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Abstract

This paper examines the performance of forecast-based nonlinear Taylor rules in a class of simple microfunded models. The paper shows that even if the policy rule leads to a locally determinate (and stable) inflation target, there exist other learnable "global" equilibria such as cycles and sunspots. Moreover, under learning dynamics, the economy can fall into a liquidity trap. By contrast, more backward-looking and "active" Taylor rules guarantee that the unique learnable equilibrium is the inflation target. This result is robust to different specifications of the role of money, price stickiness, and the trading environment.

Key words: learnability, inflation targeting, simple feedback rules, endogenous fluctuations

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1 Introduction

In the monetary policy literature forecast-based Taylor rules are widely used and recommended for policy analysis[1]. However, recent contributes have shown that forecast-based policy rules might have important drawbacks.

First, they might lead to indeterminate equilibria, as shown by Woodford (2000) and Bullard and Mitra (2001). Second, Levin, Wieland and Williams (2001) show that their performance is not robust to uncertainty about the model of the economy. Third, they are not consistent with a conventional intertemporal loss function, as proved by Svensson (2001).

Moreover, Evans, Honkapohja and Marimon (2001), and Carlstrom and Fuest (2001a) show the existence of learnable sunspot equilibria if the inflation target equilibrium is indeterminate.

In general, it is a well known result that adopting Taylor rules might generate instability in the economic system. On one side, a nonlinear Taylor rule (consistent with a zero bound on the interest rate) implies the existence of two steady states, see Benhabib et al. (2001a). One is the inflation target and the other is a low inflation/deflation equilibrium. On the other side, Benhabib et al. (2001b) show that in a simple model where money enters in the production function, linear and nonlinear Taylor rules might lead to cycles or even chaotic behavior.

The purpose of this paper is to compare the performance of forecast-based and backward-looking Taylor rules in a class of simple monetary models. Given the existence of multiple equilibria under perfect foresight (steady states, cycles and sunspots), I verify their 'robustness' to adaptive learning.

In the course of the paper I focus on those policy rules that satisfy the 'Taylor Principle' and guarantee local determinacy of the inflation target steady state[2]. I then consider the possibility of *learnable global equilibria* and how the choice of the Taylor rule affects their existence or changes their stability.

In particular, I study the existence: a) of learnable cycles and sunspots around the *determinate* inflation target equilibrium and, b) of *learnable* liquidity traps implied by the zero bound condition on the policy rule - and so far considered a mere theoretical curiosity, i.e. in McCallum (2001).

I show that forecast-based Taylor rules have a de-stabilizing effect on the economy: they lead to learnable cycles and sunspots, even in the case where the inflation target equilibrium is locally unique and stable under learning. Moreover, the economy can converge to a liquidity trap that is stable under learning. In contrast, adopting a backward-looking Taylor rule stabilizes the economy. In fact, equilibrium cycles disappear, while sunspots equilibria and the liquidity trap become unstable under learning (i.e. not robust to expectational mistakes).

The paper is structured as follows. The second section quickly reviews the model and introduces the learning algorithm. The third and fourth sections analyze the stability under learning of steady states, cycles and sunspots under a forecast-based Taylor rule and discuss the main results

[1] See, among the others, Batini and Nelson (2001).

[2] I therefore restrict the analysis to the best-case scenario where local indeterminacy does not appear.

. The fifth section compares the performance of forecast-based and backward-looking Taylor rules. Finally, in the sixth section I consider some extensions.

2 The Model

I consider a simple model of the economy with flexible prices, a discrete time version of Benhabib et al. (2001a). The model has three main components.

The representative agent's problem. Agents maximize their utility from consumption and money balances under the following budget constraint:

$$\begin{aligned} & \max_{c_t, M_t^{np}, M_t^p B_t} \sum_{t=0}^{\infty} \beta^t U\left(c_t, \frac{M_t^{np}}{P_t}\right) \\ \text{sub } M_t + B_t + P_t c_t &= M_{t-1} + R_{t-1} B_{t-1} + P_t f\left(\frac{M_t^p}{P_t}\right) \end{aligned} \tag{1}$$

where $M_t = M_t^{np} + M_t^p$ is nominal money balance (for non-productive and productive purposes respectively), B_t is bonds, R_t is the return on bonds and $f(.)$ is the production function, which depends on real money balances and a fixed input $\overline{y}$. Finally, $U(.)$ and $f(.)$ satisfy all the standard conditions. I further assume that $U_{cm} > 0$, that is consumption and money balances are Edgeworth complements, in order to guarantee local determinacy of the inflation target steady state[3].

Policy rules. The Central Bank takes policy decision by using a nonlinear Taylor rule:

$$R_t = \rho(\pi_{t+i}) = 1 + (R^* - 1)\left(\frac{\overline{\pi}_{t+i}}{\pi^*}\right)^{\frac{A}{(R^*-1)}} \tag{2}$$

where $\overline{\pi}_{t+i}$ denotes a measure of inflation: in the case of a (perfect foresight) forecast-based Taylor rule we have that $\overline{\pi}_{t+i} = E_t \pi_{t+1} = \pi_{t+1}$. In the case of a backward-looking policy rule, I set $i = 0$, while $\overline{\pi}_t$ is a weighted average of current and past inflation rates, to be defined below. Also, π^* is the inflation target chosen by the monetary authority and R^* is the interest rate consistent with the equilibrium condition:

$$\beta^{-1} \pi^* = R^*$$

Optimality conditions. The first order conditions of the representative agent's problem (1) give the following Euler equation[4]:

$$U_c\left(c_t, m_t^{np}\right) = \frac{\beta U_c\left(c_{t+1}, m_{t+1}^{np}\right)}{\pi_{t+1}} R_t \tag{3}$$

[3] See Benhabib et al. (2001b).

[4] The FOCs are obtained by subsituiting the budget constraint in the intertemporal utility function.

where m denotes real money balances, and an (implicit) money demand equation for both production and consumption money balances**:**

$$f'\left(m_t^p\right) = \frac{R_t - 1}{R_t} \tag{4}$$

$$\frac{U_{m^{np}}\left(c_t, m_t^{np}\right)}{U_c\left(c_t, m_t^{np}\right)} = \frac{R_t - 1}{R_t} \tag{5}$$

where U_x is the partial derivative with respect to the argument x.

Role for Money. I consider two cases. First, money enters only in the utility function ($M^p = 0$) and output is constant: $f(.) = \overline{y}$ (endowment economy). Second, monetary policy affects economic decisions only through the production function ($M^{np} = 0$)[5].

Finally, in the paper I do not consider the capital accumulation process. The implicit assumption is that capital is fixed in the short run[6].

2.1 Forecast-based Taylor rule

2.1.1 Money in the production function

By using the equilibrium condition in the goods market, I get the following decision rule for the representative agent:

$$U_c\left(y_t\right) = E\left\{\beta U_c\left(y_{t+1}\right)\left[\frac{R_t}{\pi_{t+1}}\right]\right\} \tag{6}$$

where $E(.)$ denotes market's expectation. Combining the expression:

$$m_t = f^{-1}\left(y_t\right)$$

obtained from the production function and the money demand function (4) gives a negative relation between output and the interest rate:

$$y_t = y(R_t), \; y' < 0 \tag{7}$$

Combining (6), (2), (7) with $i = 1$, I obtain:

$$U_c\left(y\left(\rho\left(\pi_{t+1}\right)\right)\right) = E\left\{\beta U_c\left(y\left(\rho\left(\pi_{t+2}\right)\right)\right)\left[\frac{\rho\left(\pi_{t+1}\right)}{\pi_{t+1}}\right]\right\} \tag{8}$$

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[5] In the sequel, I drop the superscript to avoid complications in the notaion.

[6] For discussion, see Woodford (2002).

Assuming perfect foresight, the solution is a well defined one-dimensional map[7]:

$$\pi_{t+2} = F(\pi_{t+1}) \tag{9}$$

2.1.2 Money in the utility function

In this case output is a constant (i.e. it does not depend on money). Using the Euler equation (3) and the goods market condition:

$$c_t = \overline{y}$$

I obtain the decision rule:

$$U_c(\overline{y}, m_t) = \frac{\beta U_c(\overline{y}, m_{t+1})}{\pi_{t+1}} R_t \tag{10}$$

From the demand equation (5) I get the following (real) money demand equation:

$$m_t = m(R_t) \;\; m' < 0 \tag{11}$$

Finally using (10), (11), and (2) to substitute for R and m , I obtain the following reduced-form solution in π:

$$U_c(\overline{y}, m(\rho(\pi_{t+1}))) = \frac{\beta U_c(\overline{y}, m(\rho(\pi_{t+2})))}{\pi_{t+1}} \rho(\pi_{t+1}) \tag{12}$$

Once again, the solution is well defined one dimensional dynamical system which takes the form of (9).

2.2 Backward dynamics and learning

In this paper I am mainly concerned with real time learning behavior and thus with the following (reduced-form) decision rule:

$$\pi_t = EG(\pi_{t+1}) \tag{13}$$

Given the form of (8) and (12), G might not be a function but a correspondence. This because π_{t+1} appears both on the RHS and on the LHS of these expressions. In fact, in the next section I show the existence of two ‘branches’ of G: this implies that for any future value of inflation, the decision rule gives two choices for current inflation.

The existence of different ‘branches’[8] of G poses the problem of how current inflation is decided, given the expectations about the future. In this paper I mainly focus on the local stability of

[7] As noted already in Benhabihb et al. (2001) and Carlstrom and Fuest (2001), the solution displays nominal indeterminacy.

[8] For an example where the branches are in the forward-looking map, see Christiano and Harrison (1999).

the perfect foresight equilibria: I assume therefore that the expectational errors are 'small' and that market participants are somewhat coordinated in a neighborhood of the perfect foresight equilibrium[9]. Nevertheless, I also consider the dynamics under learning and show the existence of other equilibria that depend on which branch of G is selected.

Notice that, under the assumption of perfect foresight, the solutions (8) and (12) can be expressed in 'backward dynamics': $\pi_t = G(\pi_{t+1})$, as opposed[10] to (9). That can be a useful approximation of the behavior under learning, provided that the agents' mistakes are not too big.

According to (13), π_t depends only on future expected values of π_{t+1}. In order to model learning behavior, I follow Guesnerie and Woodford (1991) and Evans and Honkapohja (2001a). The agents are assumed to have a (asymptotically) correct model of the economy: their perceived law of motion of the economy corresponds to the actual law of motion, *if* the learning process converges to the perfect foresight equilibrium. More specifically, the agents believe that the system is at the equilibrium, even though they do not know *which* equilibrium (steady state, cycle or sunspot) and *what values of inflation* correspond to the equilibrium.

Learning steady states and cycles. Let us first consider the deterministic case where the agents face two possible equilibria: steady states and (period-two) cycles. They expect $\pi_t = \pi_1$ for odd-t and $\pi_t = \pi_2$ for even-t, where $\pi_1 = \pi_2 = \widetilde{\pi}$ if the system is in steady state. Nevertheless, they do not observe π_1 and π_2; they estimate them recursively, updating every period their information about the state of the system. Given that at the cycle:

$$\pi_1 = G_i(\pi_2);\ \pi_2 = G_j(\pi_1)$$

where i and j denote the (possibly different) branches of G, the agents estimate the two states of the cycle by averaging the past data $G(\pi_{t-i})$ for even and odd periods separately[11]. Hence, their forecast will be $EG_i(\pi_{t+1}) = \theta_{2,t}$ if $t+1$ is odd and $EG_j(\pi_{t+1}) = \theta_{1,t}$ if $t+1$ is even, where $\theta_{l,t}$ is an estimate of π_l (for $l = 1, 2$). In order to update their estimates of the equilibrium values of inflation, market participants make use of the adaptive algorithm[12]:

$$\begin{bmatrix} \theta_{1,s} \\ \theta_{2,s} \end{bmatrix} = \begin{bmatrix} \theta_{1,s-1} \\ \theta_{2,s-1} \end{bmatrix} + \alpha_s \begin{bmatrix} G_i(\pi_{2,s}) - \theta_{1,s-1} \\ G_j(\pi_{1,s}) - \theta_{2,s-1} \end{bmatrix} \tag{14}$$

where s is such that $t = 2s + l$ and $\pi_{l,s} = \pi_{2(s-1)+l}$, allowing to *take the data in successive pairs*. This learning rule has the desirable property of being able to detect both steady states and period-two cycles[13]. In fact, if the perfect foresight equilibrium is a steady state, the two estimates

[9] Another way to express this is that the agents have *strong priors* about the equilibrium values of π.

[10] See Grandmont (1985).

[11] Note that the agents do not need to know G. They just observe past values of the RHS of (8) and (12), depending on the model. Hence, they observe past values of inflation, consumption, real money balances and the interest rate.

[12] For details, see Evans and Honkapohja (1995, 2001a).

[13] In the sense that $\theta_{1t} \to \pi_1$ and $\theta_{2t} \to \pi_2$ as $t \to \infty$.

of the states π_1 and π_2 both converge to a single constant. Also, the algorithm allows the agents to learn cycles on different branches of G.

This way of modelling the learning process has several advantages, as the fact that I do not need to assume point expectations or any prior knowledge about the nonlinear function G[14]. On the other side, (14) implies a small deviation from rationality, since the agents are assumed to have a well specified model of the economy (the MSV solution). In other words, I restrict the range of regularities that the agents are willing to extract from past data[15]. Finally, I rule out the hypothesis of heterogeneity in the forecasts and in preferences, in order to keep the analysis simple.

Concerning the monetary authority, for the case of forecast-based Taylor rules I maintain the assumption of perfect foresight. This assumption can be defended on two grounds. First, the asymmetry in information between the central bank and the market might be due also to imperfect credibility and thus it is plausible. Second, the assumption is very favorable to a good performance of the policy rule. Bounded rationality of both the policy authority and the market participants is likely to be an extra-source of bad outcomes for the policy rule.

Note that when considering a deterministic environment we assume that agents use a fixed gain algorithm: $\alpha_s = \alpha \in (0,1)$. This should capture the fact that they expect to operate in a changing environment (the market structure or Central Bank's preferences). In the case of stochastic environment, the fixed gain algorithm does not converge. I therefore assume[16] for simplicity that $\alpha_s = s^{-1}$.

