Federal Reserve Bank of New York
Staff Reports

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Federal Reserve Bank of New York Staff Reports, no. 224
October 2004; revised August 2006
JEL classification: E31, E51, E58

Abstract

This paper studies an overlapping generations economy with capital where limited communication and stochastic relocation create an endogenous transactions role for fiat money. We assume a production function with a knowledge externality (Romer-style) that nests economies with endogenous growth (AK form) and those with no long-run growth (the Diamond model). We show that the Tobin effect is always operative. Under CRRA (constant relative risk aversion) preferences, a mild degree of social increasing returns is sufficient (but not necessary) for some positive inflation to dominate zero inflation and for the Friedman rule to be suboptimal, irrespective of the degree of risk aversion.

Key words: Friedman rule, Tobin effect, monetary policy

1   Introduction

The Friedman rule, Milton Friedman's classic prescription for the optimal conduct of monetary policy, remains to date the most significant dictum in monetary theory. Friedman (1969) argued that good monetary policy is one that equates the private opportunity cost of holding money (the nominal interest rate) to its social opportunity cost (which is zero). By this logic, optimal monetary policy should never be expansionary. Critics were quick to point out potential problems with this line of thinking. Phelps (1973) argued that following a contractionary policy as proposed by Friedman may require the government to make up the lost seigniorage using distortionary means which may negate the alleged benefit of the policy. Symmetrically, others have argued that seigniorage may have enough beneficial uses to justify an expansionary policy.[1]

This paper studies a third potential limitation of Friedman's logic using an argument first articulated in Tobin (1965): what if monetary expansion caused income to rise and grow, thereby overwhelming the non-distortionary benefit of following a contractionary policy? In today's parlance, if the Tobin effect is operative, can the Friedman rule ever be optimal? In a sense, the Tobin effect and the Friedman rule represent two divergent views on the desirability of inflation. The former argues that inflation, by raising the relative return to capital, stimulates capital formation and hence growth. The latter argues that monetary expansion raises the opportunity cost of holding real balances and makes liquidity, potentially a desirable commodity, more costly. Which effect dominates?

This paper addresses this question within the context of a monetary growth model. We specify an overlapping generations model economy with capital where limited commu-

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[1]Levine (1991) considers an environment in which there are two types of infinitely-lived agents who randomly become buyers or sellers and information on agents' type is private. If buyers value consumption sufficiently more than sellers do, and if there is some randomness in the economy, then Levine shows that the optimal monetary policy is *expansionary* and not contractionary as the Friedman rule would suggest. As in our setting, lump-sum taxes that fund the contraction are imposed symmetrically on both the types. As such, a contraction hurts "an unlucky buyer" and because buyers value consumption sufficiently more than sellers do, this monetary action hurts buyers more than it benefits sellers and hence reduces overall welfare.

nication and stochastic relocation create an endogenous transactions role for fiat money.[2] At the end of each period, a fraction of agents is relocated; only fiat money is useful as a means to "communicate" with their past (hence the "limited communication"). The "stochastic relocations" act like shocks to agents' portfolio preferences and, in particular, trigger liquidations of some assets at potential losses. They have the same consequences as "liquidity preference shocks" in Diamond and Dybvig (1983), and motivate a role for banks that take deposits, hold cash reserves. The other asset is a commonly available neo-classical technology with knowledge externalities, as in Romer (1986); more specifically, the production function is given by $Y_t = A\bar{K}_t^\beta K_t^\theta L_t^{1-\theta}$, where K_t denotes the capital stock of a representative firm, L_t denotes the amount of labor hired, and $\bar{K}_t$ it the aggregate capital stock in the economy. The assumed knowledge-externality form of the production function nests economies with endogenous growth (AK form, i.e., $\theta + \beta = 1$) and those with no long run growth (i.e., $\theta + \beta < 1$ as in the classic Diamond (1965) model).

Our results are as follows. We show that the Tobin effect is always operative irrespective of the degree of risk aversion of agents. Under logarithmic utility, we show that the Friedman rule is not optimal (stationary welfare maximizing) if the steady state is dynamically efficient. In this case, we can also show that zero inflation is not optimal (indeed some amount of positive inflation dominates zero inflation). Under the more general CRRA form of preferences, we find that a sufficient (not necessary) condition for some positive inflation to dominate zero inflation is that $\theta + \beta \in (1/2, 1)$; for most realistic values of θ, this translates into a requirement that the societal production function exhibit mild increasing returns. For parameter values such that the economy is dynamically efficient under logarithmic utility, the Friedman rule is not optimal for any value of the risk aversion parameter.

These results stand in contrast to those obtained in economies with linear (fixed real

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[2] The random relocation with limited communication model was popularized by Champ, Smith, and Williamson (1997) and has been used to investigate monetary policy issues in Schreft and Smith (1998), Paal and Smith (2000), Smith (2002), Antinolfi, Huybens, and Keister (2001), among others.

return) storage technologies. Since almost all the literature thus far has focused on linear storage economies and not on neoclassical production economies, an important contribution of this paper is to highlight the fact that optimal monetary policy is strikingly different in these two kinds of economies. As discussed in Wallace (1980), a linear storage economy is one in which 1 unit invested in date-t storage (or capital) returns $x > 1$ units of date-$t+1$ units of the consumption good. By definition, such economies are dynamically efficient. Bhattacharya, Haslag, and Russell (2005), and others have demonstrated that linear storage random relocation economies, irrespective of the degree of risk aversion, always return a verdict in favor of zero inflation. Here, in contrast we are able to show, for example, that for logarithmic utility, zero inflation is suboptimal if the economy is dynamically efficient. The reason is that in economies with linear storage technologies, storage holdings of the current generation do not influence the incomes of future generations. With neoclassical production, instead, any seigniorage collected is rebated to the young which augments the deposit base of the young and, in standard cases, raises the investment in capital, and hence future incomes.

This paper is related to a vast literature on the Friedman rule (see Bhattacharya, Haslag, and Martin (2005) and the references therein). More specifically, our paper complements the work by Paal and Smith (2004) who study suboptimality of the Friedman rule in an environment with endogenous growth that shares many similarities with ours. In a money-in-the-utility-function overlapping generations economy with production, Weiss (1980) finds that the optimal policy produces positive inflation. Smith (1998) studies an overlapping generations monetary economy with production in which the rate of return dominance issue is settled by postulating a minimum size to capital investment that limits one group of agents to holding money. By focusing on the dynamically inefficient equilibria, he shows that welfare at the Friedman rule may be dominated by other feasible monetary policies. Similarly, Palivos (2005) studies an overlapping generations economy with production and heterogeneity in preference for altruism and finds that a case for pos-

itive inflation can be made even when capital does not respond to inflation. Our work also complements that of Dutta and Kapur (1998) who pose the exact question as ours in a overlapping generations economy with irreversible unobservable capital investments and uninsured liquidity preference risk (similar to ours). They find that the optimal inflation rate is positive if the Tobin effect is not operative.

