# Idiosyncratic risk and volatility bounds, or Can models with idiosyncratic risk solve the equity premium puzzle? *

Martin Lettau

Federal Reserve Bank of New York and CEPR

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*Address: Research Department, Capital Markets-3E, Federal Reserve Bank of New York, 33 Liberty St., New York, NY 10045, Martin.Lettau@ny.frb.org, http://www.newyorkfed.org/rmaghome/economist/lettau/lettau.html. An abbreviated version of this paper has been accepted for publication in *The Review of Economics and Statistics.* This paper was written during a visit at the Department of Economics at New York University, I am grateful for its hospitality. John Campbell, Mark Gertler, Blake LeBaron, Sydney Ludvigson, Anthony Lynch, Harald Uhlig, two anonymous referees, and seminar participants at Humboldt University, New York University and the Federal Reserve Bank of New York provided helpful comments. The views are those of the author and do not necessarily reflect those of the Federal Reserve Bank of New York or the Federal Reserve System.

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## Abstract

This paper uses Hansen and Jagannathan's (1991) volatility bounds to evaluate models with idiosyncratic consumption risk. I show that idiosyncratic risk does not change the volatility bounds at all when consumers have CRRA preferences and the distribution of the idiosyncratic shock is independent of the aggregate state. Following Mankiw (1986), I then show that idiosyncratic risk can help to enter the bounds when idiosyncratic uncertainty depends on the aggregate state of the economy. Since individual consumption data are not reliable, I compute an upper bound of the volatility bounds using individual income data and assume that agents have to consume their endowment. I find that the model does not pass the Hansen and Jagannathan test even for very volatile idiosyncratic income data.

JEL Classification: E44, G11, G12

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# 1 Introduction

Recently, there has been a lot of interest in macroeconomic models with incomplete markets and some form of uninsurable income risk. Aiyagari (1994), Krusell and Smith (1997) and Quadrini and Rios-Rull (1997) study the distribution of wealth in such economies while Castaneda, Diaz-Gimenez and Rios-Rull (1998) consider the effectiveness of taxation when markets are incomplete. A related question is how asset prices are affected by missing markets. Earlier papers trying to answer this question include Mankiw (1986), Aiyagari and Gertler (1991) and Weil (1992). More recently Constantinides and Duffy (1996), Heaton and Lucas (1996), den Haan (1996), Krusell and Smith (1997) and Storesletten, Telmer and Yaron (1997) study the behaviour of asset prices in more complex general equilibrium settings. Solving such models requires substantial computing time even on the most powerful computers available nowadays.

In this paper I propose a simple analytical framework to evaluate the potential of models with uninsurable income shocks to generate larger risk premia than the standard model with a representative agent and complete markets. Instead of focusing on risk premia themselves, I study how Hansen and Jagannathan's (1991), henceforth HJ, volatility bounds are affected by idiosyncratic consumption shocks. Their bounds provide a useful graphical diagnostic tool to check whether a specific stochastic discount factor (SDF) associated with a preference specification and a consumption process is consistent with some important moments of asset returns. In their original paper HJ use aggregate consumption and do not allow for any idiosyncratic uncertainty in consumption. Here, I demonstrate how such shocks can be incorporated in their framework. In the second part of the paper I evaluate the empirical significance of idiosyncratic uncertainty with respect to the Hansen-Jagannathan bounds. Since panel data of individual consumption (e.g. in the PSID data set) is unreliable, it is difficult to test these models directly. One exception is Cogley (1998). He uses data from the Consumer Expenditure Survey. Given the limitations of that data set (such as a short time-series and severe measurement error) he finds that idiosyncratic consumption shocks cannot solve the puzzle. Instead of this direct approach, I use individual income which is measured more precisely than individual consumption. By equating consumption with income, I implicitly assume that agents cannot smooth income shocks at all, they are forced to consume their endowment. If agents were allowed to trade using some restricted set of securities, they would be able to smooth, at least partially, their individual shocks. Hence, the income process provide an upper bound on the volatility of individual consumption. If models with idiosyncratic risk are not able to enter HJ bounds using income data, they will most likely not be able to perform better with consumption data. I find that the model fails to pass the HJ test even for income data. Therefore, it seems unlikely that this type of idiosyncratic uncertainty will solve the equity premium puzzle.

Hansen and Jagannathan (1991) plot the minimal volatility of the stochastic discount factor as a function of its mean. An alternative but equivalent interpretation in the case of excess returns is that the Sharpe-ratio in the standard mean-standard deviation diagram of asset returns depends on properties of the stochastic factor. This interpretations turns out to be more convenient in the presence of idiosyncratic shock. Finding a stochastic discount factor that satisfies the Hansen-Jagannathan bounds corresponds to a stochastic discount factor that generates a Sharpe-ratio that is at least as high as asset return data. Using the SP500 index to proxy for the true market portfolio yields a Sharpe-ratio of 0.54 in annual postwar data (an average return of 7.96% with a standard deviation of 14.84%). Of course the true market portfolio might yield a higher price of risk. Hence 0.54 is a lower bound for the true annual Sharpe-ratio. A model that generates a Sharpe-ratio of at least 0.54 is located inside the HJ bounds while a model with a lower Sharpe-ratio is outside the bounds. Following this interpretation I compute the Sharpe-ratio in the presence of idiosyncratic consumption risk. The goal will be to generate models with a Sharpe-ratio of around 0.54.

Hansen and Jagannathan (1991) show that a risk aversion coefficient of about 50 is needed to pass their diagnostic test using aggregate data and time-separable CRRA preferences. Equivalently, a risk aversion of 50 is needed to generate a Sharpe-ratio of 0.54. The question is whether lower levels of risk aversion are admissible once idiosyncratic shocks are allowed for. The first result of the paper is that HJ bounds and the Sharpe-ratio in an economy with idiosyncratic risk are exactly the same as in an economy with a representative agent if agents have CRRA utility and the distribution of the idiosyncratic shocks is independent of the aggregate state of the economy. The intuition is as follows. The Sharpe-ratio depends on the volatility of the stochastic discount factor as well as on the correlation of the SDF with asset returns. Idiosyncratic consumption shocks make the SDF more volatile which increases the Sharpe-ratio. On the other hand, the correlation of the SDF and asset returns decreases with idiosyncratic shocks since they are by definition uncorrelated with aggregate variables. For CRRA preferences these two effects exactly cancel as long as the distribution of idiosyncratic shocks is independent of the aggregate shock. Hence the Sharpe-ratio is not affected by idiosyncratic uncertainty. Equivalently, the HJ bounds in the presence of idiosyncratic shocks are the same as with only aggregate shocks.

Mankiw (1986) has shown that the equity premium increases if there is idiosyncratic uncertainty only in bad aggregate states. In other words, the distribution of the idiosyncratic shocks depends on the realization of the aggregate state. In this case, the irrelevance result of the effect of idiosyncratic shocks on the Sharpe-ratio and HJ bounds no longer applies. I consider two models which generalize the Mankiw-idea. Following Constantinides and Duffie

(1996), I assume that all relevant random variables are log-normally distributed. Using properties of the log-normal distribution, I obtain closed-form solutions for equity premia as well as the Sharpe-ratio. If the conditional variance of the idiosyncratic shock is negatively correlated with the aggregate consumption shock, then the Sharpe-ratio will be higher than in a model with only aggregate shock. Second, I compute the Sharpe-ratio in a model with discrete distributions. For simplicity I assume that there are only two aggregate and two idiosyncratic states. The discrete model yields virtually the same quantitative results as the log-normal model as long as risk aversion and the variances of the random variables are not too large. This suggests that the results are not too sensitive to distributional assumptions made.

Therefore, if the innovation in the variance of the idiosyncratic shock covaries enough (negatively) with the aggregate state of the economy, equity premia and Sharpe-ratios could be high even with lower risk aversion. The question is whether this mechanism is empirically relevant. First, I show that the required covariance has to be extremely negative for risk aversion coefficients under about six to substantially increase the Sharpe-ratio. The thrust of the mechanism only kicks in with higher risk aversion coefficients of around eight to ten. So there is little hope for entering HJ bounds for low risk aversion. For example, for risk aversion of unity, the covariance of the idiosyncratic variance and the aggregate shock has to be around -50.