The simple models considered above can also generate perfect foresight equilibrium cycles of order higher than two. How about their learnability? These cycles could not be detected by the learning rule above and therefore they are not learnable in this framework, but it would be straightforward to modify the algorithm to include the possibility of learning higher order cycles. Nevertheless, this extension does not seem interesting because the results are not likely to be robust to behavioral and learning heterogeneity. This intuition is reinforced by recent results from Bullard and Duffy (1998): they study the learnability of cycles in the model of Grandmont (1985), and find that under heterogeneous learning rules, the only cycles that are learnable are period-two cycles. Hence, I conjecture that even if there exists a 'representative' learning rule under which three and higher order cycles are learnable, this result might not be robust to heterogeneous expectations.

Given (13) and (14) the system can be expressed in terms of the expectations variable as[17]:

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[14] Notice that the agents observe past values of $\frac{\beta U_c}{\pi_{t+1}} R_t$ but do not necessarily know the relation between, say, output and the interest rate, or the relation between the interest rate and inflation.

[15] On this point see Grandmont (1998).

[16] The result I obtain holds also for other choices of α_t, as well known from the literature of stochastic approximation: see Evans and Honkapohja (2001a).

[17] The dynamical system is obtained by substituting the first equation for $\theta_{1,s}$ in the second equation. After this substitution, the system becomes $\theta_s = H(\theta_{s-1})$, where $\theta_s = (\theta_{1,s}, \theta_{2,s})$.

$$\begin{bmatrix} \theta_{1,s} \\ \theta_{2,s} \end{bmatrix} = \begin{bmatrix} \theta_{1,s-1} \\ \theta_{2,s-1} \end{bmatrix} + \alpha_s \begin{bmatrix} G_i\left(\theta_{2,s-1}\right) - \theta_{1,s-1} \\ G_j\left(\theta_{1,s}\right) - \theta_{2,s-1} \end{bmatrix} \tag{15}$$

which gives a two dimensional dynamical system at each branch i, j. Learning behavior is determined by the behavior of this dynamical system.

Learning sunspots. I also consider the possibility that agents learn to believe in sunspots. In fact, as shown in the next sections, under the hypothesis of rational expectations there exist sunspot equilibria. In order to assess their learnability, let us assume that the agents include in their perceived law of motion the possibility of being at a two-states sunspot equilibrium, generated by a non-fundamental exogenous 'sunspot' process s_t. Again, from Evans and Honkapohja (2001a) they estimate recursively the states π_1 and π_2 (depending on the current realization of s_t) by using the adaptive learning rule :

$$\theta_{l,t} = \theta_{l,t-1} + t^{-1}\psi_{l,t-1}q_{l,t-1}^{-1}(\pi_{t-1} - \theta_{l,t-1} + \epsilon_{t-1}) \tag{16}$$

$$q_{l,t} = q_{l,t-1} + t^{-1}(\psi_{l,t-1} - q_{l,t-1})$$

which gives the following actual law of motion:

$$\begin{aligned} \pi_t \quad = \quad & \psi_{1,t}[z_{11}G_i\left(\theta_{1,t}\right) + \left(1 - z_{11}\right)G_j\left(\theta_{2,t}\right)] + \\ & +\psi_{2,t}[(1 - z_{22})G_i\left(\theta_{1,t}\right) + z_{22}G_j\left(\theta_{2,t}\right)] \end{aligned} \tag{17}$$

where, again, i and j denote the (possibly) different branches of G. The algorithm works as follows. The agents observe a sunspot Markov process s_t with two states and a transition matrix:

$$\begin{bmatrix} z_{11} & 1 - z_{11} \\ 1 - z_{22} & z_{22} \end{bmatrix}$$

They are assumed to know the transition matrix. They update recursively each estimate of π_l (for $l = 1, 2$) depending on the current state of the sunspot: the variable $\psi_{l,t}$ is equal to one if the sunspot process is in state l and zero otherwise. Also, $q_{l,t}$ represents the fraction of observations in state l over the whole sample up to time $t-1$. Finally θ_{lt} represents the estimates of π_l in the two states of the sunspots and ϵ_t is a measurement error i.i.d, with bounded support.

Notice that the learning algorithm (16) is able to detect cycles and steady states, provided that the agents learn also about the transition probabilities[18]. In other words, the learning rule (14) is a special case of (16). For simplicity I do not consider this extension, but it should be clear that the learning rules are kept distinct only for expositional purposes.

[18] If z_{11} and z_{22} are equal to one we have a steady state, while if z_{11} and z_{22} are equal to zero we have a cycle.

3 Local Stability of Steady States

Given the dynamical system (15), we are interested in the local stability of the steady states under learning dynamics, under both the assumption of money in the utility function and money in the production function.

Consider the case of money in the production function. Linearizing the forward looking map around the active and passive steady states gives the coefficient[19]:

$$\widehat{\pi}_{t+2} = \left[1 + \frac{1}{-\sigma\epsilon_y}\left(\frac{1}{\epsilon_\rho} - 1\right)\right]\widehat{\pi}_{t+1} \tag{18}$$

where $\widehat{\pi}$ describes deviations from the steady state, ϵ_ρ is the elasticity of R with respect to π, $-\sigma = \frac{U_{cc}y}{Uc} < 0$ and $\epsilon_y = \frac{y'R}{y} < 0$. Notice that for consistency with the assumptions about the utility function described in the next sections, I assume that the intertemporal elasticity of substitution is constant.

The following proposition describes the steady states and their stability under perfect foresight and learning dynamics (all the proofs are in the Appendix).

Proposition 1 *Consider the map G implied by (8). Then:*

(i) The system (15) has two steady states, π^* and $\overline{\pi}$. At π^* monetary policy is active and $\epsilon_\rho(\pi^*) > 1$. At $\overline{\pi}$ monetary policy is passive and $\epsilon_\rho(\overline{\pi}) < 1$.

(ii) Provided $\epsilon_\rho(\pi^*)$ is such that $|-\sigma\epsilon_y| < (\frac{1}{\epsilon_\rho(\pi^*)} - 1)/2$ both the active and passive steady states are determinate in the perfect foresight dynamics and stable under learning dynamics for any $\alpha \in [0,1]$.

Remark 2 *Determinacy of the fixed point is achieved provided that the Taylor rule is 'sufficiently active' at the inflation target. In the next section I clarify for which parameters of the model the conditions in Proposition 1 hold. As mentioned in the introduction, I focus on the case where the inflation target is locally unique.*

Remark 3 *By adding small noise and letting $\alpha_t = t^{-1}$ we have that both fixed points are strongly E-Stable in the sense of Evans and Honkapohja (2001) and thus they are robust to overparametrized perceived laws of motion (i.e. if the agents model were consistent with an n-cycle, it would still converge to the fixed point)*[20].

Concerning the steady states, forecast based rules can be destabilizing even in the case where the equilibrium corresponding to the inflation target is locally unique. In fact, for suitable initial conditions (depending on the shocks hitting the economy) the economy can be driven into a liquidity trap. We should expect the existence of a "stability corridor": if the economy experiences small

[19] The result is also in Benhabib et al. (2001b)

[20] See Proposition 12.2 of Evans and Honkapohja (2001).

shocks (i.e. in expectations), than the Taylor rule drives the economy back to the inflation target. On the contrary, large shocks might lead to a liquidity trap.

In the next section I show that, if we take into account the existence of equilibrium cycles and sunspots, then the stability corridor is rather small. In fact, small shocks can drive the economy to sunspot equilibria arbitrarily close to the active steady state.

Consider the model with money in the utility function. Let us assume that $U_{cm} > 0$: under this assumption, we know from the continuos time version of this model that the active steady state is determinate[21]. Linearizing the model (12) I get the following coefficient:

$$\widehat{\pi}_{t+2} = \left[1 + \frac{1}{\epsilon_{cm}\epsilon_m}\left(\frac{1}{\epsilon_\rho} - 1\right)\right]\widehat{\pi}_{t+1} \tag{19}$$

where $\widehat{\pi}$ defines deviations from steady state, $\epsilon_{cm} = \frac{U_{cm}m}{U_c} > 0$ and $\epsilon_m = \frac{m'R}{m} < 0$.

It is immediate from (19) that the active steady state ($\epsilon_\rho(\pi^*) > 1$) is always determinate, while the determinacy of the passive steady state ($\epsilon_\rho(\overline{\pi}) < 1$) depends crucially on the parameters of the model. I also consider the existence and learnability of equilibrium stationary sunspots $\epsilon-$close to the deterministic steady states (defined $\epsilon' - SSEs$ in the sequel). Local stability and learnability of the steady states is described as follows:

Proposition 4 *Consider the map G implied by (8) (12). The map has the same steady states of (8). Consider the deterministic system (15);*

(i) the active steady state is determinate and learnable for every $\alpha \in (0,1)$;

(ii) for $\epsilon_\rho(\overline{\pi})$ such that $|\epsilon_{cm}\epsilon_m| > (\frac{1}{\epsilon_\rho(\overline{\pi})} - 1)/2$ the passive steady state is indeterminate and non-learnable for every $\alpha \in (0,1)$;

(iii) for $\epsilon_\rho(\overline{\pi})$ such that $|\epsilon_{cm}\epsilon_m| < (\frac{1}{\epsilon_\rho(\overline{\pi})} - 1)/2$ the passive steady state is determinate and learnable for every $\alpha \in (0,1)$.

Consider the stochastic system (16), (17);

(iv) for $\epsilon_\rho(\overline{\pi})$ such that $(\frac{1}{\epsilon_\rho(\overline{\pi})} - 1)/2 < |\epsilon_{cm}\epsilon_m| < (\frac{1}{\epsilon_\rho(\overline{\pi})} - 1)$ there exist $\epsilon' - SSEs$ around the (indeterminate) passive steady state.

Remark 5 *It is apparent that a sufficiently aggressive rule at the inflation target leads to a very passive response at the liquidity trap (as it is apparent from the policy rule (2)). Hence, a sufficiently aggressive rule verify the conditions for a learnable liquidity trap or a learnable sunspot. In the next section I show how the results depend on the parameters of the model and I provide examples of all possible cases, for plausible parameter values.*

Proposition (4) shows that also in the case, more common in the literature, where money enters in the utility function, forecast-based Taylor rules may lead to economic instability, even if the

[21] See Benhabib et al. (2001a).

active steady state is locally unique. In fact, *the economy can either fall into a liquidity trap or converge to a sunspot equilibria where inflation fluctuates around the passive steady state.*

Summing up, these two results partly contradict the findings of Bullard and Mitra (2001) and McCallum (2001). First, $\epsilon_\rho > 1$ (the Taylor Principle) is not a sufficient condition to stabilize the economy. The conditions in Proposition (4) show that a 'too active' [22] policy rule at the inflation target (which implies a too passive policy at the liquidity trap) might be destabilizing because it leads to learnable liquidity traps. The result is indeed confirmed in the next section, where functional forms are considered. Combine this with the result from Proposition (1) that a 'too' cautious active policy rule leads to an indeterminate and non-learnable inflation target equilibrium, while the liquidity trap is learnable. The conclusion is that uncertainty about the correct specification of the model with respect to the role of money balances makes forecast-based Taylor rules potentially destabilizing.

Second, regardless of how the model is formulated with respect to the role of money, liquidity traps are, in some cases, robust to expectational mistakes. This result contrasts with the conclusions of McCallum (2001), which dismisses the liquidity trap equilibrium as a non-learnable bubble solution and therefore non-empirically relevant [23].

4 Local Stability of Cycles and Sunspots

In order to study the global behavior of the economic system under the two models mentioned above, it is necessary to specify the functional forms for utility functions and production functions.

4.1 Money in the production function

Concerning the model with money in the production function, I follow Benhabib et al. (2001b) and specify a Cobb-Douglas utility function and a CES production function:

$$f(m_t) = \left[(1-a)\, m_t^{\mu} + a\overline{y}^{\mu}\right]^{\frac{1}{\mu}} \tag{20}$$

$$U(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma} \tag{21}$$

Given those functional forms, I calibrate the model by using the equilibrium money demand:

[22] Note that the optimal monetary policy literature advocates more active Taylor rules (i.e. ϵ_ρ) than our benchmark case, see Taylor (1999).

[23] The result also contrasts with Honkapohja and Mitra (2001). They analyze Markov sunspots equilibria and show that none is learnable in a class of models with money in the utility function and nominal rigidities.

the benchmark parametrization is shown in Table 1[24]. This section discusses the conditions for the cycles to be learnable as σ and the learning parameter $\alpha \in (0,1)$ change.

Table1 Benchmark Parametrization

β	μ	μ^{LR}	μ^{SR}	π^*	R^*	$\overline{y}$	a	A
0.996	-9	-3.5	-50	1.0103	1.0143	1	0.000350	1.522

The parameter μ is chosen to be consistent with both long run and short run log-elasticity of the money demand: μ^{LR} is the parameter consistent with the long run log-elasticity of demand and μ^{SR} is consistent with the log-elasticity in the short run[25]. The parameters σ and α are considered the 'free' parameters: in the simulations below I consider values for σ that range from 1 to 3.5, that are most commonly used in the literature.

Concerning the model with money in the production function, under the benchmark parametrization (with $\mu = -9$) the map under perfect foresight has a cycle of period 2, for (approximately) $\sigma \in (1, 2.42)$[26]. The economy fluctuates between two states of higher and lower inflation, with respect to the inflation target. Figure 1 shows the period-two cycle for a given value of σ, under both F and G.

The thick lines correspond to the branches of the (backward) map G; the thin line describes the second iterate of the map $\pi_{t+2} = F(\pi_{t+1})$ in forward dynamics and the dashed line shows the first iterate of the map in forward dynamics. As shown in Figure 1, the correspondence G -implied by the map (8)- is made up of two 'branches'. The first, G_1, intersects the 45 degree line at the passive steady state while G_2 intersects the 45 degree line at the active steady state.

Backward (G) and forward (F) maps are equal at the fixed point and at the equilibrium cycle (shown by the dotted line). Notice that each point of the cycle rests on a different branch of the backward map. Nevertheless, it is immediate to see that at the cycle:

$$G_1'(\pi_2) = \frac{1}{F'(\pi_1)};\; G_2'(\pi_1) = \frac{1}{F'(\pi_2)}$$

where π_1 corresponds to the point of the cycle where inflation is higher than the inflation target, while π_2 is the point where inflation is lower than the target. Hence, local stability of the cycle under learning can be studied from the local properties of F.

Let us consider figure 2. It describes the map F_σ^2: the different curves show how the cycle changes for varying σ.

[24] For details on how to calibrate these models, see Benhabib et al. (2001b).

[25] Notice that as I change the value of μ, I need also to change the value of a, in order to keep the model close to the data, see Benhabib et al. (2001b).

