Nonparametric Pricing of Multivariate Contingent Claims
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Abstract

In this paper, I derive and implement a nonparametric, arbitrage-free technique for multivariate contingent claim (MVCC) pricing. Using results from the method of copulas, I show that the multivariate risk-neutral density can be written as a product of marginal risk-neutral densities and a risk-neutral dependence function. I then develop a pricing technique using nonparametrically estimated marginal risk-neutral densities (based on options data) and a nonparametric dependence function (based on historical return data). By using nonparametric estimation, I avoid the pricing biases that result from incorrect parametric assumptions such as lognormality.

I apply this technique to estimate the joint risk-neutral density of euro-dollar and yen-dollar returns. I compare the nonparametric risk-neutral density with density based on a lognormal dependence function and nonparametric marginals. The nonparametric euro-yen risk-neutral density has greater volatility, skewness, and kurtosis than the density based on a lognormal dependence function. In a comparison of pricing accuracy for euro-yen futures options, I find that the nonparametric model is superior to the lognormal model.

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I. Introduction

Multivariate contingent claims (MVCCs) are derivatives whose payoff depends on more than one underlying asset. The MVCC expected payoff is a function of the multivariate probability density. So, this density function plays a crucial role in valuation of a multivariate contingent claim.

When the underlying assets follow a multivariate geometric Brownian motion, the multivariate risk-neutral density is lognormal. Pricing models using the geometric Brownian motion assumption include Margrabe (1978), Stulz (1982), Johnson (1987), Reiner (1992), and Shimko (1994). When asset prices evolve according to a multivariate binomial tree, the multivariate risk-neutral density limits to a multivariate lognormal as the step size approaches zero. Pricing models using a binomial tree assumption include Stapleton and Subrahmanyam (1984a, 1984b), Boyle (1988), Boyle, Evnine, and Gibbs (1989), and Rubinstein (1991, 1992, 1994b).

Financial prices often exhibit stochastic volatility and jumps. Both of these features result in skewed and fat-tailed density functions that are poorly approximated by a lognormal density. Univariate parametric alternatives to lognormal valuation are proposed in Sherrick, Irwin, and Forster (1990, 1992) and Longstaff (1995). Semiparametric and nonparametric estimation techniques are developed in Shimko (1993), Derman and Kani (1994), Dupire (1994), Rubinstein (1994a), and Ait-Sahalia and Lo (1998).

In the multivariate case, the lognormal density also imposes a potentially unrealistic dependence on the underlying assets. Deviations from lognormality arise from multivariate stochastic volatility (e.g. Engle and Kroner (1995) or Harvey, Ruiz, and Shepherd (1994)), multivariate extreme value distributions (e.g. Tawn (1990)), asymmetric correlations (e.g. Erb, Harvey, Viskanta (1994)), or multivariate correlated jumps.

Pricing models based on an incorrect lognormality assumption will generate biased prices. For example, these models will not be able to fit an implied volatility smile or skew. To avoid this problem, several papers have used more general density functions for valuation. Rosenberg (1998) develops a flexibly parameterized density function for MVCC valuation. However, estimation requires prices of traded multivariate claims. Ho, Stapleton, Subrahmanyam (1995) provide an approximation technique that does not impose a multivariate lognormal density.

In this paper, I derive and implement a nonparametric, arbitrage-free technique for multivariate contingent claim (MVCC) pricing. The purpose of this technique is to impose the least restrictive assumptions on the risk-neutral density function and to fully utilize the information in traded option prices and historical returns. My technique does not rely on parametric assumptions about the underlying price process. It is compatible with fat-tailed and skewed marginal densities, as well as asymmetric correlation in joint density. The procedure I develop generates MVCC prices that are consistent with univariate traded option prices, even in the presence of a volatility smile or skew.

Using results from the method of copulas, I create a multivariate risk-neutral density by combining nonparametrically estimated marginal risk-neutral densities (based on options data) with a nonparametric dependence function (based on historical returns data). I identify general conditions under which the objective and risk-neutral dependence functions are identical. This justifies the use of historical data for estimation of the dependence function.

I apply this technique to estimate the joint risk-neutral density of euro-dollar and yen-dollar returns. I then compare the nonparametric risk-neutral density with the density based on a lognormal dependence function and nonparametric marginals. I find that the risk-neutral density using the nonparametric dependence function exhibits multiple modes and asymmetries not present in the alternative density. I show that the nonparametric euro-yen risk-neutral density has greater volatility, skewness, and kurtosis than the density based on a lognormal dependence function. The nonparametric implied volatility function provides the best fit to the actual euro-yen implied volatility smile.

The remainder of the paper is structured as follows. In Section II, I describe the theory underlying nonparametric multivariate contingent claim pricing. In Section III, I present the nonparametric multivariate density estimation technique. In Section IV, I implement the nonparametric pricing technique for derivatives on euro-dollar and yen-dollar returns. Section V concludes the paper.

II. Nonparametric multivariate contingent claim pricing

II.a. A copula approach to multivariate density modeling

The method of copulas provides several important insights that I use for MVCC pricing. A copula is a function that links (or couples) two or more cumulative marginal density functions (CDFs) together to form a multivariate cumulative joint density function. Two standard references are Joe (1997) and Nelsen (1998). A useful discussion is also provided in Bouyé et. al. (2000).

Recently, there has been a growing amount of research into financial applications of copulas. Several papers that use copulas to analyze portfolio risk include Embrechts, McNeil, and Straumann (1998), Wang (1998), and Hull and White (1998). Copulas have also been used for default risk modelling in Li (2000) as well as Schönbucher and Schubert (2001). Patton (2002) uses copulas to develop a generalized class of multivariate volatility models.

The most important result in this area is Sklar's Theorem, which states that any joint density function can be represented in terms of a copula and the marginal densities. Several additional important characteristics of copulas are presented in appendix A.

For simplicity, I analyze a bivariate density using the method of copulas. Let $F_1(x)$ and $F_2(y)$ be the CDFs for random variables X and Y, $F(x,y)$ is the joint CDF of X and Y, and $C(u,v)$ is the copula of the joint density function. Then, applying Sklar's Theorem,

(1) $$F(x,y) = C(F_1(x),F_2(y))$$

In Equation (1), all of the marginal information is captured in $F_1(x)$ and $F_2(y)$, and all of the dependence information is summarized by $C(u,v)$. The copula is a joint density function, and it maps points on the unit square ($u,v \in [0,1]x[0,1]$) to values between zero and one. The copula is not affected by scaling, since it relates the quantiles (u and v) of the two distributions rather than the original variables (x and y).

One of the simplest copulas is the product copula: $C(u,v) = u*v$. Under the product copula, $F(x,y) = C(F_1(x),F_2(y)) = F_1(x)F_2(y)$, and the random variables are independent. Another important example is the normal copula. If $\Phi(x,y)$ is a joint standard cumulative normal density function and $\Phi^{-1}(u)$ and $\Phi^{-1}(v)$ are standard normal quantile functions, then the copula is $\Phi(\Phi^{-1}(u), \Phi^{-1}(v))$.[1]

Now, taking the second cross-partial of equation (1) with respect to x and y:

(2) $$f(x,y) = f_1(x)f_2(y)c(F_1(x),F_2(y))$$

[1] The normal copula is obtained using the method of inversion given in Appendix A. The quantile function is equal to value x at which the probability of an outcome less than x is equal to u. For example, $\Phi^{-1}(.05)$ corresponds to the 5th percentile or the 5% value at risk for the standard normal distribution.

This is a version of Sklar's Theorem written using probability density functions (PDFs) instead of CDFs. Equation (2) states that the joint density function is the product of the marginal densities and the derivative of the copula evaluated at the marginal CDFs. The derivative of the copula, $(\partial^2/\partial u\partial v)C(u,v)=c(u,v)$, is known as the copula density.

Since the copula is a joint CDF, the copula density is a joint PDF. The copula density is also scale-free measure of dependence. By taking cross-partials of the copula, it is apparent that the product copula density is 1, and the normal copula density is $\Phi^{-1'}(u)\Phi^{-1'}(v)\phi(\Phi^{-1}(u),\Phi^{-1}(v))$.

I define the *dependence function* as the copula density evaluated at the marginal CDFs. Solving equation (2) for the dependence function:

$$(3) \qquad c(F_1(x),F_2(y)) = f(x,y) / (f_1(x)f_2(y))$$

Equation (3) shows that the dependence function equals the ratio of the marginal and joint PDFs. The dependence function "factors out" the effect of the marginal densities on the joint, leaving only the information that determines joint behavior. However, the dependence function is not scale-free, since it is a function of x and y instead of u and v. The dependence function under the product copula is 1, and the dependence function under the normal copula density is $\Phi^{-1'}(F_1(x))\Phi^{-1'}(F_2(y))\phi(\Phi^{-1}(F_1(x)),\Phi^{-1}(F_2(y)))$.

A joint normal density has a normal copula density with $F_1(x)=\Phi(x)$ and $F_2(y)= \Phi(y)$. So, the dependence function simplifies to $\phi(x,y)/(\phi(x)\phi(y))$. This is an example of the relation given in equation (3): the dependence function is equal to the ratio of the joint and marginal densities.

II.b. Using copulas for multivariate contingent claim valuation

The "risk-neutral density" representation of the asset pricing formula is often used for nonparametric option pricing. This is a convenient valuation framework for several reasons. First, a risk-neutral density representation is guaranteed to exist in the absence of arbitrage. Second, the same risk-neutral density may be used to price all claims dependent on the same underlying asset. And, third, interpolated prices using this representation do not permit arbitrage opportunities.

A copula representation of the multivariate pricing problem allows the bivariate risk neutral density to be written as the product of the marginal risk-neutral densities and a "dependence function.” This is helpful for estimation, since the marginal risk-neutral densities can be estimated using existing techniques. The copula representation also leads to the important result that the risk-neutral and objective dependence functions are identical under general conditions. This justifies the use of historical return data for estimation of the dependence function.

It is useful to initially state the familiar risk-neutral univariate asset pricing formula as given in Equation (4). D_t is the price of a European-style derivative asset with payoff function $g(X_T)$. The underlying asset price is X_T, the option expiration date is T, and the riskless rate of interest is r. The risk-neutral density of X_T is $f_1^*(X_T)$.

$$(4) \qquad D_t = e^{-r(T-t)} \int g(X_T)\, f_1^*(X_T)\, dX_T$$

For valuation of MVCCs, the risk-neutral pricing formula has a straightforward generalization. For clarity, I will discuss the bivariate case. The payoff function for the claim now depends on two variables: X_T and Y_T. And, the risk-neutral density function is the bivariate function $f^*(X_T, Y_T)$. The bivariate risk-neutral pricing formula is:

$$(5) \qquad D_t = e^{-r(T-t)} \iint g(X_T, Y_T)\, f^*(X_T, Y_T)\, dX_T\, dY_T$$

Using the copula density representation, I write the risk-neutral density as the product of the risk-neutral marginals and the risk-neutral dependence function. I also write the risk-neutral dependence function as the ratio of the risk-neutral joint density to the product of the risk-neutral marginals. Applying equations (2) and (3),

$$(6) \qquad f^*(X_T, Y_T) = f_1^*(X_T) f_2^*(Y_T) c^*(F_1^*(X_T), F_2^*(Y_T)) \text{ and}$$
$$c^*(F_1^*(X_T), F_2^*(Y_T)) = f^*(X_T, Y_T) \,/\, (f_1^*(X_T) f_2^*(Y_T))$$

I now show that the risk-neutral dependence function is equal to the objective dependence function scaled by a dependence risk-adjustment. I use the fact that each risk-neutral density can be written as the discounted pricing kernel times the objective density. Let $M(X_T)$, $M(Y_T)$, and $M(X_T, Y_T)$ be the pricing kernel projected onto X_T, Y_T, as well as X_T and Y_T. Let B be the riskless bond price. Then:

