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Optimality of the Friedman Rule in an Overlapping Generations Model with Spatial Separation

Joseph H. Haslag
Antoine Martin

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Abstract

We examine models with spatial separation and limited communication that have shown some promise toward resolving the disparity between theory and practice concerning optimal monetary policy; these models suggest that the Friedman rule may not be optimal. We show that intergenerational transfers play a key role in this result, the Friedman rule is a necessary condition for an efficient allocation in equilibrium, and the Friedman rule is chosen whenever agents can implement mutually beneficial arrangements. We conclude that in order for these models to resolve the aforementioned disparity, they must answer the following question: Where do the frictions that prevent agents from implementing mutually beneficial arrangements come from?

Key words: Friedman rule, overlapping generations, spatial separation

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Haslag: University of Missouri-Columbia (e-mail: haslagj@missouri.edu). Martin: Federal Reserve Bank of New York (e-mail: antoine.martin@ny.frb.org). The authors thank Scott Freeman, Jordan Rappaport, Gordon Sellon, and Pu Shen for useful comments. They also thank Joydeep Bhattacharya and Steve Russell for many conversations on the topic.

1 Introduction

The question of the optimum quantity of money is of great importance to monetary theory. It is also a vexing question because of the disparity between theory and practice. Theory has shown the Friedman rule to be optimal in many different environments and under many different assumptions (see, for example, Kimbrough 1986; Chari, Christiano, and Kehoe 1996; Correia and Teles 1996). Yet, in practice, no central bank (CB) states as its objective to implement the Friedman rule, and historical episodes in which deflation occurred and interest rates approached zero have often been considered very negative.[1] A theory explaining why the Friedman rule might not be optimal would help resolve this disparity and would thus be of particular interest.

Several recent papers have shown some promise toward resolving the aforementioned disparity.[2] These papers argue the Friedman rule does not maximize a social welfare function in overlapping generations model in which money is valued because of spatial separation and limited communication. This result is thought to arise because of the careful modelling of financial intermediation.[3] In these models, some set of agents is randomly relocated. Relocated agents can only take cash with them and banks arise endogenously to help share the risk of relocation. These models exhibit a trade-off between

[1] The Great Depression and Japan in the 1990s are two such episodes.

[2] See Paal and Smith (2005), Smith (2002 a and b). Similar results arise in other models of this class, such as in Schreft and Smith (2002, 2003) but are not emphasized there.

[3] Smith (2003) writes, “As will be seen, when intermediation is analyzed seriously, the Friedman rule generally will not be optimal... .”

productive efficiency and risk-sharing. The banks' reserve-to-deposit ratio is a function of the money growth rate which is set by the CB. If the CB follows the Friedman rule the consumption of movers and nonmovers is equalized. Movers are thus fully insured but since the banks' reserves are very high, the rate of productive investment is very low. On the other hand, the CB could set a high rate of growth of the money supply. This high-money-growth-rate policy leads to high investment but also to an increase in the disparity of consumption between movers and nonmovers. [4]

Upon a closer examination, we can show that there is an intergenerational transfer that accounts for why the Friedman rule is suboptimal. Depending on the environment one is studying, the intergenerational transfer may or may not create a trade-off between efficiency and risk-sharing. We also show that the Friedman rule is necessary to achieve the efficient allocation. The efficient allocation can be achieved if the intergenerational transfer can be undone; for example, if the central bank can make loans. If the intergenerational transfers cannot be undone, the equilibrium allocation is equivalent to the solution of a problem for which there are strong restrictions imposed on how stored goods can be distributed. We conclude that the puzzle concerning the Friedman rule can be resolved in this class of models only by creating another puzzle: Where do the frictions that impose restrictions on how goods are distributed come from? Finally, we show that if agents can implement mutually beneficial arrangements, then the Friedman

[4] Paal and Smith (2005) write, "The optimal level of the nominal rate of interest in our economy is determined by trading off the benefits of bank liquidity provision (insurance) against higher rates of real growth."

rule will be chosen.

The remainder of the paper proceeds as follows: Section 2 describes the environment. Section 3 shows that financial intermediation or a trade-off between productive efficiency and risk-sharing do not play a central role in the result that the Friedman rule is sub-optimal. Instead, intergenerational transfers are key. Section 4 derives the efficient allocation and shows that that the Friedman rule is necessary to achieve this allocation in equilibrium. Section 5 shows that the Friedman rule will be chosen if agent can implement mutually beneficial arrangements. Section 6 concludes.

2 The environment

We consider an economy closely related to Schreft and Smith (2002).[5] Only a succinct description of the economic environment is provided; the interested reader is referred to Schreft and Smith (2002) for more details. Throughout we refer to this model economy as the benchmark model.

Time is divided into an infinite number of identical increments and is indexed by $t = ... - 1, 0, 1, ...$[6] The world is divided into two spatially

[5]The result that the Friedman rule does not maximize social welfare arises in many related environments. We choose this model for ease of exposition. Our results extend in a straightforward way to other environments such as those in Paal and Smith (2005), Smith (2002 a and b). Predecessors in this literature include Townsend's (1980) model with limited communication and extends through Bencivenga and Smith (1991) and Bhattacharya, Guzman, Huybens, and Smith (1997).

