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Abstract

In a simple search model of money, we study a special kind of memory that gives rise to an arrangement resembling a payment network. Specifically, we assume that agents can pay a cost to access a central database that tracks payments made and received. Incentives must be provided to agents to access the central database and to produce when they participate in this arrangement. We also study policies that can loosen these incentive constraints. In particular, we show that a "no-surcharge" rule has good incentive properties. Finally, we compare our model with that of Cavalcanti and Wallace.

Key words: payment networks, money, search

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1 Introduction

In this paper we study payment arrangements that resemble in some ways electronic payment networks. This work is motivated by dramatic changes in the U.S. payment landscape in recent years. Today, over half of non-cash payments are done in electronic form.[1] This fraction was just over 40 percent in 2000, and less than 25 percent in 1995 (Gerdes and Walton 2002). Between 2000 and 2003, the average annual growth of debit cards was over 20 percent both in volume and value. The average annual growth of credit cards was close to 10 percent in value, despite the fact that the market for credit cards is more mature (Gerdes, Walton, Liu, and Parke 2005). These changes illustrate the growing importance of electronic payments.

Electronic payments networks have also been the topic of recent policy debates. For example, in most countries credit and debit card networks enforce a 'no-surcharge' rule, under which merchants cannot charge higher prices for purchases made with payment cards rather than cash. However, the rule was prohibited for some or all credit and debit card transactions in Australia, Canada, Denmark, Mexico, the Netherlands and the U.K., and has come under examination in Mexico and for cross-border transactions in the EU (Weiner and Wright 2006).

Very little work has been done to help us understand the causes and effects of changes in electronic payments and to guide policy issues. Among the questions we are interested in are what kind of welfare benefits are there to electronic payment networks? How do such networks interact with other payment methods such as cash? What is the impact of payment network effects on optimal policy design?

Our objective is to study an environment in which frictions make payment instruments such as cash and payment networks essential.[2] We also want to adopt a mechanism design approach. For this reason, we consider a search environment of the type first studied by Kiyotaki and Wright (1991, 1993). More specifically, we use a model with divisible goods (Trejos and Wright 1995, Shi 1995).

We model an electronic payment network by assuming that agents can pay a cost to gain access to a central data base (CDB). The CDB can record the history of those agents' trades. When two agents who have access to this data base meet, they

[1] This includes credit and debit cards, ACH, and EBT.

[2] Following Neil Wallace, a payment instrument is essential if some allocations cannot be achieved without it.

can transact without money. As we argue below, our CDB resembles an electronic payment network in many respects. Apart from the CDB, agents can use money to trade.

Agents face two incentive constraints in this economy. They choose to gain access to the CDB only if their entry cost is smaller than the expected gain from having access. This is the entry constraint. Another constraint must ensure that an agent prefers to produce for another rather than lose access to the CDB. This is the no-exit constraint.

We show that agents holding money derive less benefit from having access to the CDB than agents who do not hold money. Hence, the no-exit constraint is tighter for the former type of agents. One way to relax the constraint faced by agents holding money is to impose that sellers cannot require to be paid with money if the CDB can also be used. We also show that decreasing the quantity of goods exchanged for a unit of money (increasing the price paid when using money) relaxes the incentive constraint. These results suggest that there could be benefits from the 'no-surcharge' rule of credit and debit cards. More generally, our paper emphasizes the importance of both the entry and the no-exit constraints.

We show that with heterogeneous access costs among agents, there are equilibria in which some agents access the network and also use money when needed, while other agents do not access the network and only use money. Because the benefit from having access to the CDB increases with the number of other agents who have access to it, there is a network effect. The number of agents who choose to access the CDB may be sub-optimally low. We consider policies that can relax the entry constraint without tightening the no-exit constraint. One policy is to impose a utility cost, which we interpret as a tax. Another policy is to increase the supply of money, and a third is to increase the price of goods purchased with money. We show that if the efficient allocation is such that some agents should remain out of the CDB, then it is preferable to impose a tax than to increase the money supply. In turn, it is preferable to increase the money supply rather than increase the price of goods purchased with money.

The result that a change in the money supply can provide incentives for agents to access the CDB is similar to a result obtained by Corbae and Ritter (2004). These authors show that introducing money in their economy can weaken the incentive to produce in a credit relationship and thus weaken credit partnerships. This is because money is an outside option for the parties of such partnerships. As the benefits of money increase, credit arrangements become relatively less attractive and thus more difficult to enforce. The same idea applies in our case, except that we consider a multilateral credit arrangement rather than bilateral arrangements. Studying a multilateral credit arrangement also allows us to show the importance of the size of such an arrangement. We show that if agents who refuse to produce can walk away from the CDB arrangement at no cost, then small multilateral arrangements cannot be sustained. Larger arrangements, however, can.

We also compare allocations of our economy with the no-gift allocation considered in Cavalcanti and Wallace (1999 a) (CW). We show that the benefit from the kind of memory considered by CW is at least as great as the benefit from the kind of memory we consider. Hence, at equal costs, the benefit from a few 'banks' of the type considered by CW is greater then the benefit of a small payment network of the type we consider. In contrast, the benefit from all agents having access to the CDB is the same as the benefit from all agents having public histories. When either is beneficial, then a payment network of the type we consider would be adopted if it is only slightly less expensive.

One way to think of the CDB is as a special kind of memory. Since the work of Kocherlakota (1998), it is known that memory plays a crucial role in achieving desirable allocations in economies where commitment is not possible. In particular, money can be thought of as a mnemonic device in a variety of environments. This has lead some authors to study alternative forms of memories and their interactions with money. Kocherlakota and Wallace (1998) study an economy in which there is money and a public record of all past actions that is updated with a lag. Cavalcanti, Erosa, and Temzelides (1999) consider an environment in which agents can issue notes which are redeemed at a central location. Cavalcanti and Wallace (1999 a and b) assume that some agents have public histories while other agents do not. They show that agents with public histories can issue notes that circulate and identify such agents with early banks. Corbae and Ritter (2004) assume that agents can remain in a long-term relationship as long as it is in their self-interest.

Some recent papers are interested in electronic payments. Kahn and Roberds (2005) study identity theft in a model that shares many features with ours. Monnet and Roberds (2006) examine further beneficial properties of the 'no-surcharge' rule regarding credit access for those who are cash constrained in a life-cyle model. Lotz (2005) considers electronic cash cards. See also Nosal and Rocheteau (2006) who provide a survey of the payments literature.

Other papers also examine alternatives to money in the presence of bilateral search trading frictions. He et. al. (2005, 2006) consider the role of banks and bank liabilities when there is risk of theft of cash. Camera (2000) studies how a costly multilateral trading intermediary may coexist or replace monetary bilateral trade. Telyukova and Wright (2006) develop a search theoretic model and consider the credit card debt puzzle.

The remainder of the paper proceeds as follows: Section 2 describes the model. Section 3 characterizes the ex-ante efficient allocation. Section 4 examines the entry constraint and no-exit constraint. Section 5 shows that there can be multiple equilibria. Section 6 studies policies that can loosen the entry and no-exit constraints. Section 7 compares our model with CW. Section 8 concludes.

2 The model

Time is discrete and denoted by $t = 1, 2, ...$ A mass 1 of infinitely lived agents populates the economy. There are $k > 2$ types of agents who are randomly matched in pairs in every period. There are also k types of perishable consumption goods in every period. Each agent is specialized in production and consumption. Agents of type i get period utility $u(c) > 0$ from consuming c units of good i. Agents of type i can only produce good $i+1$, modulo k, and incur a cost $c > 0$ when producing an amount c of goods. As usual, it is assumed that there exists $\hat{c}$ such that $u(\hat{c}) = \hat{c}$ and $u(c) > c$ if $c \in (0, \hat{c})$. Hence there can be no double coincidence of wants. Agents discount period utility with $\beta < 1$.

There is also a mass M of perfectly durable objects called money. Agents derive no utility from consuming money. Money comes in indivisible units and we assume that there is a storage constraint that prevents agents from holding more than one unit of money. However, we allow lotteries, as in Berentsen, Molico, and Wright (2002), so that in a single coincidence meeting in which money is used, money changes hand with probability τ.

All agents can choose to pay an entry cost to access a central data base (CDB). The CDB is a central record keeping device. It can keep track of meetings between two agents who both have access to the CDB and whether an agent produces goods for, or receives goods from, another agent. The history of an agents who have access to the CDB is only available to agents who also have access to the CDB. This is in contrast to Kocherlakota (1998) or Cavalcanti and Wallace (1999 a and b) where some agents have public histories. The type of memory we consider has limited access as opposed to a full, or public, access memory.[3]

The CDB is unable to directly monitor the behavior of agents and must rely on the reports of agents who have access. The possibility that agents do not report their actions limits the possible use of the CDB. For example, an agent who has access to the CDB and receives goods from an agent who does not have access would have an incentive to not report the meeting.

When two agents have access to the CDB, a trade is recorded if both agents send consistent messages. We assume that agents who have access to the CDB receive a number–an infinite sequence of 0 and 1–that uniquely identify them.[4] In a single coincidence meeting between two agents who have access to the CDB, the buyer agrees to reveal her identifying number to the seller. The seller sends a verifiable message to the CDB providing the identity of her trading partner and stating the amount of goods sold. This resembles transactions using credit cards where the buyer must give her card to the merchant who then sends a message to the card company identifying the buyer and describing the sale.

Agents who have access to the CDB are identifiable to each other so it is not possible to pretend not to have access. For example, they may display a sticker prominently. Agents who have access to the CDB but do not accept to produce goods when they meet another agent with access to the CDB can be punished. In other words, we assume that potential sellers cannot pretend that their connection to the CDB is not working.

The severity of the punishment that can be imposed is limited by the fact that agents have the option to walk away from the CDB arrangement. They can, however,

[3]Our CDB can be thought of as a multilateral partnership which resembles in some respect the partnerships in Corbae and Ritter's model. Unlike these authors, we assume that there is a form of public record keeping of some agents' histories but that access to this record is limited.

[4]Kahn and Roberds (2005) make a similar assumption.

remain in the economy and continue to use cash. Once they lose access to the CDB, agents can no longer join. It follows that any penalty imposed on agents with access to the CDB can be no more costly that losing their CDB access.

Note that in our model an agent's balance with the CDB does not matter. A similar assumption is made in Kahn and Roberds (2005) and in CW. Because we assume enough enforcement, if agents with access to the CDB are willing to produce goods for other agents with access, this will be true for any history of past trade. The fact that an agent's balance with the CDB is not a state variable greatly simplifies the analysis.[5]

The access cost to the CDB is paid once and for all at the beginning of the economy, before agents learn whether or not they will be money holders.[6] Agents are indexed by $i \in [0, 1]$, and the cost an agent must pay is given by $\kappa_i \geq 0$, where $\kappa_i \leq \kappa_j$ and $\kappa_{i+\Delta} - \kappa_i \leq \kappa_{j+\Delta} - \kappa_j$, if $i < j$, $\Delta > 0$; i.e., the costs are (weakly) increasing at a (weakly) increasing rate. We assume that κ_i is continuous in i.[7]

In meetings where only money can be used, the planner proposes the quantity of goods to be traded for money (with probability τ). Each participant in the meeting can agree or disagree to the proposed trade and the trade takes place if both participants agree. In meetings where both money and the CDB can be used, first the planner determines randomly which means of payment is used. The probability that the planner chooses money is θ. [8] If money is used in the meeting, then things proceed as described above. If the CDB is used, then the planner proposes a quantity to be exchanged and both participants can agree or disagree. [9] The trade

[5] Koeppl, Monnet, and Temzelides (2006) show that if settlement occurs often enough, incentive constraints associated with an agent's balance with a payment system will not bind. Allowing frequent settlement would allow us to maintain our results in an environment with less enforcement.