(7) $f_1^*(X_T) = B^{-1}M(X_T)f_1(X_T)$

$f_2^*(Y_T) = B^{-1}M(Y_T)f_2(Y_T)$

$f^*(X_T,Y_T) = B^{-1}M(X_T,Y_T)f(X_T,Y_T)$

(8) $c^*(F_1^*(X_T),F_2^*(Y_T)) = f^*(X_T,Y_T) / (f_1^*(X_T)f_2^*(Y_T))$

$= B^{-1}M(X_T,Y_T)f(X_T,Y_T)/ (B^{-1}M(X_T)f_1(X_T)B^{-1}M(Y_T)f_2(Y_T))$

$= f(X_T,Y_T)/(f_1(X_T)f_2(Y_T))\ [BM(X_T,Y_T)/(M(X_T)M(Y_T))]$

$= c(F_1(X_T),F_2(Y_T)) * [BM(X_T,Y_T)/(M(X_T)M(Y_T))]$

$= f(X_T,Y_T) / (f_1(X_T)f_2(Y_T)) * [BM(X_T,Y_T)/(M(X_T)M(Y_T))]$

Under general conditions, it is reasonable to assume that the dependence risk-adjustment, $BM(X_T,Y_T)/(M(X_T)M(Y_T))$, equals one. In this case, the risk of X_T and Y_T outcomes are priced, but the price of risk for a given X_T outcome is not affected by the Y_T outcome. For example, a general pricing kernel that satisfies the condition that the risk adjustment equals 1 is (see Appendix B):

(9) $M(X_T, Y_T) = \exp[z_1(X_T) + z_2(Y_T)]$

where $z_1(X_T)$ and $z_2(Y_T)$ are arbitrary continuous functions

Under the assumption that the dependence function equals one, the risk-neutral density is equal to the product of the risk-neutral marginals and the objective dependence function.[2]

(10) $f^*(X_T,Y_T) = f_1^*(X_T)f_2^*(Y_T)c(F_1(X_T),F_2(Y_T)) = f_1^*(X_T)f_2^*(Y_T)[f(X_T,Y_T)/(f_1(X_T)f_2(Y_T))]$

The risk-neutral pricing formula becomes:

(11) $$D_{X,Y,t} = e^{-r(T-t)} \iint g(X_T,Y_T) f_1^*(X_T) f_2^*(Y_T)[f(X_T,Y_T)/(f_1(X_T) f_2(Y_T))]dX_T dY_T$$

III. Nonparametric estimation technique for the joint risk-neutral density

[2] If the stated assumption is violated, equation (10) will produce a function that is positive, but it may not integrate to unity. For this reason, I use a scaling factor in the estimation procedure to ensure that a valid joint density is obtained.

In this section, I develop a nonparametric estimation technique for the risk-neutral density. In contrast to parametric methods, my approach does not impose restrictions that might result in pricing biases. To estimate the risk-neutral joint density, I use equation (10). The dependence function is estimated using historical return data (section III.a) and the marginals are estimated using option data (section III.b).

III.a. Nonparametric estimation technique for the objective dependence function

Equation (3) shows that the dependence function is equal to the ratio of the joint density to the product of the marginal densities. I use a sample version of this equation for estimation.[3]

$$(12)\qquad \hat{c}(F_1(x), F_2(y)) = \hat{f}(x,y)/(\hat{f}_1(x)\hat{f}_2(y))$$

I select the Nadaraya-Watson kernel estimator for nonparametric estimation of the joint and marginal densities. The kernel estimates can be interpreted as "smoothed histograms," where the degree of smoothness is determined by the magnitude of the kernel bandwidths. See Fermanian and Scaillet (2000) for a nonparametric copula estimation technique.

To estimate the probability density for asset prices X_{t+1} and Y_{t+1} ondate t, I first convert the time-series of historical prices to logarithmic returns. I match the return interval to the forecasting interval. For example, to estimate the density of prices one month in the future, I would use a time-series of monthly returns.

I set $r_{X,i}=\log(X_i/X_{i-1})$ and $r_{Y,i}=\log(Y_i/Y_{i-1})$ for returns dated i=1...t. I define $k(\cdot)$ as the Gaussian kernel, h_x and h_y as the Silverman (1996) kernel bandwidths. Equations (13) and (14) are the kernel estimators for the marginal and joint return densities.

$$(13)\qquad f_1(r_X) = (Th_X)^{-1}\sum_{i=1}^{t} k\big((r_{X,i}-r_X)/h_X\big) \qquad\qquad f_2(r_Y) = (Th_Y)^{-1}\sum_{i=1}^{t} k\big((r_{Y,i}-r_Y)/h_Y\big)$$

[3] Equation (12) provides an estimate of the unconditional or stationary dependence function. If the dependence function were time-varying, then it would be useful to model its time-series dependence. Patton (2002) has developed a time-varying parametric copula estimator. However, it is not clear how this can be generalized to the nonparametric case for the dependence function.

(14) $$f(r_X, r_Y) = (Th_X h_Y)^{-1} \sum_{i=1}^{t} k\big((r_{X,t} - r_X)/h_X\big) k\big((r_{Y,t} - r_Y)/h_Y\big)$$

At time t, the asset prices X_t and Y_t are known. So, the probability of a return of a certain size ($r_{X,t+1}$) is equal to the probability that the future price is $\exp(r_{X,t+1})X_t$. Similarly, the probability of $r_{Y,t+1}$ is the same as the probability of $\exp(r_{Y,t+1})Y_t$. The return density provides a complete description of the price density.

III.b. Nonparametric estimation technique for risk-neutral marginal densities

I apply the nonparametric technique developed in Ait-Sahalia and Lo (1998) for estimation of the risk-neutral marginal densities. Using this technique, it is possible to obtain the lognormal density as a special case when the implied volatility curve is flat. However, the estimated nonparametric density will exhibit skewness and excess kurtosis in the presence of an implied volatility smile or skew.

The most general method proposed by Ait-Sahalia and Lo (1998, equation 11) requires a kernel regression of call prices on five variables (underlying price, exercise price, time-until-expiration, riskless rate, and dividend yield). However, the empirical results reported in Ait-Sahalia and Lo (1998) are based on a dimensionality reduction approach: the call price function is given by the Black-Scholes formula with the volatility parameter replaced by a nonparametrically estimated function (Ait-Sahalia and Lo, 1998, equations 9, 13). I implement a similar dimensionality reduction approach.

Consider a collection of options (i=1..n) on an underlying asset with price X_t. Let σ_i be the Black-Scholes implied volatility, r is the riskless rate of interest, K is the option exercise price, T-t is the option time-until-expiration, and Fut_t is the price of a futures contract with identical expiration and underlying asset as the option contract.[4] Also, let $k(\cdot)$ be the Nadaraya-Watson kernel estimator with a bandwidth of h. Then:

(15) $$\hat{C}_t = BS(X_t, r, K, T-t, \hat{s}(Fut_t, K, T-t))$$

(16) $$\hat{s}(\text{Fut}_t, K, T-t) = \frac{\sum_{i=1}^{n} k_F\left((\text{Fut}_t - \text{Fut}_{t_i})/h_{\text{Fut}}\right) k_K\left((K-K_i)/h_K\right) k_{T-t}\left(((T-t)-(T-t)_i)/h_{T-t}\right) s_i}{\sum_{i=1}^{n} k_F\left((\text{Fut}_t - \text{Fut}_{t_i})/h_{\text{Fut}}\right) k_K\left((K-K_i)/h_K\right) k_{T-t}\left(((T-t)-(T-t)_i)/h_{T-t}\right)}$$

In equation (15), the call price function is the Black Scholes equation with volatility dependent on futures price, exercise price, and maturity. Equation (16) uses a kernel regression estimator of the option implied volatility function in terms of the futures price, exercise price, and option maturity.

The risk-neutral density function is obtained by taking the second derivative of the call option pricing formula with respect to the option exercise price. In equation (16) this derivative is calculated using a centered finite difference approximation. The futures price and maturity are suppressed for clarity.

(17) $$\begin{aligned} \hat{f}^*(X_T = K) &= e^{r(t-T)} \frac{\partial^2 \hat{C}_t}{\partial K^2} \\ &\cong e^{r(t-T)} \frac{\hat{C}_t(\hat{s}(K+e), K+e) - 2\hat{C}_t(\hat{s}(K), K) + \hat{C}_t(\hat{s}(K-e), K-e)}{e^2} \end{aligned}$$

This procedure can also be used with American-style options on futures. However, the calculated implied volatility from the American futures option must be the same as the Black-Scholes implied volatility for an otherwise identical European spot option. To obtain such implied volatilities, I use the BBSR (Binomial-Black-Scholes-Richardson) pricing algorithm described in Broadie and Detemple (1996) with several modifications (see Appendix C). For European options, BBSR implied volatilities converge to Black-Scholes implied volatilities as the number of time steps increases. In addition, the BBSR algorithm incorporates the effect of the early exercise premium for American options.

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[4] When the cost-of-carry model holds, $\text{Fut}_t = X_t e^{(r-\delta)(T-t)}$ so that the futures price (Fut_t) combines the effects of the spot price (X_t), riskless rate (r), and dividend yield (δ) on the call option price. The difference between the riskless rate and the dividend yield ($r-\delta$) is the cost-of-carry.

IV. Nonparametric pricing of MVCCs

In this section, I contrast the characteristics of a lognormal and nonparametric model for MVCC pricing, since the lognormal model is the benchmark model for MVCC pricing. I construct a lognormal and a nonparametric euro-dollar and yen-dollar dependence function using historical exchange return data. I also nonparametrically estimate euro-dollar and yen-dollar monthly risk-neutral return densities using futures option data. I then estimate the euro-dollar and yen-dollar joint risk-neutral density by combining the marginals and the dependence functions. I compare the characteristics of these estimated densities and their implications for pricing euro-yen cross-rate options.

IV.a. Data

I collect monthly exchange rates for the euro-dollar (EUR-USD) and yen-dollar (JPY-USD) over the period from October 1991 to September 2001. For the euro-dollar, I use the WM/Reuters closing rate after 1999, and the FTSE synthetic rate prior to 1999. For the yen-dollar, I use the WM/Reuters closing rate after 1994 and the Federal Reserve Board's H.10 noon rate prior to 1994. I calculate the euro-yen rate by taking the product of the euro-dollar rate and the inverse of the yen-dollar rate. I then construct monthly exchange returns using logarithmic differences of the end-of-month exchange rate for each series.

Table 1 presents the sample moments of these time series, and Figure 1 displays the time-series of euro-dollar and yen-dollar rates. During this period, the euro depreciated –3.59% per year against the dollar, while the yen appreciated 1.10% per year against the dollar. The euro also depreciated relative to the yen. The euro-dollar returns were the least volatile (9.8%), while the yen-dollar and euro-yen have volatilities of 12.4% and 12.8% respectively. The euro-dollar log-returns did not deviate significantly from normality in terms of skewness and kurtosis. In contrast, the yen-dollar and euro-yen series showed positive skewness and excess kurtosis. The euro-dollar and yen-dollar series have a correlation coefficient of .35.

I also use euro-dollar and yen-dollar futures option data from the International Monetary Market division of the Chicago Mercantile Exchange (CME) and euro-yen futures option data from the

FINEX division of the New York Board of Trade (NYBOT).[5] Daily settlement prices for each futures option contract and corresponding maturity futures contract are extracted along with contract type, maturity, daily trading volume, and daily open interest. I collect a one-year sample from October 2001 through September 2002 using contracts with maturities between two weeks and one-year. I also obtain a daily term-structure of U.S., Japanese, and Euro interest rates using BBA LIBOR rates recorded by Datastream.

With this data, I calculate an implied volatility for each option using the modified BBSR algorithm presented in Appendix C. I then delete all euro-dollar and yen-dollar contracts that have no trading volume. I remove futures options with unreasonably high or low implied volatilities (less than 2% or greater than 40%).

Table 2 describes the characteristics of futures option data used in the paper. There are 12,425 euro-dollar contracts, 12,206 yen-dollar contracts, and 13,100 euro-yen contracts that satisfy the screening criteria. The average implied volatility across euro-dollar contracts is 10.8% with a range of 7.6% to 15.0%, and the average across euro-dollar contracts is 10.9% with a range of 7.4% to 16.0%. For euro-dollar contracts, the proportional moneyness (log ratio of strike to futures price) covers -20.0% to 28.5%, while the yen-dollar covers –24.4% to 22.3%. The average maturity for these contracts is about three months.