[26] See Benhabib et al. (2001b).

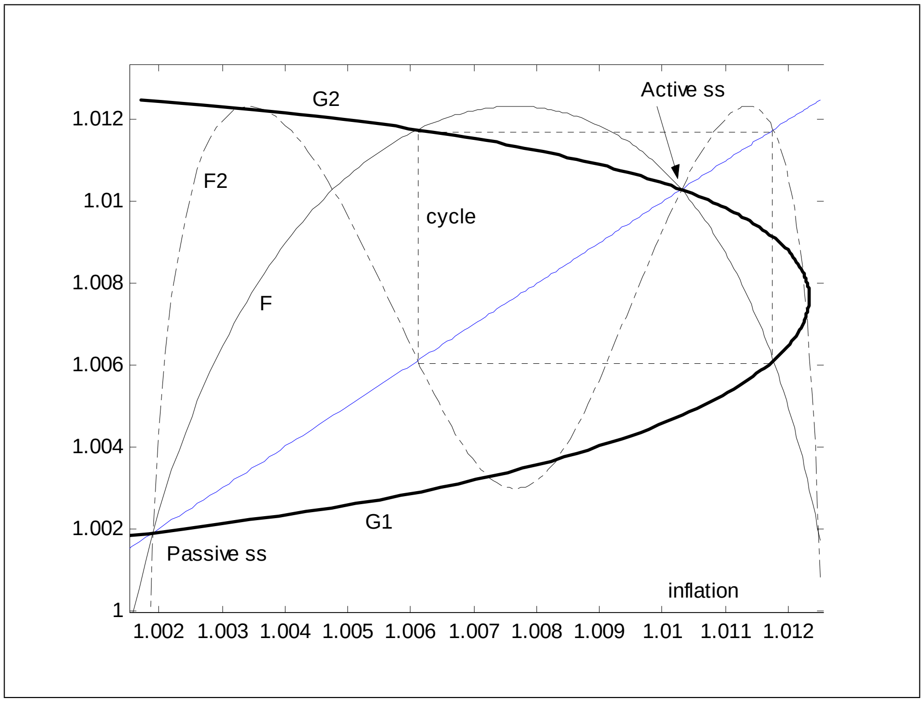

Figure 1: Figure (1)

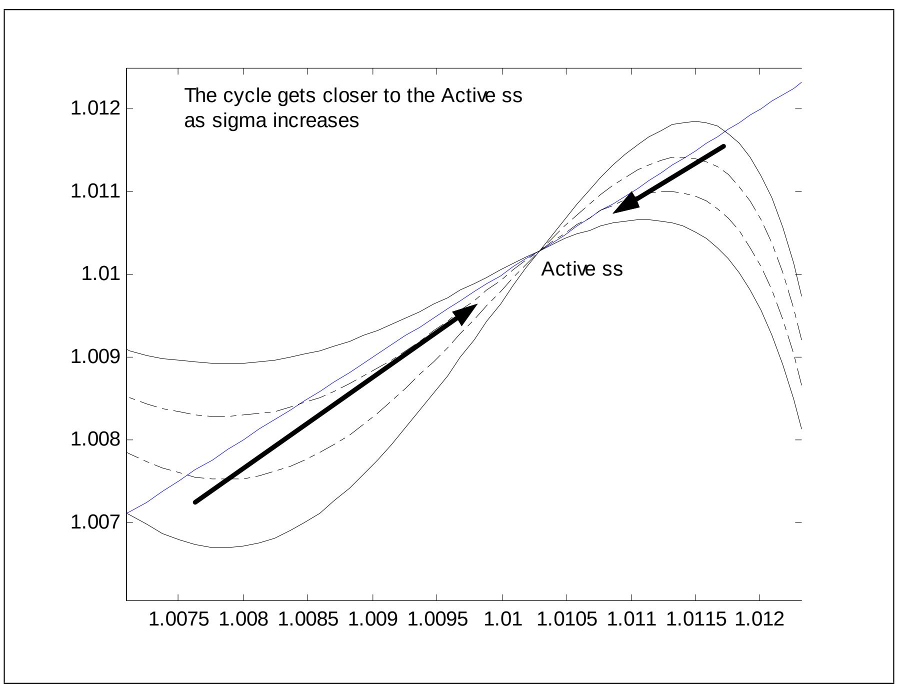

Figure 2: Figure (2)

From the picture we know that the map F_σ^2 admits a cycle for certain values of σ. Given that for $\sigma \in (1.5, 2.5)$, it exists $\overline{\sigma}$ such that for $\sigma < \overline{\sigma} < 2.5$, F_σ^2 has four fixed points. For $\sigma > \overline{\sigma}$ F_σ^2 has two fixed points.

From the observation of Figures (1) and (2) and from the form of the policy rule[27], it is possible to conclude the following:

1. at the point on the cycle where inflation is high (π_H), monetary policy is always active;

2. the degree of aggressiveness of the monetary rule at this point decreases as the two-period cycles gets closer to the active steady state and subsequently vanishes by "merging" with the fixed point;

3. as σ increases, the cycle gets closer to the fixed point and eventually disappears. Instead, for low values of σ, the point of the cycle where inflation is low (π_L) is close to the passive steady state and thus the policy rule is passive. But, as $\sigma \to \overline{\sigma}$ we have that $\epsilon_\rho(\pi_2) \to \epsilon_\rho(\pi^*)$ and it exists σ_0 such that for $\sigma < \sigma_0$, $\epsilon_\rho(\pi_2) > 1$.

Finally, notice that the qualitative features of the map, described above, are robust to small changes in the parameters. The following result is thus robust to small deviations from benchmark parameters[28].

Proposition 6 *Let G be given from (8). Given the deterministic system (15), under the benchmark parametrization, it exist $\widehat{\sigma}$ and $\widetilde{\sigma}$, ($\widehat{\sigma} < \widetilde{\sigma}$) such that for $\sigma < \widehat{\sigma}$ the cycle is stable under learning for every $\alpha \in (0,1)$. For $\sigma > \widetilde{\sigma}$ the cycle is unstable under learning for every $\alpha \in (0,1)$. For $\sigma \in [\widehat{\sigma}, \widetilde{\sigma}]$ it exists $\widehat{\alpha}_\sigma \in (0,1)$ such that for $\alpha < \widehat{\alpha}_\sigma$ the cycle is stable under learning, while for $\alpha > \widehat{\alpha}_\sigma$ the cycle is unstable under learning.*

Remark 7 *The Proposition tells us that if π_L is not too close to the inflation target, than the cycle is stable under learning, at least for some values of the parameter α. Under the benchmark parametrization $\widehat{\sigma} \approx 2$ and $\widetilde{\sigma} \approx 2.182$: for small changes of the other parameters, such as the degree of policy aggressiveness, $\widehat{\sigma}$ and $\widetilde{\sigma}$ also change. For example, we have that for $\sigma = 2.18$, $\pi_H = 4.7\%$ while $\pi_L = 3.2\%$, and $\epsilon_\rho(\pi_L) = 1.18$, which implies a mildly active policy. This equilibrium is learnable for $\alpha < 0.21$. Given that the target is 4.2%, the cycle is 'close' to the active steady state.*

Remark 8 *Given the standard parametrization a more active policy rule (say $A = 2$) implies the existence of learnable cycles for a wider range of σ (in the simulations cycles are learnable for values of $\sigma \approx 2.5$)[29].*

---

[27] Notice that changes in σ do not affect the policy rule.

[28] The result comes from the structural stability of cycles.

[29] Also, more active Taylor rule imply period three cycles and chaos for a wider range of parameters than found in Benhabib et al. (2001b).

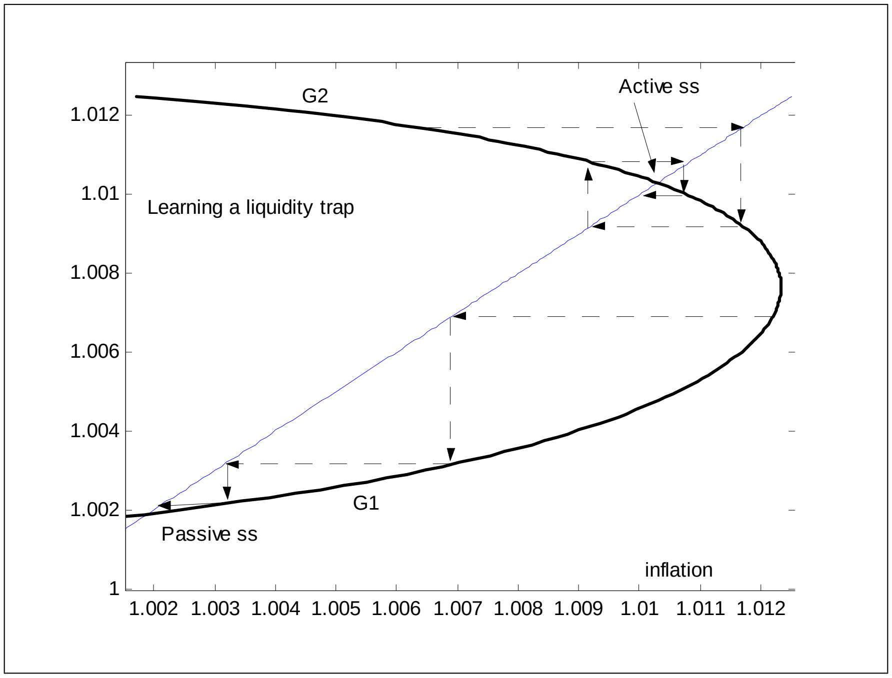

Figure 3: Figure (3)

Remark 9 *Simulations show that the result is unchanged if I consider different inflation targets. For example the (implicit) inflation target for the European Central Bank would be* $\pi^* = 2\%$ *: the Proposition holds also in this case.*

Remark 10 *It is straightforward to show that under the benchmark parametrization the inflation target steady state is determinate and learnable. In fact, as* σ *decreases both the stability conditions for the learnability of cycles and steady states are satisfied.*

The Proposition leads to two interesting results. First, it provides one example where learnable cycles exist close to a locally unique equilibrium. In fact, *both the cycle and the inflation target are learnable*. The implications for monetary policy are that small shocks are sufficient to destabilize the economy. Small deviations from the active steady state might lead the market participants to coordinate on a equilibrium cycle. Second, very active policy rules (i.e. $A > 1.5$), that are commonly obtained from optimal monetary policy design, can be destabilizing, given that they imply learnable cycles for a wider range of parameters.

Dynamics under learning. By considering explicitly the functional forms of the model, it is also possible to give an intuition about the dynamics under learning. Figure 3 shows the convergence process of (13) to the active steady state and the liquidity trap, under the hypothesis of perfect foresight, to give an idea of how the economy can fall into a liquidity trap. A shock can start the economy on the deflationary path, depending on how the agents react to their expectations (i.e. which branch of G we are considering). In this case the economy converges monotonically to the new equilibrium. Instead, convergence to the central bank's target implies fluctuations in the inflation rate.

Proposition (6) concerns the learnability of equilibrium cycles obtained under perfect foresight. But learning dynamics also generates other equilibria along bifurcation parameter values[30] of the system (15). As the cycle looses stability and σ increases, it is likely that a *saddle-node bifurcation* occurs. Hence, given $\overline{\sigma}$ as the bifurcation parameter, for some $\sigma < \overline{\sigma}$ the fixed point of the second iterate of (15) is surrounded by two other fixed point (i.e. two cycles of period 2). One of them is stable, and the other is unstable. For $\sigma > \overline{\sigma}$ the fixed point disappears (as we also see it from the graph).

Also, the cycle looses stability as α increases for given $\sigma \in [\widehat{\sigma}, \widetilde{\sigma}]$. In this case a *period-doubling bifurcation* might occur, generating a period four cycle that can be either stable or unstable. Note that *these other fixed points do not exist under perfect foresight.* They are generated by the learning behavior. Hence, forecast-based rules have the potential to further de-stabilize the economy if we take into account the dynamics under learning[31].

Proposition (6) implies two further results. First, let us add some 'small' (in the sense of sufficiently small support) noise into the model. Then Evans and Honkapohja (1995) show the existence of noisy REE cycles that are learnable if the deterministic cycle is learnable, provided the learning parameter is decreasing over time (i.e. s^{-1} is used instead of α). Hence, the results in Proposition (6) are robust to the introduction of small noise.

Second, using results from Azariadis and Guesnerie (1986), Grandmont (1986, 1987) and Evans and Honkapohja (2001a), it is possible to show that (locally) learnable sunspots equilibria exist arbitrarily close to the period two cycle and to the determinate active steady state.

Proposition 11 *Let G be given by (8), under the benchmark parametrization. Given the stochastic system (16) and (17);*

(i) for $\sigma < \widetilde{\sigma}$ (as defined in Proposition (6)) there exist learnable $\epsilon' - SSEs$ relative to the deterministic cycle;

(ii) for $\sigma < \widehat{\sigma}$ the $\epsilon' - SSEs$ are strongly E-Stable and therefore robust to model over-parametrization;

(iii) there exist learnable "liquidity trap" sunspots, where the economy fluctuates between the active and passive steady states;

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[30] Note that, given the standard parametrization of the model, the eigenvalues at the fixed points are real (and with opposite sign), while the eigenvalues at the cycle are complex when stable and real-saddle (with one eigenvalue greater than one) when unstable. In particular, the eigenvalues are real *before* becoming unstable. Also, for $\sigma \in (\widehat{\sigma}, \widetilde{\sigma})$ the cycle loses stability as α increases and, again, the change in stability involves real eigenvalues. The ustable cycle is a saddle with one eigenvalues less than minus one.

[31] These bifurcations occur only if extra conditions are satisfied, see Wiggins (1990). More precisely, the problem can be reduced to a one dimensional bifurcation through a center manifold reduction. But we do not go further in the analysis.

(iv) for any $1 < \sigma < \overline{\sigma}$, let $\pi_1(\sigma) \in (\pi^*, \pi_H(\sigma))$ and $\pi_2(\sigma) < \pi_L(\sigma)$, where $\pi_H(\sigma), \pi_L(\sigma)$ represent the equilibrium cycle, for given σ. Then there exist z_{11}, z_{22} such that the sunspot equilibrium $(\pi_1(\sigma), \pi_2(\sigma), z_{11}, z_{22})$ is learnable (whether or not the cycle is learnable).