[6] While this cost is measured in terms of utility, we could assume that at the beginning of the economy agents are endowed with a nonstorable consumption good which must be consumed before any meeting with other agents. The access cost to the CDB could be expressed in terms of this good.

[7] Allowing for discontinuities complicates the exposition without providing additional insights.

[8] To simplify notation, θ summarizes both the probability that the planner chooses money and the probability that money changes hands in the meeting.

[9] Another interpretation of our economy is that the planner always requires the CDB to be used when it can be used and, additionally, may require money to change hands with probability θ. This interpretation is possible because there is no 'cost' of using the CDB in the sense that incentives to produce for other agents in the CDB are independent of trading histories. Under this

takes place only if both participants agree. The planner can impose a punishment on participants who disagree with the proposed trade.

2.1 Value functions

We write the value functions for agents depending on whether they hold one unit of money and whether they have access to the CDB. Let $m_0 \equiv \frac{1-M}{k}$ denote the probability of a single-coincidence meeting in which the agent who can produce the good desired by the other agent does not hold a unit of money. Similarly, let $m_1 \equiv \frac{M}{k}$ denote the probability of a single-coincidence meeting in which the agent who likes to consume the good produced by the other agent holds a unit of money. We denote by V_0 and V_1 the value functions of agents who do not have access to the CDB and have no unit of money or one unit of money, respectively. Also, we denote by c_m the amount of goods exchanged for a unit of money.

$$V_0 = m_1 \left[\beta \left(\tau V_1 + (1-\tau)V_0\right) - c_m\right] + (1-m_1)\beta V_0, \tag{1}$$

$$V_1 = m_0 \left[\beta \left(\tau V_0 + (1-\tau)V_1\right) + u(c_m)\right] + (1-m_0)\beta V_1. \tag{2}$$

An agent with no money will meet an agent with money of the right type with probability m_1. In such a meeting, the agent produces, and suffers the cost c_m, in exchange for a unit of money with probability τ. With this unit of money, the agent will have value V_1 in the next period. In all other meetings, no trade can take place. It can be shown that

$$V_0 = \frac{\beta m_0 m_1 \tau \left[u(c_m) - c_m\right] - (1-\beta) m_1 c_m}{(1-\beta)\left[1-\beta+\frac{\beta\tau}{k}\right]}, \tag{3}$$

$$V_1 = V_0 + \frac{m_0 u(c_m) + m_1 c_m}{\left[1-\beta+\frac{\beta\tau}{k}\right]} > V_0. \tag{4}$$

We denote by V_0^a and V_1^a the value functions of agents who have access to the CDB and have no unit of money or one unit of money, respectively. We let c_{DB}

interpretation, the planner is not restricted to use only money or the CDB but can use both.

denote the amount of goods produced when the CDB is used.

$$\begin{aligned}
V_0^a &= (1-\lambda)m_1\left[\beta\left(\tau V_1^a + (1-\tau)V_0^a\right) - c_m\right] \\
&\quad +\lambda m_0\left[u(c_{DB}) - c_{DB}\right] + \lambda m_1\left[u(c_{DB}) - (1-\theta)c_{DB} - \theta(c_m - \beta V_1^a)\right] \\
&\quad + \left[1 - (1-\lambda)m_1 - \lambda\theta m_1\right]\beta V_0^a \qquad (5) \\
&= (1-(1-\theta)\lambda)m_1\left[\beta\tau V_1^a - c_m\right] + \lambda(m_0+m_1)\left[u(c_{DB}) - c_{DB}\right] \\
&\quad +\lambda\theta m_1 c_{DB} + \left[1 - \tau m_1(1-(1-\theta)\lambda)\right]\beta V_0^a, \qquad (6)
\end{aligned}$$

$$\begin{aligned}
V_1^a &= (1-\lambda)m_0\left[\beta\left(\tau V_0^a + (1-\tau)V_1^a\right) + u(c_m)\right] \\
&\quad +\lambda m_1\left[u(c_{DB}) - c_{DB}\right] + \lambda m_0\left[(1-\theta)u(c_{DB}) + \theta\left[u(c_m) + \beta V_0\right] - c_{DB}\right] \\
&\quad + \left[1 - (1-\lambda)m_0 - \lambda\theta m_0\right]\beta V_1^a \qquad (7) \\
&= (1-(1-\theta)\lambda)m_0\left[\beta\tau V_0^a + u(c_m)\right] + \lambda(m_1+m_0)\left[u(c_{DB}) - c_{DB}\right] \\
&\quad -\lambda\theta m_0 u(c_{DB}) + \left[1 - \tau m_0(1-(1-\theta)\lambda)\right]\beta V_1^a. \qquad (8)
\end{aligned}$$

Recall that $\theta \in [0,1]$ denotes the probability that the CDB is used in a transaction that can take place both with money or using the CDB. We assume that θ is a choice variable of the planner. An agent who has access to the CDB but does not carry one unit of money meets, with probability $1-\lambda$, an agent who does not have access to the CDB. In that case, a trade takes place only if the meeting partner wants the good produced by the agent and has a unit of money (probability m_1). With probability λ, the meeting partner has access to the CDB. If there is a single coincidence of wants but the meeting partner does not have a unit of money (probability m_0), then an exchange can only occur through the CDB. However, if there is a single coincidence of wants and the meeting partner has a unit of money (probability m_1), then an exchange can occur using money (probability θ) or the CDB.[10] In all other meetings, no exchange can occur. Similar reasoning applies for the case of an agent who has access to the CDB and holds one unit of money. We use the notation $S_i \equiv u(c_i) - c_i$, to denote the surplus from a match in which c_i goods are exchanged.

[10]Note that two kinds of single coincidence meetings can occur: Either the agent considered wants to consume the good produced by her meeting partner or the agent produces the good consumed by her meeting partner.

Rewriting the expressions above, we get

$$V_0^a = \frac{\lambda}{k}\frac{S_{DB}}{1-\beta} + (1-(1-\theta)\lambda)\left[\frac{(1-(1-\theta)\lambda)\,\beta m_0 m_1 \tau S_m - (1-\beta)m_1 c_m}{(1-\beta)\left[1-\beta+(1-(1-\theta)\lambda)\frac{\beta\tau}{k}\right]}\right]$$

$$-\lambda\theta\left[\frac{(1-(1-\theta)\lambda)\,\beta m_0 m_1 \tau S_{DB} - (1-\beta)m_1 c_{DB}}{(1-\beta)\left[1-\beta+(1-(1-\theta)\lambda)\frac{\beta\tau}{k}\right]}\right], \tag{9}$$

$$V_1^a = V_0^a + \frac{(1-(1-\theta)\lambda)\left[m_0 u(c_m) + m_1 c_m\right] - \lambda\theta\left[m_0 u(c_{DB}) + m_1 c_{DB}\right]}{\left[1-\beta+(1-(1-\theta)\lambda)\frac{\beta\tau}{k}\right]}. \tag{10}$$

From equation (10), it appears that V_1^a could be smaller than V_0^a; for example, if c_m is sufficiently small and $\theta, \lambda > 0$. The intuition is that since c_m is very small, agents who have access to the CDB would prefer to use the CDB rather than money in a single coincidence meeting. However, since $\theta, \lambda > 0$ money holders must use money in some cases. We can rule out the cases where $V_1^a < V_0^a$ by assuming that there is free disposal of money.

2.2 Welfare

In the remainder of this section, we derive expressions for the expected utility of agents depending on whether they have access to the CDB. In order to derive the expressions for expected utility, we must first know the mass of each type of agents in the economy in steady-state. Let N_0^a and N_1^a denote the steady-state mass of agents who have access to the CDB and carry zero or one unit of money, respectively. Similarly, N_0 and N_1 denote the steady-state mass of agents who do not have access to the network and carry zero or one unit of money.

Lemma 1 *$N_0^a = \lambda(1-M)$, $N_1^a = \lambda M$, $N_0 = (1-\lambda)(1-M)$, and $N_1 = (1-\lambda)M$.*

The proof is provided in the appendix.

The expected utility, net of potential access cost, associated with having access to the CDB is given by $W^a = (1-M)V_0^a + MV_1^a$, which can be written as

$$W^a = \frac{1}{k(1-\beta)}\left[u(c_{DB}) - c_{DB}\right]\lambda\left[1-\theta M(1-M)\right]$$

$$+\frac{1}{k(1-\beta)}\left[u(c_m) - c_m\right](1-(1-\theta)\lambda)M(1-M). \tag{11}$$

The expected utility associated with not having access to the CDB is given by $W = (1-M)V_0 + MV_1$, which can be written as

$$W = \frac{u(c_m) - c_m}{k(1-\beta)} M(1-M). \tag{12}$$

Lemma 2 lists some properties of W^a and W.

Lemma 2 *Let c_{DB} and $c_m \in (0, \hat{c})$, where $\hat{c}$ is such that $u(\hat{c}) = \hat{c}$.*

1. *W^a is concave in c_{DB} and c_m and reaches a maximum at $u'(c_{DB}) = 1$ and $u'(c_m) = 1$.*

2. *$\frac{\partial W^a}{\partial \theta} > 0$ if and only if $u(c_m) - c_m > u(c_{DB}) - c_{DB}$.*

3. *$\frac{\partial W^a}{\partial \lambda} > 0$ if and only if*

$$\left[u(c_{DB}) - c_{DB}\right]\left[1 - \theta M(1-M)\right] > \left[u(c_m) - c_m\right](1-\theta)M(1-M).$$

4. *W^a is concave in M and reaches a maximum at $M = 1/2$ if*

$$(1 - (1-\theta)\lambda)\left[u(c_m) - c_m\right] > \theta\lambda\left[u(c_{DB}) - c_{DB}\right],$$

   while W^a is convex in M and reaches a minimum at $M = 1/2$ if

$$(1 - (1-\theta)\lambda)\left[u(c_m) - c_m\right] < \theta\lambda\left[u(c_{DB}) - c_{DB}\right].$$

5. *W is concave in c_m and reaches a maximum at $u'(c_m) = 1$.*

6. *W is concave in M and reaches a maximum at $M = 1/2$.*

The proof is provided in the appendix. Item (2) says that if the surplus from using the CDB is greater then the surplus from using money, then the welfare of agents who have access to the CDB increases if the CDB is more likely to be used. In particular, if $c_{CB} = c_m$, W^a is independent of θ. Item (3) notes that if the surplus from using the CDB is not too small, then the welfare of agents who have access to the CDB increases when more agents have access to the CDB. Intuitively, there is a network effect. Concerning item (4) note that if $\theta = 0$ or if $u(c_{DB}) - c_{DB} > u(c_m) - c_m$, then W^a cannot be convex in M. In other words, W^a is convex in M only if agents are forced to use money in situations where the surplus of trades involving money is smaller than the surplus of trades involving the CDB. Also, item (4) takes λ as given. We will see below that the choice of λ may depend on M. The unconstrained maximum for W^a is reached at $u'(c_{DB}) = u'(c_m) = 1$, $M = 1/2$, and $\lambda = 1$.

3 The efficient allocation

Agents in the economy are identical, except possibly for their cost of access to the CDB. Hence, all agents would agree on the allocation they prefer if they could meet before the beginning of the economy, at a time when they are ignorant of the access cost that any specific individual will face. We call this allocation the efficient allocation. It is derived under the assumption that the planner can observe the cost of entry κ and can force agents to pay that cost.