The euro-yen contracts are significantly less liquid than the other two contracts. The third panel of Table 2 shows that these contracts have proportional moneyness ranging from –6.4% to 8.5%. Average euro-yen futures option trading volume and open interest are less than 1 contract per day.

IV.b. Estimation of the objective dependence function

I estimate the objective dependence function for monthly euro-dollar and yen-dollar returns using the methodology described in Section III.a. For comparison, I first estimate a lognormal dependence function by taking the ratio of a lognormal joint density and lognormal marginals. Parameters are determined by the sample moments of the exchange return series. This dependence function reflects the parametric restrictions imposed by a bivariate geometric Brownian motion.

[5] While a euro-yen futures option contract is listed on the Chicago Mercantile Exchange, it has not been traded for several years and no settlement prices are reported.

In Figure 2, I graph the lognormal marginal densities, and the greater volatility of the yen-dollar series is evident. Figure 3 plots the contours of the estimated lognormal joint density. This specification generates approximately elliptical contours. The dependence function in Figure 4 is also nearly symmetrical and is highest at the upper right and lower left.

I next estimate the nonparametric objective dependence function. Figure 5 graphs the nonparametric marginals. While the euro-dollar density function appears approximately lognormal, the yen-dollar density is positively skewed and has "fat-tails." I plot the nonparametric joint density function in Figure 6. The large yen returns in March 1995, October 1998, and August 1995 are present as modes, as is the large euro return in December 2000. In Figure 7, I graph the nonparametric dependence function. The positive dependence between euro and yen returns appears weaker than in the lognormal density function, since less of the dependence function weight is in the first or third quadrant.

IV.c. Estimation of marginal risk-neutral densities

Using the methodology described in section III.b., I nonparametrically estimate the euro-dollar and yen-dollar marginal risk-neutral densities. I evaluate each nonparametric implied volatility function for a one-month horizon (T-t=1/12) at the average exchange rate level from October 2001-September 2002 (EUR-USD = $.9193, JPY-USD = $.007983).

In Figure 8, I graph the one-month average implied volatility functions against proportional moneyness. Both functions exhibit a volatility smile. In other words, volatility is an increasing function of positive and negative moneyness. The implied volatility for positive moneyness is slightly higher on average than for negative moneyness.

I graph the corresponding one-month average risk-neutral densities in Figure 9. Prior to taking the second derivative in equation (17), I discretize the implied volatility function at 299 equally spaced points over the range from –20% to 20% and then kernel-smooth the discretized function. This additional smoothing step significantly improves the behavior of the estimated risk-neutral density function.

Panel A of Table 3 reports the moments of these densities. While these densities appear lognormal, both densities have positive skewness (EUR-USD = .34 and JPY-USD=.35) and excess kurtosis (EUR-USD = 3.8 and JPY-USD=4.1).

IV.c. Estimation of joint risk-neutral densities

To construct the joint risk-neutral density function, I combine the nonparametric marginals with either a lognormal or nonparametric dependence function. Figure 10 presents a contour plot of the risk-neutral density obtained using the lognormal dependence function. The joint density is unimodal with a density centered around the 45 degree line, reflecting positive correlation. The contour lines are not exactly elliptical, since the positive skewness of the marginals expands the density in the first quadrant.

In Figure 11, I graph the risk-neutral density obtained using the nonparametric dependence function. This density function has several modes, although the largest mode is near zero. For the nonparametric density function, the largest positive euro-dollar returns occur at small negative yen-dollar returns. In addition, the largest positive yen-dollar returns occur at euro-dollar returns near zero. Panel B of Table 3 reports that the correlation using the nonparametric dependence function (.26) is lower than the correlation using the lognormal dependence function (.37).

IV.d. Estimation of euro-yen option prices

The risk-neutral density of euro-yen rates can be derived from the joint risk-neutral density of euro-dollar and yen-dollar rates. The euro-yen rate (Z_T) is the ratio of the euro-dollar (X_T) and yen-dollar rates (Y_T). The density of a ratio of two random variables in terms of the joint density is given by:

$$(18) \qquad f_3^*(Z_T) = \int Y_T \, f_{X_T,Y_T}^*(Z_T Y_T, Y_T) dY_T$$

I apply equation (18) to calculate the euro-yen risk-neutral density using both of the previously estimated joint risk-neutral densities. These density functions are shown in Figure 12, and the moments of the densities are reported in Panel C of Table 3. The volatility, skewness, and kurtosis of the nonparametric density are larger than these moments using a lognormal dependence function.

From the Japanese perspective, the euro-yen risk-neutral density prices all European-style contingent claims with a payoff that depends on the euro-yen rate. The relevant riskless interest rate (r) for discounting is the Japanese interest rate, and the income rate (δ) is equal to the euro riskless rate. So, for a claim with a payoff function of $g(Z_T)$, the price in yen (D_t) is given by:

(19) $$D_t = e^{-(r-d)(T-t)} \int g(Z_T) f_3^*(Z_T) dZ_T$$

I use equation (19) and the two euro-yen risk-neutral densities to create a cross-section of one-month out-of-the-money euro-yen call and put option prices. Then, I numerically invert the Black-Scholes formula to obtain an implied volatility for each option. The cross-section of implied volatilities defines the euro-yen implied volatility function associated with each density.

Using the methodology from section III.b., I nonparametrically estimate the "actual" euro-yen implied volatility function using euro-yen option data from the New York Board of Trade. I do not use this implied volatility function to estimate the risk-neutral density, since these options cover a relatively narrow moneyness range.

In Figure 13, I graph the actual, nonparametric dependence, and lognormal dependence implied volatility functions. Each function is graphed over the range of –5% to 5% proportional moneyness. This range includes approximately 99% of the sample of euro-yen contracts from the NYBOT.

This figure shows that the actual at-the-money implied volatility (9.6%) is lower than the fitted values using the lognormal dependence function (12.6%) or the nonparametric dependence function (12.8%). The actual implied volatility curve is somewhat higher at the largest moneyness (11.5%) than at the smallest moneyness (11.0%). The other implied volatility functions are lower at the highest moneyness than at the smallest moneyness.

A likely explanation for the difference in implied volatility function levels is variation in the estimates of correlation between euro-dollar and yen-dollar rates. Higher levels of correlation lead to lower implied volatilities. This is one reason that the lognormal volatility function (correlation of .37) lies below the nonparametric volatility function (correlation of .26). Apparently, the market estimate of the correlation is even greater than either of these values.[6]

The actual implied volatility function also exhibits a volatility smile. The curve is approximately flat from a moneyness of –3% to 3%, and it is steeply upward sloping outside of this range. The lognormal dependence implied volatility function has a very shallow smile. The nonparametric dependence implied volatility function has a steep smile, but it is centered around 2% moneyness.

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[6] The return volatility of the log euro-yen rate is equal to the sum of the return volatility of euro-dollar and yen-dollar rates <u>minus</u> twice their correlation times the product of their standard deviations. For an empirical analysis of implied correlation, see for example, Walter and Lopez (2000).

In Figure 14, I graph each implied volatility function relative to the at-the-money implied volatility. This re-scaling eliminates the level differences and emphasizes contrasts in shape. Near-the-money, the lognormal dependence is very close to the actual, since both functions are nearly flat. Away-from-the-money, the lognormal dependence fit is much worse.

I discretize the interval and calculate the root-mean-squared-error (RMSE) between the fitted and actual relative implied volatility functions. The nonparametric estimate fits the actual function more closely (RMSE = .055) than the lognormal estimate (RMSE = .059). The RMSE of a flat implied volatility function is .065.

If the euro-dollar and yen-dollar followed a bivariate geometric Brownian motion, then the euro-yen risk-neutral density would be lognormal. The implied volatility function would be flat and the relative implied volatility function would be 1. The implied volatility function would also be flat for euro-yen options priced using the Garman-Kohlhagen (1983) model, which assumes that the euro-yen has a lognormal risk-neutral density. However, I find that a flat relative implied volatility function provides an inferior fit to either of the two alternatives.

V. Conclusion

In this paper, I develop a nonparametric technique for multivariate contingent claim valuation with several appealing features. First, this technique does not impose restrictions on the asset return process or on the functional form of the risk-neutral density. Second, fitted MVCC prices are consistent with the market prices of existing options. Third, the technique does not require data on traded multivariate claims, since the risk-neutral dependence function is estimated using historical returns data. This technique avoids the pricing errors that result from applying a lognormal model to markets with a volatility smile or skew.

I compare the implied volatility smile for euro-yen options traded on the NYBOT to estimated implied volatility curves using nonparametric and lognormal dependence functions. I find that the nonparametric model provides the most accurate prices. I also show that the nonparametric risk-neutral density exhibits asymmetries and multiple modes not present when a lognormal dependence function is used. Thus, some of the key sources of inaccuracy in the lognormal pricing model are due to the imposition of a lognormal dependence function.

Appendix A

Method of inversion

Theorem ("Method of inversion", see Nelsen (1998), Corollary 2.3.7): Suppose that F(x,y) is a joint density function with marginal quantile functions (inverse marginal cumulative density functions) given by $F_1^{-1}(u)$ and $F_2^{-1}(v)$ where u and v are cumulative probabilities. Then, the copula of F(x,y) is given by $C(u,v) = F(F_1^{-1}(u), F_2^{-1}(v))$.

The copula for any known multivariate density function can be obtained using the "method of inversion." This technique factors out the effects of the marginal densities on the dependence relation by substituting the arguments of the joint density with the quantile functions.

Invariance

Theorem ("Invariance", see Nelsen (1998), Theorem 2.4.3): Suppose that C(u,v) is the copula for X and Y. C(u,v) is also the copula for a(X) and b(Y), where these are both increasing monotonic functions.

An additional significant property of copulas relates to their invariance to the scaling of X and Y. Specifically, the copula of two random variables X and Y is the same as the copula for a(X) and b(Y), where a and b are monotonic increasing functions. For example, linear and logarithmic functions of X and Y have the same copula as X and Y.

Appendix B

Lemma: $M(x,y) = \exp[z_1(x) + z_2(y)]$ satisfies the condition that $BM(x,y)/(M(x)M(y))=1$.

Notice that the definitions of the pricing kernel projections are $M(x)=E[M(x,y)|x]$ and $M(y) =E[M(x,y)|y]$. Now, for this pricing kernel specification, $E[M(x,y)|x]=\exp[z_1(x)]E[z_2(y)]$ and $E[M(x,y)|y]=\exp[z_2(y)]E[z_1(x)]$. In addition, the unconditional expectation of the pricing kernel equals the riskless bond price, $E[h_1(x)]E[h_2(y)]=B$. So,

$$\begin{aligned} M(x)M(y) &= E[M(x,y)|x]E[M(x,y)|y] \\ &= \exp[z_1(x)]\exp[z_2(y)]E[z_1(x)]E[z_2(y)] \\ &= B\exp[z_1(x)]\exp[z_2(y)] \\ &= B\exp[z_1(x)+z_2(y)] \\ &= BM(x,y) \end{aligned}$$

Thus,

$$BM(x,y)/(M(x)M(y)) = BM(x,y)/BM(x,y) = 1.$$

Appendix C

The BBSR (Binomial-Black-Scholes-Richardson) algorithm (Broadie and Detemple (1996), pp. 1243-1245) is an enhancement to the Cox, Ross, Rubinstein (1979) binomial tree approach for pricing American or European options on the underlying asset (e.g. options on an equity index). Broadie and Detemple (1996) find the BBSR algorithm offers significant improvements in pricing accuracy over alternative algorithms.

The BBSR algorithm modifies the CRR algorithm in two ways. First, the option price tree is modified so that the continuation values at the nodes just prior to expiration are replaced with values from the Black-Scholes formula. This is referred to as the BBS algorithm (Binomial-Black-Scholes). Second, the option price is calculated using the desired number of time steps and half the number of time steps using the BBS algorithm. The BBSR (Binomial-Black-Scholes-Richardson) price is calculated using the Richardson extrapolation, which sets the BBSR price equal to twice the BBS price estimated using the desired number of time-steps minus the BBS price using half the number of time-steps. In this implementation of the algorithm, the number of time-steps is set to 100, which offers maximum precision and is the maximum number tested in Broadie and Detemple (1996).