The remainder of the paper proceeds as follows: Section 2 presents the environment, the set of primitives, the spatial and informational constraints generating limited communication and the behavior of banks. Section 3 describes the general equilibrium while Section 4 discusses optimal monetary policy under different assumptions about $\theta + \beta$. The final section includes some concluding remarks and the appendices contain proofs of all the major results.

2 The Environment

2.1 Primitives

The economy take place at infinitely many dates $t = 0, 1, \ldots, \infty$. It is populated by two-period lived overlapping generations of agents who live on two separate islands that are completely symmetric in terms of all economic activity. At each date $t > 0$, a continuum of mass 1 of agents is born on each island.[3] Young agents are endowed with 1 unit of labor which they supply inelastically while old agents have no endowment. As is standard in much of this literature, we assume agents derive utility from consuming the economy's consumption good (c) only when old. The utility function can be represented by $u(c) = c^{1-\varphi}/(1 - \varphi), \varphi > 0$; if $\varphi = 1$, then we define $u(c) = \ln c$.

The consumption good is produced by a representative firm which rents capital and

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[3]We ignore the 'initial old' in all of what follows. By optimal monetary policy, we are therefore referring to the golden rule monetary policy. Bhattacharya, Haslag, and Martin (2006) compare economies with or without an initial old. See also the discussion in Paal and Smith (2004).

hires labor from young agents. The Romer-style production function is given by

$$Y_t = F(\bar{K}_t, L_t, K_t) = A\bar{K}_t^\beta K_t^\theta L_t^{1-\theta}, \tag{1}$$

where K_t denotes the capital stock of a representative firm, L_t denotes the amount of labor hired, and $\bar{K}_t$ it the aggregate capital stock in the economy. As is standard, $\bar{K}_t$ is taken as given by individual firms. To simplify the algebra, we assume that capital depreciates completely from one period to the next. We assume that $\beta \in [0, 1-\theta]$. Hence, if $\beta = 0$, equation (1) reduces to the standard neoclassical production function as in the Diamond (1965) model. On the other hand, if $\beta = 1 - \theta$, then equation (1) takes the form of the standard endogenous growth (AK) production function. Note that this function can be expressed in terms of the capital-labor ratio, k. We assume that $k_0 > 0$ is a given.

Because of competition, factors are paid their marginal return. The rental rate on capital, ρ, and the wage rate, w, are, respectively,

$$\rho \equiv \rho(k) = A\theta k^{\beta+\theta-1}, \tag{2}$$
$$w \equiv w(k) = A(1-\theta)k^{\theta+\beta}. \tag{3}$$

2.2 Informational and spatial constraints

As in Townsend (1980) and (1987), a role for money arises in this economy because of informational and spatial constraints. Details of the nature of these constraints and the environment can be found in Schreft and Smith (1998); we only provide a brief sketch below. We assume that agents are born on two different islands and that a constant fraction α of agents on each island is randomly selected to move to the other island. These agents are called "movers." Communication between islands is limited so relocated agents can only consume if they carry money with them. As is described below, banks arise that accept deposits from agents and invest in capital and money. The banks offer money to movers so that they can consume after being relocated.

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We now describe the timing of events in each period. At the beginning of a period, firms hire labor from young agents and rent capital from banks in order to produce the consumption good. This good can be either consumed or used to produce capital for next period. Then, factors are paid and young agents deposit their entire wage income in a bank. Banks must then choose how much money and capital to hold in their portfolio. Next, agents learn their relocation status; movers withdraw cash from the bank while nonmovers wait till the following period to collect goods.

Let $0 < p_t < \infty$ denote the price level at date t. Then the gross real rate of return on money $(R_{m,t})$ between period t and $t+1$ is given by $R_{m,t} \equiv p_t/p_{t+1}$. Also, let $m_t \equiv M_t/p_t$ denote per young person real money balances at date t; $M_0 > 0$ is given. The central bank (CB) can affect the money supply in the economy through lump-sum injections or withdrawals of money. The CB chooses $z > -1$, the rate of growth of the money supply, in order to maximize the expected utility of agents. If the net money growth rate is positive then the CB uses the additional currency it issues to purchase goods, which it gives to current young agents (at the start of a period) in the form of lump-sum transfers. If the net money growth rate is negative, then the CB collects lump-sum taxes from the current young agents, which it uses to retire some of the currency. The tax $(+)$ or transfer $(-)$ is denoted τ_t. Since $M_{t+1} = (1+z)M_t$, the budget constraint of the government is given by

$$\tau_t = \frac{M_t - M_{t-1}}{p_t} = \frac{z}{1+z}m_t. \tag{4}$$

For future reference, the stationary Friedman rule for this economy involves choosing z to satisfy $1 + z^{\text{FR}} \equiv (1/\rho)$. Also note that $z^{\text{FR}} < 0$ if and only if $\rho > 1$, i.e., the net money growth rate implied by the Friedman rule is negative *if and only if* the economy is dynamically efficient. Parenthetically, note that if we replace our specification of technology with a linear storage technology with fixed gross real return of $x > 1$, then such an economy is always dynamically efficient and $z^{\text{FR}} < 0$ would *always* hold.

2.3 Banks' behavior

Banks take deposits from young agents and choose how much to invest in capital and money. The deposit contract offered to young agents allows movers to withdraw money at the end of their first period of life, just before they move. Agents are also allowed to withdraw during their second period of life. As is usual in these kinds of models, money is dominated in rate of return if the CB deviates from the Friedman rule (if $z > (1/\rho_t) - 1$). In such cases, banks want to hold as little money as possible and thus will hold just enough money to pay the movers.

Banks announce a return of d_t^m to each mover and d_t^n to each non-mover. Let s_t denote the bank's saving in the form of capital. The bank maximizes its depositors' expected utility subject to the following constraints:

$$m_t + s_t \leq w_t + \tau_t, \tag{5}$$

$$\alpha d_t^m (w_t + \tau_t) \leq m_t R_{m,t}, \tag{6}$$

$$(1 - \alpha) d_t^n (w_t + \tau_t) \leq \rho_t s_t, \tag{7}$$

and non-negativity constraints. The first equation is the bank's balance sheet constraint. The second equation states that the real balances held by the bank (from the perspective of period $t + 1$, the date at which consumption occurs) must be enough to satisfy the (predictable) liquidity demand from movers. The last constraint states that the remaining goods (which were held in the form of capital) go to the nonmovers.