Remark 12 *Result (iv) is somewhat new in the literature. I construct the sunspots on different branches of G by using results from Grandmont (1986, 1987) -see the Appendix. For example, it can be shown that for $\sigma = 1.9$, the SSE given by $\pi_1 = 1.0104$ (4.25% in annual terms), $\pi_2 = 1.0075$ (3%), $z_{11} = 0.2$, $z_{22} = 0.3$ is learnable. Note that $\pi_1 \approx \pi^*$. Moreover, assuming $\sigma = 2.35$, for which the cycle is indeterminate and non-learnable, the SSE given by $\pi_1 = 1.046$ (6%), $\pi_2 = 1.0087$(3.5%), $z_{11} = 0.19$, $z_{22} = 0.25$ is learnable and 'close' to the deterministic cycle. Note that in both the inflation states monetary policy is active.*

Remark 13 *As mentioned above for Proposition 3, Proposition 4 is proved for the benchmark parametrization, but the result is robust to small perturbations of the benchmark parameters.*

This result is important in its simplicity because *it shows the possibility of learnable equilibria, other than the inflation target, for any parameter value of σ* for which the cycle exists[32]. Even though we have local unicity and learnability of the inflation target equilibrium, the agents learning process can still converge to a sunspot equilibrium that is 'close' to it. This also means that the 'stability corridor' mentioned in the previous section is indeed extremely small if it exists at all. Hence, forecast-based Taylor rules are likely to be destabilizing, under this model of the economy.

Christiano and Rostagno (2001) and Benhabib et al. (2001c) propose the following policy. If the economy is hit by an important shock (so that we have a substantial deviation from the inflation target) the monetary authority should switch to a money rule, thus avoiding liquidity traps or other destabilizing equilibria. This solution would not be helpful in our case because, as showed in the Remarks, cycles and sunspot equilibria can be very close to the active fixed point. Thus, such a scheme would eliminate the stabilization properties of the Taylor rule.

4.2 Money in the utility function

Because of our uncertainty about the 'correct' model of the economy, it is useful to consider the behavior of forecast-based Taylor rule when money enters in the utility function[33]. In order to allow comparisons between the two models and match the data I assume that, in the model where money enters in the utility function the functional form is CES:

[32] Sunspots equilibria may seem a more empirically plausible explanation for the instability generated by the Taylor rule. In fact, the regular behavior implied by deterministic cycles is not observed in the data.

[33] Carltrom and Fuest (2001a) point out the fact that the way money enters in the utility functions has important implications for the stability properties of the Taylor rule. I consider the standard approach in this section and a different trading environment in the extensions.

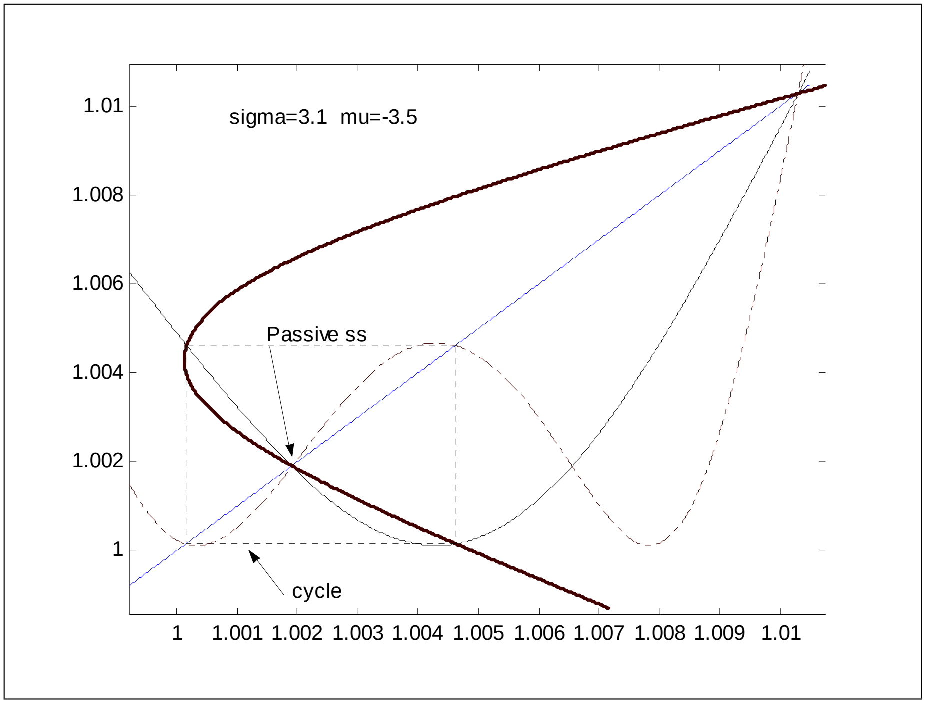

Figure 4: Figure 4

$$U\left(c_t, m_t\right) = \frac{\left(\left[\left(1-a\right) m_t^{\mu} + a c_t^{\mu}\right]^{\frac{1}{\mu}}\right)^{(1-\sigma)}}{1-\sigma} \tag{22}$$

This specification allows the money demand implied by (7) and (11) to be exactly the same. Hence, the two models have the same calibration as showed in Table 1.

We know from Proposition (4) that the stability of the fixed point, for a given Taylor rule, depends on the elasticities $\epsilon_{cm}\epsilon_m$. This is also true for the global dynamics of the system. It is easy to show that $\epsilon_{cm}\epsilon_m$ is increasing in both σ and μ. Figures 4 and 5 show the map under different values of σ and μ. First, notice that if I use values for μ close to the short run value, no cycle exists, as shown in Figure 5. This is good news, but we know from Proposition (4) that for $\epsilon_{cm}\epsilon_m$ not too low, there exist learnable sunspots around the indeterminate and non-learnable passive steady state.

For example, if we fix $\mu = -9$ and choose $\sigma > 2$ than the conditions for the existence of learnable sunspots are verified. Hence, the economy can fall into a liquidity trap for very plausible parameter values.

A period two cycle emerges *as both* μ *and* σ *increase.* Given the uncertainty about these parameters I consider a choice of $\mu < \mu^{LR}$ and $\sigma < 4$ as plausible parameter values. For example, for $\mu = -3.5$ and $\sigma = 3.1$ (and the other parameters at their benchmark values) there exists a cycle around the passive steady state, as shown in Figure 4. The result holds for higher μ and it is

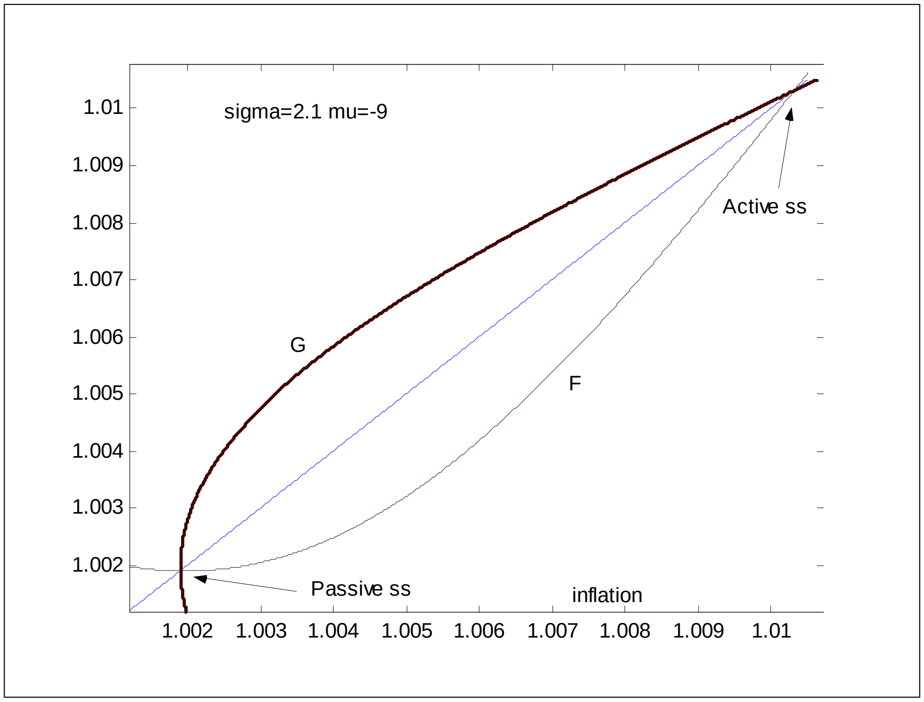

Figure 5: Figure (5)

robust to parameters' perturbation. For lower μ the values of σ required to get the cycle are not interesting. But also in this case, a too active monetary policy is further destabilizing. For example, if $A = 2$, then the cycle appears for $\mu = -9$ and $\sigma \gtrsim 1.5$. Given that $A = 2$ has been suggested to describe the policy stance of the post-Volker period[34], the existence of cycles holds for a large portion of the 'plausible' parameter space. The results are described in the following Proposition. Again, the results remain valid for small deviations from the benchmark parametrization.

Proposition 14 *Let G given by (12) and under the benchmark parametrization but with $A = 2$, such that for $\sigma > \overline{\sigma}'$ a period-two cycle exists.*

Under the deterministic system (15);

(i) for $\overline{\sigma}' < \sigma < \widetilde{\sigma}'$ the cycle is not learnable under rule (14) for any value of $\alpha \in (0,1)$;

(ii) for $\sigma \in \left(\widetilde{\sigma}', \widehat{\sigma}'\right)$, $\widetilde{\sigma}' < \widehat{\sigma}'$, there exists an $\widehat{\alpha}'$ such that for every $\alpha < \widehat{\alpha}'$, the cycle is learnable;

(iii) for $\sigma > \widehat{\sigma}'$ the cycle is learnable for every $\alpha \in (0,1)$.

Under the stochastic system (16) and (17);

(iv) for $\sigma > \widetilde{\sigma}'$ there exist learnable SSE ϵ-close to the cycle. For $\sigma > \widehat{\sigma}'$ the sunspots are robust to agent's overparametrization.

(v) there exist learnable 'liquidity trap' sunspots ϵ-close to the two fixed points.

[34] See Clarida et al. (2001).

Remark 15 *Under the benchmark parametrization $\widetilde{\sigma}' \approx 2.23$ and $\widehat{\sigma}' \approx 2.89$. Again, changing the parameters affects $\widetilde{\sigma}'$, $\widehat{\sigma}'$. It is straightforward to verify that under the benchmark parametrization, the inflation target, the liquidity trap, the cycle and the sunspots are all locally stable under learning.*

Remark 16 *Even if a formal proof is omitted for brevity, it can be shown that also for the case of money in the utility function there exist learnable sunspots that are 'close' to an indeterminate and non-learnable cycle. Assume that $\sigma = 1.8$ and $A = 2$, while the other parameters are set according to Table I. Let us consider the following equilibrium: $\pi_1 = 1.002$ (0.08%), $\pi_2 = 0.9975$ (−0.1%), with $z_{11} = 0.15$, $z_{22} = 0.15$. This is relatively close to the indeterminate and non-learnable cycle where high inflation is $\pi_H = 1.001$ (0.04%) and low inflation is $\pi_L = 0.997$ (−0.12%). The economy fluctuates among almost zero inflation and negative inflation. It can be shown, by using (54) in the Appendix, that this sunspot is learnable. Hence, also for the case of money in the utility function I conjecture the existence of learnable sunspots for all values of σ such that the cycle exists.*

The results are obtained with the same procedure as for the model with money in the production function and the proof is therefore omitted. Also in this model, forecast-based Taylor rules can be destabilizing: the economy can converge to equilibria in which inflation fluctuates below the target.

Uncertainty about the correct model of the economy and the parameter σ make a forecast-based Taylor rule an hazardous choice. In fact, the last three Propositions lead to the following conclusion. If the 'true σ', which the Central Bank is not likely to observe, is low and the economy is better approximated by a model with money in the production function, then forecast-based Taylor rules are going to be destabilizing. On the other side, if the true σ is high and the correct model has money in the utility function we get the same conclusion. Hence, *whatever is the 'estimate' of σ that it is used for calibration, there exist a model of the economy for which the forecast-based Taylor rule generates instability.*

In the case of money in the utility function, switching to a money rule after a moderate negative shock to the economy might be effective, given that the 'bad' equilibria are around the passive steady state. In fact, in a neighborhood of the steady state the rule would preserve the stabilizing properties of a Taylor rule, while for large shocks it would be optimal to shift to a money growth rule. Nevertheless, it would be appropriate to study the effects of such a rule on market expectations. How would such a more complicated rule impact on the investors' learning process? The answer is left to further research.

5 Backward-Looking Taylor Rule

In this section I argue that the instability generated by a forecast-based Taylor rule can be completely eliminated by shifting to a Taylor rule that reacts to current and past inflation. In fact, I show that such a policy rule improves on the forward looking rule in two important ways.

First, for plausible parametrizations, the economy does not have equilibrium cycles. Second, a backward-looking policy rule reverses the stability properties of the liquidity trap, leaving a unique learnable equilibrium: the inflation target.

Let us consider the case where the Central bank responds to inflation with some inertia. This is justified in Woodford (2002) and Benhabib et al. (2001a) as a more plausible description of Central Banks operating rules:

> "In practice, monetary policy will never involve feedback from an *instantaneous* rate of inflation [...], because available inflation measures will always be time-averaged over at least a period such as a month"[35]

In this case the Central Bank is assumed to react to an exponential moving average of current and past inflation rates that takes the form:

$$\overline{\pi}_t = (1-\delta)\sum_{j=0}^{\infty}\delta^j\pi_{t-j} \tag{23}$$

where $0 < \delta < 1$ defines how much weight is given to current inflation, as it is easily seen by re-writing (23) as:

$$\overline{\pi}_t = (1-\delta)\pi_t + \delta\overline{\pi}_{t-1} \tag{24}$$

By taking $\overline{\pi}_t$ as the inflation measure to which the Central Banker reacts, we can write the Taylor reaction function as:

$$R_t = 1 + (R^* - 1)\left(\frac{\overline{\pi}_t}{\pi^*}\right)^{\frac{A}{(R^*-1)}} \tag{25}$$

The next sections evaluate the properties of this rule under the two models used in the previous sections.

5.1 Money in the production function

Under the Taylor rule (25), the reduced-form model becomes[36]:

$$U_c\left(y\left(\rho\left(\overline{\pi}_t\right)\right)\right) = E\left\{\beta\left(1-\delta\right)U_c\left(y\left(\rho\left(\overline{\pi}_{t+1}\right)\right)\right)\left[\frac{\rho\left(\overline{\pi}_t\right)}{\overline{\pi}_{t+1}-\delta\overline{\pi}_t}\right]\right\} \tag{26}$$

Local stability and learnability conditions are considered in the following Proposition:

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[35] Woodford (2002), Chapter 2, p. 44.