The benefit from letting a mass λ of agents have access to the CDB is given by $\lambda\left(W^a(\lambda) - W\right)$. Since agents may have different access costs, the lowest possible cost to let a mass λ of agents have access to the CDB is given by $\int_0^\lambda \kappa_i di$. Let SW denote the social welfare function.

$$SW \equiv \lambda W^a(\lambda) + (1-\lambda)W - \int_0^\lambda \kappa_i di = W + \lambda\left[W^a(\lambda) - W\right] - \int_0^\lambda \kappa_i di. \quad (13)$$

The value of λ which characterizes the ex-ante efficient allocation maximizes SW.

Note that $W^a(\lambda) - W$ is a convex function of λ, so taking the first derivative of SW and setting it equal to zero may not provide a maximum. Whether or not it does depends on the particular shape of $\int_0^\lambda \kappa_i di$. Since we assumed that κ_i is weakly increasing at a weakly increasing rate, $\int_0^\lambda \kappa_i di$ is itself a convex function of λ. If it is sufficiently convex, then SW will be concave.

This can be illustrated by an example: Consider a particular functional form,

$$\int_0^\lambda \kappa_i di = \alpha\lambda^n, \quad (14)$$

where $\alpha > 0$ and n is a positive integer, and assume that $c_m = c_{DB} = c$. If $n = 2$, then

$$SW = W + \lambda^2\left[\frac{u(c) - c}{k(1-\beta)}\left[1 - M(1-M)\right] - \alpha\right]. \quad (15)$$

If $\alpha > \frac{u(c)-c}{k(1-\beta)}\left[1 - M(1-M)\right]$ then the term in brackets is negative and the efficient allocation is such that $\lambda = 0$. In contrast, if $\alpha < \frac{u(c)-c}{k(1-\beta)}\left[1 - M(1-M)\right]$ then the efficient allocation is such that $\lambda = 1$. If, instead, $n = 3$, then

$$SW = W + \lambda^2\left[\frac{u(c) - c}{k(1-\beta)}\left[1 - M(1-M)\right] - \alpha\lambda\right]. \quad (16)$$

For λ sufficiently small, the term in brackets is positive, so that an increase in λ increases SW. However, if $\alpha > \frac{u(c)-c}{k(1-\beta)}\left[1 - M(1-M)\right]$ then the term in brackets is negative for sufficiently high λ so that a decrease in λ increases SW. In such a case, the solution to $\max_\lambda SW$ is interior.

Abstracting from the cost of access, we can make some observations about the benefit of increasing the mass of agents having access to the CDB at the margin. This benefit is given by

$$[W^a(\lambda) - W] + \lambda W^{a\prime}(\lambda). \tag{17}$$

The first element is the benefit received by the marginal agent who obtains access to the CDB when the mass of agents having access is λ. The second element is the benefit that the mass λ of agents who already have access to the CDB receive from the addition of the marginal agent. Note that

$$\begin{aligned} [W^a(\lambda) - W] &= \lambda W^{a\prime}(\lambda) = \frac{\lambda}{k(1-\beta)}\left[u(c_{DB}) - c_{DB}\right] \\ &\quad - \frac{\lambda M(1-M)}{k(1-\beta)}\left\{\theta\left[u(c_{DB}) - c_{DB}\right] + (1-\theta)\left[u(c_m) - c_m\right]\right\}. \end{aligned} \tag{18}$$

In this economy, the benefit received by the marginal agent is exactly equal to the benefits received by all agents who have access to the CDB from the addition of the marginal agent. From lemma 2, we also know that regardless of λ the efficient allocation should set $u'(c_{DB}) = u'(c_m) = 1$ and $M = 1/2$.

4 Incentive constraints

The allocation derived in the previous section may not be achievable if the planner is unable to observe the entry cost κ and force agents to get access to the CDB. Indeed, agents in this model face two different incentive constraints. First, agents must choose whether to gain access to the CDB. Since the social benefit of the CDB is strictly greater than the private benefit, some agents may prefer not to gain access to the CDB even though the efficient allocation would call for them to do so. The entry constraint compares the expected benefit of having access to the CDB with the entry cost. Second, agents who have access to the CDB must agree to produce for other agents who have access. We assume that agents can walk away from the CDB but remain in the economy and continue to use money. Hence, the cost of any punishment is limited to the cost of permanently losing access to the CDB. The no-exit constraint compares the value of remaining in the CDB arrangement with the cost of producing.

4.1 The entry constraint

We assume that agents must decide whether or not to access the CDB at the very beginning of the economy, before they learn whether or not they will be money holders at date 1. Under this assumption, all agents are identical when they make their access decision, with the possible exception of their access cost.

The expected welfare benefit from having access to the CDB is

$$\begin{aligned} W^a - W &= \frac{\lambda}{k(1-\beta)} \left[u(c_{DB}) - c_{DB} \right] \\ &\quad - \frac{\lambda M(1-M)}{k(1-\beta)} \left\{ \theta \left[u(c_{DB}) - c_{DB} \right] + (1-\theta) \left[u(c_m) - c_m \right] \right\}. \end{aligned} \tag{19}$$

From equation (19), it appears that $W^a - W$ could be negative; for example if $\theta < 1$ and c_{DB} is sufficiently small. The intuition is that since c_{DB} is very small, the benefit from using the CDB is very small. Moreover, since $\theta < 1$, agents who have access to the CDB must use it in some single coincidence meetings when they would prefer to use money. Nobody would pay to get access to the CDB in this case so we focus on more interesting cases where $W^a - W \geq 0$.

In order to make her access decision, an agent forms beliefs about the mass λ of agents who obtain access. Based on that belief, agent i compares the benefit of having access to the CDB, $W^a(\lambda) - W$, with the cost, κ_i. The entry constraint for agent i can thus be written as $W^a(\lambda) - W \geq \kappa_i$.

Lemma 3 *Let c_{DB} and $c_m \in (0, \hat{c})$.*

1. *$W^a - W$ is concave in c_{DB} and reaches a maximum at $u'(c_{DB}) = 1$.*

2. *$W^a - W$ is convex in c_m and reaches a minimum at $u'(c_m) = 1$.*

3. *$W^a - W$ is convex in M and reaches a minimum at $M = 1/2$.*

4. *$\frac{\partial W^a - W}{\partial \theta} > 0$ if and only if $u(c_m) - c_m > u(c_{DB}) - c_{DB}$.*

The proof is provided in the appendix. Choosing c_{DB} so that it maximizes the surplus of a match relaxes the entry constraint. In contrast, the constraint is tightest when c_m is chosen to maximize the surplus of a match. The constraint can be relaxed by moving M away from $1/2$. Finally, θ does not affect the constraint provided $c_{DB} = c_m$.

4.2 The no-exit constraint

Agents who have access to the CDB may have an incentive to renege on their obligation to produce in a meeting with an agent who also has access to the CDB. This is because such agents have to pay the immediate cost of production but only receive the potential benefits from access to the CDB later. As noted above, the maximum punishment for agents who refuse to produce can be no greater than the cost of permanently losing access to the CDB.[11] The no-exit constraint compares the value of retaining access to the CDB with the cost of producing goods today. This constraint can be written as

$$\beta \left(V_i^a - V_i\right) \geq c_{DB}, i = 0, 1. \tag{20}$$

As is standard, β cannot be too small, or c_{DB} too large relative to $u(c_{DB})$, if agents are not to defect.

[11] It would be easier to sustain an arrangement such as the CDB if we assumed more severe punishments. For example, the only outside option for defecting agents could be autarky. However, more severe punishments are harder to implement as they require more monitoring of agents' behavior.

We can write

$$
\begin{aligned}
V_1^a - V_1 &= V_0^a - V_0 - \lambda\theta \frac{m_0 u(c_{DB}) + m_1 c_{DB}}{\left[1-\beta+\frac{(1-(1-\theta)\lambda)\beta\tau}{k}\right]} \\
&\quad -(1-\theta)\lambda \frac{(1-\beta)\left[m_0 u(c_m) + m_1 c_m\right]}{\left[1-\beta+\frac{(1-(1-\theta)\lambda)\beta\tau}{k}\right]\left[1-\beta+\frac{\beta\tau}{k}\right]} \qquad (21) \\
&= \frac{\lambda}{k}\frac{S_{DB}}{1-\beta} - \lambda\theta\left[\frac{(1-(1-\theta)\lambda)\,\beta m_0 m_1 \tau S_{DB} + (1-\beta)m_0 u(c_{DB})}{(1-\beta)\left[1-\beta+(1-(1-\theta)\lambda)\frac{\beta\tau}{k}\right]}\right] \\
&\quad -(1-\theta)\lambda \frac{\beta m_0 m_1 \tau S_m\left[2-(1-\theta)\lambda+\frac{(1-(1-\theta)\lambda)\beta\tau}{(1-\beta)k}\right]}{\left[1-\beta+(1-(1-\theta)\lambda)\frac{\beta\tau}{k}\right]\left[1-\beta+\frac{\beta\tau}{k}\right]} \\
&\quad -(1-\theta)\lambda \frac{(1-\beta)m_0 u(c_m)}{\left[1-\beta+(1-(1-\theta)\lambda)\frac{\beta\tau}{k}\right]\left[1-\beta+\frac{\beta\tau}{k}\right]}. \qquad (22)
\end{aligned}
$$

Inspection of equation (22) reveals that if $\lambda \to 0$, then $V_1^a - V_1 \to 0$. Hence, it is not possible to sustain very small networks as the benefits from having access to the network are not large enough to provide incentives to agents to produce when they should.

Lemma 4 *Let c_{DB} and $c_m \in (0, \hat{c})$.*

1. $V_1^a - V_1 \leq V_0^a - V_0$.
2. $V_1^a - V_1$ *is concave in* c_{DB} *and reaches a maximum at* $c_{DB}^* < c^*$*, where* c^* *is given by* $u'(c^*) = 1$.
3. $V_1^a - V_1$ *is convex in* c_m *and reaches a minimum at* $c_m^* > c^*$.
4. $V_1^a - V_1$ *is convex in* M *and reaches a minimum over* $[0,1]$ *at* $M_{min} \leq \frac{1}{2}$*, and a maximum at* $M = 1$.
5. $\left.\frac{\partial(V_1^a - V_1)}{\partial\theta}\right|_{c_m = c_{DB}} < 0$.

The first result states that the benefit from having access to the CDB is greater for agents who do not hold money than it is for agents who do hold money. This implies that we only need to be concerned about the no-exit constraint for agents who are holding a unit of money. The no-exit constraint can be relaxed by decreasing c_{DB} away from c^*. Intuitively, agents who have access to the CDB are more likely to be willing to produce a smaller amount when called upon. The constraint can also be relaxed by decreasing c_m away from c^* and by increasing M away from $M = 1/2$. Decreasing c_m, which corresponds to increasing the price of goods purchased with money, or increasing M from $M = 1/2$, hurts money holders and thus makes it more costly to lose access to the CDB. When c_{DB} and c_m are not too different, the constraint is loosened by reducing θ. This benefits buyers holding money, who like smaller values of θ.

4.3 Relaxing the constraints

In the remainder of this section, we consider how changes in some parameters can affect the entry and the no-exit constraints. From lemmas 3 and 4, both constraints can be relaxed if c_m is decreased away from c^* or increased away from $c_m^* > c^*$, and if M is increased away from $M = 1/2$. If $c_m = c_{DB}$, then $W^a - W$ is independent of θ. In that case, the entry constraint can be relaxed by increasing θ because money holders prefer to hold on to their unit of money if they can use the CDB instead.