A precise investigation into the empirical validity of these back-of-the-envelope calculations is difficult because individual consumption data is unreliable. However, it is possible to shed some light on this question using individual income data which is of higher quality. Suppose that every agent receives income which is affected by an aggregate as well as an idiosyncratic shock. Such income processes can be estimated, for example using the PSID data set. If there are complete markets, agents can completely insure against the idiosyncratic shock. If markets are incomplete, e.g. agents might be borrowing constraint, they can only insure partially against idiosyncratic risk. If there are no markets, agents are forced to consume their endowment. Thus by using the estimated income processes, an upper bound on the volatility of individual consumption can be computed. Hence, I compute asset prices and Sharpe-ratios using the idiosyncratic income shocks directly. If the model which equates consumption with income is not able to produce high Sharpe-ratios and enter HJ bounds, then there is certainly no hope that any model with some consumption smoothing will accomplish that.

Therefore I use the estimated processes for idiosyncratic income of Heaton and Lucas (1996), Krusell and Smith (1997) and Storesletten, Telmer and Yaron (1997) and compute the Sharpe-ratios as if agents had to consume had to consume their endowment. As mentioned before, the presence of idiosyncratic shocks alone does not affect asset prices. All of the papers mentioned above start with models where the distribution of the idiosyncratic shock is independent of the aggregate state. Since consumers have CRRA preferences, there is no effect on asset prices. The important parameter is the covariance of the aggregate shock and the innovations of the variance of the idiosyncratic shocks. Heaton and Lucas (1996) and Storesletten, et.al. (1997) directly estimate this parameter. Their estimates range from 0.03 to about -1. In the log-normal model, this value would imply a minimal risk aversion of about 12 to match the observed Sharpe-ratio. Next, I compute the Sharpe-ratios using the exact processes of Heaton and Lucas (1996), Krusell and Smith (1997) and Storesletten, Telmer and Yaron (1997) for individual income as consumption. Note that agents will disagree on asset prices since there is no market to equilibriate their SDFs. To compute a loose upper bound for the Sharpe-ratio I use the agent with the highest Sharpe-ratio in each period. I find that only some extreme cases in any of these studies produces a Sharpe-ratio that is in the neighborhood of 0.54. Again, it is important to stress that the model is given the best possible chance. Agents have to consume their individual income (which is very volatile) and the agent with the highest Sharpe-ratio is used. If the driving processes themselves cannot generate high Sharpe-ratios, then even extreme forms of market incompleteness cannot do any better.

This indirect strategy is only informative if consumption on the individual level is indeed smoother than individual income. Since individual consumption is not observed precisely, it impossible to verify this conjecture in the data. However, it is possible to construct theoretical conditions under which the conjecture is true. Consider a permanent income consumer who has no access to asset markets and therefore has to consume her endowment. In this case, her consumption will be smoother than her current income if her permanent income is smoother then current income. This will be the case if, roughly speaking, her income growth is negatively autocorrelated. If income growth is positively autocorrelated, the opposite will be true. The overwhelming evidence provided by the extensive literature on household labor income (e.g. Abowd and Card 1989, Carroll 1992, MaCurdy 1982, Pischke 1995, and others) points to *negative* autocorrelation. Hence permanent income is most likely smoother than current income on the household level.

The paper is organized as follows. Section 2 presents the general framework how to compute HJ bounds or equivalently Sharpe-ratios in the presence of idiosyncratic shocks. In section 3, I take the income processes of Heaton and Lucas (1996), Krusell and Smith (1997) and Storesletten, Telmer and Yaron (1997) and compute Sharpe-ratios as if agents had to consume their endowment and section 4 concludes.

# 2 The framework

## 2.1 Hansen-Jagannathan bounds and the Sharpe-ratio with idiosyncratic risk

There are many consumers indexed by $j$. Let $C_t^j$ denote consumption of consumer $j$ in period $t$. Lower case letter denote log-variables, so $c_t^j$ is log-consumption of agent $j$. As mentioned in the introduction, I will take consumption as given and focus on asset pricing given consumption. In general individual consumption has an aggregate and an idiosyncratic component. I will give a more precise definition of these components later. The idiosyncratic part can be due to some incompleteness of markets and/or uninsurable risk as modeled e.g. by Aiyagari and Gertler (1991), Lucas (1994), Heaton and Lucas (1996), Krusell and Smith (1997) and Storesletten, Telmer and Yaron (1997). The consumption processes in this paper can be understood as the post-trade allocations in these papers. Let $R_{t+1} = (P_{t+1}+D_{t+1})/P_t$ be the gross return of holding an asset from period $t$ to $t+1$. The risk-free rate is denoted $R_{t+1}^f$ and risk premia are defined as $RP_{t+1} = R_{t+1} - R_{t+1}^f$. In this paper I will focus on risk premia noting that there is also a risk-free rate puzzle as described in Weil (1989).

The link between consumption and asset returns is given by the Lucas asset pricing formula. In models with frictionless markets the Lucas equation is:

$$E_t[M_{t+1}^j R_{t+1}] = 1, \tag{1}$$

where $M^j$ is the stochastic discount factor (SDF) of agent $j$. However market frictions can break the tight link between the SDF of an agent and asset returns. He and Modest (1995) consider the effect of various frictions, such as solvency constraints, short-sale constraints and borrowing constraints. In this paper I focus on the effects of heterogeneity of agents and thus consider in particular borrowing constraints. A borrowing-constraint consumers cannot consume more than her total wealth. When this constraint is binding, (1) becomes an inequality:

$$E_t[M_{t+1}^j R_{t+1}] < 1. \tag{2}$$

Denote the set of unconstrained consumers as $J$ and the set of constraint consumers as $J^\star$. Asset prices are determined only be the unconstrained agents in $J$. In equilibrium these agents have to agree on asset prices. Since only unconstrained agents are relevant for asset prices I will focus on these agents. The Lucas asset pricing equation (1) can be expanded to yield

$$\frac{E_t[RP_{t+1}]}{\sigma_t[RP_{t+1}]} = -\rho_t[M_{t+1}^j, RP_{t+1}] \; \frac{\sigma_t[M_{t+1}^j]}{E_t[M_{t+1}^j]}, \;\; \text{for } j \in J, \tag{3}$$

where $\sigma_t$ denotes the conditional standard deviation operator and $\rho_t$ the conditional correlation operator. Hansen and Jagannathan's (HJ) (1991) volatility bounds provide a convenient way to summarize the information of the stochastic discount factor with respect to risk premia. Their analysis can be understood as follows. Assume for the moment that markets are complete and there is a representative consumer, denoted with superscript $a$. The ratio of the expected risk premium of any asset to its standard deviation is determined by the correlations of the asset with the SDF of the representative consumer and the ratio of the standard deviation of the SDF to its mean, see equation (3). Under fairly general conditions, the market portfolio $R^m$ is perfectly negatively correlated with the SDF of the representative consumer when markets are complete (see Duffie 1992). Hence for the market portfolio, (3) yields for the price of risk:

$$\frac{E_t\left[RP_{t+1}^m\right]}{\sigma_t\left[RP_{t+1}^m\right]} = \frac{\sigma_t\left[M_{t+1}^a\right]}{E_t\left[M_{t+1}^a\right]}. \tag{4}$$

The standard deviation of the SDF is a linear function of the expected value of the SDF. The slope is the price of risk of the market portfolio. For other assets this equation becomes an inequality since they are not perfectly correlated with the market portfolio:

$$\sigma_t\left[M_{t+1}^a\right] \geq \frac{E_t\left[RP_{t+1}\right]}{\sigma_t\left[RP_{t+1}\right]} E_t\left[M_{t+1}^a\right]. \tag{5}$$