Let $\gamma_t \equiv m_t/(w_t + \tau_t)$ represent the reserve to deposit ratio. Since the bank's constraints hold with equality, the banks' problem can now be rewritten as

$$\max_{\gamma_t \in [0,1]} \quad \frac{(w_t + \tau_t)^{1-\varphi}}{1 - \varphi} \left[\alpha \left(\frac{\gamma_t}{\alpha} R_{m,t} \right)^{1-\varphi} + (1 - \alpha) \left(\frac{\rho_t (1 - \gamma_t)}{1 - \alpha} \right)^{1-\varphi} \right].$$

The first order condition to this problem simplifies to

$$(d_t^m)^{-\varphi} (R_{m,t}) = (d_t^n)^{-\varphi} \rho_t. \tag{8}$$

The solution for γ_t is given by

$$\gamma_t = \gamma\left(R_{m,t}, \rho_t\right) = \frac{1}{1 + \frac{1-\alpha}{\alpha}\left(\frac{\rho_t}{R_{m,t}}\right)^{\frac{1-\varphi}{\varphi}}},$$

or, equivalently,

$$\gamma_t = \gamma\left(I_t\right) = \frac{\alpha}{\alpha + (1-\alpha)\left(I_t\right)^{\frac{1-\varphi}{\varphi}}}, \tag{9}$$

where $I_t \equiv \frac{\rho_t}{R_{m,t}}$ denotes the gross nominal interest rate between t and $t+1$. Note that I_t represents the opportunity cost of cash relative to capital. For future reference, note that when $\varphi = 1$, we write $u\left(c\right) = \ln c$, and the solution is $\gamma_t = \alpha$. Also, as is clear from (9), for all $I > 1$, $\gamma_t \gtreqless \alpha$ iff $\varphi \gtreqless 1$; specifically, $\gamma_t \in (0, \alpha)$ if $\varphi \leq 1$ and $\gamma_t \in (\alpha, 1)$ if $\varphi > 1$. Intuitively, think of the bank allocating its deposit base among two "goods", the consumption of movers and the consumption of nonmovers. When the two are complements (substitutes) a low return on money relative to capital (i.e., $I > 1$) requires that the share of the bank's portfolio allocated to consumption of movers (i.e., its money holdings) be relatively high (low).

3 General equilibrium

Since capital depreciates completely from one period to the next, capital next period is equal to savings today:

$$s_t = k_{t+1}. \tag{10}$$

The rental rate of capital, ρ_t, and the wage rate, w_t are given by equations (2) and (3), respectively. Combining the banks's budget constraint (equation 5) with equation (10), we can get an expression for k_{t+1} :

$$k_{t+1} = \left(w\left(k_t\right) + \tau_t\right) - m_t = \left(1 - \gamma_t\right)\left(w\left(k_t\right) + \tau_t\right), \tag{11}$$

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where γ_t is given by equation (9). We can use equations (4) and the definition of γ_t to obtain expressions for τ_t and m_t. These are

$$\tau_t = \frac{z\gamma_t w(k_t)}{(1+z) - z\gamma_t}, \tag{12}$$

$$m_t = \gamma_t(w_t + \tau_t) = \frac{\gamma_t w_t(1+z)}{(1+z) - \gamma_t z}. \tag{13}$$

Next, we wish to find an expression for the return on money, $R_{m,t}$. Since $\frac{m_{t+1}}{m_t} = (1+z)R_{m,t}$ holds, using (13), we have

$$R_{m,t} = \frac{\gamma_{t+1}(w(k_{t+1}) + \tau_{t+1})}{(1+z)\gamma_t(w(k_t) + \tau_t)}.$$

Finally, using (5)-(7), we can obtain expressions for d_t^m and d_t^n:

$$d_t^m = \frac{\gamma_t}{\alpha} R_{m,t} = \frac{\gamma_t}{\alpha} \frac{\gamma_{t+1}(w(k_{t+1}) + \tau_{t+1})}{(1+z)\gamma_t(w(k_t) + \tau_t)},$$

$$d_t^n = \frac{\rho_t(1 - \gamma_t)}{1 - \alpha} = \frac{f'(k_{t+1})(1 - \gamma_t)}{1 - \alpha}.$$

In steady states, we can simplify some of these expressions to get

$$R_m = \frac{1}{(1+z)}, \quad d^m = \frac{\gamma(I)}{\alpha}\frac{1}{(1+z)}, \quad d^n = \frac{f'(k)(1 - \gamma(I))}{1 - \alpha}.$$

where $I \equiv I(z) = f'(k)(1+z)$. Define $\gamma(z) \equiv \gamma(I(z))$. Then the steady state value of k may be obtained from (11) as solutions to

$$k^* = \frac{(1 - \gamma(z))(1+z)}{(1+z) - z\gamma(z)} w(k^*). \tag{14}$$

In equilibrium, z is determined by the CB by maximizing the stationary lifetime utility of a representative generation.

Formally, a stationary competitive equilibrium is a k^* that solves (14) at a value of z determined by the benevolent CB, which in turn satisfies $\gamma(z) \in [0,1]$ and the return dominance condition $z > \frac{1}{\rho(k^*)} - 1$.

For future reference, note that for logarithmic utility, using $\gamma_t = \alpha$ in (11), the expression for k_{t+1} is given by

$$k_{t+1} = \frac{(1 - \alpha)(1+z)}{(1+z) - z\alpha} A(1 - \theta) k_t^{\theta+\beta}. \tag{15}$$

Only in this case, can we derive a closed form expression for the steady state value of k and other variables:

$$d^m(z) = \frac{1}{(1+z)}, \quad d^n(z) = f'(k(z)), \tag{16}$$

$$k(z) = \left[\frac{(1-\alpha)(1+z)A(1-\theta)}{(1+z) - z\alpha}\right]^{\frac{1}{1-(\theta+\beta)}}, \quad \tau(z) = \frac{z\alpha w(k(z))}{(1+z) - z\alpha} \tag{17}$$

3.1 Characterization

In the next section, we will characterize the 'optimal' monetary policy, by which we mean the choice of z that would maximize the stationary lifetime welfare of all current and future two-period lived agents. But before we can get there, we will have to ascertain the effects of increasing the money growth rate on real money demand and the steady state capital stock. Recall that the Tobin effect is said to operative if an increase in the money growth rate raises the steady state capital stock.[4]

Proposition 1 *For any $z > -1$, and for any $\varphi > 0$, $\frac{dk^*}{dz} > 0$ holds, implying that the Tobin effect is always operative.*

[Proofs of this and other major results are in the appendix.]