One way to get a high θ is to give buyers the choice of which payment instrument they want to use. Since $\partial V_1^a/\partial\theta = \partial(V_1^a - V_1)/\partial\theta$, item 5 of lemma 4 shows that buyers who have the choice between using money or the CDB prefer to use the CDB, if c_m and c_{DB} are not too different.

We can also show that agents who can accept both money and the CDB as a mean of payment prefer to receive money.[12] Hence, if they could, sellers would have an incentive to raise the price of goods bought with money up to the point where they would be indifferent between receiving money or using the CDB.

Credit and debit card networks typically impose a 'no-surcharge' rule. This rule states that the price of a good purchased with a credit card should not be higher than price of the same good purchased with money. Our results suggest that the 'no surcharge' rule imposed by networks may have some benefits in terms of incentives.

In this economy, the planner would choose c_m and c_{DB} such that $u'(c_m) = u'(c_{DB}) = 1$, if neither incentive constrain binds. If some constraint binds, changing c_m or c_{DB} may relax that constraint. As we have seen from lemmas 3 and 4, any change in c_{DB} must tighten at least one constraint. Decreasing c_{DB} below c_{DB}^* or

[12] Note that if $c_m = c_{DB}$, then W^a is independent of θ. Since $\partial V_1^a/\partial\theta < 0$, then $\partial V_0^a/\partial\theta > 0$.

increasing c_{DB} above c^* tightens both constraints. On the other hand, decreasing c_m below c^* or increasing c_m above c_m^* relaxes both constraints. Changing c_m between c^* and c_m^* must tighten at least one constraint.

At the unconstrained optimal values of c_m and c_{DB}, a 'surcharge' for the use of the CDB would mean either an decrease in c_{DB}, which would tighten the entry constraint, or an increase in c_m, which would tighten the no-exit constraint, unless the planner increases c_m above c_m^*. However, can show that the planner would never prefer to do that.

Lemma 5 *The planner always (weakly) prefers to decrease c_m below c^* rather than increase c_m above c_m^*.*

The proof is provided in the appendix.

We can summarize these results in the following proposition.[13]

Proposition 1 *At the unconstrained optimal values of c_m and c_{DB}, deviating from the no surcharge rule must tighten at least one of the incentive constraints.*

5 Constrained efficient allocations

In this section, we describe constrained efficient allocations. These are allocations achieved by a planner who can choose M, c_{DB}, c_m, and θ but who does not observe the entry cost κ faced by each agent and who is unable to force agents to produce.

The faction λ of agents who choose to access the CDB is determined by the entry constraint. Whether agents who have access to the CDB produce for others depends on the no-exit constraint. We consider each constraint in turn.

5.1 Allocations constrained by the entry condition

Some agents who should have access to the CDB under the efficient allocation may prefer not to do so. The reason is that individual agents, when choosing to access

[13] Our paper abstracts from any cost of using cash or the CDB. Adding these costs would strengthen our results since the marginal social cost of using cash is higher than the marginal social cost of using electronic payments (see Garcia-Swartz, Hahn, and Layne-Farrar 2006).

the CDB, compare their private benefit to their cost but do not take into account the network effect.

The analysis in this section is conducted assuming that the entry constraint does not bind. Any candidate allocation with $\lambda > 0$ is not a constrained efficient allocation if the entry constraint binds.

Agent i, when considering whether or not to access the CDB, compares the cost of doing so, κ_i, with the benefit $b(\lambda) \equiv W^a(\lambda) - W$. We assume that agents who are indifferent choose to access the CDB. Equation (18) shows that the benefit from accessing the CDB is a linear function of λ. Since we assume that the distribution of entry costs is continuous, weakly increasing, and weakly concave, we can consider three cases: The graph of κ, as a function of λ, may intersect the graph of $b(\lambda)$ either 0, 1, or 2 times.

5.1.1 No intersections

There are two cases to consider: Either $\kappa_i > b(i)$, for all i, or $\kappa_i < b(i)$, for all $i > 0$.[14]

Proposition 2 *If $\kappa_i > b(i)$, for all i, then no agent accesses the CBD.*

Proof. Assume, to establish a contradiction, that a mass $\lambda > 0$ of agents choose to access the CDB. Since κ_i is continuous, then for ε sufficiently small, $\kappa_{\lambda-\varepsilon} > b(\lambda)$. Also $b(\lambda - \varepsilon) < b(\lambda)$ so all agents in the interval $[\lambda - \varepsilon, \lambda]$ prefer not to have access to the CDB. Since this will be true for any $\lambda > 0$, no agent accesses the CDB. ■

Proposition 3 *If $\kappa_i < b(i)$, for all $i > 0$, then either all agents access the CDB or no agents access the CDB.*

Proof. It is incentive compatible for all agents to access the CDB, since $b(1) > \kappa_1 \geq \kappa_i$ for all i.

If the mass of agents with cost $\kappa_i = 0$ is zero, then it is incentive compatible for no agent to access the CDB (except for a set of measure zero). Indeed, agents face cost $\kappa_i > b(0) = 0$, for all $i \in (0, 1]$. If, instead, the mass of agents with cost $\kappa_i = 0$ is positive, then it is not incentive compatible for no agent to access the CDB since we assume that agents who are indifferent choose to access the CDB. ■

[14] Since we restrict $\kappa_i \geq 0$, then it cannot be the case that $b(0) > \kappa_0$.

We can define a notion of stability of an allocation with respect to small deviations of beliefs about λ. Let $\eta \in [0,1]$ denote the mass of agents who access the CDB if all agents believe that a mass λ_η of agents access the CDB.

Definition 4 *An allocation λ is unstable if, $\forall \varepsilon > 0$, $|\lambda_\eta - \lambda| > \varepsilon \Rightarrow \eta \neq \lambda$.*

The allocation characterized by $\lambda = 0$ in the above proposition is unstable when it exists. All other allocations considered so far are stable.

5.1.2 One intersection

There are two cases to consider: First, $\kappa_i > b(i)$ for $i \in [0, \bar{\lambda})$ and $\kappa_i < b(i)$ for $i \in (\bar{\lambda}, 1]$, $0 \leq \bar{\lambda} \leq 1$. Second, $\kappa_i < b(i)$ for $i \in (0, \bar{\lambda})$ and $\kappa_i > b(i)$ for $i \in (\bar{\lambda}, 1]$, $0 \leq \bar{\lambda} \leq 1$. In the first case, the slope of the graph of κ is flatter than the slope of the graph of $b(\lambda)$ at the point at which they intersect, while the opposite is true in the second case.

Proposition 5 *If $\kappa_i > b(i)$ for $i \in [0, \bar{\lambda})$ and $\kappa_i < b(i)$ for $i \in (\bar{\lambda}, 1]$, $0 \leq \bar{\lambda} \leq 1$, then there are two stable allocations: Either all agents access the CDB, or no agent accesses the CDB (except for sets of measure zero). There is also an unstable allocation such that $\bar{\lambda}$ agents access the CDB.*

Proof. If all agents believe nobody accesses the CDB, then no agent chooses to access the CDB since $\kappa_i > b(0) \geq 0$, for all i. If all agents believe that everybody accesses the CDB, then all agents choose to access the CDB since $b(1) > \kappa_1 \geq \kappa_i$, for all i.

Now assume all agents believe that exactly $\bar{\lambda}$ agents will access the CDB. There is a mass $\bar{\lambda}$ of agents with cost $\kappa_i < \kappa_{\bar{\lambda}}$. They prefer to access the CDB since $\kappa_{\bar{\lambda}} = b(\bar{\lambda})$. For all other agents, $\kappa_i > \kappa_{\bar{\lambda}}$ and they prefer not to access the CDB.

No other belief can be supported. To see this, first consider any $\lambda \in (0, \bar{\lambda})$. By assumption, $\kappa_\lambda > b(\lambda)$, and by continuity, the same must be true in a neighborhood of λ. Hence, agents with a cost close to but smaller than λ choose not to access the CDB. Next, consider any $\lambda \in (\bar{\lambda}, 1)$. By assumption, $\kappa_\lambda < b(\lambda)$, and by continuity, the same must be true in a neighborhood of λ. Hence, agents with a cost close to but higher than λ choose to access the CDB. ■

Proposition 6 *If $\kappa_i < b(i)$ for $i \in [0, \bar{\lambda})$ and $\kappa_i > b(i)$ for $i \in (\bar{\lambda}, 1]$, $0 \leq \bar{\lambda} \leq 1$, then there is a unique stable allocation such that a mass $\bar{\lambda}$ of agents accesses the CDB. If the mass of agents with cost κ_i is equal to zero, then there is also an unstable allocation such that no agent accesses the CDB.*

Proof. The proof for the allocation characterized by $\bar{\lambda}$ is the same as in the previous proposition. The proof that this allocation is stable is omitted. The proof of existence (or lack thereof) of the no access allocation is the same as in the case where $\kappa_i < b(i)$ for all $i > 0$.

Now I show that no other belief can be supported. Suppose all agents believe that a mass $\lambda \in (0, \bar{\lambda})$ of agents access the CDB. In that case, $\kappa_\lambda < b(\lambda)$, and by continuity, the same must hold true in a neighborhood of λ. Hence, agents with a cost close to but higher than κ_λ choose to access the CDB. Suppose all agents believe that a mass $\lambda \in (\bar{\lambda}, 1]$ of agents access the CDB. In that case, $\kappa_\lambda > b(\lambda)$, and by continuity, the same must hold true in a neighborhood of λ. Hence, agents with a cost close to but lower than κ_λ choose not to access the CDB. ■

5.1.3 Two intersections

There is one case to consider: $\kappa_i > b(i)$ for $i \in [0, \bar{\lambda}_1) \cup (\bar{\lambda}_2, 1]$ and $\kappa_i < b(i)$ for $i \in (\bar{\lambda}_1, \bar{\lambda}_2)$, where $0 \leq \bar{\lambda}_1 < \bar{\lambda}_2 \leq 1$. The graph of κ_i is flatter then the graph of $b(i)$ at $\bar{\lambda}_1$, but steeper at $\bar{\lambda}_2$.

Proposition 7 *If $\kappa_i > b(i)$ for $i \in [0, \bar{\lambda}_1) \cup (\bar{\lambda}_2, 1]$ and $\kappa_i < b(i)$ for $i \in (\bar{\lambda}_1, \bar{\lambda}_2)$, where $0 \leq \bar{\lambda}_1 < \bar{\lambda}_2 \leq 1$, then either nobody accesses the CDB, or else a mass $\bar{\lambda}_1$ or mass $\bar{\lambda}_2$ of agents accesses the CDB.*

The proof of this proposition is omitted as it follows the same logic as the proofs of previous propositions. Note that the allocation with a mass $\bar{\lambda}_1$ of agents accessing the CDB is unstable while the other equilibria are stable.

5.2 Allocations constrained by the no-exit condition

We saw in section 4 that the relevant no-exit constraint is $\beta(V_1^a - V_1) \geq c_{DB}$. Equation (22) reveals that if $\theta = 1$, then $V_1^a - V_1$ is linear in λ. By continuity,

$V_1^a - V_1$ is increasing in λ for small enough values of θ. In other words, the no-exit constraint is less likely to bind for large enough networks if θ is large.

Below, we provide an example where $V_1^a - V_1$ decreases as λ increases for high values of λ. Hence, the no-exit constraint may bind for low and for high values of λ, but not for intermediate values, if θ is sufficiently small. This result is surprising since the network effect makes larger networks more attractive.

