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Abstract

A small but ambitious literature uses affine arbitrage-free models to estimate jointly U.S. Treasury term premiums and the term structure of equity risk premiums. Within this approach, this paper identifies the parameter restrictions that are consistent with a simple dividend discount model, extends the cross-section to Germany and France, averages across multiple observable-factor and market prices of risk specifications, and considers alternative samples for parameter estimation. The results produce intuitive trajectories for both sets of premiums given standard samples starting from July 1993. However, the decomposition of nominal U.S. Treasury yields, but not long-run equity risk premiums, is sensitive to data beyond 2008, which raises some questions about the net effects of unconventional monetary policy measures. Nonetheless, the rotation from sharp inversion during the financial crisis to an upward-sloping term structure of equity risk premiums more recently, with modest readings at the front end, is not inconsistent with some net moderation in required compensation for equity risk in the United States.

Key words: equity risk premium, Treasury term premium, affine arbitrage-free models

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1. Introduction

Term and the equity risk premiums are arguably the most critical unobservable variables in finance. Central bankers attempt to disentangle from yield curves and stock prices investors' expectations for the real economy and inflation as well as their perceptions and attitudes toward various risks, perhaps especially so and with increasing difficulty amid unconventional policy measures that possibly work through portfolio rebalancing or perhaps other transmission mechanisms. Investors estimate these premiums to gauge expected returns and compensation for bearing uncertainty, from the risk-free to the yardstick risky asset class. Along with active views on returns as well as variance and covariance, such estimates comprise the required inputs to quantitative portfolio optimization (e.g., Black and Litterman, 1992). Despite these strong motivations, a surprisingly small practitioner or even academic literature addresses the joint dynamics of stock and government bond prices in a comprehensive arbitrage-free framework.

Starting from a few promising exceptions (e.g., Bekaert and Grenadier, 2001; d'Addona and Kind, 2006; Koijen et al., 2010; Mamaysky, 2002; Lemke and Werner, 2009; Adrian et al., 2013), this study endeavors to extend this inclusive approach.[1] Regarding theory, the following amends Gaussian affine term structure equity models (GATSEMs) to specify the conditions under which arbitrage-free and dividend discount models (DDMs) of stock prices are consistent. With respect to empirics, the analyses cover cases besides the U.S., including Germany and France, and the underlying factors incorporate a simple "macro-finance approach" to include variables such as survey-based expectations of one-year-ahead inflation, real GDP growth, and budget deficits. Beyond an extension of the cross-section, and in addition to sample and bond maturity selection, the sensitivity analyses address model uncertainty with respect to the

[1] Not all of these studies permit time-varying risk premiums (e.g., d'Addona and Kind, 2006) or produce stock prices that are exponentially affine functions of the state variables (e.g., Bekeart and Grenadier, 2001). The methods described below most closely follow Mamayksy (2002) and, in particular, Lemke and Werner (2009).

selection of observable factors as well as alternative market price(s) of risk parameter restrictions, two subjects of notable uncertainty in the existing literature.

In general, the estimates of government bond term and equity risk premiums are consistent with common priors regarding magnitudes and trends starting from July 1993 (e.g., Lemke and Werner, 2009). However, and germane to calibration of Gaussian affine term structure models (GATSMs) in general (Durham, 2013b), the results raise questions regarding the sensitivity of long-run U.S. term premium estimates, but apparently not long-run equity risk premiums, to data through the aftermath of the global financial crisis. In addition to the elevated level of required long-run equity returns over the past few years, the prospect of higher term premiums than consensus estimates suggest perhaps raise questions about the net impact of unconventional monetary policy measures, such as large scale asset purchases (LSAPs), on risky financial asset prices, counterfactuals aside (e.g., Bernanke, 2010; Stein, 2012). On the other hand, the recent slope of the U.S. term structure of required equity returns is indeed positive, notably from moderated levels at the front end and in clear contrast to the sharp inversion observed at the height of the financial crisis, a finding that is not inconsistent with meaningful accommodation in broader financial conditions amid aggressive policy innovations. In addition, these results also highlight some additional issues in the cross section of major equity markets. For example, recent term structures of equity premiums are sharply downward-sloping for Germany and France, in contrast to the hump-shape in the U.S., which could reflect varying perceptions of monetary policy transmission or nearer-term idiosyncratic risks, such as lingering investor uncertainty that stems from EMU.