This is a somewhat startling result considering its generality. The intuition is easiest to articulate for the special case of logarithmic utility. In that case, money demand is interest-invariant; indeed the fraction of the bank's portfolio going to money or capital investment is a constant. Also, since agents care only about old-age consumption, they save their entire young-age income. A higher money growth rate unequivocally raises seigniorage which, when rebated to the young, raises their incomes and hence the bank's investment in capital.

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[4]For a good discussion of the literature on superneutrality of money or lack thereof, see Nikitin and Russell (2006). Empirical support for the Tobin effect is discussed in, among many other places, Ahmed and Rogers (2002).

More generally, money demand will respond to the interest rate and so the share of the bank's portfolio going to money will depend on the money growth rate (i.e., both income and substitution effects of a change in the nominal interest rate on money demand will be at play). A higher money growth rate will raise seigniorage (transfers to the young) only on the good side of the Laffer curve.

Using Proposition 1, we can also establish the following general equilibrium result.

Proposition 2 *If $\varphi < 1$, then $\gamma'(z) < 0$ and if $\varphi > 1$, then $\gamma'(z) > 0$.*

Proposition 2 states that when agents are sufficiently risk averse (i.e., more risk averse than that implied by logarithmic preferences), the bank's portfolio weight attached to money rises with the money growth rate, i.e., real money demand rises when the real return to money falls. Similarly, when agents are not too risk averse (i.e., less risk averse than that implied by logarithmic preferences), real money demand falls when the real return to money falls. Both $\varphi < 1$ and $\varphi > 1$ have been used in the literature; see Schreft and Smith (1998) for a defence of either assumption.

3.2   Aside on the Friedman rule

The money growth rate corresponding to the Friedman rule, call it z^{FR}, is computed from equating the return on capital to the return on money. In steady states, this reduces to

$$f'(k^*) \equiv \frac{1}{1 + z^{\text{FR}}}.$$

In general, since there is no closed form expression for k^*, we cannot derive a closed form for z^{FR}. In the case of logarithmic utility, and when $\theta + \beta < 1$, using (17), we can get

$$f'(k^*) = A\theta \, (k^*)^{\beta+\theta-1} = \frac{1}{1 + z} \Rightarrow (k^*)^{\beta+\theta-1} = \frac{(1 + z) - z\alpha}{(1 - \alpha)(1 + z)A(1 - \theta)}.$$

Then using $A\theta \, (k^*)^{\beta+\theta-1} = \frac{1}{1+z^{\text{FR}}}$, it follows that

$$z^{\text{FR}}\big|_{\varphi=1} = \frac{(1 - \theta)}{\theta} - \frac{1}{(1 - \alpha)}. \tag{18}$$

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If $\theta + \beta = 1,$ the return to capital is always $A\theta$ and so $z^{\text{FR}} = \frac{1}{A\theta} - 1$ irrespective of φ.

4 Optimal monetary policy

4.1 No long run growth, $\theta + \beta < 1$

The CB's problem is to choose z so as to

$$\max_z W(z) \equiv \frac{(w(k) + \tau(z))^{1-\varphi}}{1 - \varphi} \left[\alpha (d^m(z))^{1-\varphi} + (1-\alpha)(d^n(z))^{1-\varphi} \right]. \tag{19}$$

Using (8), we can write $\frac{d^n}{d^m} = \left(\frac{\rho}{R_m} \right)^{\frac{1}{\varphi}}$; also

$$\alpha(d^m)^{1-\varphi} + (1-\alpha)(d^n)^{1-\varphi} = (d^m)^{1-\varphi} \left[\alpha + (1-\alpha) \left(\frac{\rho}{R_m} \right)^{\frac{1-\varphi}{\varphi}} \right] = (d^m)^{1-\varphi} \frac{\alpha}{\gamma(z)},$$

where the last step comes from the definition of γ. Using $w(k) + \tau(z) = \frac{1+z}{1+z-\gamma(z)z} w(k)$ and $d^m = \frac{\gamma}{\alpha} \frac{1}{(1+z)}$, we can rewrite (19) as

$$W(z) = \frac{\alpha^\varphi}{1 - \varphi} w(k^*(z))^{1-\varphi} \left(\frac{1}{1 + z - z\gamma(z)} \right)^{1-\varphi} \gamma(z)^{-\varphi}.$$

Using (14), we get $\frac{k^*}{(1+z)(1-\gamma)} = \frac{w(k^*)}{(1+z)-z\gamma}$ which can be used to rewrite $W(z)$ as

$$W(z) = \frac{\alpha^\varphi}{1 - \varphi} \left(\frac{k^*(z)}{(1+z)(1-\gamma(z))} \right)^{1-\varphi} \gamma(z)^{-\varphi}$$

and further as

$$W(z) = \frac{\alpha^\varphi}{1 - \varphi} \left(\frac{k^*(z)}{(1+z)} \right)^{1-\varphi} \left(\frac{1}{1-\gamma(z)} \right) \left(\frac{(1-\gamma(z))}{\gamma(z)} \right)^\varphi. \tag{20}$$

Using the definition of γ, we get

$$\left(\frac{1 - \gamma(z)}{\gamma(z)} \right) = \frac{(1-\alpha)}{\alpha} (I)^{\frac{1-\varphi}{\varphi}}$$

then from (20), we have

$$W(z) = \frac{\alpha^\varphi}{1 - \varphi} \left(\frac{k^*(z)}{(1+z)} \right)^{1-\varphi} \left(\frac{1}{1-\gamma(z)} \right) \left(\frac{(1-\alpha)}{\alpha} \right)^\varphi (I)^{1-\varphi}. \tag{21}$$

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Using $I = f'(k^*(z))(1 + z)$ and $f'(k) = A\theta k^{\beta+\theta-1}$, we can rewrite (21) as

$$W(z) = \frac{(1-\alpha)^\varphi (A\theta)^{1-\varphi}}{1-\varphi} \left((k^*(z))^{\beta+\theta}\right)^{1-\varphi} \left(\frac{1}{1-\gamma(z)}\right).$$

To compute the z that maximizes $W(z)$, we evaluate the derivative $W'(z)$ as

$$W'(z) = \frac{(1-\alpha)^\varphi (A\theta)^{1-\varphi} \left((k^*(z))^{\beta+\theta}\right)^{1-\varphi}}{1-\varphi} \left(\frac{1}{1-\gamma(z)}\right) \left[(1-\varphi)(\beta+\theta)\left(\frac{1}{k^*(z)}\frac{dk^*(z)}{dz}\right) + \frac{\gamma'(z)}{1-\gamma(z)}\right].$$
(22)

Since the Tobin effect has been shown to be always operative (see Proposition 1) and since $\gamma'(z)$ changes sign depending on the size of φ (see Proposition 2), it is clear that $W'(z)$ does not have the same sign for all z.