To provide an example, we consider $V_1^a - V_1$ evaluated at $\theta = 0$, $\tau = 1$, and $c_{DB} = c_m = c$.

$$V_1^a - V_1 = \frac{\lambda}{k}\frac{u(c) - c}{1-\beta} - \lambda\left[\frac{\beta m_0 m_1 \left[u(c) - c\right]\left[(2-\lambda) + \frac{\beta}{1-\beta}\frac{1-\lambda}{k}\right] + (1-\beta) m_0 u(c)}{\left[1-\beta+\beta\frac{1-\lambda}{k}\right]\left[1-\beta+\frac{\beta}{k}\right]}\right] \tag{23}$$

In the appendix we show that this expression is concave in λ and we also obtain the partial derivative with respect to λ. Evaluated at $\lambda = 1$, the derivative is

$$\left.\frac{\partial V_1^a - V_1}{\partial \lambda}\right|_{\lambda=1} = \frac{M u(c) - c}{k(1-\beta)}. \tag{24}$$

This expression is negative if M is sufficiently small.

We now provide a numerical example showing that the no-exit constraint may bind for high and low values of λ but not for intermediate values. Consider the case where $u(c) = \sqrt{c}$. Assume that $k = 5$ and $\beta = 0.92$. The quantity $c = c_m = c_{DB}$ is chosen so as to maximize the surplus of a match, $c = 0.25$. Finally, we choose a low value of M, specifically $M = 0.1$. In this example, the constraint binds if λ is smaller than 0.75, approximately, and it also binds if λ is greater than 0.92, approximately.

The intuition for this result goes as follows. Agents who hold a unit of money are more likely to be in a match with an agent who does not have a unit of money when M is smaller. Hence, absent the CDB, they are more likely to be able to trade. As the number of agents with access to the CDB increases, the probability of meeting an agent who will accept only money decreases. Moreover, if θ is small, the CDB is used in most instances when a pair of agents could trade either with cash or with the CDB. This implies that the increase in the number of meetings in which the agent can consume using the CDB is partially offset by a decrease in the number of meetings in which the agent can consume using money. However, the increase in λ also means an increase in the number of meetings in which the agent has to produce. In summary, as λ increases, $V_1^a - V_1$ decreases because the probability of having to produce increases faster than the probability of being able to consume.

6 Achieving better allocations

In this section we consider several policies that a planner can use to relax the entry constraint. We assume that the no-exit constraint does not bind but restrict our attention to policies that either relax or do not affect the no-exit constraint.

We consider three policies: First, a utility cost, T, which we associate with a tax, can be imposed on agents in the economy.[15] The tax works as follows: Agents are taxed if they choose not to access the CDB but are not taxed if they access the CDB. Second, the money supply M can be increased above 1/2. Third, the amount of goods exchanged for a unit of money, c_m, can be reduced below the level that maximizes the surplus of a match. These policies are evaluated on their ability to improve social welfare.

6.1 If $\lambda = 1$ is optimal

First, we consider the case where $\lambda = 1$ is optimal. As the following proposition shows, the use of taxes is particularly effective when the objective is to achieve universal access to the CDB. Recall that κ_1 denotes the cost of access for the agent facing the highest cost.

Proposition 8 *Assume that the efficient allocation is such that $\lambda = 1$. If $\kappa_1 \in [W^a(1) - W, 2(W^a(1) - W)]$, then the constrained efficient allocation is different from the efficient allocation. The efficient allocation can be achieved by setting a high enough tax for agents who do not access the CDB.*

Proof. Since $\kappa_1 > W^a(1) - W$, then some agents prefer not to get access to the CDB since their access cost is greater than their private benefit. However, since

[15] As in the case of the access cost, we could assume that at the beginning of the economy agents are endowed with a nonstorable consumption good which must be consumed before any meeting with other agents. The planner could tax this good.

$\kappa_1 < 2(W^a(1) - W)$, it is desirable from the perspective of social welfare that all agents have access to the CDB. If follows that the constrained efficient allocation is different from the efficient allocation.

Assume that a tax greater than or equal to κ_1 is imposed on any agent who chooses not to gain access to the CDB. Under this threat, it is individually rational for all agents to obtain access to the CDB, regardless of what other agents do. ■

Note that since all agents obtain access to the CDB, no tax is paid. It is the case that agents whose cost of access is higher than $\frac{u-c}{k(1-\beta)}[1 - M(1-M)]$ are made worse off by gaining access to the CDB than they would have been if they did not gain access and did not have to face the tax. However, before agents know their types, they would all agree on the desirability of the tax.

The key to the result is the assumption that $\kappa_1 \leq 2\frac{u-c}{k(1-\beta)}[1 - M(1-M)]$, so that all agents should have access to the CDB in order to achieve the efficient allocation. The use of taxes is not as effective when $\kappa_1 > 2\frac{u-c}{k(1-\beta)}[1 - M(1-M)]$, since in that case it is not desirable that all agents have access.

It may not be possible to obtain the efficient allocation by changing the money supply or the amount of goods exchanged for a unit of money. Indeed, the marginal social benefit of adding the last agent to the CDB is always greater than the private benefit given by $W^a(1) - W$, for any values of M or c_m. This is shown formally in the following proposition.

Proposition 9 *If $\kappa_1 \in (\frac{u(c_{DB})-c_{DB}}{k(1-\beta)}, 2\frac{u(c_{DB})-c_{DB}}{k(1-\beta)}]$, then the efficient allocation is such that $\lambda = 1$, however it is not possible to achieve $\lambda = 1$ only by changing M or c_m (or both).*

Proof. From section 3, we know that the efficient allocation calls for $\lambda = 1$ if $\kappa_1 < 2\frac{u-c}{k(1-\beta)}$. Assume all agents join the CDB and consider the agent with the highest cost. This agent will choose to access the CDB if $[W^a(1) - W] \geq \kappa_1$. From equation (3), $[W^a(1) - W]$ can be no greater than

$$\frac{1}{k(1-\beta)}[u(c_{DB}) - c_{DB}]. \tag{25}$$

This maximum is reached if $M = 1$ (and, by symmetry, if $M = 0$) or if $c_m = 0$ and $(1-\theta) = 1$. It follows that if $\kappa_1 > \frac{u(c_{DB})-c_{DB}}{k(1-\beta)}$ then the agent with the highest cost will choose not to access the CDB. By continuity of the access cost schedule, the mass of agents joining the CDB is strictly less than 1. In this case, choosing the money supply does not make it possible to achieve the ex-ante efficient allocation. ■

Finally, note also that since $W^a(0) - W = 0$, it is always an equilibrium for no agents to access the CDB. Indeed, much as with money, if nobody expects the CDB to be used, then nobody has an incentive to use it. Changing M or c_m does not affect $W^a(0) - W$ and thus does not impact the no access equilibrium. In contrast, proposition 8 shows that a high enough tax will eliminate that equilibrium.

6.2 If $\lambda < 1$ is optimal

In this section, we focus on the case where the efficient allocation is such that $\lambda < 1$. The problem is to choose T, M, and c_m to maximize the social welfare function

$$SW = \lambda W^a + (1-\lambda)(W - T) - \int_0^\lambda \kappa_i di, \tag{26}$$

taking into account the fact that λ solves

$$\begin{aligned} \kappa_\lambda - T &= W^a(\lambda) - W \qquad (27) \\ &= \frac{\lambda}{k(1-\beta)}\left[u(c_{DB}) - c_{DB}\right] \\ &\quad - \frac{\lambda M(1-M)}{k(1-\beta)}\left\{\theta\left[u(c_{DB}) - c_{DB}\right] + (1-\theta)\left[u(c_m) - c_m\right]\right\}. \qquad (28) \end{aligned}$$

The question we ask is: What is the least costly way to provide incentives for $\tilde{\lambda}$ agents to access the CDB? Notice that the access cost must be the same regardless of the policy chosen, since

$$\int_0^{\tilde{\lambda}} \kappa_i di$$

is independent of the policy choice. Also, the social welfare function can be rewritten

$$SW = W - T + \tilde{\lambda}\left[W^a(\tilde{\lambda}) - (W - T)\right] - \int_0^{\tilde{\lambda}} \kappa_i di. \tag{29}$$

By equation (27), it must be the case that $W^a(\tilde{\lambda}) - (W - T) = \kappa_{\tilde{\lambda}}$ regardless of which policy is chosen. Hence, when comparing two policies, (T, M, c_m) and (T', M', c_m'), it is enough to compare $W(T, M, c_m) - T$ with $W(T', M', c_m') - T'$, subject to constraint (27).

We want to compare three sets of policies:

1. $(T = \tilde{T} > 0, M = 1/2, c_m = c^*)$,

2. $(T = 0, M = \tilde{M} \neq 1/2, c_m = c^*)$,

3. $(T = 0, M = 1/2, c_m = \tilde{c}_m \neq c^*)$,

where c^* is defined by $u'(c^*) = 1$. We also assume throughout this section that $c_{DB} = c^*$.

The following proposition states that it is preferable to tax agents who do not access the CDB rather than change the money supply. In turn, changing M is preferable to changing c^*. Note that a linear combination of the policies considered cannot be better than the policy of only taxing agents who do not access the CDB.

Proposition 10 $W(\tilde{T}, 1/2, c^*) \geq W(0, \tilde{M}, c^*) \geq W(0, 1/2, \tilde{c}_m)$.

Proof. First, note that these expressions are given by

$$W(\tilde{T}, 1/2, c^*) = \frac{u(c^*) - c^*}{4k(1-\beta)} - \tilde{T}, \tag{30}$$

$$W(0, \tilde{M}, c^*) = \frac{u(c^*) - c^*}{k(1-\beta)} \tilde{M}(1 - \tilde{M}), \tag{31}$$

$$W(0, 1/2, \tilde{c}_m) = \frac{u(\tilde{c}_m) - \tilde{c}_m}{4k(1-\beta)}. \tag{32}$$

We can use equation (27) to obtain

$$\tilde{T} = \kappa_{\tilde{\lambda}} - \tilde{\lambda} \frac{3\left[u(c^*) - c^*\right]}{4k(1-\beta)}. \tag{33}$$

and

$$\frac{u(c^*) - c^*}{k(1-\beta)} \tilde{M}(1 - \tilde{M}) = \frac{u(c^*) - c^*}{k(1-\beta)} - \frac{\kappa_{\tilde{\lambda}}}{\tilde{\lambda}}. \tag{34}$$

We can show that $W(\tilde{T}, 1/2, c^*) \geq W(0, \tilde{M}, c^*)$ since

$$W(\tilde{T}, 1/2, c^*) - W(0, \tilde{M}, c^*) = \frac{1 - \tilde{\lambda}}{\tilde{\lambda}} \left[\frac{3\tilde{\lambda}\left[u(c^*) - c^*\right]}{4k(1-\beta)} - \kappa_{\tilde{\lambda}} \right] = \frac{1 - \tilde{\lambda}}{\tilde{\lambda}} \tilde{T} \geq 0. \tag{35}$$

Now we want to show that $W(0, \tilde{M}, c^*) \geq W(0, 1/2, \tilde{c}_m)$. From equation (27) we can get

$$\frac{\kappa_{\tilde{\lambda}}}{\tilde{\lambda}} k(1-\beta) = \left[u(c^*) - c^*\right] \left[1 - \tilde{M}(1 - \tilde{M})\right], \tag{36}$$

from policy $(0, \tilde{M}, c^*)$, and

$$\frac{\kappa_{\tilde{\lambda}}}{\tilde{\lambda}} k(1-\beta) = [u(c^*) - c^*] - \frac{1}{4}\left[\theta\left(u(c^*) - c^*\right) + (1-\theta)\left(u(\tilde{c}_m) - \tilde{c}_m\right)\right], \tag{37}$$

from policy $(0, 1/2, \tilde{c}_m)$. Combining these two expressions we get

$$[u(c^*) - c^*]\,\tilde{M}(1-\tilde{M}) = \frac{1}{4}\left[\theta\left(u(c^*) - c^*\right) + (1-\theta)\left(u(\tilde{c}_m) - \tilde{c}_m\right)\right] \geq \frac{1}{4}\left(u(\tilde{c}_m) - \tilde{c}_m\right). \tag{38}$$

This completes the proof. ■

The intuition for this result is that the wedge that must be created to provide incentives for the marginal agent to obtain access to the CDB is the same whether a tax is imposed or whether M or c_m is modified. In the case of the tax, however, only agents who do not access the CDB pay the cost associated with this wedge. In the case of a change in M or c_m, all agents must pay that cost. Also, note that $W(0, \tilde{M}, c^*) = W(0, 1/2, \tilde{c}_m)$ if $\theta = 0$. If the CDB is always used when money is also available, agents who have access to the CDB are not affected by the change in c_m unless money is the only payment method available.