The BBSR algorithm is designed to price options on the underlying asset, so several minor adjustments must be made to the BBSR algorithm so that it is appropriate for pricing futures options. To price futures options, the up and down parameters must reflect futures returns (u_F, d_F) rather than the underlying asset returns (u and d). The appropriate formulas are $u_F = u*e^{-c\Delta}$ and $d_F = d*e^{-c\Delta}$, where Δ is the amount of time represented by one time-step, and u and d are the parameters from the BBSR algorithm.

These parameters are derived as follows. The cost-of-carry model states that $Fut_t = e^{c(T-t)}X_t$, where Fut_t is the futures price, X_t is the spot price, T-t is the number of years until the futures contract expires, and c is the continuously compounded annual cost of carry. For equity index options, the cost of carry equals the difference between the continuously compounded annual riskless rate and the dividend yield.

$$u_F = Fut_u / Fut_t = e^{c(T-t-\Delta)}X_u / e^{c(T-t)}X_t = u*e^{-c\Delta}$$
$$d_F = Fut_d / Fut_t = e^{c(T-t-\Delta)}X_d / e^{c(T-t)}X_t = d*e^{-c\Delta}$$

To price futures options, we can also write the risk-neutral probabilities (p, 1-p) in terms of the futures returns rather than the underlying returns to obtain p_F and 1-p_F. The appropriate formula is $p_F = (1-d_F)/(u_F - d_F)$. This parameter is derived as follows. The BBSR algorithm defines p as $p = (e^{c\Delta} - d) / (u - d)$. Using the definitions above, $u=u_F e^{c\Delta}$ and $d=d_F e^{c\Delta}$. Substituting into p to define p_F, we have $p_F = (1-d_F)/(u_F - d_F)$. The cost-of-carry (c) is calculated as the logarithm of the ratio between the futures and spot price divided by the time until expiration.

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Table 1 - Exchange return data

Monthly logarithmic returns, 1991.10 - 2001.09

	EUR-USD	JPY-USD	EUR-JPY
N	120	120	120
Mean (annualized)	-3.59%	1.10%	-4.63%
Std. dev. (annualized)	9.82%	12.41%	12.83%
Skewness	-0.03	0.84	-0.69
Kurtosis	3.02	5.89	4.43

*Return correlation: (EUR-USD,JPY-USD) = 0.35

Table 2 - Futures option data

Euro futures options (Chicago Mercantile Exchange)
Daily settlements, October 2001 - September 2002

	N	Mean	Std. dev.	Minimum	Maximum
Implied volatility (annual %)	12425	10.79%	1.43%	7.64%	15.03%
Proportional moneyness (%)	12425	-0.30%	5.48%	-19.98%	28.52%
Time until expiration (years)	12425	0.24	0.19	0.04	0.99
Trading volume (contracts per day)	12425	47.09	112.01	1	3102
Open interest (contracts per day)	12425	520.89	598.88	0	4834
Option price (in dollars)	12425	$0.0100	$0.0119	$0.0001	$0.1370

Yen futures options (Chicago Mercantile Exchange)
Daily settlements, October 2001 - September 2002

	N	Mean	Std. dev.	Minimum	Maximum
Implied volatility (annual %)	12206	10.85%	1.47%	7.43%	15.95%
Proportional moneyness (%)	12206	0.33%	5.20%	-24.44%	22.37%
Time until expiration (years)	12206	0.27	0.21	0.04	0.99
Trading volume (contracts per day)	12206	47.96	328.25	1	19942
Open interest (contracts per day)	12206	944.48	2682.24	0	21004
Option price (in dollars)	12206	$0.000112	$0.000143	$0.000001	$0.002125

Euro-yen futures options (New York Board of Trade)
Daily settlements, October 2001 - September 2002

	N	Mean	Std. dev.	Minimum	Maximum
Implied volatility (annual %)	13100	9.97%	1.69%	5.56%	15.06%
Proportional moneyness (%)	13100	0.15%	2.47%	-6.43%	8.49%
Time until expiration (years)	13100	0.18	0.10	0.04	0.45
Trading volume (contracts per day)	13100	0.01	0.57	0	50
Open interest (contracts per day)	13100	0.74	5.30	0	50
Option price (in yen)	13100	2.11	1.52	0.01	9.42

Table 3 - Risk-neutral densities

Panel A: Characteristics of the marginal risk-neutral densities

	Marginal densities	
	EUR-USD	JPY-USD
Mean	1.97%	1.98%
Standard deviation	11.31%	11.16%
Skewness	0.34	0.35
Kurtosis	3.81	4.07

Panel B: Characteristics of the joint risk-neutral density

	Dependence function	
	Lognormal	Nonparametric
Correlation	0.3705	0.2634
Covariance	0.0004	0.0003

Panel C: Characteristics of JPY-EUR risk-neutral density

	Dependence function	
	Lognormal	Nonparametric
Mean	-0.68%	0.64%
Standard deviation	12.65%	13.50%
Skewness	0.10	0.26
Kurtosis	3.21	4.41

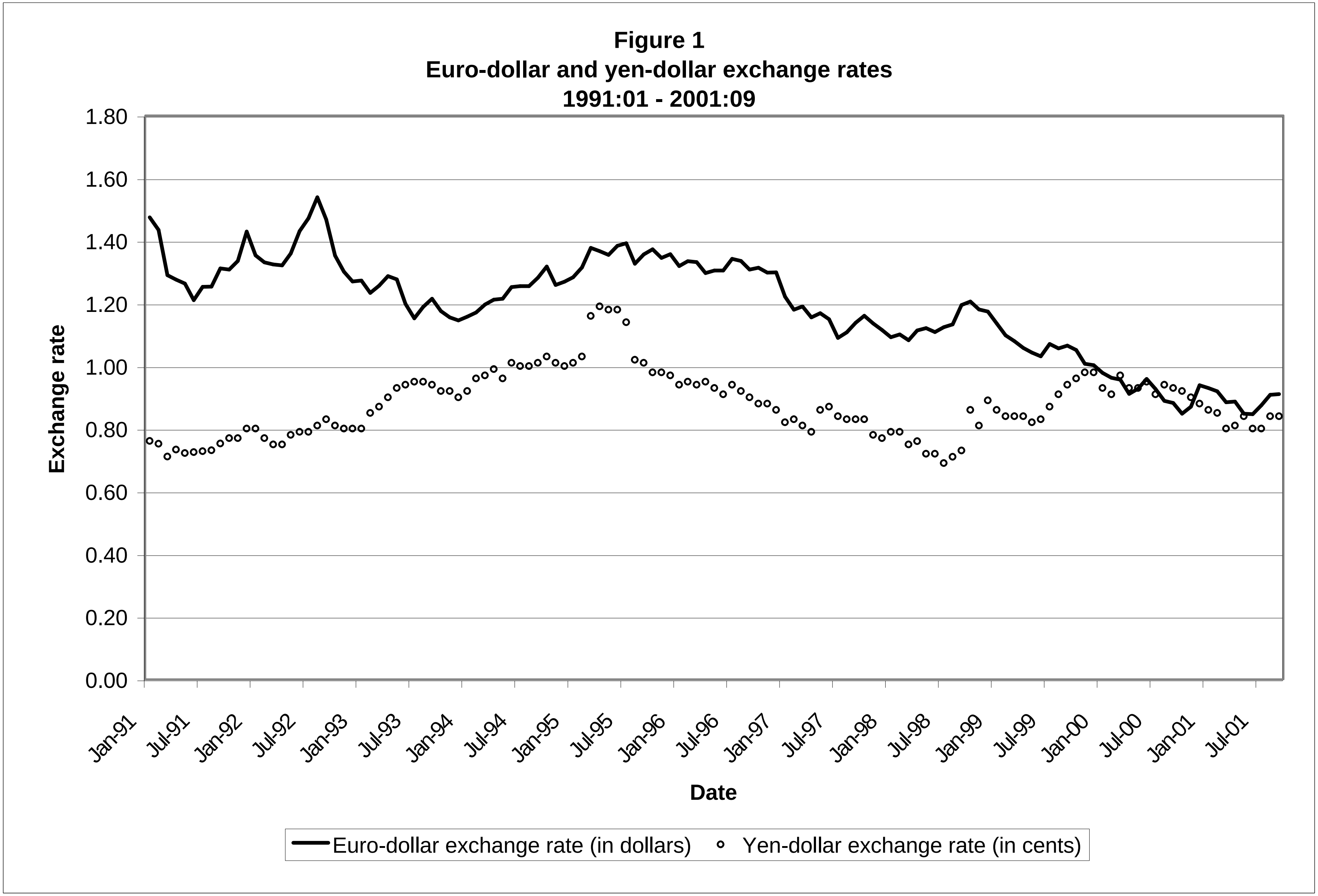


Figure 1
Euro-dollar and yen-dollar exchange rates
1991:01 - 2001:09
Exchange rate
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0.00
Jan-91
Jul-91
Jan-92
Jul-92
Jan-93
Jul-93
Jan-94
Jul-94
Jan-95
Jul-95
Jan-96
Jul-96
Jan-97
Jul-97
Jan-98
Jul-98
Jan-99
Jul-99
Jan-00
Jul-00
Jan-01
Jul-01
Date
Euro-dollar exchange rate (in dollars)
Yen-dollar exchange rate (in cents)

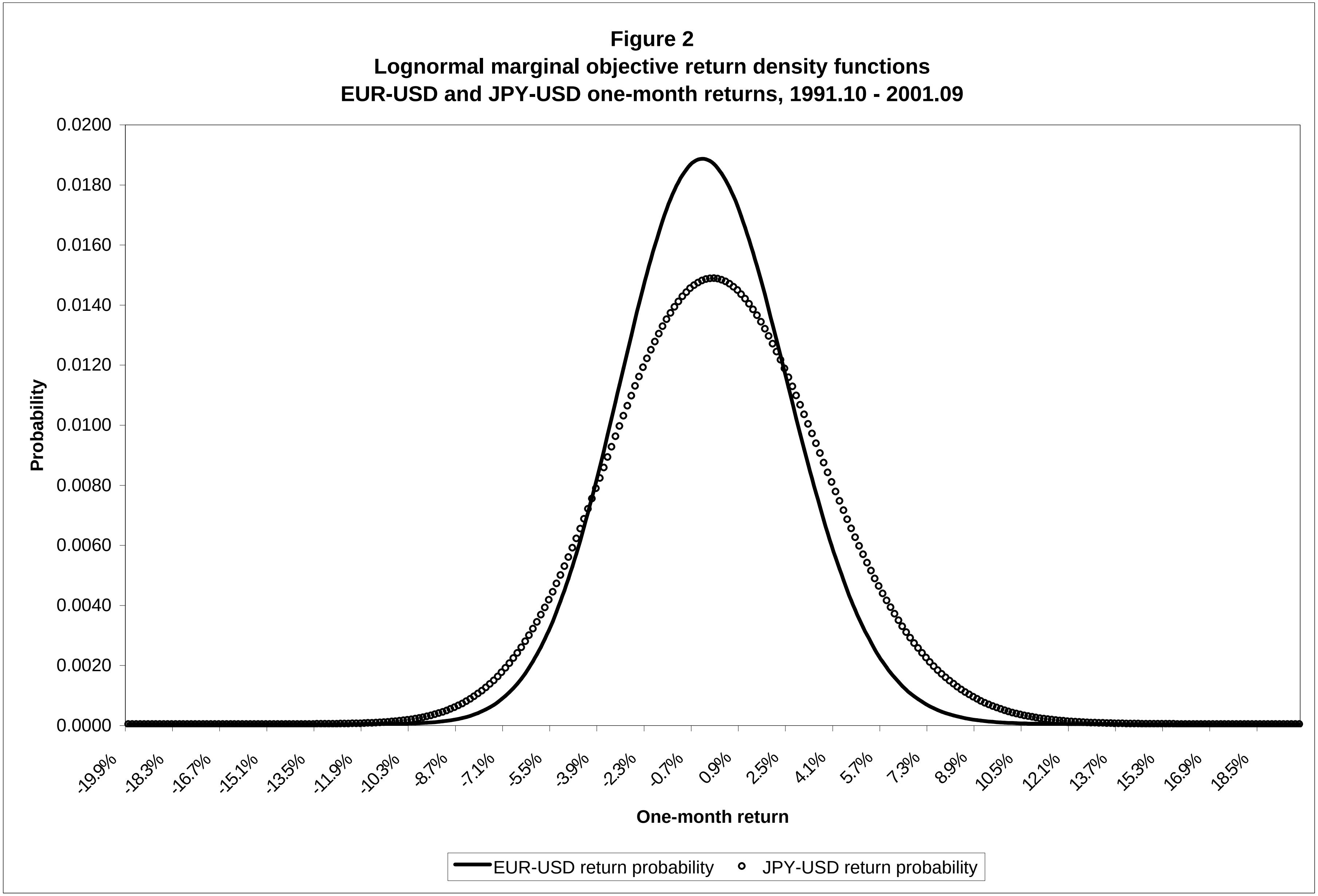


Figure 2
Lognormal marginal objective return density functions
EUR-USD and JPY-USD one-month returns, 1991.10 - 2001.09
0.0200
0.0180
0.0160
0.0140
0.0120
0.0100
0.0080
0.0060
0.0040
0.0020
0.0000
Probability
-19.9%
-18.3%
-16.7%
-15.1%
-13.5%
-11.9%
-10.3%
-8.7%
-7.1%
-5.5%
-3.9%
-2.3%
-0.7%
0.9%
2.5%
4.1%
5.7%
7.3%
8.9%
10.5%
12.1%
13.7%
15.3%
16.9%
18.5%
One-month return
EUR-USD return probability
JPY-USD return probability