Section 2 briefly reviews recent joint estimations of bond and stock market dynamics. Section 3 extends the framework and identifies the parameter restrictions that impose consistency between a simple DDM and this arbitrage-free approach. Section 4 describes the general research design and outlines the two-step estimation procedure. Section 5 describes the general results, and Section 6 includes some implications for the contemporary environment. Section 7 concludes.

2. A Brief Review of the Model

Contrary to most equity risk premium measures,[2] GATSEMs (e.g., Mamaysky, 2002; Lemke and Werner, 2009) afford an ex-ante time-varying term structure as opposed to a fixed estimate over an undefined investment horizon. Similar to Gaussian affine term structure models (GATSMs) based on Vasicek (1977), the *n* factors, denoted by the $n \times 1$ vector, *X*, follow a mean-reverting process, as in

$$X_{t+1} = a + \kappa X_t + \Sigma \eta_{t+1} \tag{1}$$

where *a* is an $n \times 1$ vector, κ is an $n \times n$ matrix, Σ is an $n \times n$ matrix, and η is an $n \times 1$ normally distributed vector of shocks, as in $\eta \sim i.i.d., N(0, I)$. Also, the dynamics of the nominal pricing kernel or stochastic discount factor, *M*, follow

$$M_{t+1} = \exp\left(-r_t - \frac{1}{2}\lambda_t^{'}\lambda_t - \lambda_t^{'}\eta_{t+1}\right) \tag{2}$$

where *r* is the instantaneous nominal risk-free short rate,[3] an affine function of the underlying factors following

$$r_t = \delta_0 + \delta_1^{'} X_t \tag{3}$$

where δ_0 is a scalar, and δ_1 is an $n \times 1$ vector. The market price of risk, λ, is an $n \times 1$ vector and a linear function of the state variables following

[2] For recent surveys of equity risk premium estimates that do not adopt an arbitrage-free approach, see Mehra (2008) and Hammond et al. (2011).
[3] Lemke and Werner (2009) delineate the real rate and inflation processes in the models.

$$\lambda_t = \lambda_0 + \Lambda_1 X_t \tag{4}$$

where is λ_0 an $n \times 1$ vector, and Λ_1 is an $n \times n$ matrix.

Equations (1)–(4) resemble the rudiments of GATSMs, and turning to the inclusion of equities, the no arbitrage condition implies that the product of the stochastic discount factor and the price of a financial asset is a martingale. Following Lemke and Werner (2009), the payoff of a stock comprises the price, *V*, at time *t + 1* as well as the dividend, *D*, paid during the interval, and therefore given the standard arbitrage-free pricing relation follows

$$V_t = E_t\left\{M_{t+1}\left(V_{t+1} + D_{t+1}\right)\right\} = E_t\left\{M_{t+1}\Gamma_{t+1}V_{t+1}\right\} \tag{5}$$

with $\Gamma_t V_t = V_t + D_t$ and γ, the dividend or payout yield,[4] following

$$\gamma_t = \frac{D_t}{V_t} \approx \ln \Gamma_t \tag{6}$$

As further discussed below, in the observable-factor case with the empirical dividend yield included in *X*, γ is trivially an affine function of the underlying state variables, as in

$$\gamma_t = \rho_0 + \rho_1{}' X_t \tag{7}$$

with $\rho_0 = 0$.[5] Similar to bonds, the index level stock price is an exponentially affine function of the state variables following

$$V_t = \exp\left[c\left(t - t_0\right) + D' X_t\right] \tag{8}$$

where t_0 is a free parameter, and as noted in Appendix 1, *c* is a scalar and *D* is a $n \times 1$ vector that follow, respectively

[4] As Lemke and Werner (2009) argue, the factor representing the yield captures the payout of the stock index divided by its price. However, as firms let stock holders participate in profits by other means (e.g. stock buybacks), the yield subsumes all payments to investors, not just dividends.