[6]It is standard in this literature to have an initial period and an initial old generation but to ignore the welfare of that initial generation. To simplify the exposition we choose instead to have no initial period. Bhattacharya, Haslag, and Martin, forthcoming, contrast economies with or without an initial date.

separated locations. Each location is populated by a continuum of agents of unit mass. Agents live for two periods and receive an endowment of ω units of the single consumption good when young and nothing when old. Let c_t denote old-age consumption of the members of the generation born at date t; their lifetime utility is given by $u(c_t) = \frac{c_t^{1-\rho}}{1-\rho}$, where $\rho \in (0,1)$.

After depositing their after-tax endowment into a bank, agents learn whether they must move to the other location or not. Let π denote the probability that an individual will be relocated. We assume a law of large numbers holds so π is also the measure of agents that are relocated. π is the same on both islands so that moves across location are symmetric. Movers redeem their bank deposits in the form of money as this is the only way for them to acquire goods in the new location. In contrast, nonmovers redeem their deposits in the form of goods.

Goods deposited in the bank can be used to acquire money from old agents belonging to the previous generation or put into storage. Each unit of the consumption good put into storage at date t yields $x > 1$ units of the consumption good at date $t+1$, where x is a known constant.

The CB chooses the sequence $\{\sigma_t\}_{t=-\infty}^{\infty}$, the rate of growth of the money supply at each date. It follows that the money supply evolves according to $M_t = \sigma_t M_{t-1}$. When we consider steady-states, $\sigma_t = \sigma$, for all t and $p_t = \sigma p_{t-1}$. To implement monetary policy, the CB can levy lump-sum taxes τ_t on the endowment of agents. To remove money from the economy, the CB taxes goods which are later exchanged for cash. A lump-sum subsidy is received in the form of a money injection. In short, τ_t can be either positive or negative and is given by

$$\tau_t = \frac{-M_t + M_{t-1}}{p_t} = -\frac{\sigma_t - 1}{\sigma_t} m_t, \tag{1}$$

where $m_t = M_t/p_t$.

2.1 Bank behavior

Agents deposit their entire after-tax/transfer endowments with a bank. The bank chooses a gross real return d_t^m to pay to movers and d_t^n to pay to nonmovers. In addition, the bank chooses values m_t and s_t standing for the real value of money balances and storage investment, respectively.

These choices must satisfy the bank's balance sheet constraint

$$m_t + s_t \leq \omega - \tau_t. \tag{2}$$

Banks behave competitively, so they take as given the return on their investments. In particular, the return on real money balances is p_t/p_{t+1}. If $x > p_t/p_{t+1}$ banks will want to hold as little liquidity as possible since money is dominated in rate of return. If $x = p_t/p_{t+1}$, banks are indifferent between money and storage and there are multiple equilibria. We are not interested in the multiplicity of equilibria because it is not robust in the sense that for p_t/p_{t+1} arbitrarily close to but strictly smaller than x the equilibrium is unique. For this reason, we consider the limiting economy as $p_t/p_{t+1} \to x$.

Banks must have sufficient liquidity to meet the needs of movers. This is captured by the following expression,

$$\pi d_t^m (\omega - \tau_t) \leq m_t \frac{p_t}{p_{t+1}}. \tag{3}$$

A similar condition for nonmovers, who consume all the proceeds from the storage technology, is given by

$$(1-\pi)d_t^n(\omega-\tau_t) \leq x s_t. \tag{4}$$

Banks maximize profits. Because of free entry banks choose, in equilibrium, their portfolio in a way that maximizes the expected utility of a representative depositor. After substitution, the bank's problem is written as

$$\frac{(\omega-\tau_t)^{1-\rho}}{1-\rho}\left\{\pi\left(d_t^m\right)^{1-\rho}+(1-\pi)\left(d_t^n\right)^{1-\rho}\right\} \tag{5}$$

subject to equations (2), (3), and (4).

The time subscript is dropped in what follows because we focus on steady-state allocations. Let $\gamma=\frac{m}{\omega-\tau}$ denote the bank's reserve-to-deposit ratio. Then, since equations (2), (3), and (4) hold with equality, the bank's objective function is to choose γ to maximize

$$\frac{(\omega-\tau)^{1-\rho}}{1-\rho}\left\{\pi^{\rho}\left[\frac{\gamma}{\sigma}\right]^{1-\rho}+(1-\pi)^{\rho}\left[(1-\gamma)x\right]^{1-\rho}\right\}. \tag{6}$$

Lemma 1 *(i) the bank's optimal reserve-to-deposit ratio is inversely related to the money growth rate; (ii) in the limit, as* $\frac{1}{\sigma}\rightarrow x$*,* $\gamma\rightarrow\pi$ *and* $c^m\rightarrow c^n$*.*

The proof is provided in the appendix. The reserve-to-deposit ratio chosen by the bank increases as σ decreases. Hence, as the rate of growth of the money supply approaches the Friedman rule, banks increase their holding of money which implies they invest less in the storage technology. Since the Friedman rule implies full insurance against the risk of being relocated, this is the sense in which there is a trade-off between efficiency and risk-sharing.[7]

2.2 The optimum quantity of money

The CB chooses $\sigma \geq 1/x$ in order to maximize equation (6) subject to the government budget constraint. The Friedman rule corresponds to the limit as $\sigma \to 1/x$. In this case, the rate of return of money is equal to the rate of return of storage. This definition is consistent with Friedman's (1969) dictum.