One important caveat to this result is that it assumes the no-exit constraint (20) is not binding. If the no-exit constraint is binding, then there might be a role for choosing $M > 1/2$. The key idea is that agents must have incentives to both access the CDB and produce under the CDB arrangement. Proposition 10 concerns the access decision, assuming agents are willing to produce.

If agents can be taxed when they decide not to produce, then it is optimal to set $M = 1/2$ and $c_m = c^*$ and use taxes to ensure that the entry constraint holds. However, if one assumes that agents cannot be taxed if they refuse to produce, then the only way to loosen the entry constraint may be to change the money supply or the amount of goods exchanged for a unit of money. The general point here is that changing the money supply is an easy way to affect all agents even if it is difficult to keep track of them, while using taxes requires an ability to keep track of agents.

7 Comparison with Cavalcanti-Wallace

In this section, we compare some allocations of economies with limited memory studied in the previous sections with the 'no-gift' allocations studied in Cavalcanti and Wallace (1999 a). To facilitate the comparison, we assume that both economies share the environment described in section 2, except that in one case agents may have access to a CDB while in the other they can make their histories public information. Also, we assume that the no-exit constraint is not binding in the two environments and that the amount of goods exchanged is the same in all meetings, and is denoted by c. Note that an important difference between the two economies is that in the case of a CW economy, the money supply is endogenous, while in the economies considered in this paper, it is exogenously given.

In a no-gift allocation, agents whose histories are public may issue notes that are used as a medium of exchange. Production always occurs in a single-coincidence-of-want meeting between two agents whose histories are public. In a meeting between two agents whose histories are private, production occurs if the buyer holds a note and the seller does not. In a meeting between an agent whose history is private and an agent whose history is public, production occurs if the agent whose history is private either wants to consume and holds a note or can produce and does not hold a note. Let λ^p denote the fraction of agents whose histories are public information. We can write the value functions as

$$\begin{aligned} V_0^{CW} &= (1-\lambda^p)\left[m_1\left[\beta V_1^{CW} - c\right] + (1-m_1)\beta V_0^{CW}\right] \\ &\quad +\lambda^p\left[\frac{1}{k}\left[\beta V_1^{CW} - c\right] + (1-\frac{1}{k})\beta V_0^{CW}\right], \qquad (39) \\ V_1^{CW} &= (1-\lambda^p)\left[m_0\left[\beta V_0^{CW} + u(c)\right] + (1-m_0)\beta V_1^{CW}\right] \\ &\quad +\lambda^p\left[\frac{1}{k}\left[\beta V_0^{CW} - c\right] + (1-\frac{1}{k})\beta V_1^{CW}\right]. \qquad (40) \end{aligned}$$

Taking into account the fact that $m_1 = m_0 = 1/2k$, it can be shown that

$$\begin{aligned} V_0^{CW} &= \frac{\frac{1+\lambda^p}{2k}\left[\frac{1+\lambda^p}{2k}\beta(u(c)-c) - (1-\beta)c\right]}{(1-\beta)\left[1-\beta+\beta\frac{1+\lambda^p}{k}\right]}, \qquad (41) \\ V_1^{CW} &= V_0 + \frac{\frac{1+\lambda^p}{2k}(u(c)+c)}{\left[1-\beta+\beta\frac{1+\lambda^p}{k}\right]} > V_0. \qquad (42) \end{aligned}$$

The welfare of these agents, denoted by W_{CW} is

$$W_{CW} = \frac{1+\lambda}{4}\frac{u(c)-c}{k(1-\beta)}. \qquad (43)$$

There is no individual state variable for agents whose histories are public. These agents' welfare, denoted by W^p_{CW}, is given by

$$W^p_{CW} = \beta W^p_{CW} + (1-\lambda^p)\left[m_1 u(c) - m_0 c\right] + \lambda^p \frac{1}{k}(u(c)-c) = 2W_{CW}. \tag{44}$$

To compare social welfare in both economies, we assume that the distributions of costs are identical. Since we have no good guide to inform us about how these costs might differ, this assumption allows us to limit the differences between the two environments to the type of memory available.

The social welfare function in a CW economy can thus be verified to be

$$SW_{CW} = \lambda^p W^p_{CW} + (1-\lambda^p) W_{CW} - \int_0^{\lambda^p} \kappa_i di = \frac{(1+\lambda^p)^2}{4}\frac{u(c)-c}{k(1-\beta)} - \int_0^{\lambda^p} \kappa_i di. \tag{45}$$

For a given λ^a, the social welfare function in the economies studied in the paper is maximized at $M = 1/2$. For such M, it is given by

$$SW_{CDB} = \lambda^a W^a + (1-\lambda^a) W - \int_0^{\lambda} \kappa_i di = \frac{1+3(\lambda^a)^2}{4}\frac{u(c)-c}{k(1-\beta)} - \int_0^{\lambda^a} \kappa_i di. \tag{46}$$

It can be seen that $SW_{CW} \geq SW_{CDB}$ if and only if $1 \geq \lambda \geq 0$. Hence, for all λ, welfare in a CW economy is at least as high as in an economy with a CDB. In fact, it is strictly greater if $\lambda \in (0,1)$. If $\lambda = 0$, both economies are identical and all trades require money. If $\lambda = 1$, then both economies are also identical, but in that case money is unnecessary.

We can summarize this result with the following proposition.

Proposition 11 *If $\lambda^a = \lambda^p = \lambda$, then $SW_{CW} \geq SW_{CDB}$ for all λ.*

One might think that proposition 11 implies that there will always be at least as large a fraction of agents with public histories in a CW economy as of agents with access to the CDB in an economy of the type we study; i.e., $\lambda^p \geq \lambda^a$. This turns out not to be the case. The condition for the above conjecture to hold is $W^p_{CW} - W_{CW} \geq W^a - W$ or, equivalently, $\lambda \geq \left[3 - 4M(1-M)\right]^{-1}$. If $M = 1/2$, this condition is verified for all values of λ. However, if $M \neq 1/2$, then the condition may not hold for large values of λ.

In the economies we study, if the entry constraint does not bind, then $M = 1/2$ and $\lambda^p \geq \lambda^a$ must hold. However, if the entry constraint binds, then it may be desirable to choose $M > 1/2$, in which case $\lambda^p < \lambda^a$ could occur. Note that $SW_{CW} \geq SW_{CDB}$ would still hold.

An interesting difference between the two economies is that small but positive values of λ^a cannot be supported as an equilibrium in the economies studied in this paper, while small positive values of λ^p can be supported as an equilibrium of a CW economy. This can be seen by looking at the entry constraint.

For the economies studied in this paper, we have already pointed out that $V_1^a - V_1 \to 0$ as $\lambda \to 0$, so that the entry constraint cannot hold. In a CW economy, we assume that agents with public histories become indistinguishable from agents whose histories are not public, if they refuse to produce when they should. Under this assumption, the entry constraint is given by $\beta\left(W_{CW}^p - W_{CW}\right) \geq c$. Since

$$W_{CW}^p - W_{CW} = \frac{u(c) - c}{k(1-\beta)} \frac{1+\lambda}{4},$$

the entry constraint can hold in a CW economy even for very small values of λ^p.

CW interpret agents with public histories as playing the role of early banks. We interpret the CDB as having some features of credit card networks. The results presented in this section suggest that in economies in which the cost of memory is high there are more benefits to be gained from having a few banks than from having a limited network resembling credit cards. Indeed, such a network may not be sustainable. On the other hand, in economies where the cost of memory is not too high, the benefits from having many agents with public information is not much greater than having many agents with access to the CDB. Hence, if the cost of the latter kind of memory is even slightly smaller than the cost of the former, it might be beneficial to adopt something resembling a credit card network

8 Conclusion

This paper considers an economy where agents can pay a cost to access a central data base. This CDB is a form of memory that keeps track of individual histories and allows agents who have access to it to engage in transactions that would otherwise not be possible without money. This kind of memory has features that resemble those of some payment networks such as credit cards.

We show that agents holding money derive less benefit from having access to the CDB than agents who do not hold money. Thus it is more difficult to convince the former type of agents to trade using the CDB. One way to loosen the entry constraint faced by agents holding money is to impose that sellers cannot require to be paid with money if the CDB can also be used. Another way is to reduce the amount of goods exchanged for money (increase the price of goods purchased with money). This suggests that the 'no surcharge rule' may have benefits. More generally, our paper emphasizes the fact that both access to the CDB and continued participation in the network are important and that the incentives for each may be different.

We show that a network effect is present since the benefits of having access to the CDB is greater when more agents have access to it. Because of the network effect, fewer agents may access the CDB in equilibrium than would be efficient. We consider policies that can affect the entry condition: Imposing a utility cost, which can be interpreted as a tax, increasing the money supply, or increasing the price of goods purchased with money. We show that if it is efficient for all agents to access the CDB, then imposing a high enough tax on agents who do not obtain access can achieve the efficient allocation. This cannot be done by changing only the money supply.

We also compare our model with that of Cavalcanti and Wallace (1999 a), who consider an economy in which some agents have public histories. The type of memory that these authors consider provides greater benefits that the memory we study. This is particularly so when comparing an economy with few agents who have public histories with an economy with few agents having access to the CDB. However, if all agents have access to the CDB the benefits from that type of memory is the same as when all agents have public histories.

9 Appendix

Proof of lemma 1

First, note that by definition,

$$
\begin{aligned}
N_0^a + N_1^a &= \lambda. && (47)\\
N_0 + N_1 &= 1-\lambda. && (48)\\
N_0^a + N_0 &= 1-M. && (49)\\
N_1^a + N_1 &= M. && (50)
\end{aligned}
$$

Now we need the transition probabilities between different types. First note that having access to the CDB is a permanent, once and for all decision. Hence we can consider agents having access separately from those who do not have access to the CDB. We start with the latter type.

An agent who does not have access to the CDB and is not holding a unit of money today could have been either an agent who was holding a unit of money yesterday and spent it (probability $(1-M)/k$), or an agent who was not holding a unit of money yesterday and did not acquire money (probability $1-(M/k)$). Thus we can write

$$N_{0,t} = N_{1,t-1}\frac{1-M}{k} + N_{0,t-1}\left(1-\frac{M}{k}\right). \tag{51}$$

Similarly, an agent who does not have access to the CDB and is holding a unit of money today could have been either an agent who did not have a unit of money yesterday but acquired one (probability M/k), or an agent who did have a unit of money yesterday but was unable to buys goods (probability $1-[(1-M)/k]$). Thus we can write

$$N_{1,t} = N_{0,t-1}\frac{M}{k} + N_{1,t-1}\left(1-\frac{1-M}{k}\right). \tag{52}$$

In steady state, either of these equations yields $N_0M = N_1(1-M)$. This, combined with $N_0 + N_1 = 1-\lambda$, implies $N_0 = (1-\lambda)(1-M)$, and $N_1 = (1-\lambda)M$.