This yields the HJ bound for the volatility of the SDF as a function of its expected value for the case of risk premia (see e.g. Cochrane and Hansen (1992), Figure 1). However, it turns out that an equivalent but slightly different interpretation of the HJ bounds is more convenient in the presence idiosyncratic shocks. Equation (3) can be used to compute the Sharpe-ratio in the standard finance textbook diagram with the standard deviation of returns on the $x$-axis and expected returns on the $y$-axis:

$$\begin{aligned}
\mathrm{SR}(M_{t+1}^j) &\equiv \max_{\text{all assets}} \frac{E_t\left[RP_{t+1}\right]}{\sigma_t\left[RP_{t+1}\right]} && (6)\\
&= \left(\max_{\text{all assets}} -\rho_t(M_{t+1}^j, RP_{t+1})\right) \frac{\sigma_t\left[M_{t+1}^j\right]}{E_t\left[M_{t+1}^j\right]} && (7)\\
&= -\rho_t(M_{t+1}^j, RP_{t+1}^m) \frac{\sigma_t\left[M_{t+1}^j\right]}{E_t\left[M_{t+1}^j\right]}, \quad \text{for } j \in J. && (8)
\end{aligned}$$

The Sharpe-ratio is defined as the price of risk of the market portfolio and depends on the volatility of the SDF and its correlation with the market portfolio. Note that in an economy with complete markets, the SDF of the representative agent is perfectly negatively correlated with the market portfolio, so that the Sharpe-ratio is determined only by the volatility of the SDF. This is, however, not longer true in a model with idiosyncratic shocks since returns on aggregate assets cannot be correlated with idiosyncratic shocks. Hence despite the fact that the SDF of an individual might be very volatile, the net effect on the Sharpe-ratio, and therefore on the HJ bounds, is ambiguous. In particular, using $\sigma_t\left[M^j_{t+1}\right]/E_t\left[M^j_{t+1}\right]$ as an upper bound for the Sharpe-ratio is misleading because it ignores the effect on the correlation with the SDF of an individual. In the next section I will show that the two effects cancel out for a wide class of models when consumers have CRRA preferences. This result implies that the HJ bounds do not necessarily change in the presence of idiosyncratic risk.

Since the true market portfolio is not observable, a proxy has to be used. For the SP500 index, the ratio of the mean to standard deviation is about 0.54 (= 7.96%/14.84%) in annual post-war data, or 0.27 in quarterly data. Of course, the true Sharpe-ratio is likely to be higher, so 0.54 represents a lower bound. Hence the challenge is to build models which generate Sharpe-ratios (using equation (8) of that magnitude. Equivalently, a model that generates Sharpe-ratios higher than these numbers will be inside the original HJ bounds for risk premia. Of course, the HJ bounds for raw returns are much tighter, so a model that is inside the HJ bounds for excess returns might still fail the HJ diagnostic test for raw returns.

## 2.2 Aggregate and idiosyncratic uncertainty

Aggregate consumption is defined as $\bar{C}_t = \bar{E}[C^j_t]$ where $\bar{E}[.]$ denotes the *cross-sectional* mean across agents. Note that in general $\bar{M}_{t+1} \neq \beta\left(\bar{C}_{t+1}/\bar{C}_t\right)^{-\gamma}$ so that the SDF computed with aggregate consumption cannot be used to price assets. Individual consumption depends on two shocks. First, there is an aggregate shock $\epsilon$ which is common to all agents. Second, there is an idiosyncratic shock $\eta^j$ to the consumption of each agent $j$ which is independent across agents and $\bar{E}\left[\eta^j\right] = 0$. For now I leave the distribution of the shocks unspecified. The set $I_t$ is the information set available to consumers at time $t$. Both $t+1$ shocks are independent of $I_t$. Aggregate consumption is determined by the aggregate shock alone (as well as possibly some other state variable which I do not specify for notational convenience): $C^j_{t+1} = C^j_{t+1}\left(\bar{C}_{t+1}(\epsilon_{t+1}), \eta^j_{t+1}\right)$. In particular, I assume that the consumption growth of consumer $j$ is of the form

$$\Delta c^j_{t+1} = E_t \Delta c^j_{t+1} + \epsilon_{t+1} + \eta^j_{t+1}. \tag{9}$$

## 2.3 Dividends

A key assumption is that dividends on any asset depend only on the aggregate state of the economy: $D = D(\epsilon)$. In particular dividends do not depend on any idiosyncratic shock. This is a natural assumption since idiosyncratic shocks are by their very nature uninsurable by the individual. If these shocks were correlated with aggregate asset return, these assets could be used to hedge against the shocks. Hence, the assumption that idiosyncratic shocks are uncorrelated with aggregate returns is almost an identifying restriction of the shocks. [1]

## 2.4 Independent shocks

First, I consider the case in which the random variables $\epsilon$ and $\eta^j$ are independent. In the next section I allow the *distribution* of $\eta^j$ to depend on the realization of the aggregate shock $\epsilon$ following Mankiw (1986) and Constantinides and Duffie (1996). For now I exclude this case and return to it in section 2.5.

In Mankiw (1986) and Constantinides and Duffie (1996) risk premia are not affected by the presence idiosyncratic shocks when this shock is independent of the aggregate shock. Their models assume specific distributions of the shocks. Mankiw's model has only two discrete states while Constantinides and Duffie assume lognormality. While these are clearly to interesting cases, it is useful to check whether the result holds for other distributions as well. Next, I show that risk premia are indeed not affected by idiosyncratic shocks for *any* distribution as long as the idiosyncratic shocks are independent of the aggregate shock.

**Proposition 1** *If preferences are CRRA and the idiosyncratic consumption shocks are independent of the aggregate shock, HJ bounds and the Sharpe-ratio are unaffected by idiosyncratic consumption shocks. The risk-free rate depends on the distribution of the idiosyncratic shocks. It can be lower or higher than in a model without idiosyncratic risk depending on the distribution of the idiosyncratic shock.*

The proposition is proven in the appendix. To understand the intuition, consider the Sharpe-ratio in (6). Idiosyncratic shocks make the SDF more volatile, hence $\sigma_t\left[M_{t+1}^j\right]/E_t\left[M_{t+1}^j\right]$ increases. On the other hand, idiosyncratic shocks reduce the correlation of the SDF with risk premia (since returns only depend on aggregate shocks). As the proposition shows these effects exactly cancel for CRRA preference *irrespective* of the distribution of the shocks.

## 2.5 Dependence

The above section showed that the presence of idiosyncratic uncertainty does not necessarily affect risk premia. More assumptions on the structure of these shocks are needed in order

[1] It is worth noting that this assumption is usually violated in models with only two classes of agents where each class is exposed to idiosyncratic uncertainty. If one class of agents receives a idiosyncratic shock, half of the total population is exposed to the same shock. But if half the population receives the same shock, aggregate variables will be affected. Hence there can be a correlation of the 'idiosyncratic' shock and aggregate asset returns. Den Haan (1996) analyzes the difference between models with two classes of agents and many individuals in detail.

to improve the ability of the model to create higher risk premia and Sharpe-ratios. The argument in the proof of proposition 1 depends crucially on the independence of the idiosyncratic and aggregate shocks. This suggests a possibility for idiosyncratic shocks to affect risk premia if the distribution of the idiosyncratic shocks depends on the aggregate state in the right way. Mankiw (1986) suggests such a model. He presents a model in which there is an uninsurable risk of becoming unemployed only in bad aggregate states. He shows that this mechanism leads to higher risk premia.

Constantinides and Duffie (1996) consider a more sophisticated model building on the intuition of Mankiw. In particular they show that a no-trade equilibrium in a model with idiosyncratic shocks to labor income can be supported if the labor shocks are persistent. In their model, the variance of the idiosyncratic shock enters the Lucas-equation of each consumer. In this section, I extend their model somewhat and decompose the process of the variance in an expected and an unexpected component. I show that only the unexpected component has an effect on risk premia and volatility bounds. In other words, if the variance of the idiosyncratic shock varies over time, but the variance of the time-$t$ shock is known as time $t-1$, then risk premia are again unaffected by the idiosyncratic shock. This restricts the class of variance processes for which risk premia are affected. GARCH models, for example, imply that the variance is know with certainty one-period ahead. Hence a model in which the variance of the idiosyncratic shock follows a GARCH process will exhibit the same risk premia as a model with complete markets.