It is shown in the appendix that welfare is maximized at $\sigma = 1$. We can summarize this result in the following proposition,

Proposition 1 *The Friedman rule does not maximize social welfare. The maximizing rate of growth of the money supply is strictly greater than* $1/x$.

Schreft and Smith (2003) show the results of this section are unaffected when x is a random variable, rather than a known constant.

3 The role of intermediation and the trade-off between efficiency and risk-sharing

This section considers separately the role of financial intermediation and the trade-off between productive efficiency and risk-sharing. We ask if either is necessary for the result that the Friedman rule does not maximize social welfare. We consider slight variations of the benchmark model. In the

[7] It follows from equations (3) and (4) that $d^m = d^n$ when $\gamma = \pi$. Consumption for movers and non-movers are equal if the returns are equal.

first alternative model, financial intermediation plays no role. The second alternative model is a pure exchange economy, so there can be no trade-off between efficiency and risk-sharing. Yet in both of these models, the Friedman rule does not maximize social welfare. We argue that an intergenerational transfer is key for the result that the Friedman rule is suboptimal. This intergenerational transfer creates a trade-off between efficiency and risk-sharing in some environments but does not in others.

3.1 An economy without intermediation

This section describes a world similar to the one presented in section 2. However, the absence of idiosyncratic uncertainty means that intermediation plays no role. We show proposition 1 also holds in this economy.

Time is indexed by $t = ... - 1, 0, 1, ...$ and the world is divided in two spatially separated locations. Each location is populated with a continuum of agents of unit mass, who live for two periods. Agents receive an endowment ω of a location-specific good when young and nothing when old. Only old-age consumption is valued. Let c_h and c_a denote steady-state old-age consumption of home-location and away-location goods, respectively. Lifetime utility is given by

$$u(c_h, c_a) = \lambda^{\rho} \frac{c_a^{1-\rho}}{1-\rho} + (1-\lambda)^{\rho} \frac{c_h^{1-\rho}}{1-\rho}, \quad \rho, \lambda \in (0, 1),$$

where λ is a weight in the utility function.

There is no idiosyncratic uncertainty in this economy and, consequently, no role for intermediation. After receiving their endowment, consumers can sell some goods to old agents who moved from the other location in exchange for money. They invest the remainder of their goods in a storage technology. Each unit of the location-specific good put into storage at date t yields $x > 1$ units of the same good at date $t + 1$, where x is a known constant. In old age, agents receive the return from their investments and consume it. Later during that period, they travel to the other location and can buy location-specific goods from young agents. Young agents cannot travel and old agents from one location never meet the old agents from the other location.

Monetary policy is conducted as in the previous section. The CB chooses the rate of growth of the money supply to maximize social welfare which is given by the utility of a member of generation $t \geq 1$. Let m denote the real amount of money acquired by a consumer and s the amount stored by that consumer,

$$m + s \leq \omega - \tau. \tag{7}$$

Money can be used in the next period to acquire away-location goods.

$$c_a \leq m\frac{1}{\sigma} = (\omega - \tau)\frac{\gamma}{\sigma}. \tag{8}$$

Home-location consumption can be no greater than the proceeds from storage.

$$c_h \leq xs = (\omega - \tau)(1 - \gamma)x. \tag{9}$$

Substituting these quantities into the agent's utility function yields

$$\frac{(\omega-\tau)^{1-\rho}}{1-\rho}\left\{\lambda^{\rho}\left[\frac{\gamma}{\sigma}\right]^{1-\rho}+(1-\lambda)^{\rho}\left[(1-\gamma)x\right]^{1-\rho}\right\}, \tag{10}$$

which corresponds exactly to equation 6 whenever $\lambda = \pi$. This establishes the following proposition.

Proposition 2 *Proposition 1 holds in this environment.*

3.2 A pure exchange economy

In this section we show that proposition 1 also holds in an economy without production. Time is indexed by $...-1, 0, 1, ...$ and the world is divided in two spatially separated locations. Each location is populated with a continuum of agents of unit mass, who live for two periods. Agents receive an endowment ω_1 of the consumption good when young and ω_2 when old. Goods are perishable, cannot be moved between islands and cannot be stored. Monetary policy is conducted as in the models above. The CB chooses the rate of growth of the money supply to maximize social welfare, which is given by the expected utility of a member of generation $t \geq 1$. Also as above, the Friedman rule is associated with the money growth rate that equalizes the consumption of movers and nonmovers.

With probability π a young agent must move to the other island. The mass of agent who must move is assumed to be π as well. Movers cannot receive their endowment when old; however, they can exchange claims on their endowment for money. As in the model of section 2, banks will arise to insure agents against the risk of relocation. In the absence of a storage technology, the CB need not be concerned with productive efficiency in setting the optimal monetary policy.

Agents value consumption according to

$$u(c,c') = \frac{c^{1-\rho}}{1-\rho} + \beta\frac{(c')^{1-\rho}}{1-\rho}, \quad \rho,\beta \in (0,1), \tag{11}$$

where c denotes consumption when young, and c' denotes consumption when old. These agents face the following budget constraint

$$m + c \leq \omega_1 - \tau. \tag{12}$$

The money they have acquired, m, as well as the claims on their future endowment, are deposited in a bank. After they have learned they must relocate, movers go to the bank and withdraw cash.[8] Movers face the following constraint,

$$\pi c'^m \leq \frac{m}{\sigma}, \tag{13}$$

while nonmovers face the constraint

$$(1-\pi)c'^n \leq \omega_2. \tag{14}$$

The expected utility of a member of a representative generation is thus

$$U(t) = \frac{(\omega_1 - \tau - m)^{1-\rho}}{1-\rho} + \frac{\beta}{1-\rho}\left\{\pi^\rho\left[\frac{m}{\sigma}\right]^{1-\rho} + (1-\pi)^\rho\left[\omega_2\right]^{1-\rho}\right\}. \tag{15}$$

The following proposition is proved in the appendix.