Lemma 1 *The sign of $W'(z)$ depends only on the sign of $-[1 - (\theta + \beta)][(1+z)(1 - \gamma(z;\varphi))] + (\beta + \theta)$.*

As discussed earlier, many authors using a model identical to ours but with a linear storage technology, have established at least two condition-free results: a) zero inflation is optimal, and b) the Friedman rule is not optimal. Next we investigate if these results extend to models with a concave neoclassical technology.

4.1.1   Zero inflation

Not having a closed form expression for γ at $z = 0$ is a stumbling block towards using Lemma 1 directly to get the sign of $W'(0)$; specifically, it is not possible to derive general necessary and sufficient conditions for zero inflation to be optimal. Instead, we take a different approach and seek sufficient conditions. Using Lemma 1, it follows that for $W'(0) > 0$, it is necessary and sufficient that

$$W'(0) > 0 \Leftrightarrow \frac{(\beta + \theta)}{[1 - (\theta + \beta)]} > (1 - \gamma(0)).$$
(23)

Clearly, since $\gamma(0) \in (0, 1)$, a sufficient condition for (23) to hold is that $\frac{(\beta+\theta)}{[1-(\theta+\beta)]} > 1$ or $(\beta + \theta) \in \left(\frac{1}{2}, 1\right)$.

Proposition 3 *If $(\beta + \theta) \in \left(\frac{1}{2}, 1\right)$, then zero inflation $(z = 0)$ is not optimal and positive inflation dominates zero inflation irrespective of the degree of risk aversion.*

In the case of logarithmic utility, we can derive a necessary and sufficient condition for zero inflation to not be optimal. Notice that for logarithmic preferences, $\gamma = \alpha$ for all z. Then (23) reduces to

$$\frac{(\beta + \theta)}{[1 - (\theta + \beta)]} > (1 - \alpha). \tag{24}$$

Also, if the steady state is dynamically efficient, $z^{\text{FR}}|_{\varphi=1} < 0$ must hold; then (18) implies that $(1 - \alpha) < \frac{\theta}{(1-\theta)}$. It is easy to check that $(1 - \alpha) < \frac{\theta}{(1-\theta)}$ implies (24).

Corollary 1 *In the case of logarithmic utility, for positive inflation to dominate zero inflation, it is sufficient that the steady state be dynamically efficient.*

The upshot of this analysis is that when the knowledge externality (β) is sufficiently high, it is welfare maximizing to set a positive money growth rate. If, as is standard, we set $\theta = 0.4$ (see Cooley, 1995; ch. 1, page 20), then $\beta > 0.1$ is sufficient (not necessary) for zero inflation to be suboptimal.

Example 1 *Let $A = 1$, $\theta = 0.4$, $\alpha = 0.08$, $\beta = 0.08$. Then $(\beta + \theta) < 1/2$ and $\frac{(\beta+\theta)}{[1-(\theta+\beta)]} > (1 - \alpha)$. Then $z > 0$ dominates $z = 0$ for both $\varphi = 0.95$ and $\varphi = 1.1$.*

Example 1 illustrates that $(\beta + \theta) \in \left(\frac{1}{2}, 1\right)$ is not necessary for zero inflation to be suboptimal and that $\frac{(\beta+\theta)}{[1-(\theta+\beta)]} > (1 - \alpha)$ may be enough to ensure the dominance of positive z over $z = 0$ for a range of φ *around* 1.

4.1.2 Friedman rule

Using Lemma 1, it follows that the Friedman rule would not be optimal if and only if

$$- [1 - (\theta + \beta)] \left[(1 + z^{\text{FR}}) (1 - \alpha) \right] + (\beta + \theta) > 0$$

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was true. If the steady state is dynamically efficient, then $(1 + z^{\text{FR}}) < 1$ holds; therefore a sufficient condition for the Friedman rule to not be optimal would be

$$\frac{(\beta + \theta)}{[1 - (\theta + \beta)]} > (1 - \alpha)$$

which is the same as (24).

Proposition 4 *If the steady state for the model under logarithmic utility is dynamically efficient, then the Friedman rule for the model with a generic $\varphi \neq 1$ is not optimal irrespective of φ, the degree of risk aversion.*

In the special case of logarithmic utility, we know that $z^{\text{FR}}|_{\varphi=1} = \frac{(1-\theta)}{\theta} - \frac{1}{(1-\alpha)}$. Then it can be shown that

$$- [1 - (\theta + \beta)] \left[(1 + z^{\text{FR}}|_{\varphi=1})(1-\alpha) \right] + (\beta + \theta) > 0$$

reduces to $(\theta + \beta) \frac{(1-\alpha)}{\theta} > 0$ which always holds.

Corollary 2 *For logarithmic utility, the Friedman rule is never optimal.*

The upshot of the above discussion is that when $\varphi \geq 1$, a sufficient (by no means necessary) condition for *neither* the Friedman rule *nor* zero inflation to be optimal (and for positive inflation to be optimal) is (24). For the US, depending on the specifics of how α is measured (i.e., whether it is measured as M2/GDP or the reserves to deposit ratio at commerical banks), $\alpha \in (0.06, 0.1)$ and so $(1 - \alpha)$ has an upper bound of 0.9. Then (24) requires $(\beta + \theta) > 0.47$ or if we set $\theta = 0.4$, for positive inflation to be optimal, it is enough that there be a mild degree of social increasing returns $(\beta > 0.07)$.

A sufficient (but not necessary) condition for *neither* the Friedman rule *nor* zero inflation to be optimal (and for positive inflation to be optimal) *irrespective of the degree of risk aversion* is $(\beta + \theta) > 1/2$.