[36] In both models, the solution for the backward looking Taylor rule is a non invertible (in both the forward and backward dynamic) one dimensional difference equation. Also, by computing the map numerically, it is possible to notice that both steady state belong to the same branch. The second branch either does not exist or it exists for values of the inflation rate that do not make economic sense.

Proposition 17 *Let G be given by (26). Given the system (15). Assume;*

$$\sigma < \max\left\{-\frac{\left(\epsilon_{\rho}\left(\pi^{*}\right)\epsilon_{y}\left(\overline{\pi}\right)\right)^{-1}}{(1-\delta)}, -\frac{1}{2\epsilon_{y}\left(\overline{\pi}\right)}\left(1+\frac{1}{\epsilon_{\rho}\left(\pi^{*}\right)}\right)\right\} \tag{27}$$

(i) Then the inflation target is determinate and learnable for every α, while the passive steady state is indeterminate and non-learnable for every α. Moreover, condition (27) is verified for every parameter value that gives a locally determinate inflation target under the forecast-based Taylor rule.

(ii) given the benchmark parametrization, for $\mu \in \left(\mu^{SR}, \mu^{LR}\right)$, no cycles exist around the active steady state;

Given the system (16), (17);

(iii) There exist $\epsilon' - SSEs$ around the indeterminate passive steady state but they are unstable under learning.

(iv) for any σ, it exists δ^{*} such that (27) holds;

Remark 18 *Note that under the (implausible) case where $\delta = 0$, statements (i)-(iii) hold for values of $\mu \in \left(\mu^{SR}, \mu^{LR}\right)$ and for $3.5 > \sigma > 1$, provided*[37] $\frac{A}{R^{*}} < 2.3$.

Remark 19 *From the linearized equation it is possible to check that the economy converges monotonically to the inflation target, for benchmark parameters (provided $\sigma < -\frac{\left(\epsilon_{\rho}(\pi^{*})\epsilon_{y}(\overline{\pi})\right)^{-1}}{(1-\delta)}$).*

Remark 20 *From the condition (27), the set of parameters which lead to local determinacy under the backward-looking Taylor rule is larger than for the forecast-based Taylor rule. Also, the set of parameters for which (i), (ii) and (iii) hold increases for more backward-looking policy rules ($\delta \to 1$).*

Remark 21 *By simulating the map numerically, it is clear that no cycles exist for the mentioned parameter values.*

The Proposition says that if the policy rule is sufficiently backward-looking, then it guarantees a unique learnable equilibrium, i.e. a unique equilibrium that it is robust to expectational mistakes. Hence, *it is always possible to find a policy rule that is enough backward-looking to stabilize the economy.* Also a contemporaneous Taylor rule guarantees a unique equilibrium, for plausible parameter values.

[37] Notice that the last condition must hold only in the worse case where $\mu = \mu^{LR}$ and $\sigma = 3.5$. With these parameter values the elasticity of output with respect to the interest rate at the passive steady state is very high in absolute value. Under the standard parametrization the condition becomes $\frac{A}{R^{*}} < 12$!

5.2 Money in the utility function

The reduced-form under the backward-looking rule (25) becomes:

$$U_c(\overline{y}, m(\rho(\overline{\pi}_t))) = \frac{\beta U_c(\overline{y}, m(\rho(\overline{\pi}_{t+1})))(1-\delta)}{\overline{\pi}_{t+1} - \delta\overline{\pi}_t}\rho(\overline{\pi}_t) \tag{28}$$

The stability properties of the Taylor rule are discussed in the following Proposition.

Proposition 22 *Let G be given by (28). Given the system (14) we have that for any parameter value;*

(i) the active steady state is determinate and learnable for any $\alpha \in (0,1)$;

(ii) the passive steady state is indeterminate and non-learnable for any $\alpha \in (0,1)$;

(iii) no cycles exist around the active steady state;

Given the system (16), (17);

(iii) there exist $\epsilon' - SSE$ sunspot equilibria around the passive steady state but they are not learnable.

Remark 23 *The result is quite strong in that it holds for any parameter value. Shifting to such a rule guarantees stability, under the assumptions of this model. It actually holds for also in the case $U_{cm} < 0$, provided $\epsilon_{cm} > \frac{1}{\epsilon_m(1-\delta)\epsilon_\rho(\pi^*)}$. Hence, by choosing the appropriate δ it is possible to obtain a stabilizing Taylor rule even in the case $U_{cm} < 0$.*

Remark 24 *As for Proposition (17), convergence to the inflation target is monotonic.*

Remark 25 *By simulations it is possible to show that no cycles exist, given that the map G is monotonic.*

Also for the case of money in the utility function, *if the central bank responds to a weighted average of current and past inflation rates it exists a unique equilibrium that is stable under learning: the inflation target.*

Concluding, from the Propositions 6 and 7 it is apparent that more backward-looking Taylor rules should be preferred to forecast-based rules for the following reasons: 1) they eliminate the possibility that adverse shocks might lead the economy to a liquidity trap, where the inflation target is systematically missed; 2) they reduce possible sources of instability in the economic system, generated by equilibrium fluctuations around the inflation target; 3) they increase model robustness, given that the parameter values for which we obtain a unique learnable equilibrium is increased. Notice also that the results are obtained allowing uncertainty about what the correct model of the economy should be, at least concerning the role of money: in fact the results are robust to both specifications of the model. In the next section I verify if the results hold under different assumptions on the model of the economy.

6 Extensions

6.1 Cash In Advance Constraint (CIA)

Let us consider a different trading environment as in Carlstrom and Fuest (2001b). They show that under CIA timing the stability of the model under a Taylor rule dramatically changes. Under a forecast-based Taylor rule determinacy is achieved by a passive policy stance. The goal of this paragraph is to show that, on the contrary, the previous results are robust to a modification in the trading environment.

In the simplest case of an endowment economy, the budget constraint becomes:

$$M_{t+1} = M_t + \tau + B_{t-1}R_{t-1} - B_t - P_t c_t + P_t y$$

and the CIA constraint is:

$$A_t = M_t + \tau + B_{t-1}R_{t-1} - B_t$$

where A_t denotes the liquidity available to the representative agent in the current period. Substituting back the CIA constraint in the utility functional (defined over current consumption and real liquidity) and solving the optimization under the resource constraint we obtain a money demand equation:

$$a_t = H\left(R_t\right) \quad H' < 0$$

and the following reduced-form:

$$U_c\left(c, H\left(R_t\right)\right) = U_c\left(c, H\left(\rho\left(R_{t+1}\right)\right)\right)\frac{\beta}{\pi_{t+1}}\rho\left(R_{t+1}\right) \tag{29}$$

In the following Proposition I compare the stability properties of forecast-based and backward-looking Taylor rules. As above, I consider the *most favorable* case where the rules lead to local determinacy and I study the possibility of learnable equilibria other than the inflation target. Where needed, the functional form for the utility function is CES. This leads to a demand for money that is equivalent to the case of money in the production function.

Proposition 26 *Let G be given by the CIA model. Assume:*

$$\left|\epsilon_{cm}\epsilon_H\right| > \frac{1}{2}\left(\frac{1}{\epsilon_\rho\left(\pi^*\right)} + 1\right) \tag{30}$$

where ϵ_H denotes the elasticity of the demand for money. Under the forecast-based Taylor rule;

(i) the active steady state is determinate and learnable for every α, under the learning rule (14);

(ii) the passive steady state is determinate and learnable for every α;

(iii) it exists $\widehat{\epsilon_\rho}\left(\pi^*\right)$ consistent with (30) such that for $\epsilon_\rho\left(\pi^*\right) > \widehat{\epsilon_\rho}\left(\pi^*\right)$ there exist learnable cycles and $\epsilon' - SSE$ around the active steady state, under the learning rules (14) and (16) respectively;

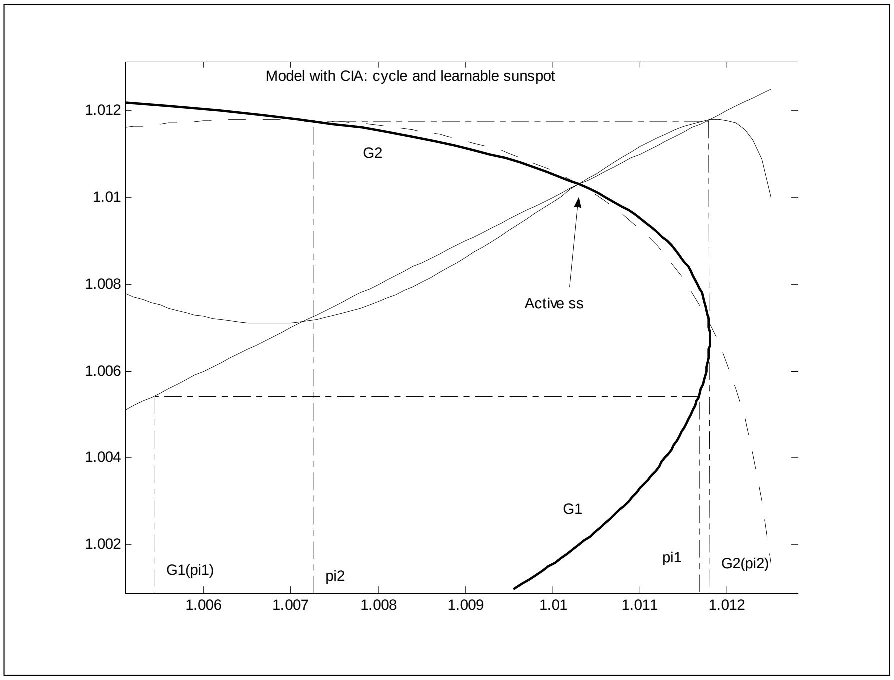

Figure 6: Figure (6)

Under the backward-looking Taylor rule;

(iv) the active steady state is determinate and learnable for every α and no cycles or sunspots exist around it for $\mu \in \left(\mu^{SR}, \mu^{LR}\right)$, under the learning rules (14) and (16) respectively;

(v) the passive steady state is indeterminate and non-learnable for every α under rule (14); there exist $\epsilon' - SSE$ sunspots around it but they are not learnable under (16).

(vi) for any value of $\epsilon_{cm}\epsilon_H$, it exists δ^* such that (iv) and (v) hold;

Remark 27 *A forecast-based policy rule that is aggressive enough to satisfy (30) exists provided the elasticity of money demand is not too low (i.e. $\mu \to \mu^{LR}$). Condition (30) is verified for quite high values of A. Under the most favorable condition ($\sigma \to 1$ and $\mu \to \mu^{SR}$) there exists $A < 3$ for which the condition holds.*

Remark 28 *As for Proposition 5, it is possible to show the existence of learnable sunspots around indeterminate and non-learnable cycles. Consider the following example, as shown in Figure (6). Set $\mu = \mu^{SR}$, $\sigma = 1.01$, $A = 3.2$ and the other parameters as in Table I. Then the sunspot equilibrium $\pi_1 = 1.0117$ (4.7%), $\pi_2 = 1.0068$ (2.7%) and $z_{11} = 0.027$, $z_{22} = 0.19$ is learnable. It is relatively close to the indeterminate and non-learnable cycle where high inflation $\pi_H = 1.0118$ (4.8%) and low inflation $\pi_L = 1.0071$ (2.8%)*[38]*.*

[38] In this case the stability conditions are calculated numerically, given that the map is not invertible both in backward and forward dynamics.

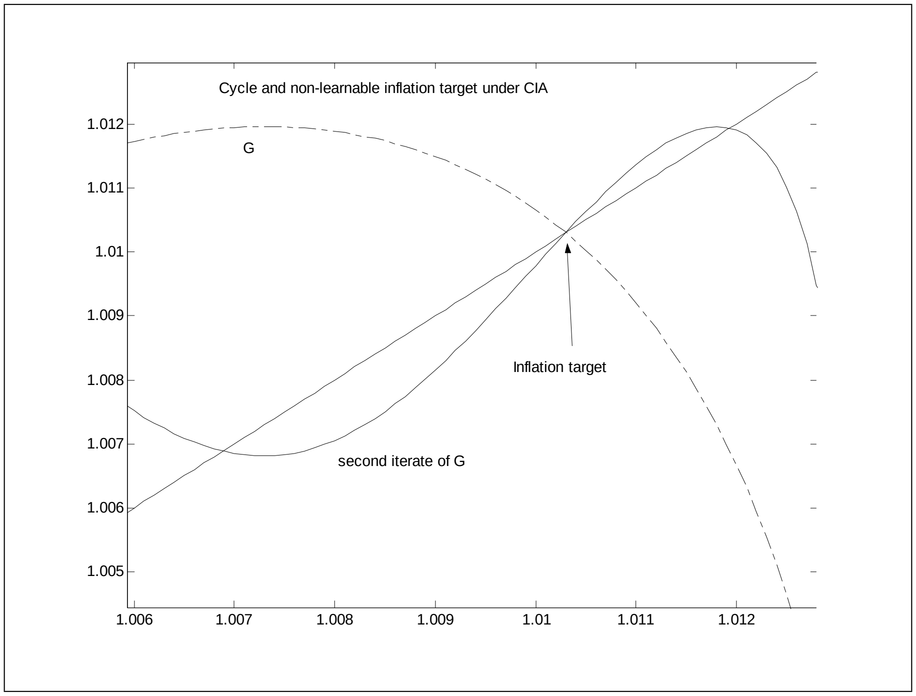

Figure 7: Figure (7)

Remark 29 *Figures (7),(8) make clear the result in (iv), for a given parametrization. It is immediate to see that the cycle appears only as the active steady state becomes indeterminate and thus (30) is violated.*

Remark 30 *Also in this case, the properties of the backward-looking Taylor rule are preserved for a wider set of parameter values than implied by condition (30): in other words, a backward-looking Taylor rule is stabilizing also for parameter values that imply indeterminacy under a forecast-based rule. Also, a more backward-looking rule (higher δ) increases the parameter space for which (iv) and (v) hold -see the proof of (vi).*

Figures (6), (8) give an example of the possible equilibria in the two cases. It is apparent that also in a different trading environment, forecast-based Taylor rules generate instability[39]: a too passive policy generates local indeterminacy, while a too active policy leads to multiple learnable equilibria around the locally determinate and stable inflation target equilibrium. Instead, shifting to a sufficiently back-ward looking rule guarantees a unique equilibrium. Hence, changing the trading environment does not reverse my results.

