest rate model involves only one factor, we need only one instrument to hedge the swap. For simplicity of exposition, we choose to replicate the IAR swap's payoffs using forward contracts instead of the more typically used futures contracts In our example, the forward rate implied by the initial term structure is 9 991 percent on one-year contracts maturing at time 1.

We choose the first hedge at time 0 to offset the two possible time 1 swap values The time 1 swap values are composed of two elements the actual cash flows paid or received on the swap and the expected value of the time 2 payments or receipts. The actual cash flows from the swap are the value of the time 0 payment (-$ 263) at time 1 plus the time 1 amount (+$ 663 in the up-state, or -$ 631 in the

Table 2

Fixed Rate Payer's Cash Flows from the IAR Swap

	Cash Flows		
Path	Time 0	Time 1	Time 2
1	$100*(10%-F)	$90*(11%-F)	$90*(12%-F)
2	100*(10%-F)	90*(11%-F)	72*(10%-F)
3	100*(10%-F)	50*(9%-F)	40*(10%-F)
4	100*(10%-F)	50*(9%-F)	0*(8%-F)

Table 3

Fixed Rate Payer's Cash Flows from the IAR Swap with a Fixed Rate of 10.26 Percent

	Cash Flows		
Path	Time 0	Time 1	Time 2
1	$-0 263	$0 663	$1 563
2	-0 263	0 663	-0 189
3	-0 263	-0 631	-0 105
4	-0 263	-0 631	0 0

down-state).[6] The expected remaining value of the swap is \$ 612 in the up-state, and -\$.048 in the down-state.

If the dealer combines the \$100 swap with -\$97 5 of the forward contract, the portfolio's value will be equal to zero at time 1 whether rates rise to 11 percent or fall to 9 percent [7] At time 1, the dealer follows the same type of calculation, keeping track of the time 2 values of the swap and the previous hedge The new hedge amounts are -\$87.0 if we are in the up-state or +\$5.2 if we are in the down-state. The process of readjusting hedges through time is known as "dynamic hedging."

If we adopt these hedge amounts, the outcome from hedging the swap along each path offsets the payoffs from the swap along that path Table 4 illustrates the calculations of the hedged swap's value along the first path The hedged swap's value along the other three paths will also equal zero at time 2.

Another hedging method computes the change in the swap's value for changes in each forward rate This "bucket" hedge method involves (1) the initial purchase of a series of forward contracts in amounts that offset the recomputed swap's value and (2) the dynamic adjustment of the hedge through purchases or sales of additional forward contracts in the future.[8] Because bucket hedging allows for nonparallel shifts in the yield curve, it implicitly assumes multiple sources of risk, it thus requires multiple hedging instruments Bucket hedging is useful if interest rate dynamics are more complicated than the single factor model assumes

Issues

In this example, the hedges perfectly offset the swap if any of the four modeled interest rate paths is realized Although it is simplistic to assume that interest rates will follow one of these four paths, the example illustrates potential issues that can arise when valuing and hedging interest-rate-dependent derivatives In particular, the pricing and hedging of any interest rate derivative security depend on decisions at several levels concerning·

- the interest rate model How many factors are relevant? What type of process do they follow—for example, normal, lognormal?
- the parameters of the model· What are the volatilities? If the model includes more than one factor, what are the correlations?
- the implementation of the model: How small are the time steps? Is it a binomial or trinomial tree? How many simulations are used?

If assumptions about the model and the parameters of the model are incorrect, the hedging cannot offset realized gains and losses. In our example, hedging depends on the forward rates implied by our interest rate tree. If these rates are not realized, the cash flows from the hedges cannot perfectly offset the cash flows from the swap These rates can be wrong because the short rate process is in fact not well represented by a single factor normal distribution with constant volatility Valuing the swap using other interest rate models—for example, a two factor lognormal interest

[6] The total cash flow from the swap in the up-state is therefore \$ 372, which equals -\$ 263*(1 11) + \$ 663

[7] The hedge amounts are essentially (the negative of) the derivative of the swap's value with respect to interest rates In our example, the first hedge amount, \$97 5, equals \$ 983 (the swap's value in the up-state) less \$- 966 (the value in the down-state), divided by 02 (the difference in the interest rates)

[8] In our example, we would initially sell \$80 1 of the forward contract maturing at time 1 and \$19 1 of the contract maturing at time 2 If rates rise to 11 percent, we would sell \$68 1 of the contracts maturing at time 2 at the new forward rate, if rates fall to 9 percent, we would buy \$24 5 of the time 2 forward contracts

Table 4

Payment Stream for the First Path

	Cash Flows				
	Time 0	Time 1	Time 2		Future Value
Swap at time 0	– 263	– 263(1 11)	– 263(1 11)(1 12)	=	– 327
Swap at time 1		663	663(1 12)	=	– 743
Swap at time 2			1 563	=	1 563
Hedge entered at time 0		–97 5(11- 0999)	–97 5(11- 0999)(1 12)	=	–1 102
Hedge entered after up-jump at time 1			–87 0(12- 1099)	=	– 878
Value of the hedged swap at the end of time 2					0 0

rate model—can give a different fixed rate and different hedges.

Different assumptions about the parameter values also affect the fixed rate. In our example, if the volatility is 1.5 percentage points instead of 1.0, the fixed rate will increase from 10.26 percent to 10.60 percent. The differences across models and parameter values can be considerable, and careful judgment should be used when testing the sensitivity of the results to different assumptions.

The fixed rate and the subsequent hedging also depend on how the model (with its assumptions) is implemented. The goal in implementing the model is to approximate numerically a stochastic process. If we shorten the time steps, we will find a different fixed rate than we find with annual time steps. The appropriate time step for valuation is the one in which the fixed rates have converged on a value. In our example, the hedge ratios at time 1 are significantly different when the rates rise to 11 percent than they are when the rates fall to 9 percent. If we shorten the time steps and update the hedge ratios more often, the hedging will change more gradually than is illustrated by our example.

Actual models are more complex than our example at all levels: volatilities are not necessarily constant, the initial term structure is not conveniently flat, and models are implemented with higher frequencies. Adjustments need to be incorporated for nonparallel shifts in the yield curve because nonparallel shifts will affect the swap's value. Making errors at any of these levels will potentially result in a misvalued instrument.