Proposition 3 *The Friedman rule does not maximize social welfare in this economy.*

[8]We focus on the case where liquidity is scarce, so movers receive all the money held by the bank.

3.3 The role of intergenerational transfers

We have shown that the Friedman rule may be suboptimal in environments closely related to the benchmark model presented in section 2 but where neither intermediation nor a trade-off between efficiency and risk-sharing play a role. In this section we show that an intergenerational transfer is key to the sub-optimality of the Friedman rule. This transfer creates a trade-off between efficiency and risk-sharing in some environments but not in others.

To make this argument we consider the benchmark model presented in section 2. In particular, we study the effect of changing the rate of growth of the money supply at one specific date. Formally, consider an economy for which monetary policy is given by the sequence $\{\sigma_t\}_{t=-\infty}^{\infty}$. We study the effect of changing this sequence at one date, $t = t*$, so that $\sigma_{t*} > \tilde{\sigma}_{t*} > 1/x$. At all other dates, the money growth rate is unchanged; so the sequence $[...\sigma_{t*-1}, \sigma_{t*}, \sigma_{t*+1}...]$ is compared with $[...\sigma_{t*-1}, \tilde{\sigma}_{t*}, \sigma_{t*+1}...]$. To simplify the analysis, we assume that $u(c) = ln(c)$. With log utility, it can be verified that $\gamma_t = \pi$. [9]

By the definitions of the reserve-to-deposit ratio and τ_t, we have

$$M_t = p_t\gamma_t\left(\omega - \tau_t\right) = p_t\gamma_t\omega - \gamma_t\left(M_{t-1} - M_t\right). \tag{16}$$

[9] With log utility, the income effect of a change in prices is exactly offset by the substitution effect so that the reserve-to-deposit ratio is the same for any value of p_{t+1}/p_t. If $u(c) = \left(c^{1-\rho}\right)/(1-\rho)$, with $\rho \in (0,1)$, the analysis is more complicated because γ_t changes with p_{t+1}/p_t. In this case, the substitution effect dominates the income effect and a decrease in p_{t+1}/p_t leads to an increase in the reserve-to-deposit ratio. However, the intuition is the same as in the case of log utility.

With log utility, this equation can be rewritten as

$$M_t \left[1 - \pi \left(\frac{\sigma_t - 1}{\sigma_t}\right)\right] = p_t \pi \omega. \tag{17}$$

From equation 17, it is clear that m_t changes only at date t^* when σ_{t*} is decreased. Taking into account the fact that $M_t = \sigma_t M_{t-1}$, we can write

$$\frac{M_{t-1}}{p_t} = \frac{\pi \omega}{\sigma_t (1 - \pi) + \pi}. \tag{18}$$

Thus a decrease of σ_t at date t^* increases the value of money held at the end of date $t^* - 1$. From equation (3), we can see that the movers of the previous generation, that is, the old money-holders at date t^*, are able to increase their consumption. With log utility, the reserve-to-deposit ratio is unchanged and investment decreases. It can be shown that in the case where $u(c) = \left(c^{1-\rho}\right)/(1-\rho)$, with $\rho \in (0, 1)$, the reserve-to-deposit ratio will increase as the money growth rate decreases, but investment will decrease nonetheless. Hence, with $\sigma_t * > \tilde{\sigma}_{t*}$, output declines. If the change in $\tilde{\sigma}_{t*}$ is unexpected from the perspective of agents born at date $t^* - 1$, then the decline in the money growth rate does not influence the perceived risk-sharing between movers and nonmovers of that generation. In other words, when the change is unexpected, there is no trade-off between efficiency and risk-sharing. Note further that in steady-state, the transfers from each generation cancel each other out, but the decrease in investment remains.

Suppose instead that the change $\sigma_t * > \tilde{\sigma}_{t*}$ is an anticipated by agents of generations $t^* - 1$. The $t^* - 1$ generation benefits both from the transfer and from the increase in risk-sharing between movers and nonmovers. In steady-state, the benefit from increased risk-sharing must be weighted against the reduction in investment.

In both the nonstoachastic model and the pure-exchange model, there is no trade-off between efficiency and risk-sharing. Rather, it is the transfer between generations that accounts for the welfare loss we have studied here. In some environments, the aforementioned trade-off arises as a byproduct of the intergenerational transfer. To confirm this explanation, the next section studies an economy where the transfer can be undone. In such cases, we find that the Friedman rule is once again optimal.[10]

4 The efficient allocation

We derive the efficient allocation of the model of section 2. Then we show that this allocation can be achieved if the CB is able to make loans.[11] The efficient allocation maximizes the steady-state expected utility of a representative generation subject to a feasibility constraint. It solves

$$\max \pi u(c_t^m) + (1-\pi) u(c_t^n) \tag{19}$$

subject to

[10] Bhattacharya, Haslag, and Martin (2005) study the role of these kinds of transfers in other types of economies.