Figure 3
Lognormal joint objective return density function
EUR–USD and JPY–USD one–month returns, 1991.10 – 2001.09
Contour plot

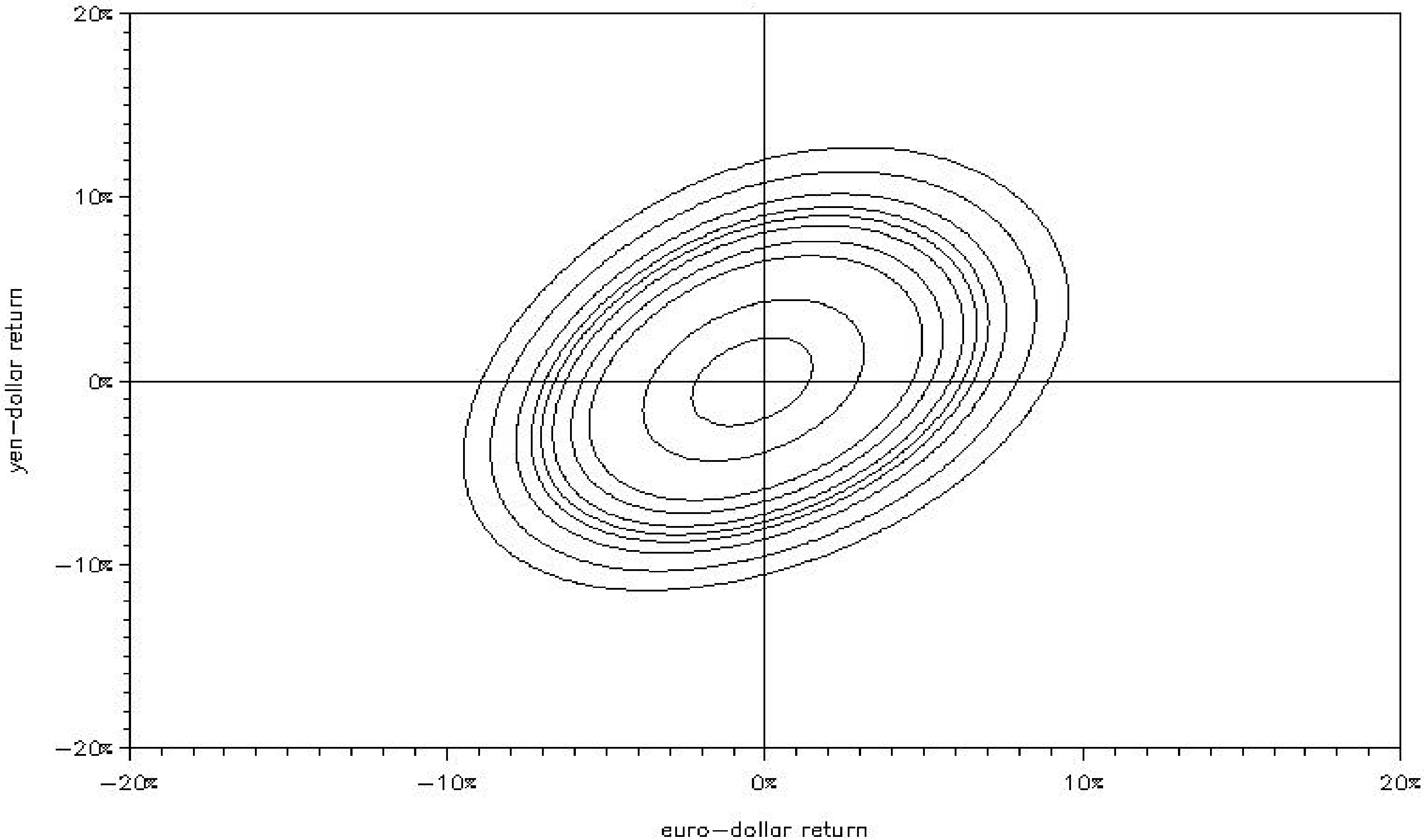


Figure 4
Lognormal dependence function
EUR–USD and JPY–USD one-month returns, 1991.10 – 2001.09
Contour plot

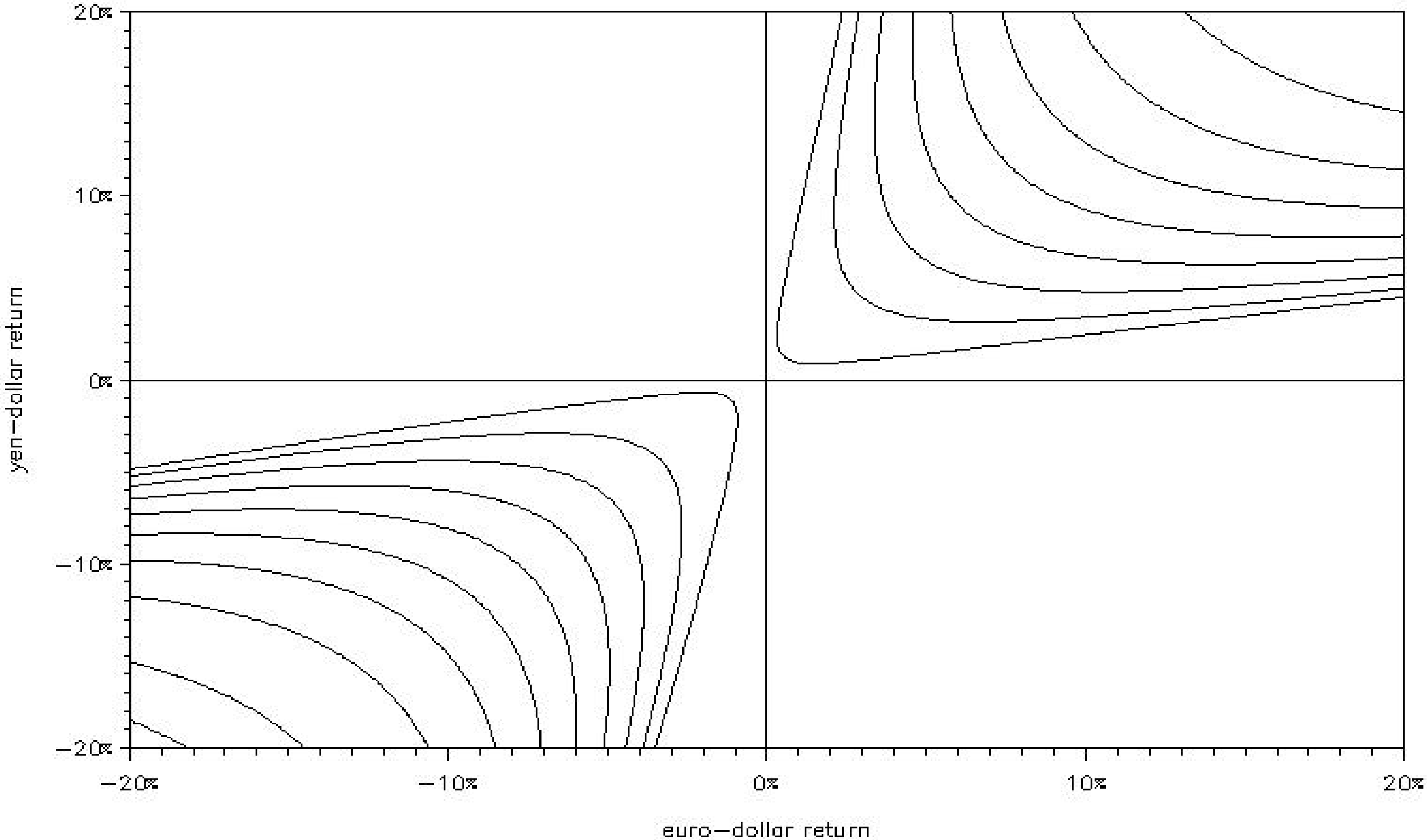


Figure 5
Nonparametric marginal objective return density functions
EUR-USD and JPY-USD one-month returns, 1991.10 - 2001.09

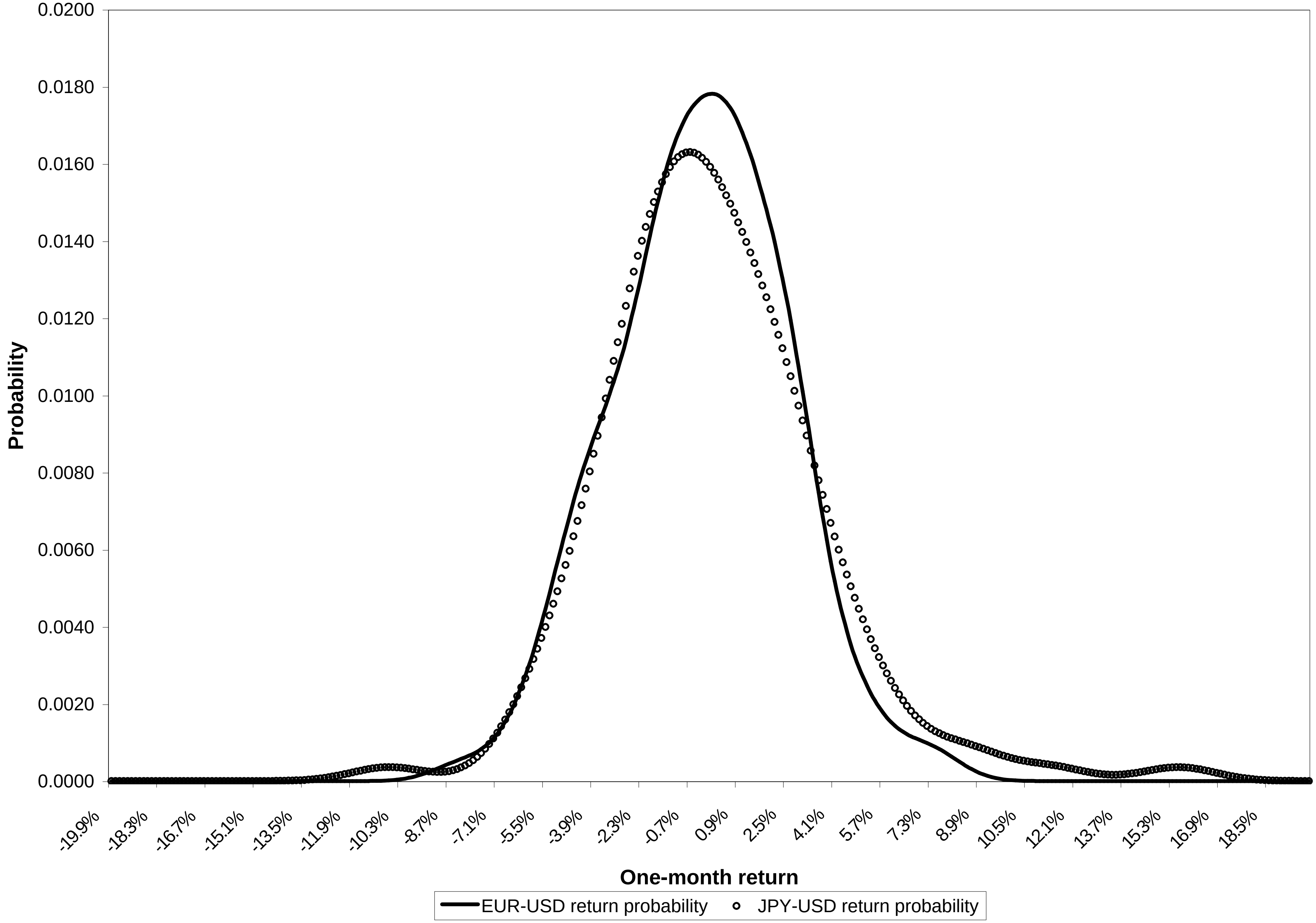


Figure 6
Nonparametric objective return density function
EUR–USD and JPY–USD one-month returns, 1991.10 – 2001.09
Contour plot