[39] We know from Carlstrom and Fuest (2001b) that a passive forward-looking Taylor rule achieves local determinacy. Nevertheless, it can be shown that a too passive Taylor rule leads to learnable 'inflationary traps'. The economy converges to a sunspot equilibrium where inflation fluctuates at hyperinflationary levels.

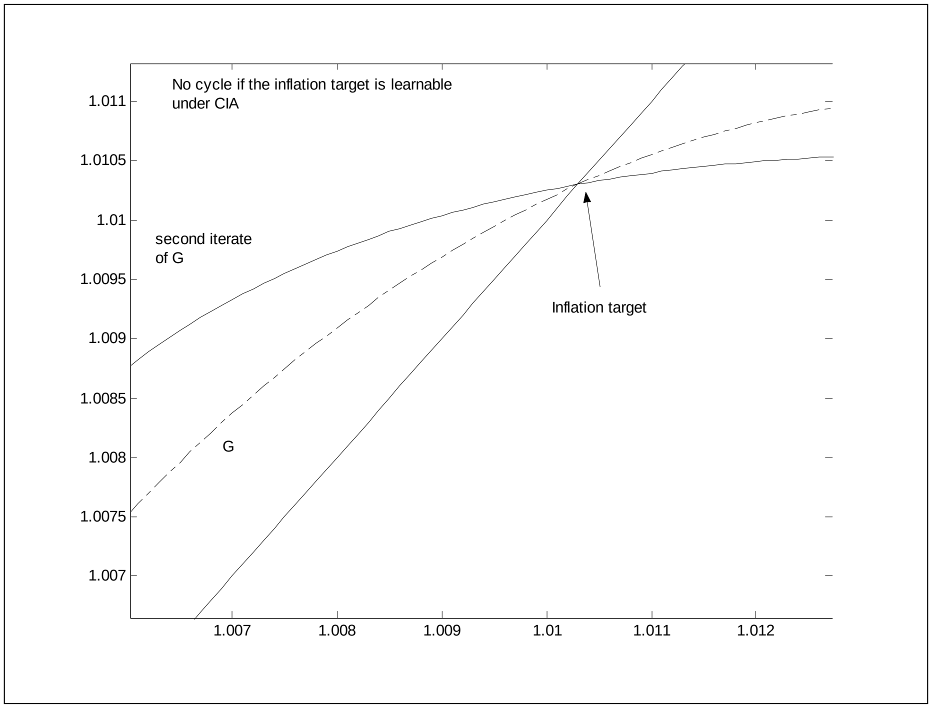

Figure 8: Figure (8)

6.2 The model with sticky prices

Let us consider endogenous labor supply, monopolistic competition and price stickiness. In this section I show that a *contemporaneous* Taylor rule ($\delta = 0$) maintains the 'good' properties described in the previous sections. I analyze the latter case for analytical simplicity, but it is also the least favorable parametrization to study the stability properties of a backward-looking Taylor rule, if prices are flexible. Nevertheless, Benhabib et al. (2002) find, in a continuos time version of the sticky price model, that instability can arise under perfect foresight, even if backward-looking rules are adopted. A study of the discrete time model with $\delta > 0$ implies a three dimensional system and it is left to further research[40].

I use a simplified version of the standard model considered in the literature on Taylor rules, i.e. Woodford (1999)[41]. The consumer-producer solves the following intertemporal problem:

$$\max_{a_t, m_t, h_t, p_t^j} \sum_{t=0}^{\infty} \beta^t \left[U\left(c_t, m_t\right) + V\left(h_t\right) - \frac{\theta}{2} \left(\frac{p_t^j}{p_{t-1}^j} - \pi^* \right)^2 \right] \tag{31}$$

[40] It is not immediate to extend the result in continous time to the discrete time model, because of timing issues. Also, there is the question of the learnability of the other equilibria that are discussed in Benhabib et al. (2002).

[41] The model is a discrete time version of Benhabib et al. (2001a,b).

$$a_t + c_t = \frac{R_{t-1}}{\pi_t} a_{t-1} + \frac{1 - R_{t-1}}{\pi_t} m_{t-1} + \frac{p_t^j}{p_t} Y^d d(\frac{p_t^j}{p_t}) - \tau_t$$

where $V(.)$ represents the dis-utility from labor and θ captures the costs of price revisions. Finally τ is a transfer from the government. As in the previous sections, I consider both cases of money in the production function and money in the utility function.

6.2.1 Money in the utility function

Given the equilibrium condition:

$$f(h_t, m_t) = Y^d d(\frac{p_t^j}{p_t}) = c_t \tag{32}$$

and the problem (31) we obtain the following first order conditions:

$$U_c(c_t, m_t) = \beta \frac{U_c(c_{t+1}, m_{t+1}) R_t}{\pi_{t+1}} \tag{33}$$

$$U_m(c_t, m_t) = U_c(c_t, m_t) \frac{(R_t - 1)}{R_t} \tag{34}$$

$$\begin{aligned} \theta(\pi_t - \pi^*) \pi_t &= \theta(\pi_{t+1} - \pi^*) \pi_{t+1} + \\ &+ c_t U_c(c_t, m_t) \eta \left(\frac{1+\eta}{\eta} - \frac{z'(h_t)}{f'(h_t) U_c(c_t, m_t)} \right) \end{aligned} \tag{35}$$

which define respectively the consumption Euler equation, the money demand and the price setting equation (which generates a version of the Phillips curve). From (32), (33) and (34) it is possible to express c_t as a function of m_t and R_t. By substituting it in both the Euler equation and the Phillips curve, and by using the contemporaneous Taylor rule, I obtain a two dimensional dynamical system in π_t and c_t. It is possible to show that the system is well defined under perfect foresight, under the functional form specification described above[42]. But, as in the case of flexible prices, the backward dynamics is not well defined. For this reason, I restrict the analysis to the local stability of the system under learning.

6.2.2 Money in the production function

In this case the first order conditions become:

$$U_c(c_t) = \beta \frac{U_c(c_{t+1}) R_t}{\pi_{t+1}} \tag{36}$$

[42] Notice that the money demand equation that we obtain from the FOC has the same specification that the one obtained under flexible prices, thus allowing for the same calibration.

$$U_c(c_t)\frac{(R_t-1)}{R_t}=\frac{z'(h_t)}{f_h(h_t,m_t)}f_m(h_t,m_t) \tag{37}$$

$$\begin{aligned}\theta(\pi_t-\pi^*)\pi_t &= \theta(\pi_{t+1}-\pi^*)\pi_{t+1}+ \\ &\quad +U_c(c_t)c_t\eta\left(\frac{1+\eta}{\eta}-\frac{z'(h_t)}{f'(h_t)U_c(c_t)}\right)\end{aligned} \tag{38}$$

By using the equilibrium condition $c_t = f(h_t, m_t)$, and equations (36) and (37) it is possible to express h_t and m_t both as a function of c_t and R_t. By substituting these expressions into the Euler equation and the Phillips curve and using the Taylor rule we obtain a two dimensional dynamical system in π_t and c_t. Also in this case, it is possible to show that it is well defined in forward dynamics[43] but not in backward dynamics.

6.2.3 Learning and Simulation Results

Unlike in the previous sections, it is not possible to give analytical results. Therefore, I study the stability properties of the system by numerical simulations. I consider a grid of plausible parameters' values: it is described in the Table 2 below[44]. The other parameters are left at benchmark values.

Table 2

Param.	Min	Max
η	-50	-3.5
σ	1.001	5
θ	35	350
A	1.4	3

Cycles and complex behavior. Under perfect foresight there is no evidence of local bifurcations and cycles around the active and passive steady states, in both cases of money in the production function and money in the utility function, for the parameter values described in the Table 2. As in the continuos time case[45], the active steady state is a *source* and the passive steady state is a *saddle*. For no parameter values the active steady state or the passive steady state reverse their stability so that I exclude local bifurcations. Under perfect foresight there is global indeterminacy as in Benhabib et al. (2001), so that, after a small shock, the economy might converge to a liquidity trap. Are these equilibria robust to expectational mistakes?

[43] This is achieved by imposing the restriction that the dis-utlility from labor. If this restriction is not satisfied, then also the forward map is a corrispondence.

[44] Notice that these parameter values include the case where $U_{cm<0}$.

[45] Described in Benhabib et al. (2001b).

Learnability of the steady states. The model under backward dynamics can be represented as follows:

$$\begin{aligned} c_t &= G_1\left(c_{t+1}, \pi_{t+1}\right) \\ \pi_t &= G_2\left(c_{t+1}, \pi_{t+1}\right) \end{aligned}$$

Having excluded the existence of cycles under perfect foresight, I simplify the analysis by restricting to the possibility of learning steady states. Hence, under the (asymptotically correct) agents' perceived low of motion, the economy evolves as $(c_t, \pi_t) = (\widetilde{c}, \widetilde{\pi})$, where the $\widetilde{}$ denotes a steady state value. Since the steady state values are unknown, the agents use the following simple rule to estimate them recursively:

$$\begin{aligned} \theta_t^c &= \theta_{t-1}^c + \alpha\left(G_1\left(\theta_{t-1}^c, \theta_{t-1}^\pi\right) - \theta_{t-1}^c\right) \\ \theta_t^\pi &= \theta_{t-1}^\pi + \alpha\left(G_2\left(\theta_{t-1}^c, \theta_{t-1}^\pi\right) - \theta_{t-1}^\pi\right) \end{aligned} \tag{39}$$

where θ^c is the estimate for $\widetilde{c}$, θ^π is the estimate for $\widetilde{\pi}$, and α is the learning parameter, as in the last sections. In order to study the learnability of the steady states, I linearize the backward map around the active and passive steady state and I evaluate the stability under learning for various parameter values (including α). The linearized system can be written as[46]:

$$\begin{bmatrix} \widehat{c}_t \\ \widehat{\pi}_t \end{bmatrix} = DGE_{t-1} \begin{bmatrix} \widehat{c}_{t+1} \\ \widehat{\pi}_{t+1} \end{bmatrix}$$

and local stability under learning depends on:

$$J = \begin{bmatrix} 1 + \alpha\left(DG_{11} - 1\right) & \alpha DG_{12} \\ \alpha DG_{21} & 1 + \alpha\left(DG_{22} - 1\right) \end{bmatrix} \tag{40}$$

Simulations show that the active steady state is learnable for every $\alpha \in (0, 1)$. On the other side, *the passive steady state is never learnable* for any $\alpha \in (0, 1)$. The liquidity trap that the model predicts under the assumption of perfect foresight is not robust to learning. Moreover, using a result from Honkapohja and Mitra (2001) I can exclude, from the numerical simulations, the existence of learnable sunspots around the passive steady states[47].

By converse, different results are obtained under forward looking Taylor rules. From simulations I obtain the following results:

[46] Note that this stability condition is weaker than E-stability.

[47] The result states that learnablility of sunspot equilibria around an indeterminate steady state is excluded if DG has at least one eigenvalue that has real part greater than one, as it is verified in this model for all the plausible parameter values.

1) under perfect foresight, there are parameter values for which the stability of the active steady state changes in both models with money in the utility function and money in the production function;

2) under learning, there are parameter values for which the stability of the active steady state under learning changes as α changes. In both cases the system is likely to undergo an Hopf bifurcation[48], thus generating instability even in the case of a locally determinate or learnable inflation target;

3) liquidity traps are non learnable for the model with $\theta > 0$. But in the case that prices are fixed for a *finite* time[49], the model can be approximated (asymptotically) by the solution of (31) with $\theta = 0$, and liquidity traps become learnable under both elastic labor supply and monopolistic competition.

Summing up, under more general assumptions about the market structure and market frictions the results discussed in the previous section seem to hold, for plausible parameter values. Taking into account more sources of uncertainty about the 'correct' model of the economy does not seem to change the predictions of the simple version of the model.

7 Conclusions

The paper shows that forecast-based Taylor rules can lead to bad outcomes, even in the case of local stability of the inflation target equilibrium and after excluding equilibria that are not robust to expectational mistakes. I study a very simple model with flexible prices, under different assumptions about the role of money. I find that in both cases where money enters in the utility or in the production function forecast-based Taylor rules are de-stabilizing in the following sense. Even if the inflation target (or active) steady state is locally unique and stable under learning, there exist other 'global' equilibria that are also stable under learning. In particular, the economy can be driven into a liquidity trap, under both model specifications. Also, the economy can converge to learnable cycles and sunspots that can be either close to the inflation target or to the liquidity trap equilibrium, depending on the role of money. These results show that the multiplicity of equilibria generated by the forecast-based Taylor rule are not just a theoretical curiosity but they seem to be *robust*, at least to expectational mistakes.

On the other hand, more backward-looking policy rules stabilize the economy in the following sense. The equilibria that exist under forecast-based Taylor rules either cease to exist or are not robust to expectational mistakes. In fact, backward-looking Taylor rules lead to a unique learnable

[48] In this case, for nearby parameters we have that the active steady state is sourraunded by an attracting closed curve. In different models, Grandmont et al. (1998) and Bloise (2001) show the possibility of constructing sunspots that might be learnable. DeVilder (1995) shows, under the assumption of perfect foresight, that a Hopf bifurcation might lead to higher period cycles that also might be learnable.

[49] see Carltrom and Fuest (2000).

equilibrium: the inflation target. Deterministic cycles, sunspots equilibria or the liquidity trap cease to exist or are not stable under learning.

Moreover, the results are robust to different model specification. I consider the case of a different trading environment, as in Carlstrom and Fuest (2001), and I show that my results do not change. This is somewhat surprising, given that Carlstrom and Fuest (2001) show how conclusions about determinacy can change dramatically as we change the timing of the model. I also extend the model with endogenous labor supply, monopolistic competition and sticky prices and show that a *contemporaneous* (reacting only to the current inflation rate) Taylor rule stabilizes the economy, as for the case of the simple model.

Hence, the paper shows more evidence of the superiority of Taylor rules based on current and past information rather than forecasts.

8 Appendix

Proof of Proposition (1)

(i) I just re-state the results in Benhabib et al. (2001a, 2001b). Notice that the fixed points of F and (15) are exactly the same.