[5] As the following describes, the dividend yield is measured with error in the estimation. Also, as further noted below, the restriction that $\rho_0 = 0$ could be relaxed with an exclusive latent-factor estimation.

$$c = \delta_0 - \rho_0 - \left(\rho_1 + D\right)^{'}\left(a - \Sigma\lambda_0\right) - \frac{1}{2}\left(\rho_1 + D\right)^{'}\Sigma\Sigma^{'}\left(\rho_1 + D\right)$$
$$D^{'} = \left[\rho_1^{'}\left(\kappa - \Sigma\Lambda_1\right) - \delta_1^{'}\right]\left(I - \kappa + \Sigma\Lambda_1\right)^{-1} \tag{9}$$

Gross one-period stock returns, $R^{S\&P,(1)}$, include price (de)appreciation and the payout, as in

$$R_{t+1}^{S\&P,(1)} = \frac{V_{t+1} + D_{t+1}}{V_t} = \frac{V_{t+1}}{V_t}\Gamma_{t+1} \tag{10}$$

As such, and simply taking logs of (10) and the affine solution (8), one-period log-returns, $r^{S\&P,(1)}$, equal the capital gain (i.e., the change of ex-dividend log stock prices, Δv) plus the dividend yield in the next period, following

$$\begin{aligned} r_{t+1}^{S\&P,(1)} &= \Delta v_{t+1} + \gamma_{t+1} \\ &= c + D^{'}\Delta X_{t+1} + \rho_0 + \rho_1^{'} X_{t+1} \end{aligned} \tag{11}$$

Taking expectations of (11) shows that conditionally anticipated returns are also affine functions of the state vector,[6] following

$$\begin{aligned} E\left\{r_{t+1}^{S\&P,(1)}\right\} &= c + \rho_0 + \left(D^{'} + \rho_1^{'}\right)a + \left[D^{'}\left(\kappa - I\right) + \rho_1^{'}\kappa\right]X_t \\ &= f_1 + F^{'}X_t \end{aligned} \tag{12}$$

where *f* and *F* denote the fixed and time-varying components of expected returns, respectively. To consider periods beyond *t+1*, under the assumption of reinvestment, multi-period returns comprise the *N*-period capital gain plus the average of dividends over the horizon, as in (scaled in single-period terms)

$$r_{t+N}^{S\&P,(N)} = \frac{1}{N}\left(v_{t+N} - v_t + \sum_{i=1}^{N}\gamma_{t+i}\right) \tag{13}$$

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[6] Note that

$$\begin{aligned} E\left\{\Delta X_{t+1}\right\} &= E\left\{X_{t+1}\right\} - X_t \\ &= E\left\{a + \kappa X_t + \Sigma\eta_{t+1}\right\} - X_t \\ &= a + \left(\kappa - I\right)X_t \end{aligned}$$

The expected *N*-period returns comprise average capital gains and payouts in discrete time over the full investment horizon, which similar to the derivation of (12) are affine functions of the factors, following

$$\begin{aligned} E\left\{r_{t+N}^{S\&P,(N)}\right\} &= E\left\{\frac{1}{N}\sum_{i=1}^{N}\left(\Delta v_{t+i}+\gamma_{t+i}\right)\right\} \\ &= f_N + F_N^{'} X_t \end{aligned} \tag{14}$$

where f_N and F_N are detailed in Appendix 2. Finally, define the *N*-period equity risk premium, θ, as the additional required return over the *N*-horizon (model-implied) comparable maturity government bond yield, following

$$\begin{aligned} \theta_t^{(N)} &= E_t\left\{r_{t+N}^{S\&P(N)}\right\} - \hat{y}_t^N \\ &= f_N + A_N + \left(F_N^{'} + B_N^{'}\right) X_t \end{aligned} \tag{15}$$

Therefore, notably the model produces a complete term structure of the equity risk premium (i.e., $\theta_t^{(1)}, \theta_t^{(2)}, \ldots, \theta_t^{(N)}$) for any (sample) point *t*.