To proceed, assume that the shocks are log-normally distributed: $\epsilon_{t+1} \sim N(-\sigma_\epsilon^2/2, \sigma_\epsilon^2)$ and $\eta_{t+1}^j \sim N(-\sigma_\eta^2/2, \sigma_{\tilde{\eta}}^2)$. [2] As in Constantinides and Duffie I let the variance of the idiosyncratic shock in period $t+1$ depend on the realization of the aggregate state: $\sigma_\eta^2 = f(\epsilon_{t+1})$. It is useful to rewrite the Lucas asset pricing equation (1) in terms of conditional moments. Since asset returns do not depend on the idiosyncratic shock, (1) can be rewritten as

$$E\left[E\left[M^j(\epsilon_{t+1}, \eta_{t+1}^j)|I_t, \epsilon_{t+1}\right] RP(\epsilon_{t+1})|I_t\right] = 0. \tag{10}$$

Using the properties of the log-normal distribution, the expectation of the SDF conditional on the realization of the aggregate shock equals

$$\begin{aligned} E\left[M^j(\epsilon_{t+1}, \eta_{t+1}^j)|I_t, \epsilon_{t+1}\right] &= E\left[\beta\left(\frac{\bar{C}_{t+1}(\epsilon_{t+1})}{\bar{C}_t(\epsilon_t)}\right)^{-\gamma}\left(\frac{1+\eta_{t+1}^j}{1+\eta_t^j}\right)^{-\gamma}|I_t, \epsilon_{t+1}\right] && (11) \\ &= \beta E\left[e^{-\gamma \eta_{t+1}^j}|I_t, \epsilon_{t+1}\right] e^{-\gamma \Delta \bar{c}_{t+1}} && (12) \\ &= \beta e^{\frac{\gamma(\gamma+1)}{2}\sigma_\eta^2(\epsilon_{t+1})} e^{-\gamma \Delta \bar{c}_{t+1}} && (13) \end{aligned}$$

[2] The means of the log variables are chosen to correct for Jensen's effect to insure that the expected growth rate of the variables in levels are unity for each agent: $E\left[e^{\epsilon_{t+1}}\right] = e^{-\sigma_\epsilon^2/2+\sigma_\epsilon^2/2} = 1$.

This is the SDF for a consumer in Constantinides and Duffie (1996). Equation (13) shows that the variance of the idiosyncratic shocks enters the SDF and hence can effect risk premia. To extend their analysis, I ask exactly what properties of the variance process are required to increase risk premia. Wlog. the variance of $\eta$ can be decomposed as follows:

$$\sigma^2_{\eta,t+1} = E_t\sigma^2_{\eta,t+1} + \varsigma_{t+1} \tag{14}$$

Only the properties of the unexpected part $\varsigma_{t+1}$ will play a role for risk premia while the expected component only affects the risk-free rate. Assume that $\varsigma_{t+1} \sim N(0, \sigma^2_\varsigma)$. [3] I allow $\varsigma_{t+1}$ and $\epsilon_{t+1}$ to be correlated, let the correlation be $\rho_{\varsigma\epsilon}$. This correlation will turn out to be crucial for risk premia. Equation (13) can be used to compute the relevant expressions for the Sharpe-ratio as follows.

$$\frac{\sigma[E\left[M^j|I_t,\epsilon_{t+1}\right]|I_t]}{E[E\left[M^j|I_t,\epsilon_{t+1}\right]|I_t]} = \frac{\sigma\left[e^{\frac{\gamma(\gamma+1)}{2}\sigma^2_{\eta,t+1}-\gamma\Delta\bar{c}_{t+1}}|I_t\right]}{E\left[e^{\frac{\gamma(\gamma+1)}{2}\sigma^2_{\eta,t+1}-\gamma\Delta\bar{c}_{t+1}}|I_t\right]} \tag{15}$$

$$= \left(e^{\mathrm{Var}\left[\frac{\gamma(\gamma+1)}{2}\sigma^2_{\eta,t+1}-\gamma\Delta\bar{c}_{t+1}|I_t\right]}-1\right)^{1/2}. \tag{16}$$

The conditional correlation equals

$$-\rho(E\left[M^j|I_t,\epsilon_{t+1}\right], RP_{t+1}|I_t) = -\rho\left(e^{\frac{\gamma(\gamma+1)}{2}\sigma^2_{\eta,t+1}-\gamma\Delta\bar{c}_{t+1}}, e^{rp_{t+1}}|I_t\right) \tag{17}$$

$$= -\frac{e^{\mathrm{Cov}\left[\frac{\gamma(\gamma+1)}{2}\varsigma_{t+1}-\gamma\Delta\epsilon_{t+1},\xi_{t+1}\right]}-1}{\left(e^{\mathrm{Var}\left[\frac{\gamma(\gamma+1)}{2}\sigma^2_{\eta,t+1}-\gamma\bar{c}_{t+1}|I_t\right]}-1\right)^{1/2}\left(e^{\sigma^2_\xi}-1\right)^{1/2}}. \tag{18}$$

Combining (16) and (18) and noting that the market portfolio is perfectly correlated with a claim to aggregate consumption yields for the Sharpe-ratio

$$\mathrm{SR}(M^j_{t+1}) = -\frac{e^{\frac{\gamma(\gamma+1)}{2}\sigma_{\varsigma\epsilon}-\gamma\sigma^2_\epsilon}-1}{\left(e^{\sigma^2_\epsilon}-1\right)^{1/2}} \tag{19}$$

$$\lesssim \gamma\sigma_\epsilon\left(1-\frac{1+\gamma}{2}\frac{\sigma_\varsigma}{\sigma_\epsilon}\rho_{\varsigma\epsilon}\right). \tag{20}$$

Equation (19) implies that a negative correlation between the innovation in returns and the innovation in the cross-sectional variance of idiosyncratic consumption is needed to increase

[3] Note that this assumption allows for the possibility of negative variances since the normal distribution has an unbounded support. The proper way to correct this would be to truncate the normal distribution at zero and compute the moments using the truncated distribution. Of course, this has to be done numerically and analytical solution are not possible. The equations given in the text yield solutions which very close to the results using the truncated distribution for reasonable parameter values, such as the estimates presented later. I therefore choose to work with the normal distribution.

the Sharpe-ratio. The approximation (20) makes this even clearer. The Sharpe-ratio no longer equals (approximately) the product of relative risk aversion and the standard deviation of aggregate consumption. The extra term involves the covariance between aggregate consumption and the *innovation in the variance of idiosyncratic risk*. The Sharpe-ratio is decreasing in this covariance, in particular a negative covariance increases the Sharpe-ratio compared to the case without idiosyncratic shocks. Note that the covariance of $\eta$ and $\xi$ is multiplied with $(1+\gamma)/2$ which is a Jensen's inequality term and occurs because the model is written in logs. It is important to stress that the time-series of the cross-sectional variance of idiosyncratic consumption has not only to be changing over time but must have an unexpected component. Only the unexpected part of the time-series variation which correlated with aggregate consumption has an effect on the Sharpe-ratio. Hence assuming e.g. a GARCH process for $\sigma^2_{\eta,t+1}$ will not have an effect on risk premia because $\sigma^2_{\eta,t+1}$ will be known with certainty as of time $t$. Instead, a heteroskedastic process like stochastic volatility is required. The open question whether this possibility is empirically relevant will be dealt with in the next section.

### 2.6 A calibration

In post-war quarterly data, the Sharpe-ratio is at least 0.27 using the SP500 index as a proxy for the market return. Of course the true market portfolio probably implies a higher Sharpe-ratio but 0.27 it a useful lower bound. The quarterly standard deviation of aggregate consumption is equal to 0.56% in post-war data. To match the point estimate of the Sharpe-ratio of 0.27, the standard complete markets model without idiosyncratic shocks requires a risk aversion coefficient of around 50. This is an alternative way to state the HJ bounds. The challenge for models since Mehra and Prescott (1986) has been to build models which require lower risk aversion. This section asks whether models with idiosyncratic risk can create higher risk premia and Sharpe-ratios for reasonable parameter values.