(ii) In order to verify the local stability of the system (15), under the backward map implied by (8), these conditions on the Jacobian need to be verified:

$$|D| = \left|(1-\alpha)^2\right| < 1 \tag{41}$$

$$|T| = \left|G'(\widetilde{\pi})^2 \alpha^2 - 2\alpha + 2\right| < |1+D| \tag{42}$$

where $\widetilde{\pi}$ denotes the fixed point. It is straightforward to show that (41), (42) imply the following necessary and sufficient condition for the learnability of the steady state:

$$\left|G'(\pi)\right| < 1$$

We do not need to know explicitly on which branch of G the fixed point is. In fact, I use the fact that around the fixed point:

$$G'(\pi) = \frac{1}{F'(\pi)}$$

Hence, determinacy implies learnability and indeterminacy implies non-learnability. Given that the condition in the Proposition implies determinacy of both steady states (as it is easy to see from (18)), (ii) is proved.

Proof of Proposition (4)

(i)-(iii) the proof follows the same steps as in Proposition (1) and it is therefore omitted.

(iv) this follows directly from Evans and Honkapohja (2001b): if $G'(\pi) < -1$ then there exist E-Stable (and therefore learnable under (16) and (17)) stationary sunspot equilibria around the fixed point. This condition is verified exactly for $-(\frac{1}{\epsilon_\rho(\overline{\pi})} - 1) < \epsilon_{cm}\epsilon_m < -(\frac{1}{\epsilon_\rho(\overline{\pi})} - 1)/2$.

Proof of Proposition (6)

First, notice again that although we cannot provide an analytical expression for G_i we can use 1) the fact that locally the map is conjugate to a linear map, and; 2) its relation with the forward looking map F.

The conditions for stability of the cycle under learning are:

$$|D| = \left|(1-\alpha)^2\right| < 1 \tag{43}$$

$$|T(\sigma)| = \left|\nu(\sigma)\alpha^2 - 2\alpha + 2\right| < |1+D| \tag{44}$$

where:

$$\begin{aligned}\nu(\sigma) &= \frac{1}{F'\left(\pi_H(\sigma)\right)F'\left(\pi_L(\sigma)\right)} \\ &= \Pi_{i=1}^{2}\left[1+\frac{1}{-\sigma\epsilon_{yi}(\sigma)}\left(\frac{1}{\epsilon_{\rho i}(\sigma)}-1\right)\right]\end{aligned}$$

and where $-\sigma, \epsilon_{y_i}$ and $\epsilon_{\rho i}$are the elasticities computed at the two values of π over the cycle. From (43), (44) it is straightforward to obtain a necessary and a sufficient condition for local stability of deterministic cycles of period 2 [50]. We have:

$$\text{local stability} \Longrightarrow \Pi_{i=1}^{2} G_i' < 1 \tag{45}$$

$$\left|\Pi_{i=1}^{2} G_i'\right| < 1 \Longrightarrow \text{local stability} \tag{46}$$

where i denotes the branch where the derivative is evaluated.

A sufficient condition for stability under learning, for every value of α is:

$$\begin{aligned} & \left|G_2'\left(\pi_H(\sigma)\right)G_1'\left(\pi_L(\sigma)\right)\right| \\ = \; & |\nu(\sigma)| < 1\end{aligned}$$

which implies $|1/v(\sigma)| = |F'\left(\pi_H(\sigma)\right)F'\left(\pi_L(\sigma)\right)| > 1$. Now, we know that at $\pi_H(\sigma)$:

$$\Delta_1(\sigma) = \frac{1}{-\sigma\epsilon_{y1}(\sigma)}\left(\frac{1}{\epsilon_{\rho 1}(\sigma)}-1\right) < -2 \tag{47}$$

given the requirement that the active fixed point is stable. The stability condition can be rewritten as:

$$|1+\Delta_1(\sigma)+\Delta_2(\sigma)+\Delta_1(\sigma)\Delta_2(\sigma)| > 1$$

where:

$$\Delta_2(\sigma) = \frac{1}{-\sigma\epsilon_{y2}(\sigma)}\left(\frac{1}{\epsilon_{\rho 2}(\sigma)}-1\right)$$

Let us start with the negative inequality:

$$\begin{aligned} 1/v(\sigma) &< -1 \Longrightarrow \\ \Delta_2(\sigma) &> -\frac{2+\Delta_1(\sigma)}{1+\Delta_1(\sigma)} \end{aligned} \tag{48}$$

[50] The result is obtained by Guesnerie and Woodford (1991), pag. 118 for a similar learning algorithm.

From the Figure 2 and (47) we have that as σ increases $\Delta_2(\sigma)$ decreases, as the policy rule becomes more active at $\pi_L(\sigma)$. Conversely $\Delta_1(\sigma)$ increases as $\pi_H \to \pi^*$. Hence, the RHS of (48) increases as σ increases (using the fact that $\Delta_1(\sigma) < -2$). From continuity of F with respect to σ it exists $\widehat{\sigma}$ such that :

$$\Delta_2(\widehat{\sigma}) = -\frac{2 + \Delta_1(\widehat{\sigma})}{1 + \Delta_1(\widehat{\sigma})} > -1 \tag{49}$$

and thus for $\sigma < \widehat{\sigma}$ we have stability under learning for every admissible α.

Let us consider the case where:

$$-1 < 1/v(\sigma) < 0 \tag{50}$$

Under (50) the sufficient condition (46) is not satisfied, and $\sigma > \widehat{\sigma}$. But, from (45), the necessary conditions for stability are:

$$v(\sigma)^{-1} < 0 \quad v(\sigma)^{-1} > 1$$

of which only the latter is also sufficient. It is easy to verify that $v(\sigma)^{-1} < 0$ holds for $\Delta_2(\sigma) > -1$. Since we know that $\Delta_2(\sigma)$ is decreasing in σ, from continuity it exists $\widetilde{\sigma} > \widehat{\sigma}$ (where $\Delta_2(\widetilde{\sigma}) = -1$) such that for $\sigma \in (\widehat{\sigma}, \widetilde{\sigma})$ the necessary condition is satisfied.

Now it is clear from (43), (44) that in this case there exists an $\widehat{\alpha}_\sigma$ of low enough value such that the stability conditions are verified.

In order to show what happens in the case $\sigma \in [\widetilde{\sigma}, \overline{\sigma})$ I use the second necessary (and sufficient) condition implied by (46):

$$1/v(\sigma) > 1$$

It is easy to see that this condition is verified if $F'(\pi_L)$ is large and negative: in other words, if the policy rule at $\pi_L(\sigma)$ is sufficiently active. Given that $\Delta_2(\sigma)$ is decreasing in σ, we could have a $\widetilde{\sigma} < \sigma < \overline{\sigma}$ where the cycle is stable. We now prove that this is not possible.

We have that for $\sigma = \widetilde{\sigma} + \epsilon$, for $\epsilon > 0$ arbitrarily small, $v(\sigma) > 1$. But we know that this implies that the stability conditions (43), (44) are violated for any α. As $\sigma \to \overline{\sigma}$ we have that $v(\sigma) \to 1 = v(\overline{\sigma})$, where $|T(\overline{\sigma})| = |1 + D|$. But this is the point where the cycle disappears. Hence, there is no $\sigma < \overline{\sigma}$ such that $1/v(\sigma) > 1$ and the cycle exists. Hence, for $\sigma > \widetilde{\sigma}$ we have instability under learning, for every parameter of α and this completes the proof.

Proof of Proposition (11)

(i) first using Proposition 12.5 of Evans and Honkapohja (2001a) the existence of sunspots ϵ-close to the deterministic cycle is guaranteed, provided $G_2(\pi_H(\sigma))G_1(\pi_L(\sigma)) \neq 1$. Since we know that this condition is violated only in the case where the cycle is destroyed ($\sigma = \overline{\sigma}$), then it is possible to construct stationary sunspots on the deterministic cycle for all $\sigma < \overline{\sigma}$.

Second, by using Proposition 12.7 and 12.9 in Evans and Honkapohja (2001a) we have that necessary and sufficient condition for learnability of sunspots with the algorithm (14) is that:

$$G_2'(\pi_H(\sigma))G_1'(\pi_L(\sigma)) < 1$$

But we know from Proposition (6) that this condition holds for every $\sigma < \widetilde{\sigma}$.

(ii), from Proposition 12.2 of Evans and Honkapohja (2001a)[51] we have that strong E-Stability is obtained for $|G_2'(\pi_H(\sigma))G_1'(\pi_L(\sigma))| < 1$. From Proposition6 this is verified if $\sigma < \widehat{\sigma}$.

(iii) this is an application of Proposition 12.7 and 12.9 in Evans and Honkapohja (2001a): sunspots constructed on the active and passive steady state exists and are learnable if and only if $|G_2'(\pi_1(\sigma))| < 1$ and $|G_1'(\pi_2(\sigma))| < 1$ which is verified under the condition in Proposition (1).

(iv) If we consider explicitly the map in its backward dynamics, it is actually possible to show that sunspot equilibria exist outside a neighborhood of the deterministic cycle, both in the cases where the cycle is learnable and not. The result depends on the existence of the two branches G_1 and G_2.

These sunspots can be constructed by using the global methods of Grandmont (1987): a sunspot equilibrium exists if and only if $\pi_1(\sigma)$ and $\pi_2(\sigma)$ are included in the open interval formed by their images:

$$G_i(\pi_1(\sigma)) < \pi_1(\sigma) < \pi_2(\sigma) < G_j(\pi_2(\sigma))$$

or

$$G_i(\pi_2(\sigma)) < \pi_1(\sigma) < \pi_2(\sigma) < G_j(\pi_1(\sigma)) \tag{51}$$

where i and j denote the branch of G. Figure 9 gives an example: as $\sigma = 2.35 > \widehat{\sigma}$, the cycle is indeterminate in forward dynamics, and non-learnable. By continuity of the eigenvalues, also sunspots in the neighborhood of the cycle are not learnable. But it is possible to construct sunspots relatively close to the cycle by choosing the appropriate branch of G.

The Figure shows how to construct such a cycle: it is easy to verify that condition (51) is satisfied.

Since it is possible to show numerically that the correspondence G keep the same form for $\sigma \in (1, \overline{\sigma})$ - under the benchmark parametrization -, it is evident that it is also possible to construct sunspot equilibria on the two branches for any admissible σ.

Let us consider the learnability of such sunspots. First, given a two-state sunspot equilibrium (I limit my analysis to those equilibria) we have that:

$$\begin{aligned} \pi_1(\sigma) &= z_{11}G_1(\pi_1(\sigma)) + (1 - z_{11})G_2(\pi_2(\sigma)) \\ \pi_2(\sigma) &= (1 - z_{22})G_1(\pi_1(\sigma)) + z_{22}G_2(\pi_2(\sigma)) \end{aligned} \tag{52}$$

where (52) says that, given $\pi_1(\sigma)$ the agents choose the current π using the first branch, while given $\pi_2(\sigma)$ the decision is taken using the second branch, as in Figure 9. From (52) it is possible to recover the value of the transition probabilities as a function of the states $\pi_1(\sigma), \pi_2(\sigma)$:

[51] pg. 303.

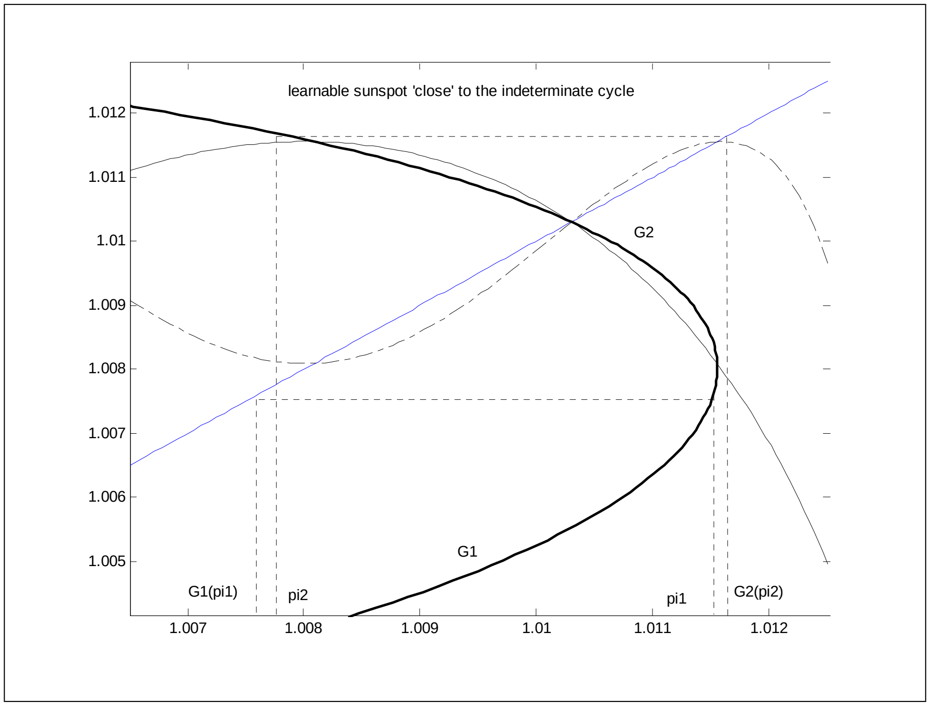

Figure 9: Figure (9)

$$
\begin{aligned}
z_{11} &= [G_2(\pi_1(\sigma)) - G_1(\pi_2(\sigma))]^{-1} [\pi_1(\sigma) - G_1(\pi_2(\sigma))] \\
z_{22} &= [G_2(\pi_1(\sigma)) - G_1(\pi_2(\sigma))]^{-1} [G_2(\pi_1(\sigma)) - \pi_2(\sigma)]
\end{aligned}
\tag{53}
$$

The condition (51) guarantees that the values found for z_{11} and z_{22} are probabilities.

Second, it is well known from Evans and Honkapohja (1995) that local convergence under learning can be studied by considering E-Stability conditions[52].

This is verified if the eigenvalues of the following matrices are less than one:

$$
\begin{bmatrix}
z_{11}G_1'(\pi_1(\sigma)) & (1-z_{11})G_2'(\pi_2(\sigma)) \\
(1-z_{22})G_1'(\pi_1(\sigma)) & z_{22}G_2'(\pi_2(\sigma))
\end{bmatrix}
\tag{54}
$$

Now, using trace and determinant, stability conditions are:

$$
|D(\sigma)| = \left|(z_{11} + z_{22} - 1)G_1'(\pi_1(\sigma))G_2'(\pi_2(\sigma))\right| < 1 \tag{55}
$$

$$
\left|-z_{11}G_1'(\pi_1(\sigma)) - z_{22}G_2'(\pi_2(\sigma))\right| < |1 + D(\sigma)| \tag{56}
$$

[52] For details see Evans and Honkapohja (2001), Evans and Honkapohja (1995).