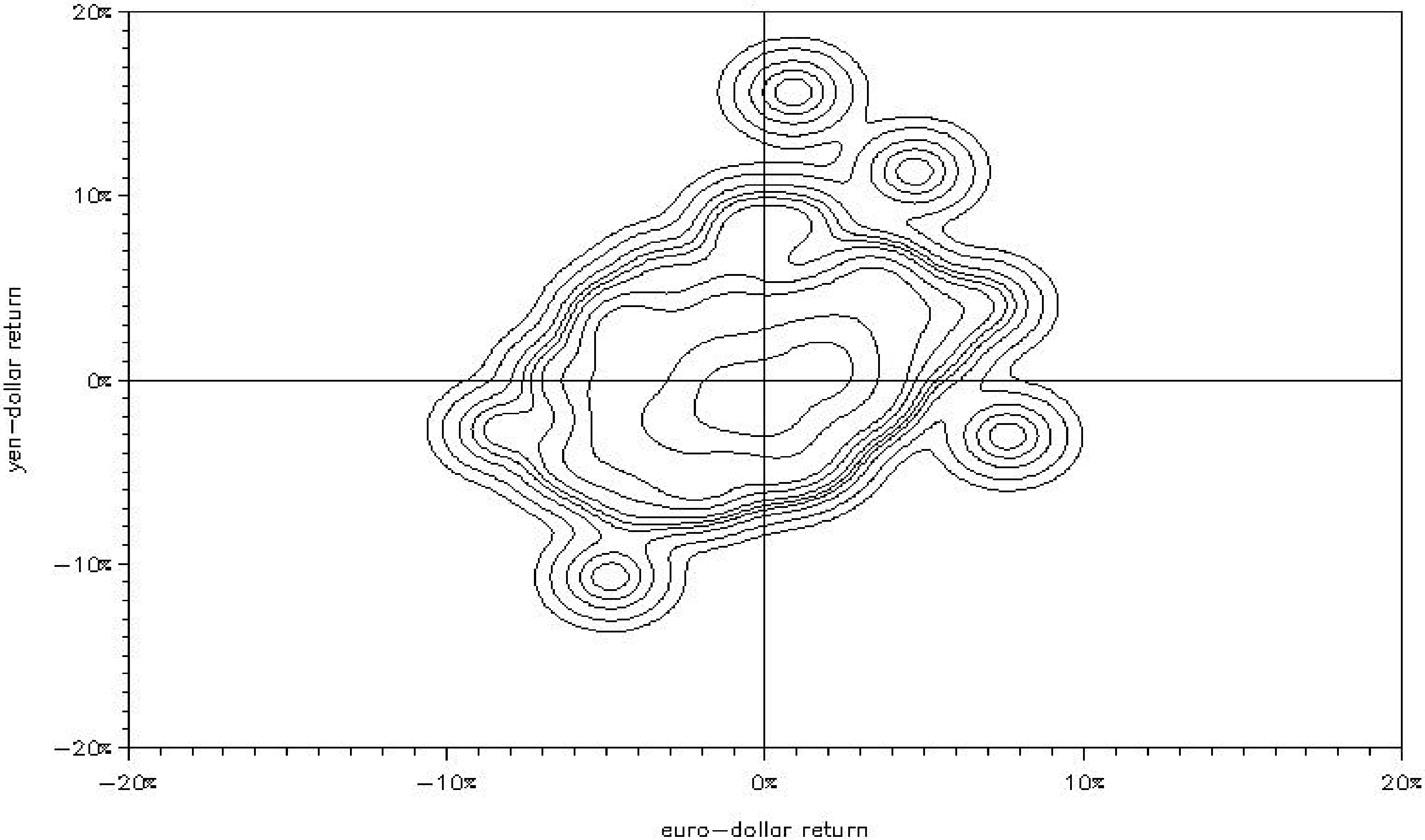


Figure 7
Nonparametric dependence function
EUR–USD and JPY–USD one–month returns, 1991.10 – 2001.09
Contour plot

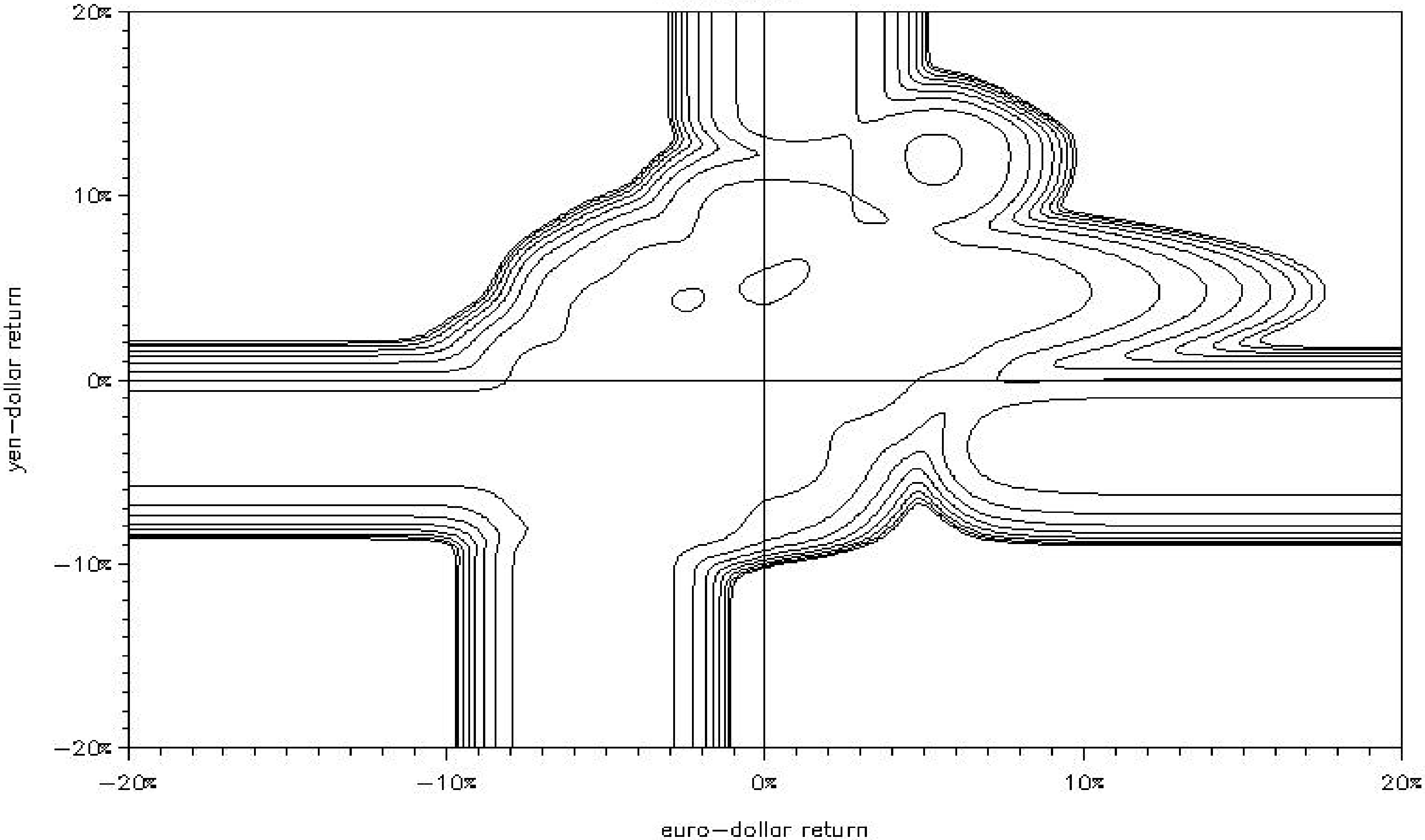


Figure 8
Nonparametric implied volatility functions
EUR-USD and JPY-USD one-month returns, 2001.10 - 2002.09

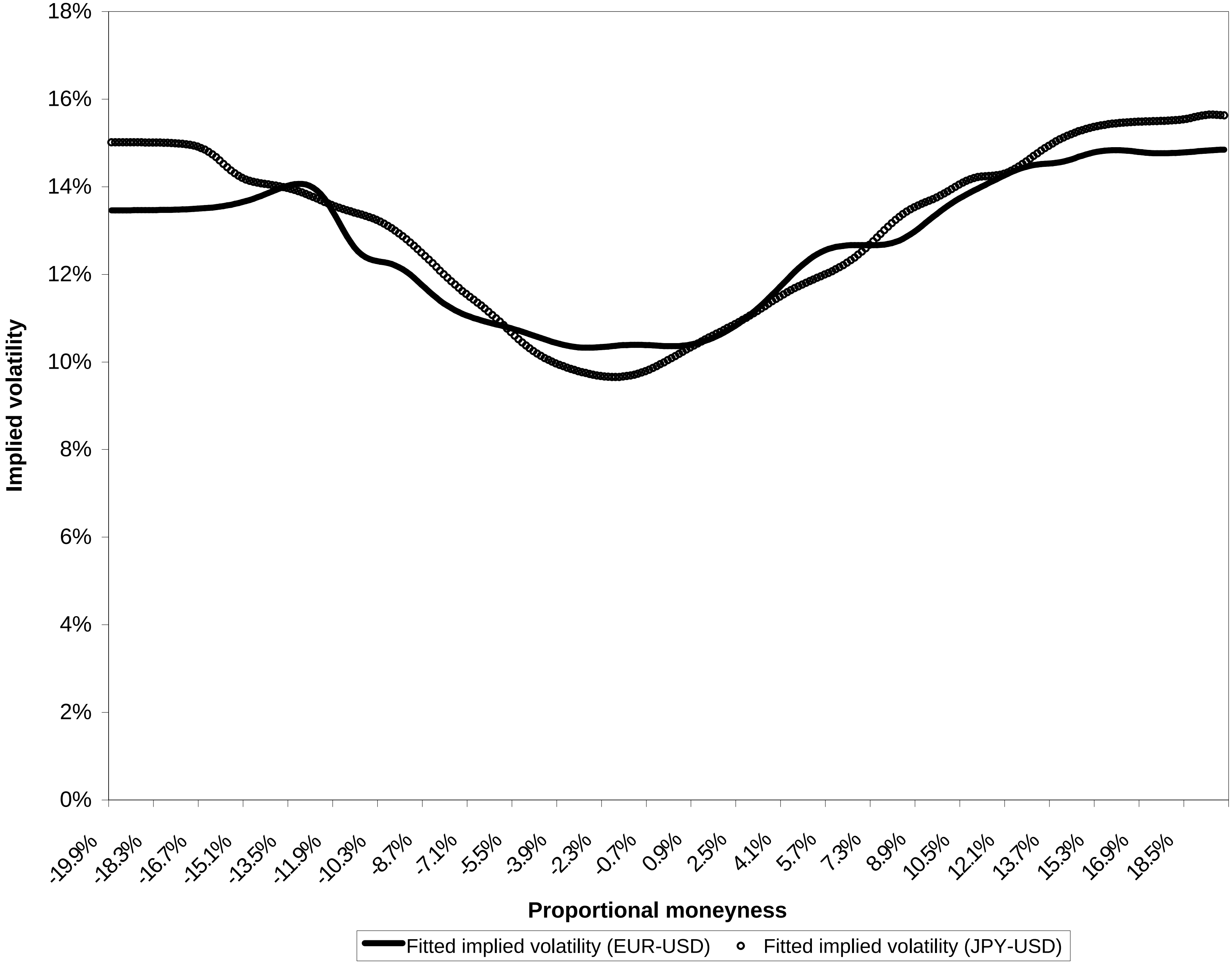


Figure 9
Nonparametric marginal risk-neutral return density functions
EUR-USD and JPY-USD one-month returns, 2001.10 - 2002.09