Consider the model in which the distribution of the idiosyncratic shocks depends on the aggregate state as discussed in section 2.5. In particular I assume that the variance of the idiosyncratic shock is assumed to be linear in the innovation of the aggregate state:

$$\sigma^2_{\eta,t+1} = a_0 + a_1 \epsilon_{t+1}. \tag{21}$$

Note, that this is the most favorable case since innovations in the cross-sectional variance are perfectly correlated with the aggregate shock. Of course, this assumption is not likely to be true in the data. The model in section 2.5 implies that a negative $a_1$ increases the Sharpe-ratio and risk premia. If there is more idiosyncratic uncertainty when aggregate times are bad, consumers require higher compensation to hold risk. Figure 1 plots the Sharpe-ratio as a function of $a_1$. Risk aversion in the top panel is set to unity while the lower panel sets $\gamma$ to ten. $\sigma_\epsilon$ is set to 0.56%. The intercept term $a_0$ affects the risk-free rate but not risk premia and is chosen large enough to make negative variances unlikely. Each panel contains three graphs: the exact and approximate Sharpe-ratios in the lognormal model ((19) and (20)), as well as a model with only two discrete states for the aggregate and idiosyncratic shocks (the moments where chosen to match the lognormal model). For small risk aversion, the graphs coincide are virtually indistinguishable. The Sharpe-ratio increases linearly with more negative $a_1$. Note, however, that the Sharpe-ratio is still an order of magnitude smaller than that in the data even for fairly negative values of $a_1$.

The effect of $a_1$ are significantly larger when risk aversion is higher, as shown in the bottom panel, since the square of risk aversion multiplies the idiosyncratic variance term in (19) and (20). Note, that the approximation (20) is still close to the exact term (19) while the discrete model is significantly nonlinear.

What value of $a_1$ is required to achieve a Sharpe-ratio of 0.27 as in the data? Taking $\sigma_\epsilon = 0.56\%$ as given, Figure 2 plots the required value of $a_1$ as a function of $\gamma$ to achieve a Sharpe-ratio of 0.27. When risk aversion is around unity, $a_1$ has to be about -45. Such negative values seem unplausible. As risk aversion increases, the implied $a_1$ increases. For $\gamma = 10$ an $a_1$ of -0.7 is needed. This shows that the dependence of the variance of the idiosyncratic shock on the aggregate state only increases the Sharpe-ratio substantially in conjunction with fairly high levels of risk aversion. Of course, it is an empirical question to pin down a realistic value of $a_1$. I will discuss the empirical evidence in detail in the next section.

# 3 Volatility Bounds with Income Data

Next, I evaluate the empirical importance of the mechanism presented in the previous section. The problem with a direct empirical analysis of this channel is that individual consumption data (e.g. in the PSID) is known to be of poor quality. In addition consumption data in the PSID span only a short time series. Hence estimating equations like (21) directly from consumption data is flawed with measurement error, at least with currently available data sets. However, individual income data in the PSID is of much higher quality and there is a large empirical literature that estimates income processes for individual households (e.g., Abowd and Card 1989, Carroll 1992 and MaCurdy 1982, just to name a few). In the next section I use these income processes instead of consumption to evaluate the volatility bounds on an individual level. This strategy is appropriate if individual consumption is smoother than individual income. Of course, since there is no data set with a long time series of high quality household consumption data, it is impossible to check directly whether individual consumption is indeed smoother than income. However, we can find theoretical conditions under which we expect this to be the case. I will argue next that these conditions are likely to be fulfilled in the data.

Start with an individual who consumes her permanent income. This is a useful point of departure since much of the consumption literature is cast in this framework (see Deaton 1992 for a survey). Consider the following general process for income growth of household $j$, $\Delta y_{jt}$:

$$\Delta y_{jt} = A(L)\epsilon_t + B(L)n_{jt} + C(L)u_{jt}, \tag{22}$$

where $\epsilon_t$ represents an aggregate shock, $n_{jt}$ is a persistent idiosyncratic shock, $u_{jt}$ is a transitory idiosyncratic shock and $A(L), B(L)$ and $C(L)$ are distributed lag operators. This process has a Wold representation $\Delta y_{jt} = D(L)\vartheta_{jt}$. A permanent income consumer sets her consumption equal to her permanent income. Hence her consumption growth is equal to the growth in permanent income which can be shown to be:

$$\Delta py_{jt} = \frac{r}{1+r}\sum_{k=0}^{\infty}(1+r)^{-k}(E_t - E_{t-1})y_{t+k}, \tag{23}$$

where $r$ is the discount rate. It is straightforward to show that $\Delta py_{jt} = D(\frac{1}{1+r})\vartheta_{jt}$. Hence the innovation in consumption of individual $j$ will be less volatile than the innovation of her income if $D(\frac{1}{1+r}) < 1$. Whether this condition is satisfied depends on the properties of the income process (22). The empirical labor literature, e.g., Abowd and Card (1989), Carroll (1992) and MaCurdy (1982), suggests that household income data are well described by the the following special case of (22):

$$\Delta y_{jt} = \epsilon_t + n_{jt} + u_{jt} - u_{jt-1}. \tag{24}$$

The aggregate shock has a permanent effect as does one of the idiosyncratic shocks. In addition there is completely transitory idiosyncratic shock. In this case the growth of permanent income is smother than the growth in current income since

$$\Delta py_{jt} = \epsilon_t + n_{jt} + \frac{r}{1+r}u_{jt}. \tag{25}$$

Intuitively, consumption is smoother than income because income growth is negatively autocorrelated so that a shock to income has less than a one-to-one effect on permanent income. On the other hand, if income growth is positively autocorrelated, then permanent income is more volatile than current income. For example, consider

$$\Delta y_{jt} = \epsilon_t + n_{jt} + u_{jt} + u_{jt-1}. \tag{26}$$

In this case

$$\Delta py_{jt} = \epsilon_t + n_{jt} + \frac{2+r}{1+r} u_{jt} \tag{27}$$

and therefore consumption is more volatile than income. Since theoretically, permanent income can be more or less volatile than income, it is an empirical question which case is more realistic. The evidence in the empirical literature points overwhelmingly in the direction of *negatively* autocorrelated income growth on the household level (Abowd and Card 1989, Carroll 1992, MaCurdy 1982, Pischke 1995). In this case, permanent income is smoother than current income, and therefore consumption of a permanent income consumer will be smoother than her income.

In contrast to the empirical labor literature, papers that study the effects of idiosyncratic income shocks on asset price usually use levels instead of growth rates. For example, Heaton and Lucas (1996) and Storesletten, et. al. (1997) estimate individual income processes of the form (net of aggregate shocks)

$$y_{it} = z_{it} + u_{it} \text{ where } z_{it} = \rho z_{it-1} + \upsilon_{it}. \tag{28}$$

In growth rates, this process can be rewritten as

$$\Delta y_{it} = \frac{1-L}{1-\rho L} \upsilon_{it} + u_{it}. \tag{29}$$

It is easy to see that (29) is negatively autocorrelated as long as $\rho \leq 1$. Heaton and Lucas estimate $\rho$ to be around 0.53 while Storesletten, et. al. obtain $\rho = 0.92$ and hence permanent income is smoother than current income in either case.

Given the empirical evidence favoring negative serial correlation of individual income growth, and the implication that permanent income is smoother than current income, the strategy in this paper to use HJ bounds with income processes as a lower bound to the bounds with consumption data is justified.

One possible objection to the arguments used above, is that consumption behaviour in models with incomplete markets might deviate from permanent income behavior. But even in more complex models, it is unlikely that consumption is more volatile than current income if income growth is negatively autocorrelated. Consider the example of borrowing constraints. Agents will in general be able to achieve a limited amount of consumption smoothing by building up buffer stock savings and hence consumption tends to be smoother than income on the individual level (Carroll 1992 and Ludvigson and Michaelides 2000). Even in the extreme case of no buffer stock, consumption will not be more volatile than current income since agents will just consume their current income ("rule of thumb" consumers in the Campbell and Mankiw 1989 sense).