[11] For example, as suggested by Smith (2002 a) we can think of the CB as operating a discount window or engaging in open market operations. We consider a CB which makes loan at a gross rate of interest of 1. See also Antinolfi and Keister (forthcoming) who study a discount window policy in this kind of model.

$$\begin{aligned} \pi c_t^m + (1-\pi)c_t^n &\leq \omega_{t+1} - s_{t+1} + x s_t, & (20) \\ s_t &\leq \omega_t, \forall t. & (21) \end{aligned}$$

The first constraint states that the resources which can be divided between the members of generation t, at date $t+1$, can be no greater than the endowment of the generation $t+1$, minus the goods stored by the generation $t+1$, plus the goods stored by generation t. We impose the second constraint because we are interested in steady-state allocations.

We use the fact that movements between islands are symmetric to write the above problem. The idea is that there is one planner for both islands. The planner is subject to the same constraints as agents in that goods cannot be moved across islands. However, the planner can give goods stored on an island to agents moving from the other island.

In steady-state, the feasibility constraint becomes

$$\pi c^m + (1-\pi)c^n \leq \omega + (x-1)s. \tag{22}$$

We call feasible allocations any allocation $\{c^m, c^n, s\}$ such that equation 22 is satisfied. The right hand side of equation (22) is maximized for $s = \omega$ since $x > 1$. Also, since depositors are risk-averse, expected utility is maximized if $c^n = c^m$. In words, the efficient allocation gives the same consumption to movers and nonmovers and all the endowment of each generation is stored. It is thus apparent that the allocation obtained under $\sigma = 1$ cannot be efficient.

Proposition 4 *In this economy, the Friedman rule is a necessary condition for an allocation to be efficient.*

This is clear since $c^n > c^m$ away from the Friedman rule. The Friedman rule is not a sufficient condition since the efficient allocation also requires $s = \omega$.

In the remainder of the section, we show that it is possible to achieve the efficient allocation if the CB makes loans. Specifically, we assume that at date t, banks can borrow money from the CB for movers born at date t. Then, at date $t+1$, banks sell goods to agents who moved from the other island in order to obtain the money necessary to repay the CB loan. CB loans are made at a net interest rate of zero. Let b denote the loan received from the CB and assume that there is a cap $\bar{b}$ on the amount a bank can borrow. Equations (3) and (4) become, respectively,

$$\begin{aligned} \pi d^m(\omega-\tau) &\leq \frac{m}{\sigma}+\frac{b}{\sigma}, & (23)\\ (1-\pi)d^n(\omega-\tau) &\leq xs-\frac{b}{\sigma}=x(\omega-\tau)-xm-\frac{b}{\sigma}. & (24) \end{aligned}$$

The equality in expression (24) follows from equation (2). For any σ such that $x > 1/\sigma$, banks can relax constraint (24) without altering constraint (23) by increasing b and decreasing m by an equal amount. With $x > 1/\sigma$, banks borrow as much money as they can from the CB and $b = \bar{b}$. As $\bar{b}$ increases, m decreases and tends to zero while s tends to ω. By increasing $\bar{b}$, banks increase the amount of goods stored without altering the relative consumption of movers and nonmovers. As $\sigma \to 1/x$, $c^m \to c^n$ so that in the limit movers and nonmovers consume the same. We can summarize this argument in the following proposition.

Proposition 5 *The Friedman rule is optimal in this economy if the CB can make big enough loans. The efficient allocation can be achieved.*

Note that if $m = 0$, money does not circulate between generations. In that case the efficient allocation is achieved but it is not clear what it means for the CB to follow the Friedman rule. Indeed, the CB lends money to banks at date t and retires all the money at date $t + 1$. Also, in all periods the price level is indeterminate. However, the consumption enjoyed by all generations is the same for any strictly positive, finite price level. If $\bar{b}$ is not too big and $m > 0$, then some money circulates between generations and the price level is determinate. Thus we can approximate the efficient allocation arbitrarily closely by letting $\bar{b}$ increase in such a fashion that $m \to 0$. For all such allocations, the Friedman rule is necessary for $c^n = c^m$.

Our result differs from Smith (2002) because we focus on the limiting economy as $\sigma \to 1/x$. As mentioned above, we do not think that the multiplicity of equilibria at $\sigma = 1/x$ is particularly interesting since the equilibrium is unique for any $\sigma > 1/x$. A similar point is made by Antinolfi and Keister (forthcoming) who show that there is no multiplicity of equilibria if the CB makes loans at a net interest rate arbitrarily close to, but strictly greater than 1.

4.1 Remark

The efficient allocation can be achieved under alternative assumptions as well. For example, if lump-sum taxes and subsidies are feasible. Also, we can further illustrate the role of intergenerational transfers, by relating our model to that of Freeman (1993). Consider the benchmark economy of section 2 with an initial date. Assume that each agent cares about the utility of his/her unique offspring. As in Freeman (1993), the natural choice of a welfare function in this environment is the utility of a member of the initial old generation. It follows from proposition 6 (below), that the Friedman rule is optimal in this environment. Indeed, either the initial old does not want to make a bequest or bequests are positive: In the former case the Friedman rule maximizes the value of a unit money and thus the consumption of the initial old while in the latter case the Friedman rule maximizes the utility of an agent's offspring, for any given level of bequest. Once again, the key is the role played by inter-generational transfers. With a decrease in the growth rate of the money supply, the value of money increases and so does the value of that transfer. In the models of sections 2 and 3, the CB has an incentive to limit the size of that transfer, because it does not have a way to offset it. Bequests are a way to offset the transfer; indeed, bequests redistribute goods from one generation to the next in a way that offsets the redistribution created by a higher value of money.