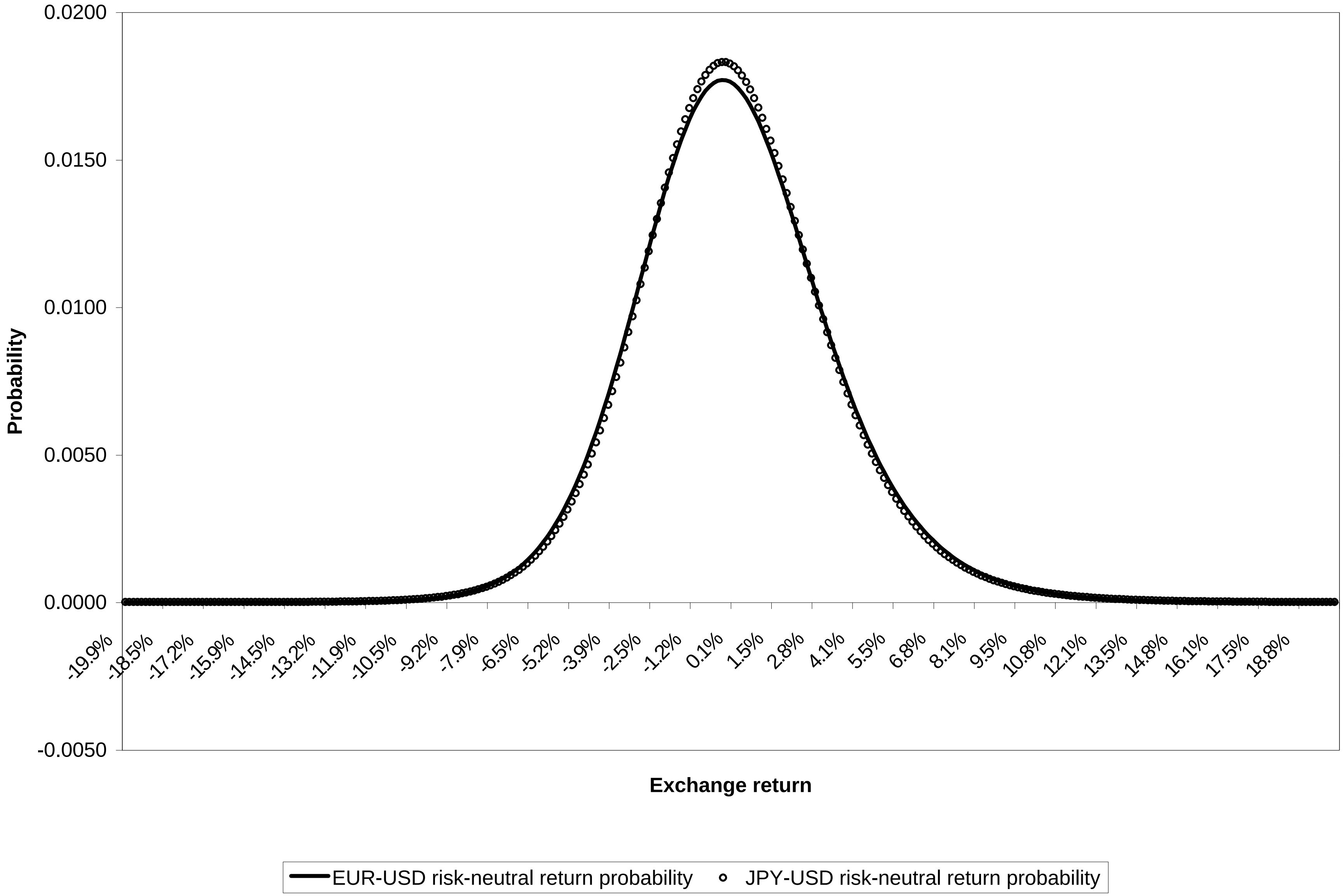


Figure 10
Lognormal joint risk-neutral density
EUR-USD and JPY-USD one-month returns
Contour plot

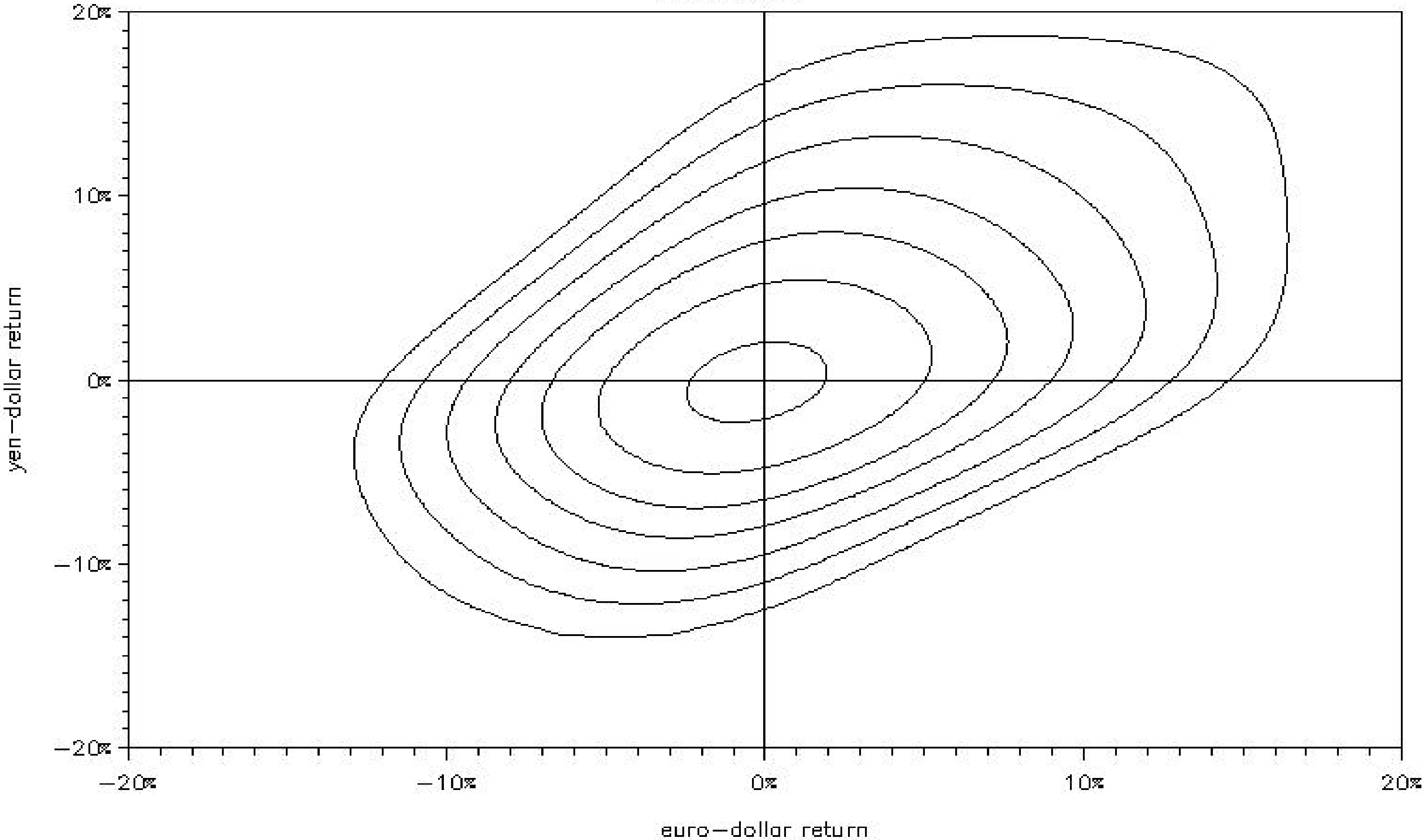


Figure 11
Nonparametric joint risk–neutral density
EUR–USD and JPY–USD one–month returns
Contour plot

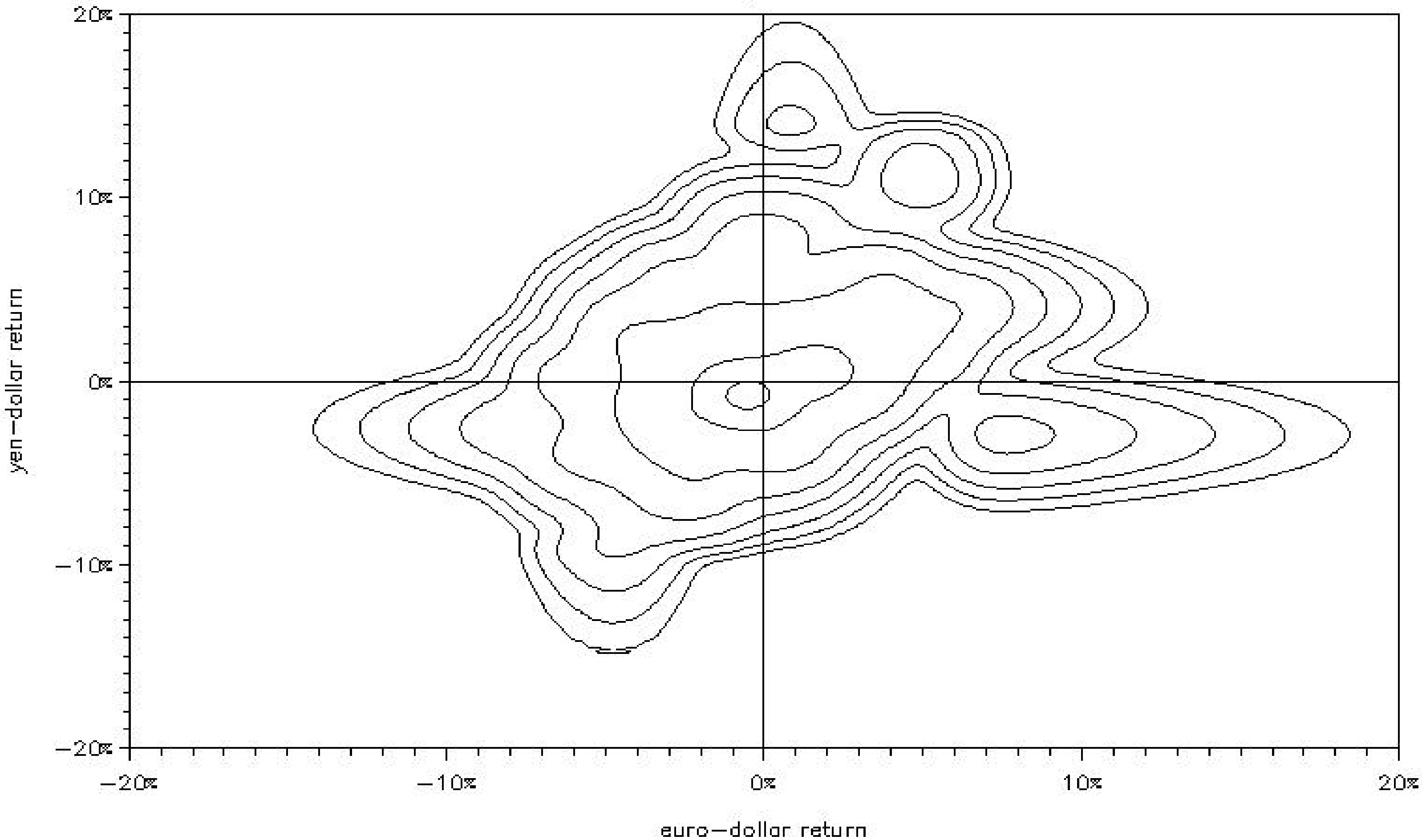


Figure 12
Nonparametric marginal risk-neutral return density functions
EUR-JPY one-month returns, 2001.10 - 2002.09