3. Toward an Arbitrage-Free Dividend Discount Model

The preceding model produces bond and stock prices under the absence of arbitrage, and the question arises as to whether the affine-model-implied solutions are formally consistent with other valuation frameworks, namely a DDM or the final phase of a multi-period model. The key is to express the express the DDM components as affine functions of the underlying factors of the arbitrage-free model and solve for the necessary parameter restrictions that guarantee equivalency.

Following the discussion in Durham (2013a), consider a very distant-horizon, *t* + *T*, expected value for share prices or the terminal phase of a multi-stage DDM. The standard pricing formula includes expected future dividends; the equity cost of capital, which comprises the nominal risk-free (forward) Treasury rate and the equity risk premium; and expected steady-state dividend growth, g, as in,

$$\gamma_{t+T} = y_{t+T} + \theta_{t+T} - E\{g_{t+T}\} \tag{16}$$

Relaxing the pure expectations hypothesis for interest rates, the distant-horizon forward Treasury rate is the sum of the expected nominal short rate and a term premium, P. Also, assume that at this future juncture earnings and dividends cannot grow faster than the underlying economy, and therefore that the anticipated nominal short rate—perhaps reminiscent of an equilibrium Taylor rule in which the policy rate is quite close to the sum of anticipated inflation and potential real GDP growth—must be equivalent to expected nominal earnings growth. As such, the familiar (final phase of the) DDM simplifies as follows

$$\begin{aligned} \gamma_{t+T} &= E\{r_{t+T}\} + \mathrm{P}_{t+T} + \theta_{t+T} - E\{g_{t+T}\} \\ &= \mathrm{P}_{t+T} + \theta_{t+T} \end{aligned} \tag{17}$$

with $E\{r_{t+T}\} \approx E\{g_{t+T}\}$. Thus, under the DDM, the dividend yield is equal to the sum of the Treasury term premium and the equity risk premium. In addition, note the identities of the forward term and equity risk premiums as follows

$$\begin{aligned} \gamma_{t+T} &= \left(y_{t+T}^{UST} - E\{r_{t+T}\}\right) + \left(E\left\{r_{t+T}^{(S\&P,1)}\right\} - y_{t+T}^{UST}\right) \\ &= E\left\{r_{t+T}^{(S\&P,1)}\right\} - E\{r_{t+T}\} \end{aligned} \tag{18}$$

Therefore, the future dividend yield can also be expressed as the spread between expected returns on shares and anticipated risk-free short rates. Under the arbitrage-free model, both expectations are affine functions of the state variables following (14) , (7), and (3), and the DDM implies

$$\rho_0 + \rho_1^{'} X_{t+T} = \left(f_1 + F^{'} X_{t+T}\right) - \left(\delta_0 + \delta^{'} X_{t+T}\right) \tag{19}$$

which requires $\rho_0 = f_1 - \delta_0$ and $\rho_1 = F - \delta$. Note that just as (8) implies a solution for *c* and *D* that precludes arbitrage, (19) also embeds a solution for *c* and *D*, given (12), that is also consistent with the DDM, $V_t = \exp\left[c_{DDM}\left(t - t_0\right) + D_{DMM}^{'} X_t\right]$, say.