Now consider agents who have access to the CDB. An agent not holding money today could have been either an agent not holding money yesterday who did not acquire money (probability $1-\left[\left(N_{1,t-1} + (1-(1-\theta))N_{1,t-1}^a\right)/k\right]$) or an agent who did hold a unit of money yesterday but spent it (probability $\left(N_{0,t-1} + (1-(1-\theta))N_{0,t-1}^a\right)/k$).

Thus we can write

$$N_{0,t}^a = N_{0,t-1}^a \left(1 - \frac{N_{1,t-1} + (1-(1-\theta))N_{1,t-1}^a}{k}\right) + N_{1,t-1}^a \frac{N_{0,t-1} + (1-(1-\theta))N_{0,t-1}^a}{k}. \tag{53}$$

An agent holding a unit of money today could have been an agent not holding a unit money yesterday and who acquired it (probability $\left(N_{1,t-1} + (1-(1-\theta))N_{1,t-1}^a\right)/k$) or an agent who was holding a unit of money yesterday and could not buy goods (probability $1 - \left[\left(N_{0,t-1} + (1-(1-\theta))N_{0,t-1}^a\right)/k\right]$). Thus we can write

$$N_{1,t}^a = N_{0,t-1}^a \frac{N_{1,t-1} + (1-(1-\theta))N_{1,t-1}^a}{k} + N_{1,t-1}^a \left(1 - \frac{N_{0,t-1} + (1-(1-\theta))N_{0,t-1}^a}{k}\right). \tag{54}$$

In steady state, either of these equations yields $N_0^a N_1 = N_1^a N_0$, or, using the expressions for N_0 and N_1, $N_0^a M = N_1^a(1-M)$. This, with $N_0^a + N_1^a = \lambda$, implies $N_0^a = \lambda(1-M)$, and $N_1^a = \lambda M$.

Proof of lemma 2

Item 1 follows from

$$\frac{\partial W^a}{\partial c_{DB}} = \frac{1}{k(1-\beta)}\left[u'(c_{DB}) - 1\right]\lambda\left[1 - \theta M(1-M)\right], \tag{55}$$

$$\frac{\partial^2 W^a}{\partial c_{DB}^2} = \frac{1}{k(1-\beta)} u''(c_{DB})\lambda\left[1 - \theta M(1-M)\right] < 0, \tag{56}$$

and

$$\frac{\partial W^a}{\partial c_m} = \frac{1}{k(1-\beta)}\left[u'(c_m) - 1\right](1-(1-\theta)\lambda)M(1-M), \tag{57}$$

$$\frac{\partial^2 W^a}{\partial c_m^2} = \frac{1}{k(1-\beta)} u''(c_m)(1-(1-\theta)\lambda)M(1-M) < 0. \tag{58}$$

Item 2 follows from

$$\frac{\partial W^a}{\partial \theta} = \frac{1}{k(1-\beta)}\lambda M(1-M)\left\{\left[u(c_m) - c_m\right] - \left[u(c_{DB}) - c_{DB}\right]\right\}. \tag{59}$$

Item 3 follows from

$$\frac{\partial W^a}{\partial \lambda} = \frac{\left[u(c_{DB}) - c_{DB}\right]\left[1 - \theta M(1-M)\right] - \left[u(c_m) - c_m\right](1-\theta)M(1-M)}{k(1-\beta)}. \tag{60}$$

Item 4 follows from

$$\begin{aligned}
\frac{\partial W^a}{\partial M} &= \frac{(1-2M)}{k(1-\beta)}\left\{(1-(1-\theta)\lambda)\left[u(c_m)-c_m\right]-\theta\lambda\left[u(c_{DB})-c_{DB}\right]\right\}, & (61)\\
\frac{\partial^2 W^a}{\partial M^2} &= -\frac{2}{k(1-\beta)}\left\{(1-(1-\theta)\lambda)\left[u(c_m)-c_m\right]-\theta\lambda\left[u(c_{DB})-c_{DB}\right]\right\}. & (62)
\end{aligned}$$

Item 5 follows from

$$\begin{aligned}
\frac{\partial W}{\partial c_m} &= \frac{1}{k(1-\beta)}\left[u'(c_m)-1\right]\lambda M(1-M), & (63)\\
\frac{\partial^2 W}{\partial c_m^2} &= \frac{1}{k(1-\beta)}u''(c_m)\lambda M(1-M)<0. & (64)
\end{aligned}$$

Item 6 is a consequence of

$$\frac{\partial W}{\partial M} = \frac{(1-2M)}{k(1-\beta)}\left[u(c_m)-c_m\right], \quad (65)$$

$$\frac{\partial^2 W}{\partial M^2} = -\frac{2}{k(1-\beta)}\left[u(c_m)-c_m\right]<0. \quad (66)$$

Proof of lemma 3

Item 1 follows from

$$\begin{aligned}
\frac{\partial(W^a-W)}{\partial c_{DB}} = \frac{\partial W^a}{\partial c_{DB}} &= \frac{1}{k(1-\beta)}\left[u'(c_{DB})-1\right]\lambda\left[1-\theta M(1-M)\right], & (67)\\
\frac{\partial^2(W^a-W)}{\partial c_{DB}^2} = \frac{\partial^2 W^a}{\partial c_{DB}^2} &= \frac{1}{k(1-\beta)}u''(c_{DB})\lambda\left[1-\theta M(1-M)\right]<0. & (68)
\end{aligned}$$

Item 2 follows from

$$\begin{aligned}
\frac{\partial(W^a-W)}{\partial c_m} &= \frac{-\lambda M(1-M)}{k(1-\beta)}(1-\theta)\left[u'(c_m)-1\right], & (69)\\
\frac{\partial^2(W^a-W)}{\partial c_m^2} &= \frac{-\lambda M(1-M)}{k(1-\beta)}(1-\theta)u''(c_m)>0. & (70)
\end{aligned}$$

Item 3 follows from

$$\begin{aligned}
\frac{\partial(W^a-W)}{\partial M} &= -\frac{\lambda(1-2M)}{k(1-\beta)}\left\{\theta\lambda S_{DB}+(1-\theta)S_m\right\}, & (71)\\
\frac{\partial^2(W^a-W)}{\partial M^2} &= \frac{2\lambda}{k(1-\beta)}\left\{\theta\lambda S_{DB}+(1-\theta)S_m\right\}>0. & (72)
\end{aligned}$$

Item 4 follows from

$$\frac{\partial(W^a - W)}{\partial(1-\theta)} = \frac{\partial W^a}{\partial\theta} = \frac{1}{k(1-\beta)}\lambda M(1-M)\left\{S_m - S_{DB}\right\}. \tag{73}$$

Proof of lemma 4

Item 1 is immediate from inspection of equation (21).

Item 2 follows from

$$\begin{aligned}\frac{\partial(V_1^a - V_1)}{\partial c_{DB}} &= \frac{\lambda\left[u'(c_{DB})-1\right]}{k(1-\beta)} - \frac{\lambda\theta(1-(1-\theta)\lambda)\beta m_0 m_1\tau\left[u'(c_{DB})-1\right]}{(1-\beta)\left[1-\beta+(1-(1-\theta)\lambda)\frac{\beta\tau}{k}\right]} \\ &\quad - \frac{\lambda\theta m_0 u'(c_{DB})}{\left[1-\beta+(1-(1-\theta)\lambda)\frac{\beta\tau}{k}\right]}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2(V_1^a - V_1)}{\partial c_{DB}^2} &= \frac{\lambda}{k(1-\beta)}u''(c_{DB}) - \frac{\lambda\theta(1-(1-\theta)\lambda)\beta m_0 m_1\tau}{(1-\beta)\left[1-\beta+(1-(1-\theta)\lambda)\frac{\beta\tau}{k}\right]}u''(c_{DB}) \\ &\quad - \frac{\lambda\theta m_0}{\left[1-\beta+(1-(1-\theta)\lambda)\frac{\beta\tau}{k}\right]}u''(c_{DB}) \\ &= \frac{\lambda u''(c_{DB})}{k}\left[(1-\theta(1-M)) + \frac{(1-(1-\theta)\lambda)\beta\tau}{(1-\beta)k}(1-\theta M(1-M))\right] < 0.\end{aligned}$$

Note that $\frac{\partial(V_1^a - V_1)}{\partial c_{DB}} < 0$ at c^* and positive as $c_{DB} \to 0$. So $V_1^a - V_1$ is maximized for some value of $c_{DB} \in (0, c^*)$.

Item 3 follows from

$$\begin{aligned}\frac{\partial(V_1^a - V_1)}{\partial c_m} &= -\frac{(1-\theta)\lambda\beta m_0 m_1\tau\left[u'(c_m)-1\right]\left[2-(1-\theta)\lambda+\frac{(1-(1-\theta)\lambda)\beta\tau}{(1-\beta)k}\right]}{\left[1-\beta+\frac{\beta}{k}\right]\left[1-\beta+\frac{(1-(1-\theta)\lambda)\beta\tau}{k}\right]} \\ &\quad - \frac{(1-\theta)\lambda(1-\beta)m_0 u'(c_m)}{\left[1-\beta+\frac{\beta}{k}\right]\left[1-\beta+\frac{(1-(1-\theta)\lambda)\beta\tau}{k}\right]}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2(V_1^a - V_1)}{\partial c_m^2} &= -\frac{(1-\theta)\lambda\beta m_0 m_1\tau u''(c_m)\left[2-(1-\theta)\lambda+\frac{(1-(1-\theta)\lambda)\beta\tau}{(1-\beta)k}\right]}{\left[1-\beta+\frac{\beta}{k}\right]\left[1-\beta+\frac{(1-(1-\theta)\lambda)\beta\tau}{k}\right]} \\ &\quad - \frac{(1-\theta)\lambda(1-\beta)m_0 u''(c_m)}{\left[1-\beta+\frac{\beta}{k}\right]\left[1-\beta+\frac{(1-(1-\theta)\lambda)\beta\tau}{k}\right]} > 0.\end{aligned}$$

Note that $\frac{\partial(V_1^a - V_1)}{\partial c_m} < 0$ at $u'(c^*) = 1$, so that value of c_m that minimizes $V_1^a - V_1$ is greater than c^*.