4.2 Another puzzle?

We can show that the equilibrium allocation of the model of section 2, under the "optimal" monetary policy $\sigma = 1$ is the solution to a different problem. The problem is to maximize the objective given in (19), subject to the following constraints

$$\pi c^m \leq \omega - s, \tag{25}$$

$$(1-\pi)c^n \leq xs. \tag{26}$$

The solution to this problem is

$$c^m = \frac{\omega}{\pi + (1-\pi)x^{\frac{1-\rho}{\rho}}}, \tag{27}$$

$$c^n = x^{\frac{1}{\rho}} c^m. \tag{28}$$

The constraints imposed in this problem allow only the goods provided to nonmovers to be stored. The goods provided to movers must come directly from the endowment and thus increasing the consumption of movers must decrease investment.

Under these constraints, not only can the planner not transfer goods between islands, but the planner is not allowed to give goods stored on one island to the movers from the other island. It is not clear to us where such a restriction may come from. Hence, in order to solve the puzzle of the disparity between theory and practice concerning optimal monetary policy, this model must impose strong restrictions on how goods can be stored and given to different kind of agents, in the form of constraints (25) and (26). This raises another puzzle: What are the frictions that force these constraints? In other words, what prevents arrangements, such as the discount window studied above, which undo the intergenerational transfer.

5 Allowing mutually beneficial arrangements and the role of commitment

In this section, we show that if agents in this economy are able to implement mutually beneficial arrangements, and if they are able to commit, then they will choose that the CB implement the Friedman rule. Here, the focus is not on implementing a particular allocation, such as the efficient allocation, but in finding properties of allocations that can be achieved if agents that are alive at date t can bargain over the choice of σ_t.

We again consider the economy of section 2. We show that at any date t, if a feasible allocation is such that $\sigma_t > 1/x$, then it is possible to find a Pareto improving allocation with σ'_t which is feasible and such that $\sigma_t > \sigma'_t > 1/x$. Since this is true for all σ_t, only the limiting allocation as $\sigma_t \to 1/x$ cannot be improved upon.

Proposition 6 *If commitment is possible, then any feasible allocation such that $\sigma_t > 1/x$ is Pareto dominated by a feasible allocation with $\hat{\sigma}_t$ such that $\sigma_t > \hat{\sigma}_t > 1/x$.*

Proof. An allocation of the economy of section 2 is completely characterized by a sequence $\{\sigma_t\}_{t=-\infty}^{\infty}$. Consider a feasible allocation such that

$\sigma_t > 1/x$ at date t^*. Call this the *initial* allocation. An *alternative* allocation can be obtained by lowering σ_{t^*}. The sequence of σ_t is unaltered for all other $t \neq t^*$. As we have seen in section 3.3, with $\hat{\sigma}_{t^*} < \sigma_{t^*}$, there is an increase consumption by money holders in generation t^*-1 to the detriment of agents in generation t^*. It follows that all agents alive at date t^* would agree to lower σ_{t^*} if money holders from generation $t^* - 1$ agree to keep their consumption constant and give the extra goods they can purchase to members of generation t^*. If the appropriate transfers are made, any value of σ_{t^*} leaves all agents alive at date t^* indifferent. Now assume that at date $t^* - 1$, agents of generation $t^* - 1$ anticipate the change to σ_{t^*}. This is possible because these agents know that young agents alive at date t^* can be made indifferent between σ_{t^*} and $\hat{\sigma}_{t^*}$ with an appropriate transfer, as argued above. Agents who are young at date t^*-1 strictly prefer a lower value of σ_{t^*} since it increases risk-sharing. Note that this requires that agents of generation t^*-1 be able to commit to making the transfer. Indeed, at date t^* nonmovers from generation $t^* - 1$ would prefer to not make the required transfer. ■

If at date t^* nonmovers from generation t^*-1 can renege on their promise to make a transfer to agents of generation t^*, then movers can still obtain a decrease in σ_{t^*}. However, any increase in the value of money must be exactly offset by the transfer from those in generation $t^* - 1$ to agents in generation t^*. With appropriate transfers in place, there is no increase in risk-sharing since nonmovers do not share the cost of the transfer.

The intuition for proposition 6 is that some surplus is created when the difference between c^m and c^n is reduced without changing the level of investment. The proposition does not specify how this surplus is redistributed. The distribution of the surplus is likely to depend on political economy considerations that are beyond the scope of the paper. Proposition 6 implies that the Friedman rule will be chosen in the economy of section 2 unless there is some friction that prevents agents from implementing mutually beneficial arrangements.

In the remainder of this section we provide an example of what such an arrangement might look like.[12] Assume agents alive in a given period can vote to modify the rules under which the CB operates. The set of such rules is called the CB's *charter*. In this model, it specifies σ and τ, as well as, possibly, other lump-sum taxes or transfers. The following proposition shows the CB will be required to implement the Friedman rule.

Proposition 7 *Any charter will require the CB to follow the Friedman rule.*

Proof. Suppose it is not the case, then it is possible to write a new charter according to which the CB follows the Friedman rule and old money holder make transfers to banks. According to proposition 6, this new charter can be designed to be unanimously accepted. ■

[12] Other arrangements are possible. See, for example, Bhattacharya, Haslag, and Russell (2005).