4.2 Long run endogenous growth, $\theta + \beta = 1$

With $\theta + \beta = 1$, the production function takes the AK form implying the possibility of long run growth. For analytical convenience, henceforth we assume logarithmic utility. Then, from (15) it follows that on a balanced growth path,

$$\frac{k_{t+1}}{k_t} = \frac{(1-\alpha)(1+z)}{(1+z) - z\alpha} A(1-\theta) \equiv g(z)$$

implying the rate of growth of the economy now depends on the money growth rate. Since

$$\frac{\partial}{\partial z} \left(\frac{1+z}{1+z(1-\alpha)} \right) = \frac{\alpha}{(1+z(1-\alpha))^2} > 0$$

it follows that $g'(z) > 0$ and hence the growth rate of the economy rises with an increase in the money growth rate. This is the growth-analog of the standard Tobin effect in levels. Hence, with logarithmic utility, the Tobin effect in growth rates is always operative thereby complementing our result in the previous subsection. Also notice that $g'(z) > 0$ implies that the growth-maximizing money growth rate, if it exists, is certainly not the Friedman rule.

Note $\frac{m_{t+1}}{m_t} = \frac{w(k_{t+1})}{w(k_t)} = \frac{k_{t+1}}{k_t}$ and so real balances are also growing along the *same* balanced growth path. Then along this balanced growth path, the return on money is given by

$$\frac{p_t}{p_{t+1}} = \left[\frac{m_{t+1}}{(1+z)m_t} \right] \Rightarrow \frac{p_t}{p_{t+1}} = \frac{(1-\alpha)A(1-\theta)}{(1+z) - z\alpha}$$

For logarithmic utility,

$$d_t^m = \frac{p_t}{p_{t+1}} = \frac{(1-\alpha)}{(1+z) - z\alpha} A(1-\theta), \quad d_t^n = \rho = A\theta. \tag{25}$$

Note that $\frac{k_{t+1}}{k_t} = g(z)$ implies that $k(t) = (g(z))^t k_0$. Welfare at t is given by

$$W_t(z) \equiv \alpha \ln (d_t^m (w_t + \tau_t)) + (1-\alpha) \ln (d_t^n (w_t + \tau_t)) = \ln (w_t + \tau_t) + \alpha \ln d_t^m + (1-\alpha) \ln d_t^n \tag{26}$$

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It is easy to check that

$$w_t + \tau_t = A \left(1 - \theta\right) k_t \left(\frac{\left(1 + z\right)}{\left(1 + z\right) - z\alpha} \right)$$

Then it follows from (25)-(26) that $W_t\left(z\right)$ is given by

$$W_t\left(z\right) = \ln\left[A\left(1 - \theta\right) k_t \left(\frac{\left(1 + z\right)}{\left(1 + z\right) - z\alpha} \right) \right] + \alpha \ln \left[\frac{\left(1 - \alpha\right) A \left(1 - \theta\right)}{\left(1 + z\right) - z\alpha} \right] + \left(1 - \alpha\right) \ln\left[A\theta \right]$$

which simplifies to

$$W_t\left(z\right) = \ln\left[A\left(1 - \theta\right) k_0 \right] + \ln\left[\left(g\left(z\right)\right)^t \right] + \ln\left[\left(\frac{\left(1 + z\right)}{\left(1 + z\right) - z\alpha} \right) \right] \qquad (27)$$

$$+ \alpha \ln \left[\frac{\left(1 - \alpha\right) A \left(1 - \theta\right)}{\left(1 + z\right) - z\alpha} \right] + \left(1 - \alpha\right) \ln\left[A\theta \right]$$

We posit that the central bank maximizes $\mathcal{W}\left(z\right) = \sum_{t=0}^{\infty} \phi^t W_t\left(z\right)$ where $\phi \in \left(0, 1\right)$ is a discount factor. It is clear from (27) that $\mathcal{W}\left(z\right)$ is convergent for each z.

Proposition 5 *Under logarithmic utility, when $\theta + \beta = 1$, i.e., there is endogenous growth, and $A\theta > 1$, then $\mathcal{W}'\left(z^{FR}\right) > 0$ implying the Friedman rule is not optimal.*

Analogous to our earlier results, the Friedman rule is not welfare maximizing even in the presence of endogenous long run growth. Additionally, it is inconsistent with maximum growth. As is well known, models of endogenous growth ala Romer produce equilibria with inefficiently low levels of investment because the social return to capital investment is higher (due to the knowledge externality) than the private return. As argued by Smith (1998), the Friedman rule cannot cure this inefficiency. Raising the money growth rate via the Tobin effect fosters private capital investment and hence improves welfare.

5   Concluding remarks

Most of the literature interested in optimal monetary policy in random relocation models has studied models with a storage technology. In this paper, we show that optimal monetary policy looks very different across random relocation models with concave production

functions and those with linear storage technologies. Many authors have demonstrated that dynamically efficient linear storage random relocation economies, irrespective of the degree of risk aversion, always support zero inflation as the golden rule. Here in contrast we show, for example, that for logarithmic utility, zero inflation is never optimal if the economy is dynamically efficient. The reason for this difference lies in the power of the Tobin effect. In economies with linear storage technologies, storage holdings of the current generation do not influence the incomes of future generations. In contrast, with neoclassical production, any seigniorage collected is rebated to the young which augments the deposit base of the young, and in standard cases, raises the investment in capital (the Tobin effect) and hence future incomes.

A question that is at the heart of many analyses of optimal monetary policy is, why do central banks in the real world never implement the Friedman rule? To the list of answers to this question, we add neoclassical production (specifically, the Tobin effect) as one possible explanation.