Item 4 follows from

$$\begin{aligned}
\frac{\partial(V_1^a - V_1)}{\partial M} &= -\lambda\theta \frac{(1-(1-\theta)\lambda)\frac{\beta\tau}{k^2}(1-2M)S_{DB} - (1-\beta)\frac{1}{k}u(c_{DB})}{(1-\beta)\left[1-\beta+\frac{(1-(1-\theta)\lambda)\beta\tau}{k}\right]} \\
&\quad -(1-\theta)\lambda \frac{\frac{\beta\tau}{k^2}(1-2M)S_m\left[2-(1-\theta)\lambda+\beta\frac{1-(1-\theta)\lambda}{(1-\beta)k}\right] - (1-\beta)\frac{1}{k}u(c_m)}{\left[1-\beta+\frac{\beta}{k}\right]\left[1-\beta+\frac{(1-(1-\theta)\lambda)\beta\tau}{k}\right]}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2(V_1^a - V_1)}{\partial M^2} &= 2\lambda\theta \frac{(1-(1-\theta)\lambda)\frac{\beta\tau}{k^2}S_{DB}}{(1-\beta)\left[1-\beta+\frac{(1-(1-\theta)\lambda)\beta\tau}{k}\right]} \\
&\quad 2(1-\theta)\lambda \frac{\frac{\beta\tau}{k^2}S_m\left[2-(1-\theta)\lambda+\beta\frac{1-(1-\theta)\lambda}{(1-\beta)k}\right]}{\left[1-\beta+\frac{\beta}{k}\right]\left[1-\beta+\frac{(1-(1-\theta)\lambda)\beta\tau}{k}\right]} > 0
\end{aligned}$$

Note that $\frac{\partial(V_1^a - V_1)}{\partial M} < 0$ at $M = 1/2$, so that value of M that minimizes $V_1^a - V_1$ is greater than c^*. Inspection of equation (4) shows that this maximized at $M = 1$, corresponding to $m_0 = 0$, since in this case all negative terms drop out.

Finally, for item 5, if $c_{DB} = c_m = c$, then

$$\begin{aligned}
V_1^a - V_1 &= \frac{\lambda}{k}\frac{u(c)-c}{1-\beta} \\
&\quad -\lambda \frac{\beta\tau m_0 m_1 [u(c)-c]\left[\left(1-\beta+\frac{\beta\tau}{k}\right)(1-(1-\theta)\lambda)+(1-\beta)(1-\theta)\right]}{(1-\beta)\left[1-\beta+\frac{\beta\tau}{k}\right]\left[1-\beta+\beta\tau\frac{1-(1-\theta)\lambda}{k}\right]} \\
&\quad -\lambda \frac{m_0 c\left[\left(1-\beta+\frac{\beta\tau}{k}\right)-(1-\theta)\frac{\beta\tau}{k}\right]}{\left[1-\beta+\frac{\beta\tau}{k}\right]\left[1-\beta+\beta\tau\frac{1-(1-\theta)\lambda}{k}\right]}. \qquad (74)
\end{aligned}$$

After some algebra, we get

$$\left.\frac{\partial(V_1^a - V_1)}{\partial\theta}\right|_{c_{DB}=c_m} = -\lambda(1-\lambda)\frac{\beta\tau m_0\left[(1-M)u(c)+Mc\right]}{k\left[1-\beta+\beta\tau\frac{1-(1-\theta)\lambda}{k}\right]^2} < 0.$$

Concavity and partial derivative of equation (23) with respect to λ

First, note that equation (23) can be written in the following way:

$$V_1^a - V_1 = \frac{\lambda\left[u(c) - c\right]}{k(1-\beta)} - \lambda\left[A + B + C\right],$$

where A, B, and C are given by

$$\begin{aligned}
A &= \frac{\beta m_0 m_1 \left[u(c) - c\right]}{(1-\beta)\left[1 - \beta + \frac{\beta}{k}\right]}, \\
B &= \frac{(1-\lambda)\beta m_0 m_1 \left[u(c) - c\right]}{\left[1 - \beta + \frac{\beta}{k}\right]\left[1 - \beta + \frac{\beta(1-\lambda)}{k}\right]}, \\
C &= \frac{(1-\beta) m_0 u(c)}{\left[1 - \beta + \frac{\beta}{k}\right]\left[1 - \beta + \frac{\beta(1-\lambda)}{k}\right]}.
\end{aligned}$$

It can be verified that

$$\begin{aligned}
\frac{\partial A}{\partial \lambda} &= \frac{\partial^2 A}{\partial \lambda^2} = 0, \\
\frac{\partial B}{\partial \lambda} &= -\frac{(1-\beta)B}{(1-\lambda)\left[1 - \beta + \frac{\beta(1-\lambda)}{k}\right]}, \\
\frac{\partial^2 B}{\partial \lambda^2} &= 2\frac{\beta \frac{\partial B}{\partial \lambda}}{k\left[1 - \beta + \frac{\beta(1-\lambda)}{k}\right]}, \\
\frac{\partial C}{\partial \lambda} &= \frac{\beta C}{k\left[1 - \beta + \frac{\beta(1-\lambda)}{k}\right]}, \\
\frac{\partial^2 C}{\partial \lambda^2} &= 2\frac{\beta \frac{\partial C}{\partial \lambda}}{k\left[1 - \beta + \frac{\beta(1-\lambda)}{k}\right]}.
\end{aligned}$$

The partial derivative of equation (23) with respect to λ is

$$\begin{aligned}
\frac{\partial\left(V_1^a - V_1\right)}{\partial \lambda} &= \frac{\lambda\left[u(c) - c\right]}{k(1-\beta)} - \left[A + B + C\right] - \lambda\left[\frac{\partial A}{\partial \lambda} + \frac{\partial B}{\partial \lambda} + \frac{\partial C}{\partial \lambda}\right] \\
&= \frac{\lambda\left[u(c) - c\right]}{k(1-\beta)} - \frac{\beta m_0 m_1 \left[u(c) - c\right]}{(1-\beta)\left[1 - \beta + \frac{\beta}{k}\right]} \\
&\quad - \frac{\beta m_0 m_1 \left[u(c) - c\right]\left[(1-\lambda)\left(1 - \beta + \frac{\beta(1-\lambda)}{k}\right) - \lambda(1-\beta)\right]}{\left[1 - \beta + \frac{\beta}{k}\right]\left[1 - \beta + \frac{\beta(1-\lambda)}{k}\right]^2} \\
&\quad - \frac{(1-\beta) m_0 u(c)}{\left[1 - \beta + \frac{\beta(1-\lambda)}{k}\right]^2}.
\end{aligned}$$

Evaluated at $\lambda = 1$, this equation gives us the expression in the text.

Next we show that equation (23) is concave with respect to λ.

$$\begin{aligned}
\frac{\partial^2 \left(V_1^a - V_1\right)}{\partial \lambda^2} &= -2\left[\frac{\partial A}{\partial \lambda} + \frac{\partial B}{\partial \lambda} + \frac{\partial C}{\partial \lambda}\right] - \lambda \left[\frac{\partial^2 A}{\partial \lambda^2} + \frac{\partial^2 B}{\partial \lambda^2} + \frac{\partial^2 C}{\partial \lambda^2}\right] \\
&= 2\frac{(1-\beta)\beta m_0 m_1 \left[u(c) - c\right]}{\left[1 - \beta + \frac{\beta(1-\lambda)}{k}\right]^3} - 2\frac{(1-\beta)\beta m_0 u(c)}{k\left[1 - \beta + \frac{\beta(1-\lambda)}{k}\right]^3} \\
&= -2\frac{(1-\beta)\beta m_0 \left[(1-M)u(c) - Mc\right]}{\left[1 - \beta + \frac{\beta(1-\lambda)}{k}\right]^3} < 0.
\end{aligned}$$

Proof of lemma 5

Welfare in this economy increases with $u(c_m) - c_m$. The tightness of the entry constraint decreases with $u(c_m) - c_m$, while the tightness of the no-exit constraint increases with

$$\beta \tau m_1 \left[u(c_m) - c_m\right]\left[2 - (1-\theta)\lambda + \frac{(1-(1-\theta)\lambda)\beta\tau}{(1-\beta)k}\right] + (1-\beta)u(c_m).$$

If two values of c_m, $\bar{c}_m > \underline{c}_m$ give the same tightness for the entry constraint, then welfare is the same in both cases. However, since $\bar{c}_m > \underline{c}_m$ the planner (weakly) prefers $\underline{c}_m$.

Similarly, if two values of c_m, $\bar{c}_m > \underline{c}_m$ give the same tightness for the no-exit constraint, then

$$u(\underline{c}_m) - \underline{c}_m > u(\bar{c}_m) - \bar{c}_m.$$

The planner then prefers $\underline{c}_m$ since welfare is higher.

References

[1] Berentsen, A., M. Molico, and R. Wright, 2002. "Indivisibilities, Lotteries, and Monetary Exchange," *Journal of Economic Theory* 107, 70 - 94.

[2] Camera, Gabriele, 2000. "Money, Search, and Costly Matchmaking," *Macroeconomic Dynamics* 4, 289-323.

[3] Cavalcanti, R. de O., A. Erosa, and T. Temzelides, 1999. "Private Money and Reserve Management in a Random Matching Model," *Journal of Political Economy* 107, 929-945.

[4] Cavalcanti, R. de O. and N. Wallace, 1999a. "A Model of Private Bank-Note Issue," *Review of Economic Dynamics* 2, 104-136.

[5] Cavalcanti, R. de O. and N. Wallace, 1999b. "Inside and Outside Money as Alternative Media of Exchange," *Journal of Money, Credit, and Banking* 31, part 2, 443-457.

[6] Corbae, D. and J. Ritter, 2004. "Decentralized Credit and Monetary Exchange without Record Keeping," *Economic Theory*, 24, 933-951

[7] Garcia-Swartz, D. D., R. W. Hahn, and A. Layne-Farrar, 2006. "The Move Toward a Cashless Society: Calulating the Costs and Benefits," *Review of Network Economics* 5, 199-228.

[8] Gerdes, G. R. and J. K. Walton, 2002. "The Use of Checks and Other Noncash Payment Instruments in the United States," *Federal Reserve Bulletin* 88, 360-374.

[9] Gerdes, G. R., J. K. Walton, M. X. Liu, and D. W. Parke, 2005. "Trends in the Use of Payment Instruments in the United States," *Federal Reserve Bulletin* 91, 180-201.

[10] He, P., L. Huang, and R. Wright, 2005. "Money and Banking in Search Equilibrium," *International Economic Review* 46, 637-670.

[11] He, P., L. Huang, and R. Wright, 2006. "Money, Banking, and Monetary Policy," manuscript.

[12] Kahn, C. M. and W. Roberds, 2005. "Credit and identity theft," Federal Reserve Bank of Atlanta *working paper* 2005-19.

[13] Kiyotaki, N. and R. Wright, 1991. "A Contribution to the Pure Theory of Money," *Journal of Economic Theory* 53, 215-235.

[14] Kiyotaki, N. and R. Wright, 1993. "A Search-Theoretic Approach to Monetary Economics," *American Economic Review* 83, 63-77.

[15] Kocherlakota, N., 1998. "Money is Memory," *Journal of Economic Theory* 81, 232-251.

[16] Kocherlakota, N. and N. Wallace, 1998. "Incomplete Record-Keeping and Optimal Payment Arrangements," *Journal of Economic Theory* 81, 272-289.

[17] Koeppl, T., C. Monnet, and T. Temzelides, 2006. "A Dynamic Model of Settlement," manuscript.

[18] Monnet, C. and W. Roberds, 2006. "Credit and the No-Surcharge Rule," manuscript.

[19] Lotz, S., 2005. "A Model of Electronic Cash Cards," manuscript.

[20] Nosal, E., and G. Rocheteau, 2006. "The Economics of Payments," Federal Reserve Bank of Cleveland *Policy Discussion Paper* 14.

[21] Shi, S., 1995. "Money and Prices: A Model of Search and Bargaining," *Journal of Economic Theory* 67, 467-498.

[22] Telyukova, I. A. and R. Wright, 2006. "A Model of Money and Credit, with Application to the Credit Card Debt Puzzle," manuscript.

[23] Trejos, A. and R. Wright, 1995. "Search, Bargaining, Money and Prices," *Journal of Political Economy* 103, 118-141.

[24] Weiner, S. E. and J. Wright, 2005. "Interchange Fees in Various Countries: Developments and Determinants," *Review of Network Economics* 4, 290-323.