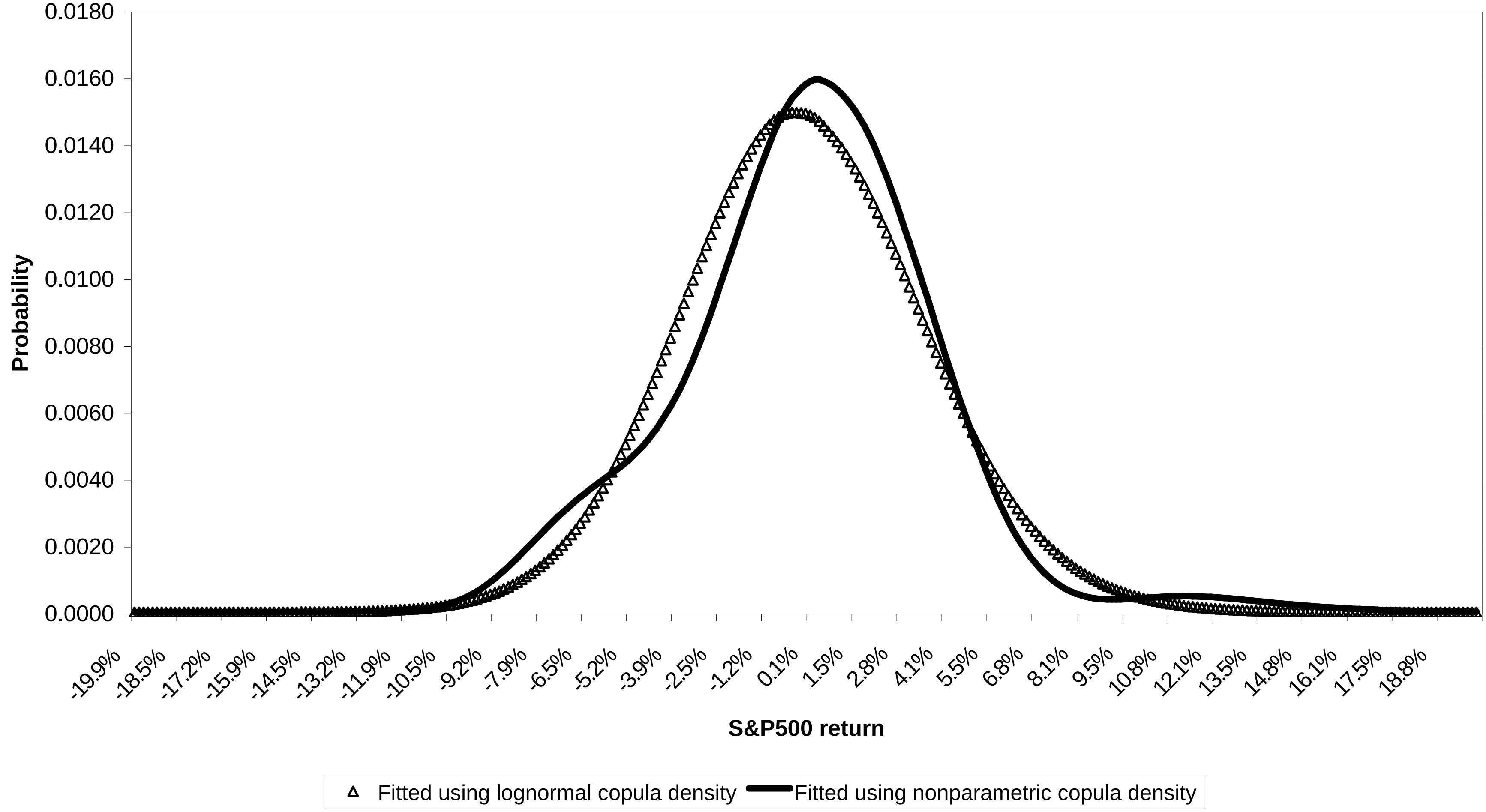
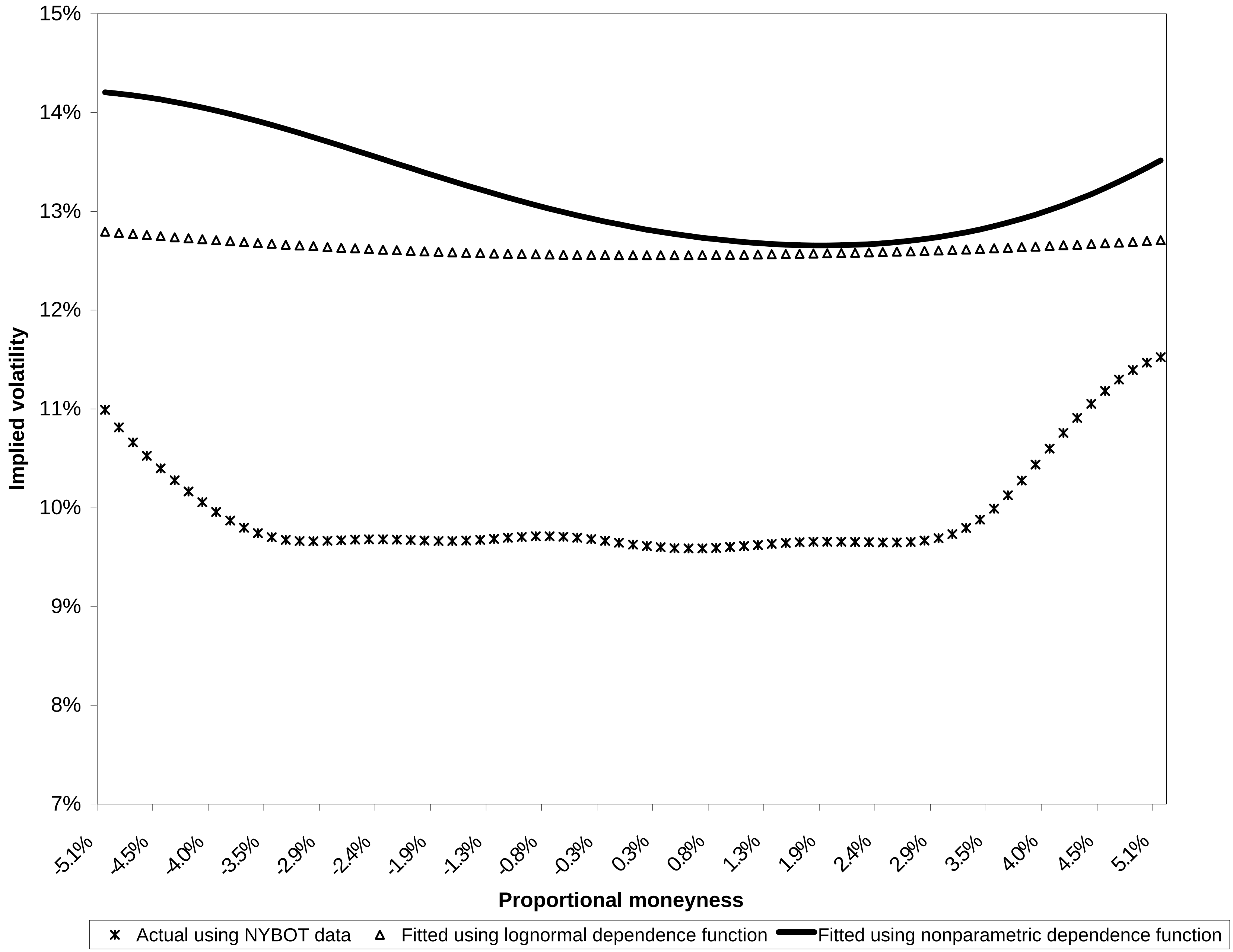


Figure 13
Implied volatility functions for EUR-JPY



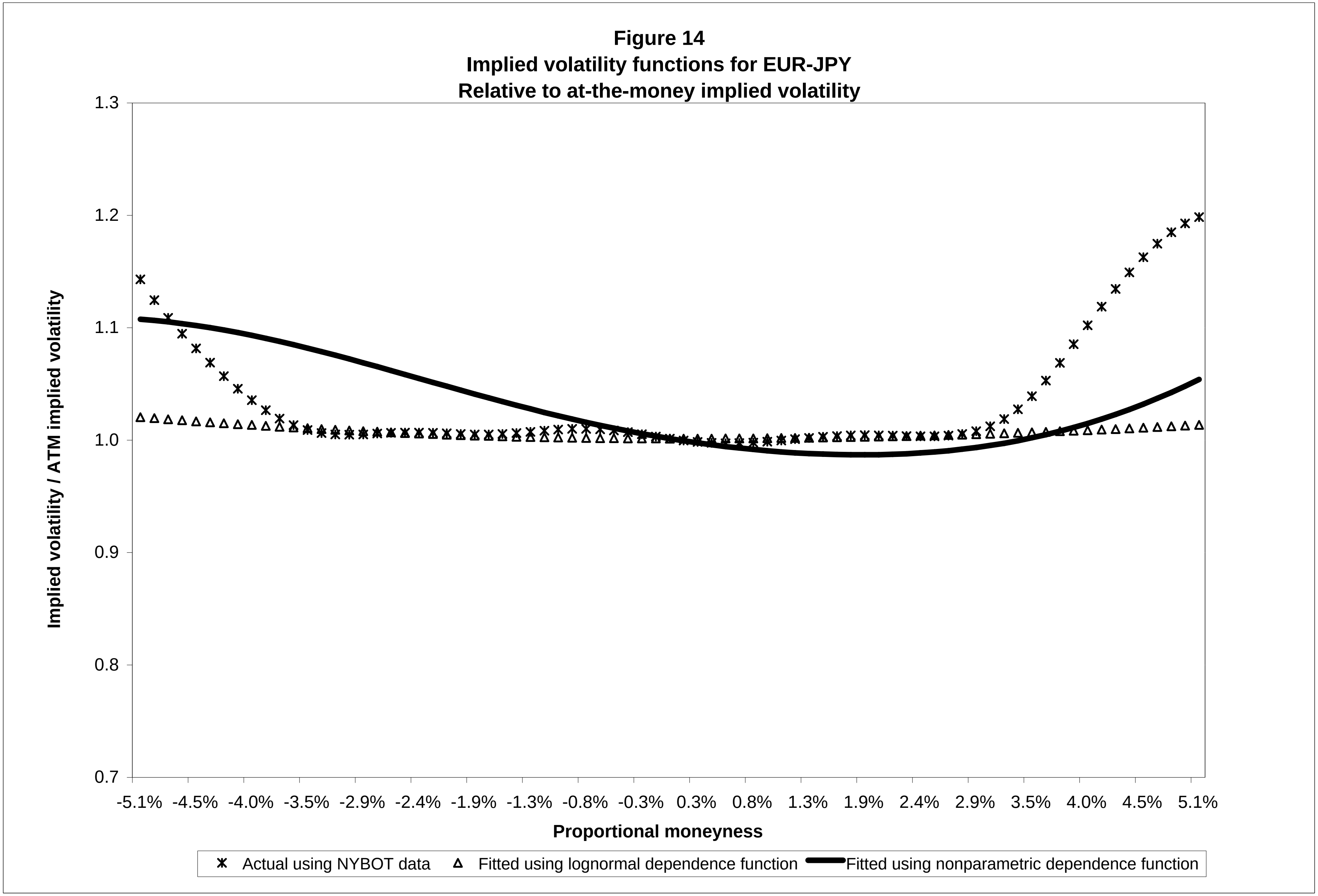


Figure 14
Implied volatility functions for EUR-JPY
Relative to at-the-money implied volatility
Implied volatility / ATM implied volatility
1.3
1.2
1.1
1.0
0.9
0.8
0.7
-5.1% -4.5% -4.0% -3.5% -2.9% -2.4% -1.9% -1.3% -0.8% -0.3% 0.3% 0.8% 1.3% 1.9% 2.4% 2.9% 3.5% 4.0% 4.5% 5.1%
Proportional moneyness
Actual using NYBOT data
Fitted using lognormal dependence function
Fitted using nonparametric dependence function