Appendix

A   Proof of Proposition 1

Since $z > -1$, and $\gamma \in [0,1]$, the numerator and denominator of (14) are positive and hence k^* is a differentiable function of z. Straightforward differentiation of (14) yields

$$\frac{1}{k^*}\frac{dk^*(z)}{dz} = \frac{-\gamma'(z)}{(1-\gamma(z))} + \frac{1}{(1+z)} - \left[\frac{1-\{\gamma(z)+z\gamma'(z)\}}{(1+z)-z\gamma(z)}\right] + \frac{1}{w(.)}w'(.)\frac{dk^*(z)}{dz}$$

which reduces to

$$\frac{dk^*(z)}{dz}\left[\frac{1}{k^*} - \frac{w'(k^*)}{w(k^*)}\right] = \frac{-\gamma'(z)}{(1-\gamma(z))} + \frac{1}{(1+z)} - \left[\frac{1}{(1+z)-z\gamma(z)}\right]\left[1-\gamma(z)\left\{1+\frac{z\gamma'(z)}{\gamma(z)}\right\}\right] \tag{28}$$

Next we seek an expression for $\frac{z\gamma'(z)}{\gamma(z)}$. Since $I(z) \equiv (1+z)f'(k(z))$, we have

$$\frac{dI}{dz} = f'(k) + (1+z)f''(k)\frac{dk}{dz}.$$

Since $f'(k) = A\theta k^{\beta+\theta-1}$ and $f''(k) = \theta A(\theta+\beta-1)k^{\beta+\theta-2}$, we have $f'' = (\theta+\beta-1)\frac{I}{k(1+z)}$ and so $\frac{dI}{dz}$ reduces to

$$\frac{dI}{dz} = I\left[\frac{1}{(1+z)} - (1-(\theta+\beta))\frac{1}{k}\frac{dk}{dz}\right] \tag{29}$$

Using (9), it is easy to check that

$$\frac{d\gamma}{dz} = -\left(\frac{1-\varphi}{\varphi}\right)\gamma(1-\gamma)\left(\frac{1}{I}\frac{dI}{dz}\right) \tag{30}$$

which, using (29) reduces to

$$\gamma'(z) = -\left(\frac{1-\varphi}{\varphi}\right)\gamma(1-\gamma)\left[\frac{1}{(1+z)} - (1-(\theta+\beta))\frac{1}{k}\frac{dk}{dz}\right] \tag{31}$$

from where it follows that

$$\frac{z\gamma'(z)}{\gamma(z)} = -z\left(\frac{1-\varphi}{\varphi}\right)(1-\gamma)\left[\frac{1}{(1+z)} - (1-(\theta+\beta))\frac{1}{k}\frac{dk}{dz}\right]. \tag{32}$$

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Since, $\frac{kw'(k)}{w(k)} = (\theta + \beta)$ holds, then (28) along with (31)-(32) implies

$$\frac{1}{k^*}\frac{dk^*(z)}{dz}\left[1 - (\theta + \beta)\right] = \frac{-1}{(1 - \gamma(z))}\left(-\left(\frac{1-\varphi}{\varphi}\right)\gamma(1-\gamma)\left[\frac{1}{(1+z)} - (1 - (\theta + \beta))\frac{1}{k}\frac{dk^*(z)}{dz}\right]\right)$$
$$+ \frac{1}{(1+z)} - \left[\frac{1}{(1+z) - z\gamma(z)}\right]\left[1 - \gamma(z)\left\{1 - z\left(\frac{1-\varphi}{\varphi}\right)(1-\gamma)\left[\frac{1}{(1+z)} - (1 - (\theta + \beta))\frac{1}{k}\frac{dk}{dz}\right]\right\}\right].$$

Repeated rearrangement yields

$$\left[1 - (\theta + \beta)\right]\frac{1}{k^*}\frac{dk^*(z)}{dz}\left[1 + \left(\frac{1-\varphi}{\varphi}\right)\gamma - \frac{z\gamma(1-\gamma)}{[(1+z) - z\gamma]}\left(\frac{1-\varphi}{\varphi}\right)\right]$$
$$= \frac{1}{(1+z)}\left[\left(\frac{1-\varphi}{\varphi}\right)\gamma - \frac{z\gamma\left(\frac{1-\varphi}{\varphi}\right)(1-\gamma)}{[(1+z) - z\gamma]} + \left(1 - \frac{(1-\gamma)(1+z)}{(1+z) - z\gamma}\right)\right]$$

which reduces to

$$\frac{1}{k^*}\frac{dk^*(z)}{dz} = \frac{1}{(1+z)\left[1 - (\theta + \beta)\right]}\left[\frac{\gamma\left(\frac{1}{\varphi}\right)}{(1+z) + \gamma\left\{\left(\frac{1-\varphi}{\varphi}\right) - z\right\}}\right] \tag{33}$$

So the sign of $\frac{dk^*(z)}{dz}$ is the same as the sign of $1 + z(1-\gamma) + \gamma\left(\frac{1-\varphi}{\varphi}\right)$. Notice though that

$(1+z) + \gamma\left\{\left(\frac{1-\varphi}{\varphi}\right) - z\right\} = (1+z)(1-\gamma) + \frac{\gamma}{\varphi} > 0.$ ∎

B   Proof of Proposition 2

Using (31) and (33), we get

$$\frac{d\gamma}{dz} = -\left(\frac{1-\varphi}{\varphi}\right)\gamma(1-\gamma)\left[\frac{1}{(1+z)} - \frac{1}{(1+z)}\left[\frac{\gamma\left(\frac{1}{\varphi}\right)}{(1+z) + \gamma\left\{\left(\frac{1-\varphi}{\varphi}\right) - z\right\}}\right]\right]$$

which upon rearrangement yields

$$\frac{d\gamma}{dz} = -\left(\frac{1-\varphi}{\varphi}\right)\frac{\gamma(1-\gamma)}{(1+z)}\left[\frac{(1+z) + \gamma\left\{\frac{1}{\varphi} - (1+z) - \left(\frac{1}{\varphi}\right)\right\}}{(1+z) + \gamma\left\{\left(\frac{1-\varphi}{\varphi}\right) - z\right\}}\right]$$

and finally to

$$\frac{d\gamma}{dz} = -\left(\frac{1-\varphi}{\varphi}\right)\gamma(1-\gamma)^2\underbrace{\left[\frac{1}{(1+z) + \gamma\left\{\left(\frac{1-\varphi}{\varphi}\right) - z\right\}}\right]}_{>0}$$

The rest is immediate.∎

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C   Proof of Lemma 1

From (22), we know

$$W'(z) = \frac{(1-\alpha)^\varphi (A\theta)^{1-\varphi} \left((k^*(z))^{\beta+\theta}\right)^{1-\varphi}}{1-\varphi} \left(\frac{1}{1-\gamma(z)}\right) \left[(1-\varphi)(\beta+\theta)\left(\frac{1}{k^*(z)}\frac{dk^*(z)}{dz}\right) + \frac{\gamma'(z)}{1-\gamma(z)}\right]$$

Using (31), one can simplify the term in square parenthesis above down to

$$(1-\varphi)\left\{\left(\frac{1}{k^*(z)}\frac{dk^*(z)}{dz}\right)\left[(\beta+\theta) + \left(\frac{1}{\varphi}\right)\gamma(1-(\theta+\beta))\right] - \left(\frac{1}{\varphi}\right)\frac{\gamma}{(1+z)}\right\}.$$