6 Conclusion

This paper studies models with spatial separation and limited communication that have shown some promise toward resolving the disparity between theory and practice concerning optimal monetary policy. We have argued that the key to the result that the Friedman rule is suboptimal is the fact that changes in the rate of growth of the money supply create intergenerational transfers. Such transfers may or may not result in a trade-off between productive efficiency and risk-sharing.

We also derive the efficient allocation of the model of section 2 and show that the Friedman rule is a necessary condition for an allocation to be efficient. We show that if the CB is able to make sufficiently big loans, then the efficient allocation can be achieved. We have also shown that the equilibrium allocation of section 2 is the solution to a different problem where consumption of movers is restricted to come from endowment goods rather than from stored goods. This raises another puzzle: What a the frictions which impose such strong restrictions?

Finally, we study some properties of equilibrium allocations when agent are able to implement mutually beneficial arrangements. We show that if at any date t, $\sigma_t > 1/x$, then it is possible to increase the utility of all agents by lowering σ_t and making a transfer of goods from old agents of the previous generation to the current generation. We conclude that we should expect the Friedman rule to be chosen in these economies unless there is some impediment to making mutually beneficial arrangements.

Appendix

Proof of Proposition 1

Let $\Omega(\sigma) := \omega - \tau = \frac{\omega - g}{1 - \frac{\sigma-1}{\sigma}\gamma(\sigma)}$, and $\Gamma(\sigma) := \pi^\rho \left[\frac{\gamma}{\sigma}\right]^{1-\rho} + (1-\pi)^\rho \left[(1-\gamma)x\right]^{1-\rho}$, where g denotes government purchases of goods and services. Throughout our analysis, we assume that $g = 0$. Welfare is given by

$$W(\sigma) := \frac{\Omega(\sigma)^{1-\rho}}{1-\rho}\Gamma(\sigma). \tag{29}$$

The expression for the banks' reserve to deposit ratio is obtained by taking the derivative of equation 6 with respect to γ and setting it to zero.

$$\gamma(\sigma) = \left\{1 + \left(\frac{1-\pi}{\pi}\right)[\sigma x]^{\frac{1-\rho}{\rho}}\right\}^{-1} \tag{30}$$

The expression for $\gamma'(\sigma)$ is given by

$$\gamma'(\sigma) = \frac{1-\rho}{\rho}\frac{\gamma(\sigma)}{\sigma}\left(\gamma(\sigma) - 1\right), \tag{31}$$

First we show $\Gamma'(\sigma) < 0$. Recall,

$$\Gamma(\sigma) = \pi^\rho \left[\frac{\gamma(\sigma)}{\sigma}\right]^{1-\rho} + (1-\pi)^\rho \left[(1-\gamma(\sigma))x\right]^{1-\rho}. \tag{32}$$

Thus,

$$\begin{aligned}
\Gamma'(\sigma) &= \pi^\rho(1-\rho)\left[\frac{\gamma'(\sigma)}{\sigma}\left(\frac{\gamma(\sigma)}{\sigma}\right)^{-\rho} - \frac{1}{\sigma}\left(\frac{\gamma(\sigma)}{\sigma}\right)^{1-\rho}\right] \\
&\quad -(1-\rho)\gamma'(\sigma)\left(\frac{1-\pi}{1-\gamma(\sigma)}\right)^\rho x^{1-\rho} \\
&= -\pi^\rho(1-\rho)\frac{1}{\sigma}\left(\frac{\gamma(\sigma)}{\sigma}\right)^{1-\rho} < 0,
\end{aligned}$$

Since $\pi^\rho \frac{1}{\sigma}\left(\frac{\gamma(\sigma)}{\sigma}\right)^{-\rho} - \left(\frac{1-\pi}{1-\gamma(\sigma)}\right)^\rho x^{1-\rho} = 0$.

Next, we show $W(\sigma)$ reaches a maximum at $\sigma = 1$. Note $W'(\sigma) > 0$ if and only if

$$\Omega'(\sigma)\frac{\sigma}{\Omega(\sigma)} > -\frac{1}{1-\rho}\Gamma'(\sigma)\frac{\sigma}{\Gamma(\sigma)}. \tag{33}$$

$$\begin{aligned}
\Omega'(\sigma) &= \frac{(\omega - g)\left[\sigma - \gamma(\sigma)(\sigma-1) - \sigma + \sigma\gamma(\sigma) + \sigma\gamma'(\sigma)(\sigma-1)\right]}{\left[\sigma - \gamma(\sigma)(\sigma-1)\right]^2} \\
&= \Omega(\sigma)\left[\frac{1}{\sigma} - \frac{1-\gamma(\sigma)+\gamma'(\sigma)(\sigma-1)}{\sigma-\gamma(\sigma)(\sigma-1)}\right],
\end{aligned}$$

which implies

$$\Omega'(\sigma)\frac{\sigma}{\Omega(\sigma)} = \frac{\gamma(\sigma)+\sigma\gamma'(\sigma)(\sigma-1)}{\sigma-\gamma(\sigma)(\sigma-1)}. \tag{34}$$

Substituting for $\gamma'(\sigma)$, we get

$$\Omega'(\sigma)\frac{\sigma}{\Omega(\sigma)} = \frac{\gamma(\sigma)\left[1+\frac{1-\rho}{\rho}(\sigma-1)(\gamma(\sigma)-1)\right]}{\sigma-\gamma(\sigma)(\sigma-1)}. \tag{35}$$