For $\sigma < \widehat{\sigma}$ the conditions are satisfied because $|G_1'(\pi_1(\sigma))G_2'(\pi_2(\sigma))| < 1$: it is sufficient to choose the states inflation values close to the cycle.

Let us consider the case $\sigma > \widehat{\sigma}$. First notice that $-1 < G_2'(\pi_2(\sigma)) = \frac{1}{F'(G_2(\pi_2(\sigma)))} < 0$ because $G_2(\pi_2(\sigma)) > \pi^*$, for every σ. Fix $\pi_2(\sigma)$ such that $\pi_2(\sigma) \leq \pi_L$ (i.e. the low inflation rate at the equilibrium cycle).

Second, fix $\pi_1(\sigma)$ such that $G_1'(\pi_1(\sigma)) > 0$. This value exists for every admissible σ: in fact, as $\pi_1(\sigma) \to \overline{\pi}$ $G_1'(\pi_1(\sigma)) = \frac{1}{F'(G_1(\pi_1(\sigma)))}$ turns positive and decreases in absolute value, given that $F'(\overline{\pi}) > 1$ for every σ. Moreover, I checked numerically[53] (and it is clear from the picture, for a given σ) that $0 < G_1'(\pi^*) < 1$, for every σ. Hence, it is possible to choose $\pi_1(\sigma) \in (\pi^*, \pi_H(\sigma))$ such that $0 < G_1'(\pi^*) < 1$. But this implies that both (55) and (56) are satisfied and the sunspot is learnable.

Proof of Proposition (17):

(i) The linearization gives the following difference equation:

$$\widehat{\pi}_{t+1} = \left[\frac{\left(-\sigma\epsilon_y - \frac{(1-\delta)\epsilon_\rho + \delta}{(1-\delta)\epsilon_\rho}\right)}{\left(-\sigma\epsilon_y - \frac{1}{(1-\delta)\epsilon_\rho}\right)} \right] \widehat{\pi}_t \tag{57}$$

Given that at the active ss $\epsilon_\rho > 1$, local uniqueness is clearly guaranteed by (27). Learnability follows as in Proposition (1). On the other hand, for $\epsilon_\rho < 1$ the result is reversed.

(ii) let us consider the map G_σ under the contemporaneous rule. From the linearized equation, it is apparent that for $|-\sigma\epsilon_y| < \frac{1}{(1-\delta)\epsilon_\rho}$ no bifurcation occurs. In fact, at the active steady state $0 < G_\sigma'(\pi^*) < 1$.For $\widehat{\sigma}$ such that $|-\widehat{\sigma}\epsilon_y| = \frac{1}{(1-\delta)\epsilon_\rho}$ the derivative is not defined and there is a discontinuity. For $-\frac{(1-\delta)\epsilon_\rho+\delta}{\epsilon_y(\overline{\pi})(1-\delta)\epsilon_\rho} > \sigma > \widehat{\sigma}$ we have that $G_\sigma'(\pi^*) < 0$. Hence, $G_\sigma'(\pi^*)$ does not cross 1. Instead, for $|\sigma| = -\frac{1}{2\epsilon_y(\overline{\pi})}\left(1 + \frac{1}{\epsilon_\rho(\pi^*)}\right)$ it is straightforward to check that the system undergoes a flip bifurcation (it exists a σ^* such that $G_{\sigma^*}'(\pi^*) = -1$). The next Lemma, shows that this bifurcation is sub-critical: this means that as the active steady state is determinate (and thus (27) holds), no cycle around it exists.

Lemma *At each $\mu \in \left(\mu^{SR}, \mu^{LR}\right)$ there exists $\sigma(\mu) = -\frac{1}{2\epsilon_y(\mu)}\left(1 + \frac{1}{\epsilon_\rho}\right)$ such that a period-doubling bifurcation occurs. For $\sigma < \sigma(\mu)$ the active steady state is determinate and for $\sigma > \sigma(\mu)$ it is indeterminate. (2) For $\sigma > \sigma(\mu)$ and sufficiently close to the bifurcation point a period two cycle exists. For any $\sigma < \sigma(\mu)$ no cycle exists.*

Proof of Lemma. From Devaney (1989) we know that the condition for a period doubling bifurcation is:

$$e = \left(\frac{1}{2}\left(\frac{\partial^2 F}{\partial \pi^2}\right)^2 + \frac{1}{3}\left(\frac{\partial^3 F}{\partial \pi^3}\right) \right) \neq 0 \text{ at } (\pi^*, \sigma(\mu))$$

[53] I checked numerically that for $\sigma \geq 1.001$ and $\sigma \leq 2.5$ ($\geq \overline{\sigma}$,given that the cycle does not exists for that values of σ) the derivative has values ranging from 0.15 to 0.85.

Moreover, given

$$d = \frac{\partial F_{\sigma(\mu)}}{\partial \sigma} \text{ at } \pi^*$$

we have that if $\frac{d}{e} > 0$ then the bifurcation is sub-critical (the cycle exists when the fixed point is stable). It is trivial to verify from the linearized equation that $d < 0$. It is also possible to show numerically[54] that $e < 0$ for $\mu \in (\mu^{SR}, \mu^{LR})$. Moreover, from the linearized equation we have that at $\sigma = \sigma(\mu)$ F' is equal to -1. Hence (1) and (2) follow. This proves that no cycles exist around the active steady state.

ii) the linearized equation implies that under condition (27), we have:

$$G'(\overline{\pi}) > 1$$

where G represent the backward map under the contemporaneous rule. This is the condition for the non-learnability of sunspots equilibria, as in Propositions 12.6 and 12.10 in Evans and Honkapohja (2001a).

(iv) trivial from (27). This completes the proof.

Proof of Proposition (22):

(i) After linearization, the local behavior of the system is determined by the linear difference equation:

$$\widehat{\pi}_{t+1} = \left[\frac{\left(\epsilon_{cm}\epsilon_m - \frac{(1-\delta)\epsilon_\rho + \delta}{(1-\delta)\epsilon_\rho} \right)}{\left(\epsilon_{cm}\epsilon_m - \frac{1}{(1-\delta)\epsilon_\rho} \right)} \right] \widehat{\pi}_t \tag{58}$$

which gives determinacy (and learnability) under active monetary policy, for any parameter values. By converse, passive policy imply indeterminacy (and non-learnability).

(ii) follows the same steps as (ii) in Proposition (17).

(iii) it is apparent that $\frac{\left(\epsilon_{cm}\epsilon_m - \frac{(1-\delta)\epsilon_\rho + \delta}{(1-\delta)\epsilon_\rho} \right)}{\left(\epsilon_{cm}\epsilon_m - \frac{1}{(1-\delta)\epsilon_\rho} \right)}$ is always positive. Hence, at the passive steady state $\frac{\left(\epsilon_{cm}\epsilon_m - \frac{(1-\delta)\epsilon_\rho + \delta}{(1-\delta)\epsilon_\rho} \right)}{\left(\epsilon_{cm}\epsilon_m - \frac{1}{(1-\delta)\epsilon_\rho} \right)} > 1$ and Proposition 12.6 and 12.10 in Evans and Honkapohja (2001a) allow to exclude learnability of the sunspots equilibria.

Proof of Proposition (26)

(i)-(v) In the case of backward-looking, the linearization leads to:

$$\widehat{\pi}_t = \left[1 + \frac{(1-\delta)(\epsilon_\rho - 1)}{\epsilon_{cm}\epsilon_H\epsilon_\rho (1-\delta) - \delta} \right] \widehat{\pi}_{t+1}$$

In the case of forward looking:

[54] The Maple program is available on request.

$$\widehat{\pi}_{t+2} = \frac{\left(\epsilon_{cm}\epsilon_H + \frac{1}{\epsilon_\rho}\right)}{(\epsilon_{cm}\epsilon_H + 1)}\widehat{\pi}_{t+1}$$

The proof follows following the same steps as for Proposition (6), Proposition (11) and Proposition (17). Concerning (iv): it is possible to show numerically, as I do for the Lemma, that the change of stability of the inflation target leads to a subcritical bifurcation for $\mu \in \left(\mu^{LR}, \mu^{SR}\right)$. Figure 7 shows the bifurcation cycle for a given value of μ.

(vi) it is easy to verify that (iv) and (v) hold if:

$$|\epsilon_{cm}\epsilon_H| > \frac{(\epsilon_\rho - 1)}{2\epsilon_\rho} - \frac{\delta}{\epsilon_\rho (1-\delta)}$$

Then, by letting $\delta \to 1$ the statement is proved.

References

[1] Azariadis, C. and R. Guesnerie, 1986, Sunspots and Cycles, The Review of Economic Studies 53, 725-737.

[2] Batini, N. and E. Nelson, 2001, Optimal Horizon for Inflation Targeting, Journal of Economic Dynamics and Control 25, 891-910.

[3] Benhabib, J., Schmitt-Grohe', S. and M. Uribe, 2001a, Monetary Policy and Multiple Equilibria, American Economic Review 91, 167-185.

[4] Benhabib, J., Schmitt-Grohe', S. and M. Uribe, 2001b, Chaotic Interest Rate Rules, mimeo, New York University.

[5] Benhabib, J., Schmitt-Grohe', S. and M. Uribe, 2001c, Avoiding liquidity traps, mimeo, New York University.

[6] Benhabib, J., Schmitt-Grohe', S. and M. Uribe, 2002, Designing Monetray Policy: Backward-Looking Interest-Rate Rules, New York University.

[7] Bernanke, B. S. and M. Woodford, 1997, Inflation Forecasts and Monetary Policy, Journal of Money, Credit and Banking 29, 653-684.

[8] Bloise, G., 2001, A Geometric Approach to Sunspot Equilibria, Journal of Economic Theory 101, 519-539.

[9] Bullard, J. and J. Duffy, 1998, On Learning and the Stability of Cycles, Macroeconomic Dynamics 2, 22-48.

[10] Bullard, J. and K. Mitra, 2001, Learning About Monetary Policy Rules, Federal Reserve Bank of St. Louis Working Paper.

[11] Carlstrom, C. and T. Fuerst, 2000, Forward vs. Backward-Looking Taylor Rules, Working Paper, Federal Reserve Bank of Cleveland.

[12] Carlstrom, C. and T. Fuerst, 2001a, Learning And the Central Bank, Federal Reserve Bank of Cleveland Working Paper.

[13] Carlstrom, C. and T. Fuerst, 2001b, Timing And Real Indeterminacy in Monetary Models, Journal of Monetary Economics 47, 285-298.

[14] Christiano, L. J.and S.Harrison, 1999, Chaos, Sunspots and Automatic Stabilizers, Journal of Monetary Economics 44, 3-31.

[15] Christiano, L.J., and M. Rostagno, 2001, Money Growth Monitoring and the Taylor Rule, NBER Working Paper 8539.

[16] Clarida, R., Gali', J. and M. Gertler, 1997, Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory, mimeo New York University.

[17] Devaney, R. L., 1989 An Introduction to Chaotic Dynamical Systems (Addison-Wesley).

[18] Evans, G.W. and S. Honakapohja, 1995, Local Convergence of Recursive Learning to Steady States and Cycles in Stochastic Nonlinear Models, Econometrica 63, 195-206.

[19] Evans, G.W. and S. Honakapohja, 2001a, Learning and Expectations in Macroeconomics, manuscript.

[20] Evans, G.W. and S. Honakapohja, 2001b, Existence of Adaptively Stable Sunspot Equilibria near an Indeterminate Steady State, Working Paper, University of Helsinki.

[21] Evans, G. W, Honkaophja S and R. Marimon, 2001, Stable Sunspot Equilibria in a Cash in Advance Economy, Working Paper, University of Helsinki.

[22] Eusepi, S., 2001, Optimal Monetary Policy Under Bounded Rationality, mimeo Universita' "Tor Vergata" of Rome.

[23] Grandmont, J.M., 1985, On Endogenous Competitive Cycles, Econometrica 53, 995-1045.

[24] Grandmont, J. M.,1986, Stabilizing Competitive Business Cycles, Journal of Economic Theory 40, 57-79.

[25] Grandmont, J. M., 1989, Local Bifurcations and Stationary Sunspots in: Barnett W., Geweke J. and K. Shell, eds, Economic Complexity, Chaos, Sunspots, Bubbles and Nonlinearity (Cambridge University Press)

[26] Grandmont, J.M., 1998, Expectations Formation and Stability of Large Socioeconomic Systems, Econometrica 66, 741-781.

[27] Grandmont, J.M, Pintus P. and de R. Vilder, 1998, Capital-Labor Substitution and competitive nonlinear endogenous business cycles, Journal of Economic Theory 80, 14-59.

[28] Guesnerie, R. and M. Woodford, 1991, Stability of Cycles With Adaptive Learning Rules, in: Barnett, W., Cornet, B., D'Aspremont, C., J., G., and A. Mas-Colell, eds, Equilibrium Theory and Applications (Cambridge University Press).

[29] Honkapohja, S. and K. Mitra, 2001, Are Non-Foundamental Equilibria Learnable in Models of Monetary Policy?, Working Paper, University of Helsinki.

[30] Levin, A., Wieland, V. and J.C. Williams, 2001, The Performance of Forecast-Based Monetary Policy Rules Under Model Uncertainty, ECB Working Paper 68.

[31] McCallum, B. T., 2001, Inflation Targeting and the Liquidity Trap, NBER Working Paper 8225.

[32] Svensson, L. E. O., 2001, Requiem for Forecast-Based Instrument Rules, Working Paper, Princeton University.

[33] Taylor, B. J., ed., 1999 Monetary Policy Rules (University of Chicago Press).

[34] de Vilder, R., 1994, Endogenous Business Cycles, Tinbergen Institute Research Series book no. 96.

[35] Wiggins, S., 1990, Introduction to Applied Nonlinear Dynamical Systems and Chaos (Springer Verlag).

[36] Woodford, M., 1999, Optimal Monetary Policy Inertia, NBER Working Paper n. 7261.

[37] Woodford, M., 2000, Pitfalls of Forward-Looking Monetary Policy, American Economic Association Papers and Proceedings 96 (2), 100-104.

[38] Woodford, M., 2002, Interest and Prices: Foundations of a Theory of Monetary Policy, manuscript, Princeton University.