If consumption is indeed smoother than income, then risk premia and volatility bounds computed using income data provide a useful lower bound. If risk premia are low for income data, then there is surely little hope in generating higher risk premia using smoother consumption data. Hence, this approach provides a straightforward benchmark to evaluate the potential of models with idiosyncratic risk. The advantage is that there is no need for complicated numerical solutions since the analytical framework in sections 2.4 and 2.5 can be used. The punchline is that risk premia are too small even if individual income data are used to compute them. The reason is that the covariance of the volatility of idiosyncratic income risk and the aggregate state is too small to increase risk premia, even when income is used instead of consumption.

Since agents receive different income in each period and they cannot trade with each other, they will in general disagree on asset prices. One agent might require an expected return of 5% for some asset given her particular income process while another agents might require only a 3% expected return for the same asset. If there were open asset markets, they would trade the asset and smooth their consumption paths so that in equilibrium they agree on expected returns of all assets. Since I abstract from any asset trading in this paper I compute the expected returns implied by the consumption process (which is equal to the income process by assumption) of each individual agent. In order to obtain an upper bound for the ability of idiosyncratic risk to increase risk premia, I consider the highest expected return of a given asset across all agents. In terms of the Sharpe-ratio, using the individual income processes as consumption will result in different Sharpe-ratios for different agents. An upper bound for the Sharpe-ratio that would result from trading is given by the highest Sharpe-ratio computed from the individual income processes. To give the models the best possible shot, I report this maximal Sharpe-ratio across agents:

$$\mathrm{SR}_t^{max} = \max_j \mathrm{SR}_t^j. \tag{30}$$

Next, I will compute risk premia and volatility using three representative income processes that have been used in asset pricing models with incomplete markets.: Heaton and Lucas (1996), Krusell and Smith (1997) and Storesletten, et. al. (1997).

## 3.1 Income process from Heaton and Lucas

Heaton and Lucas (HL) (1996) present a model with idiosyncratic labor income shocks, transaction costs, borrowing and short-sale constraints. There are two types of agents. The shocks of the two types are dependent and sum to one. If one type receives a positive shock, the second type receives an opposite shock. They calibrate the idiosyncratic income processes with PSID data. Income of type $j$ is given by

$$Y_t^j = D_t^a/2 + \eta_t^j(Y_t^a - D_t^a) \tag{31}$$

$$= Y_t^a\left(\delta_t + \eta_t^j(1-\delta_t)\right), \tag{32}$$

where $Y^a$ is aggregate income, $D^a$ are aggregate dividends, $\delta = D^a/Y^a$ is the share of dividends in aggregate income and $\eta^j$ is the fraction of total income received by type $j$. Let the growth rate of aggregate income be given by $\gamma^a$. Heaton and Lucas (1997) estimate a discrete first-order Markov process for the state vector $[\log(\gamma^a), \log(\delta^a), \log(\eta^j)]$. Each variable can take two values so that there are eight states. See HL for the exact transition probabilities. The important parameters for the analysis here are as follows. The conditional standard deviation of aggregate income growth is 0.0278 and 0.0536 for dividend growth. HL consider two processes for idiosyncratic labor income. The first version has a constant conditional standard deviation of 0.251. The second lets the standard deviation depend on aggregate income growth

$$\sigma_{\eta,t+1} = \tilde{a}_0 + \tilde{a}_1 \log(\gamma_{t+1}^a) \tag{33}$$

Using PSID data, HL estimate the parameters as $\tilde{a}_0 = 0.29$ and $\tilde{a}_1 = -1.064$. However, in their actual model HL use $\tilde{a}_1 = -4.5$, more than four times the estimated value.

Some differences to model considered in this paper are worth pointing out. First, the income process (31) is a mixture of the additive and multiplicative shocks. The dividend component adds the additive term to the multiplicative distribution of labor income. To allow a closer comparison to the analysis of the multiplicative model, I also compute asset prices without dividend income. Second, HL write the *standard deviation* of the idiosyncratic shock as a linear function of the aggregate growth rate. The model presented in section 2.5 shows that the Sharpe-ratio is approximately linear in the coefficient of the *variance* of the idiosyncratic shock. Squaring (33) yields for the variance

$$\sigma_{\eta,t+1}^2 = \tilde{a}_0^2 + 2\tilde{a}_0\tilde{a}_1 \log(\gamma_{t+1}^a) + \tilde{a}_1^2(\log(\gamma_{t+1}^a))^2. \tag{34}$$

Hence the coefficient $a_1$ in (21) corresponds approximately to $2\tilde{a}_0\tilde{a}_1$ since $(\log(\gamma_{t+1}^a))^2$ is small. The estimated values of $\tilde{a}_0$ and $\tilde{a}_1$ in HL imply $a_1 = -0.62$, while their actually used values imply $a_1 = -2.61$.

Table 1 summarizes the results for the Heaton and Lucas income shocks. The numbers are computed as follows. I use the processes for the idiosyncratic labor shocks from HL. Three sets of parameters for the standard deviation of the idiosyncratic shock are used: $\tilde{a}_0 = 0.251, \tilde{a}_1 = -0$; $\tilde{a}_0 = 0.360, \tilde{a}_1 = -1.064$ and $\tilde{a}_0 = 0.290, \tilde{a}_1 = -4.450$. The second set of parameters corresponds to the PSID regressions results. The aggregate shocks are chosen to match the moments of aggregate consumption. In each of the eight states, I compute the

Sharpe-ratio for both types of agents. Then I compute the average Sharpe-ratio (weighted with the probabilities of the stationary distribution) across states using the higher of the two Sharpe-ratios. The transitions probabilities are taken from HL. In addition, I report the corresponding Sharpe-ratios in the log-normal model using the same model parameters [4].

For low risk aversion of unity, the picture looks very bleak. None of the three case reach a Sharpe-ratio of 0.1. Note that the effect of dividend uncertainty is very small. It is clear that higher risk aversion is needed to match the Sharpe-ratio in the data. The bottom panel uses $\gamma = 5$. Here, Sharpe-ratios increase substantially when $\tilde{a}_1$ is negative. For the case corresponding to the PSID data, the Sharpe-ratio is around 0.18. This value is still only about one third of the Sharpe-ratio in the data. Only if $\tilde{a}_1 = -4.45$, i.e. more than four times the PSID value, is the Sharpe-ratio roughly high enough to be consistent with the data.

Is this a success? As mentioned before, this analysis only computes a very loose upper limit for the Sharpe-ratio. Consumers cannot trade against their idiosyncratic labor shocks at all, the Sharpe-ratio of the poor agents is used despite the fact that those agents are likely to be borrowing constraint and hence do not determine asset prices, and lastly, the required value for $\tilde{a}_1$ is about four times the value found in the data. Only in this case does the model generate a reasonable Sharpe-ratio. Thus is not surprising the HL found low Sharpe-ratios and risk premia once they allow agents to trade. The model is not able to produce higher risk premia even given the exogenous processes.

## 3.2 Income process from Krusell and Smith

Krusell and Smith (KS) (1997) propose a model where agents have a higher probability to become unemployed when aggregate times are bad. This yields again a negative covariance of idiosyncratic uncertainty and state aggregate that has the potential to increase risk premia. Let aggregate output per capita be given by $Y^a$. If an agent becomes unemployed, she receives unemployment insurance of $gY^a$. Hence, an employed agent's income is $(1-g)Y^a$. Aggregate income can take on two values, so that there are four possible states for each agent.

KS calibrate the model as follows. The unemployment rate in the good aggregate state is 4% and 10% in the low state. Unemployed agents receive $g = 9\%$ unemployment insurance. The Markov transition probabilities are chosen that unemployment shocks are fairly persis-

[4]The differences between the HL income process and the log-normal model are due to the Markov transition probabilities. Agents in different states have different probabilities to move to some state in the next period. This leads to state-dependent Sharpe-ratios. Since I take the average of the higher Sharpe-ratio, they cannot be directly compared to the log-normal model.

tent, i.e. an unemployed agent today has a smaller probability to find a job tomorrow than an agent who is employed today. Again, I refer to KS for the exact transition probabilities. All other parameters are taken from KS. They calibrate the model for quarterly data.