Then

$$W'(z) = (1-\alpha)^\varphi (A\theta)^{1-\varphi} \left((k^*(z))^{\beta+\theta}\right)^{1-\varphi}\left(\frac{1}{1-\gamma(z)}\right) \times \tag{34}$$

$$\left\{\left(\frac{1}{k^*(z)}\frac{dk^*(z)}{dz}\right)\left[(\beta+\theta) + \left(\frac{1}{\varphi}\right)\gamma(z)(1-(\theta+\beta))\right] - \left(\frac{1}{\varphi}\right)\frac{\gamma(z)}{(1+z)}\right\}.$$

Using (33) in (34), we note that the sign of $W'(z)$ depends only on the sign of

$$\left(\frac{1}{\varphi}\right)\frac{\gamma}{(1+z)}\left\{\frac{\left[(\beta+\theta) + \left(\frac{1}{\varphi}\right)\gamma(1-(\theta+\beta))\right]}{[1-(\theta+\beta)]}\left[\frac{1}{(1+z)+\gamma\left\{\left(\frac{1-\varphi}{\varphi}\right)-z\right\}}\right] - 1\right\}$$

It is tedious but routine to check that

$$\left\{\frac{\left[(\beta+\theta) + \left(\frac{1}{\varphi}\right)\gamma(1-(\theta+\beta))\right]}{[1-(\theta+\beta)]}\left[\frac{1}{(1+z)+\gamma\left\{\left(\frac{1-\varphi}{\varphi}\right)-z\right\}}\right] - 1\right\}$$

$$= \frac{1}{[1-(\theta+\beta)]}\left\{\frac{\left[(\beta+\theta) + \left(\frac{1}{\varphi}\right)\gamma(1-(\theta+\beta))\right] - [1-(\theta+\beta)]\left[(1+z)+\gamma\left\{\left(\frac{1-\varphi}{\varphi}\right)-z\right\}\right]}{(1+z)+\gamma\left\{\left(\frac{1-\varphi}{\varphi}\right)-z\right\}}\right\}$$

$$= \frac{(\beta+\theta)\left[1+(1+z)(1-\gamma)\right] - (1+z)(1-\gamma)}{(1+z)+\gamma\left\{\left(\frac{1-\varphi}{\varphi}\right)-z\right\}}$$

and since $z > -1$, the sign of $W'(z)$ depends only on the sign of

$$(\beta+\theta) - [1-(\beta+\theta)](1+z)(1-\gamma).$$

∎

D Proof of Proposition 5

From (27), it follows that $\mathcal{W}(z)$ is given by

$$\sum_{t=0}^{\infty} \phi^t \left\{ \ln\left[A\left(1-\theta\right)k_0\right] + \ln\left[\left(g\left(z\right)\right)^t\right] + \ln\left[\left(\frac{\left(1+z\right)}{\left(1+z\right)-z\alpha}\right)\right] + \alpha\ln\left[\frac{\left(1-\alpha\right)A\left(1-\theta\right)}{\left(1+z\right)-z\alpha}\right] + \left(1-\alpha\right)\ln A\theta \right\}$$

(35)

Notice $\sum_{t=0}^{\infty} \phi^t \ln\left[\left(g\left(z\right)\right)^t\right] = \sum_{t=0}^{\infty} \phi^t t \ln g\left(z\right) = \ln g\left(z\right) \sum_{t=0}^{\infty} \phi^t t$ and so (35) implies

$$\frac{\left(1-\phi\right)}{\phi}\mathcal{W}\left(z\right) = \ln g\left(z\right)\frac{\left(1-\phi\right)}{\phi}\sum_{t=0}^{\infty}\left(\phi^t t\right) + \ln\left[A\left(1-\theta\right)k_0\right] + \ln\left[\left(\frac{\left(1+z\right)}{\left(1+z\right)-z\alpha}\right)\right]$$

$$+\alpha\ln\left[\frac{\left(1-\alpha\right)A\left(1-\theta\right)}{\left(1+z\right)-z\alpha}\right] + \left(1-\alpha\right)\ln\left[A\theta\right];$$

then it is clear that optimal choice of z depends only on the following terms:

$$\mathcal{W}\left(z\right) = \ln g\left(z\right)\frac{\left(1-\phi\right)}{\phi}\sum_{t=0}^{\infty}\phi^t t + \ln\left[\left(\frac{\left(1+z\right)}{\left(1+z\right)-z\alpha}\right)\right] + \alpha\ln\left[\frac{\left(1-\alpha\right)A\left(1-\theta\right)}{\left(1+z\right)-z\alpha}\right]$$

Since $\sum_{t=0}^{\infty} \phi^t t = \frac{\phi}{\left(1-\phi\right)^2}$, using the expression for $g(z)$, we get

$$\mathcal{W}\left(z\right) = \frac{1}{\left(1-\phi\right)}\ln\left[\frac{\left(1+z\right)}{\left(1+z\right)-z\alpha}\right] + \ln\left(\frac{\left(1+z\right)}{\left(1+z\right)-z\alpha}\right) - \alpha\ln(1+z) - z\alpha$$

$$+\frac{1}{\left(1-\phi\right)}\ln\left[A\left(1-\alpha\right)^2\right] + \alpha\ln\left(1-\alpha\right)\left(1-\theta\right)A$$

and finally relevant terms,

$$\mathcal{W}\left(z\right) = \frac{1}{\left(1-\phi\right)}\ln\left[\frac{\left(1+z\right)}{\left(1+z\right)-z\alpha}\right] + \ln\left(\frac{\left(1+z\right)}{\left(1+z\right)-z\alpha}\right) - \alpha\ln\left[\left(1+z\right)-z\alpha\right]$$

Note that

$$\frac{\partial}{\partial z}\left(\frac{1+z}{1+z\left(1-\alpha\right)}\right) = \frac{\alpha}{\left(1+z\left(1-\alpha\right)\right)^2}$$

Then it follows that

$$\mathcal{W}'\left(z\right) = \frac{1}{\left(1+z\left(1-\alpha\right)\right)}\left[\frac{\alpha}{\left(1+z\right)}\frac{1}{\left(1-\phi\right)} + \frac{1}{\left(1+z\right)} - \alpha(1-\alpha)\right]$$

Since $f'\left(k\right) = A\theta = 1/\left(1+z^{\text{FR}}\right)$, it follows that $1/\left(1+z^{\text{FR}}\right) > \alpha(1-\alpha)$ if $A\theta > 1$. ∎

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