We also need the expression for $\Gamma'(\sigma)\frac{\sigma}{\Gamma(\sigma)}\left(-\frac{1}{1-\rho}\right)$. It is given by

$$\Gamma'(\sigma)\frac{\sigma}{\Gamma(\sigma)}\left(-\frac{1}{1-\rho}\right) = \frac{\pi^{\rho}\left(\frac{\gamma(\sigma)}{\sigma}\right)^{1-\rho}}{\pi^{\rho}\left(\frac{\gamma(\sigma)}{\sigma}\right)^{1-\rho} + (1-\pi)^{\rho}\left(1-\gamma(\sigma)\right)^{1-\rho}x^{1-\rho}} = \gamma(\sigma), \tag{36}$$

since $\pi^{\rho}\frac{1}{\sigma}\left(\frac{\gamma(\sigma)}{\sigma}\right)^{-\rho} - \left(\frac{1-\pi}{1-\gamma(\sigma)}\right)^{\rho}x^{1-\rho} = 0$. It follows that $W(\sigma)' > 0$ if and only if

$$\frac{1+(1-\rho)\rho(\sigma-1)(\gamma(\sigma)-1)}{\sigma-\gamma(\sigma)(\sigma-1)} > 1. \tag{37}$$

This last expression is equivalent to $\sigma > 1$. Thus, $W(\sigma)$ is maximized in the limit as $\sigma \to 1$.

Note that the value of σ which maximizes $W(\sigma)$ does not depend on x. The above result holds if x is a random variable rather than a known constant. Assume $\bar{x} \geq x \geq \underline{x} > 0$. Let F denote the cumulative distribution function of x and f the associated probability distribution function. Assume $\int_{\underline{x}}^{\bar{x}} x f(x) dx = \hat{x} > 1$. In the above expressions, $x^{1-\rho}$ is replaced by $\int_{\underline{x}}^{\bar{x}} x^{1-\rho} f(x) dx$. Nothing else is modified and the result goes through.

Proof of Proposition 3

To find the optimal amount of real balances, m_t, agents choose to acquire, we take the partial derivative of $U(t)$ with respect to m_t and set it equal to zero.

$$m_t = (\omega_1 - \tau) \frac{\pi \beta^{\frac{1}{\rho}} \sigma^{\frac{\rho-1}{\rho}}}{1 + \pi \beta^{\frac{1}{\rho}} \sigma^{\frac{\rho-1}{\rho}}}. \tag{38}$$

The CB then chooses σ to maximize $U(t)$.

$$\frac{\partial U(t)}{\partial \sigma} = (\omega_1 - \tau - m_t)^{-\rho} \left(\frac{\partial \tau}{\partial \sigma} - \frac{\partial m_t}{\partial \sigma} \right) + \beta \pi^{\rho} \left\{ \frac{1}{\sigma^{1-\rho} m_t^{\rho}} \frac{\partial m_t}{\partial \sigma} - \frac{m_t^{1-\rho}}{\sigma^{2-\rho}} \right\}, \tag{39}$$

where

$$\frac{\partial m_t}{\partial \sigma} = \frac{\rho - 1}{\rho} \frac{1}{\sigma} \frac{m_t (\omega_1 - \tau)}{1 + \pi \beta^{\frac{1}{\rho}} \sigma^{\frac{\rho-1}{\rho}}}, \tag{40}$$

$$\frac{\partial \tau}{\partial \sigma} = \frac{1}{\sigma^2} m_t. \tag{41}$$

We can thus write

$$\frac{\partial U(t)}{\partial \sigma} = (\omega_1 - \tau - m_t)^{-\rho} \left[\frac{m_t}{\sigma^2} - \frac{\rho - 1}{\rho} \frac{1}{\sigma} \frac{m_t (\omega_1 - \tau)}{1 + \pi \beta^{\frac{1}{\rho}} \sigma^{\frac{\rho-1}{\rho}}} \right] \tag{42}$$

$$+ \beta \pi^{\sigma} \left[\left(\frac{m_t}{\sigma} \right)^{1-\rho} \frac{\rho - 1}{\rho} \frac{1}{\sigma} \frac{(\omega_1 - \tau)}{1 + \pi \beta^{\frac{1}{\rho}} \sigma^{\frac{\rho-1}{\rho}}} - \frac{1}{\sigma} \left(\frac{m_t}{\sigma} \right)^{1-\rho} \right] \tag{43}$$

$$= m_t^{1-\rho} \left(\frac{\beta \pi^{\rho}}{\sigma^{1-\rho}} \right) \frac{1}{\sigma} \left[\frac{1}{\sigma} - 1 \right]. \tag{44}$$

It is clear from this last equation that $U(t)$ is maximized at $\sigma = 1$. However, this will in general not correspond to the Friedman rule. The

Friedman rule, which equates the consumption of movers and nonmovers, requires $\sigma\omega_2 = m_t$. It can be shown that this implies

$$\sigma = \left[\frac{\omega_1 - \omega_2}{\omega_2}\right] \pi^\rho \beta. \tag{45}$$

Hence, there exists a $\sigma > 0$ which corresponds to the Friedman rule whenever $\omega_1 > \omega_2$. This value of σ is equal to 1 only for a set of measure zero in the parameter space.

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