Table 2 reports the Sharpe-ratios in this economy. Recall that the Sharpe-ratio in quarterly data is 0.27. Unemployed agents have a much higher Sharpe-ratio than employed agents since their current and expected future income is much lower. Note that the Sharpe-ratio of both types is higher in good aggregate times, i.e. it is procyclical. This comes from the specific choice of the transition probabilities. However, the lowest Sharpe-ratio (of the unemployed agents in bad times) just matches the Sharpe-ratio in the data.

### 3.3 Income process from Storesletten, Telmer and Yaron

Storesletten, Telmer and Yaron (STY) (1997) study an OLG model with idiosyncratic income shocks and borrowing constraints. They calibrate various Markov processes for idiosyncratic shocks to match the PSID data. Labor supply of an agent $j$, $N^j$ is determined by (potentially multiple) idiosyncratic shocks: $N_t^j = \exp(\eta_t^j)$. The labor input is supplied inelastically and combined with a aggregate Cobb-Douglas production function to produce the consumption good.

STY propose different processes for $\eta_t^j$. Their first three processes differ mainly in the persistence of the idiosyncratic shocks. This turns out to be important for the risk-sharing in their OLG model. In their fourth version, they allow the aggregate state to affect the variance of the idiosyncratic shock. They estimate equation (21) using PSID data. Using two different estimation techniques they estimate the coefficients as $\hat{a}_0 = 0.054, \hat{a}_1 = -0.011$ and $\hat{a}_0 = 0.054, \hat{a}_1 = -0.046$. Both slope coefficients are not negative enough to increase the Sharpe-ratio, as sees in Figure 3. Finally, STY propose a different technique to estimate cross-sectional dispersion. They estimate a model with a high and low idiosyncratic variance regime depending on whether the aggregate economy is above or below trend. STY find that the variance of idiosyncratic shocks is 0.032 when the aggregate economy is above trend and 0.184 when it is below trend. Table 4 reports the Sharpe-ratio for all five cases considered by STY. Again, all other parameters are taken from STY.

The case without any idiosyncratic shocks generates a Sharpe-ratio of 0.0594. The following three cases produce the same Sharpe-ratio as without idiosyncratic risk. Here, the distribution of the idiosyncratic shocks are assumed to be independent of the aggregate state. Hence, as shown in proposition 1 idiosyncratic shocks to not change asset prices. The next two versions incorporate the dependence of the variance of the idiosyncratic shock on the aggregate state as estimated from the PSID data set. Since the estimated $a_1$'s are negative, the Sharpe-ratio increases. But the increase is only minute since the parameters are not negative enough. The Sharpe-ratio in the case with the changing variance regime is about 0.1 which is still only one-fifth of the required value. None, of the income processes considered by STY can produce high Sharpe-ratios even if agents cannot smooth them.

# 4 Conclusion

In this paper I study the effect of idiosyncratic consumption risk on Hansen and Jagannathan's (1991) volatility bounds for stochastic discount factors. Instead of plotting the original HJ bounds, I compute the Sharpe-ratio in the presence of idiosyncratic risk. A model that matches the Sharpe-ratio of the data also passes the HJ test since both formulations are equivalent. First, I show that when consumers have CRRA preferences and the distribution of the idiosyncratic shock is independent of the aggregate state, Sharpe-ratios and therefore HJ bounds are not affected at all by idiosyncratic uncertainty. The reason is that while the SDF of an individual consumer is increasing with idiosyncratic uncertainty, the correlation of the her SDF and aggregate assets is decreasing. These two effects cancel exactly for CRRA preferences.

This irrelevance results does not hold anymore when the distribution of idiosyncratic risk depends on the realization of the aggregate state. Following Mankiw (1986) and Constantinides and Duffie (1996) I show that Sharpe-ratios increase when the variance of the idiosyncratic shock is negatively correlated with aggregate shocks. I compute the Sharpe-ratio in a log-normal model and a discrete 4-state model and show that the quantitative results are approximately the same when risk aversion is not too large.

To evaluate the empirical relevance of this mechanism I compute Sharpe-ratios using individual income processes from the PSID data set. Using income data provides an upper bound for individual consumption data in the absence of reliable consumption data. I take the estimated income processes from recent studies of Heaton and Lucas (1996), Krusell and Smith (1997) and Storesletten, Telmer and Yaron (1997) and compute Sharpe-ratios as if agents had to consume their endowment. Since there is no market that equilibriates the SDF of different agents, they will disagree on asset prices. To obtain an upper bound on Sharpe-ratios, I compare the Sharpe-ratio in the data to that of the maximal Sharpe-ratio across agents. I find that Sharpe-ratios are generally smaller than in the data, only some extreme cases just reaches the data equivalent. Hence it seems unlikely that idiosyncratic consumption risk can generate SDF that pass the Hansen-Jagannathan bounds test.

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# A Appendix: Proof of Proposition 1

**Proof:** *It is useful to rewrite the Lucas asset pricing equation (1) in terms of conditional moments. Since asset returns do not depend on the idiosyncratic shock, (1) can be rewritten as*

$$E\left[E\left[M^j(\epsilon_{t+1},\eta_{t+1})|I_t,\epsilon_{t+1}\right]RP(\epsilon_{t+1})|I_t\right]=0. \tag{35}$$

*Using these conditional expectations the equity premium can be expressed as*

$$
\begin{aligned}
E[RP(\epsilon_{t+1})|I_t] \;=\;& -\sigma[RP(\epsilon_{t+1})|I_t] \;\; \rho[E\left[M^j(\epsilon_{t+1},\eta_{t+1})|I_t,\epsilon_{t+1}\right], RP(\epsilon_{t+1})|I_t] \\
& \times \frac{\sigma[E\left[M^j(\epsilon_{t+1},\eta_{t+1})|I_t,\epsilon_{t+1}\right]|I_t]}{E[E\left[M^j(\epsilon_{t+1},\eta_{t+1})|I_t,\epsilon_{t+1}\right]|I_t]}.
\end{aligned} \tag{36}
$$

*The expectation of the SDF of agent $j$ conditional on the aggregate shock can be simplified further if the agent has CRRA preferences:*

$$
E\left[M^j(\epsilon_{t+1},\eta_{t+1})|I_t,\epsilon_{t+1}\right] = E\left[\beta\left(\frac{\bar{C}_{t+1}(\epsilon_{t+1})}{\bar{C}_t(\epsilon_t)}\right)^{-\gamma}\left(\frac{1+\eta^j_{t+1}}{1+\eta^j_t}\right)^{-\gamma}|I_t,\epsilon_{t+1}\right] \tag{37}
$$

$$
= \beta\left(\frac{\bar{C}_{t+1}(\epsilon_{t+1})}{\bar{C}_t(\epsilon_t)}\right)^{-\gamma} E\left[\left(\frac{1+\eta^j_{t+1}}{1+\eta^j_t}\right)^{-\gamma}\right]. \tag{38}
$$

*Substituting this expression into (35) shows immediately that the idiosyncratic component cancels out since $\eta^j_{t+1}$ does not depend on $\epsilon_{t+1}$ and $I_t$. Hence asset prices are determined only by aggregate consumption. Note that no assumption on the distribution of the idiosyncratic shocks are necessary (other than the existence of expected growth in the idiosyncratic shock of course). It follows immediately that the Sharpe-ratio is not affected either.*

*The risk-free rate depends on the aggregate shock as well as on the distribution of the idiosyncratic component:*

$$
R^f_{t+1} = 1/E_t[M^j_{t+1}] \tag{39}
$$

$$
= \left(\beta\left(\frac{\bar{C}_{t+1}(\epsilon_{t+1})}{\bar{C}_t(\epsilon_t)}\right)^{-\gamma} E\left[\left(\frac{1+\eta^j_{t+1}}{1+\eta^j_t}\right)^{-\gamma}\right]\right)^{-1}. \tag{40}
$$

*Note that the risk-free rate can decrease or increase in the presence of idiosyncratic shocks.* •

**Table 1: The Sharpe-ratio with Heaton-Lucas income shocks**

| income process | Heaton and Lucas | |
|---|---|---|
| | with dividends | no dividends |
| | $\gamma = 1$ | |
| HL1: $\tilde{a}_1 = 0$ | 0.0110 | 0.0111 |
| HL2: $\tilde{a}_1 = -1.064$ | 0.0299 | 0.0301 |
| HL3: $\tilde{a}_1 = -4.450$ | 0.0767 | 0.0769 |
| | $\gamma = 5$ | |
| HL1: $\tilde{a}_1 = 0$ | 0.0552 | 0.0553 |
| HL2: $\tilde{a}_1 = -1.064$ | 0.1813 | 0.1820 |
| HL3: $\tilde{a}_1 = -4.450$ | 0.5773 | 0.5785 |

Note: Table reports the average Sharpe-ratio across the eight states of the economy weighted with the probabilities of the stationary distribution. In each state the maximal Sharpe-ratio across agents is used. The model is calibrated for annual data.

**Table 2: The Sharpe-ratio with Krusell-Smith income shocks**

| agent | aggregate state | |
|---|---|---|
| | bad | good |
| employed | 0.0641 | 0.1088 |
| unemployed | 0.1759 | 0.2840 |
| no employment shocks | 0.0037 | 0.0037 |

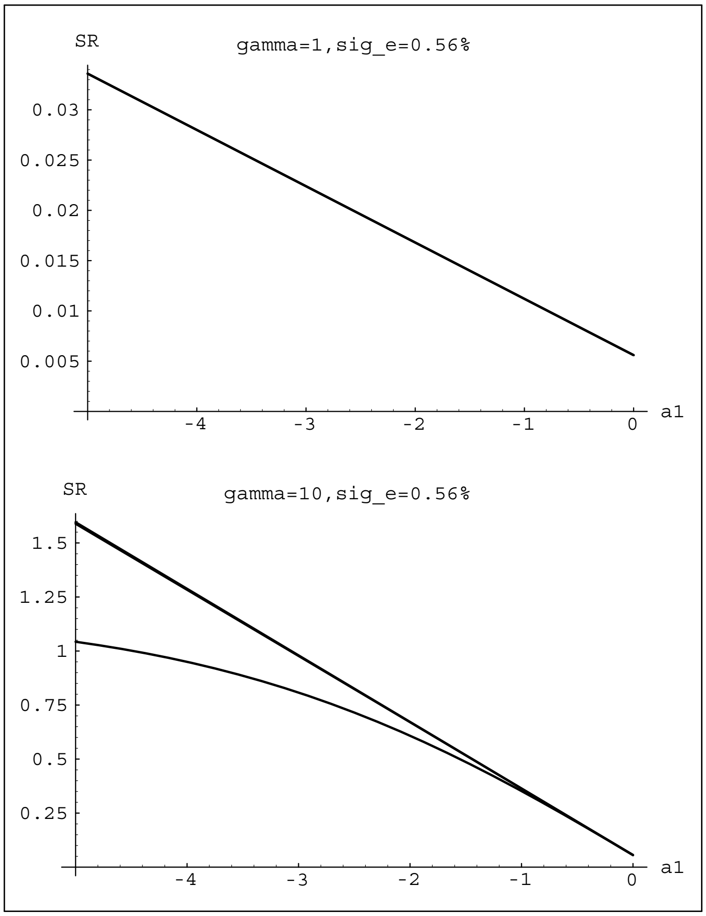
Note: Table reports the Sharpe-ratio with KS unemployment shocks. The model is calibrated for quarterly data.

**Table 3: The Sharpe-ratio with Storesletten, et.al. income shocks**

| income process | Sharpe-ratio |
|---|---|
| no idio. shocks | 0.0594 |
| unit root | 0.0594 |
| high persistence | 0.0594 |
| moderate persistence | 0.0594 |
| low persistence | 0.0594 |
| PSID $a_1 = -0.011$ | 0.0604 |
| PSID $a_1 = -0.046$ | 0.0635 |
| high/low variance regimes | 0.1013 |

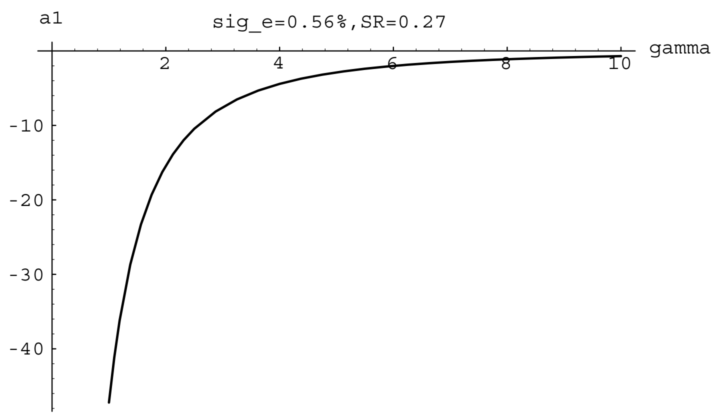
Note: Table reports the Sharpe-ratio for income processes considered in Storesletten, Telmer and Yaron (1997). The parameters for the different AR1 income shocks are as follows. STY approximate the processes with discrete Markov chains. Let $\rho$ be the AR1 coefficient and $\sigma_\eta$ the innovation standard deviation. Unit root: $\rho = 1, \sigma_\eta = 0.201$, high persistence: $\rho = 0.929, \sigma_\eta = 0.230$, moderate persistence: $\rho = 0.529, \sigma_\eta = 0.251$. The two PSID cases use the unit root parameters and the estimated coefficients in (21). The last case assumes a variance of 0.032 when the economy is the high aggregate state and 0.184 when the aggregate state is low. The model is calibrated for annual data.

Figure 1: Sharpe-ratio with dependence



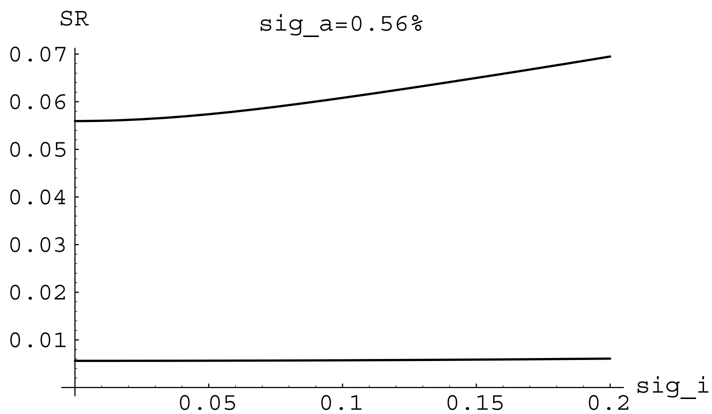

Note: The figure shows the Sharpe-ratio in the model with dependence. Each plot contains a graph for the exact and linearized log-normal model as well a graph for the 4-state model. The variance of the idiosyncratic shock is assumed to be linear in the aggregate state: $\sigma^2_{\eta,t+1} = a_0 + a_1\epsilon_{t+1}$. The top panel plots the Sharpe-ratio as a function of $a_1$ when risk aversion is unity, the lower panel assumes $\gamma = 10$. In both plots $\sigma_\epsilon$ is set to the value from the data, 0.56%.

Figure 2: Required $a_1$




Note: The figure shows the required value for $a_1$ to achieve a Sharpe-ratio of 0.27 as a function of $\gamma$ in the log-normal model. $\sigma_\epsilon$ is set to the value from the data, 0.56%.

Figure 3: Sharpe-ratio with additive shocks




Note: The figure shows the Sharpe-ratio for the case of additive shocks as a function of the standard deviation of the idiosyncratic shock. The standard deviation of aggregate shocks is set to 0.56%. The lower graph is for $\gamma = 1$, the upper graph for $\gamma = 10$.