To guarantee that the solution to the model is consistent with both the no arbitrage condition and the DDM, the parameter estimates must be consistent with (19). Denoting the last element of the underlying (observable) state vector as the dividend yield, assuming again a steady-state dividend yield with no anticipated changes from *t* + *T* to *t* + *T* + *1*), it follows that

$$\begin{aligned} \gamma_{t+T} &= \left(f_1 - \delta_0\right) + \left(F^{'} - \delta^{'}\right) X_{t+T} \\ &= 0 + \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ \gamma \end{bmatrix}_{t+T} \\ &= 0 + e_N^{'} X_{t+T} \end{aligned} \tag{20}$$

as $F^{'} - \delta^{'}$ "picks out" the long-run dividend yield to satisfy the steady-state DDM condition. Simple rearranging from (19) and (20) implies that $F^{'} - \delta^{'} = e_N^{'}$, or in terms of the root parameters

$$D_{DDM}^{'} = \left(e_N^{'} + \delta^{'} - \delta_\gamma^{'} \kappa\right)\left(\kappa - I\right)^{-1} \tag{21}$$

For both the arbitrage-free and the DDM conditions to be equivalent—(9) and (21)—the required condition is $D_{DDM}^{'} = D^{'}$, or

$$\left(e_N^{'} + \delta^{'} - \delta_\gamma^{'} \kappa\right)\left(\kappa - I\right)^{-1} = \left[\delta_\gamma^{'}\left(\kappa - \Sigma\Lambda_1\right) - \delta_1^{'}\right]\left(I - \kappa + \Sigma\Lambda_1\right)^{-1} \tag{22}$$

And, the first term in (20) implies that $f_1 = \delta_0$, or

$$c_{DDM} = \delta_0 - \left(\delta_\gamma + D\right)^{'} a \tag{23}$$

Similar to (22), another required condition is, $c_{DDM} = c$, as in

$$0 = -\frac{1}{2}\left(\delta_\gamma + D\right)' \Sigma\Sigma' \left(\delta_\gamma + D\right) + \left(\delta_\gamma + D\right)' \Sigma\lambda_0 \tag{24}$$

Therefore, (22) and (24) produce a solution for stock prices that is both arbitrage-free and consistent with the long-run implications of a simple DDM.

4. Research Design: Parameter Estimation and Sensitivity Analyses

The specification and estimation of the model differs from previous GATSEMs, and to some degree more common GATSMs, in a number of ways. As noted, coverage includes Germany and France in addition to the U.S., which obviously affords inferences across markets, and the underlying state vector exclusively includes observable factors, broadly similar to Ang et al. (2006) and Li and Wei (2012) among GATSMs but distinct from Lemke and Werner (2009), who delineate among real variables and inflation and also estimate two latent factors that govern the real yield curve in their GATSEM.

Also, this application considers alternative specifications of observable variables in *X*. Similar to the set of "free" variables in an "extreme bound analysis" (EBA) (e.g., Leamer, 1983), each model includes the level of the yield curve, measured by the 5-year yield (and alternatively the first principal component); the slope, measured by the spread between the 5-year and 3-month yields (and alternatively the second principle component); and the aggregate equity index dividend yield.[7] The remaining two observable factors—or the "doubtful" variables in EBA parlance—include every linear combination among four additional variables. These include a proxy for the curvature of the term structure (or alternatively the third principal component) as

[7] The U.S. Treasury yields in the estimation follow Gürkaynak et al. (2007), and non-U.S. data are from Bloomberg. Dividend series and stock returns are based on Datastream country indices, with the exception of the S&P 500 for the U.S.

well as other factors broadly common to “macro-finance” approaches (e.g., Ang and Piazzesi, 2003)—i.e., the 1-year-ahead (Consensus Economics) mean forecasts of headline consumer price inflation, real GDP growth, and the federal or central government budget deficit ratio to GDP. Combinatorics therefore requires six alternative 5-factor models or specifications of *X*.

Besides observable factor specification, there is hardly agreement in the literature on the market price(s) of risk parameter restrictions. For example, Kim and Wright (2005) and Kim and Orphanides (2012) specify a full matrix for Λ_1, whereas Lemke and Werner impose a diagonal matrix, and Li and Wei include restrictions such that the supply factors are not priced directly, which follows some intuition. Finally, the specification in Cochrane and Piazzesi (2008) is the most restrictive. In short, to insure that the results are not sensitive to alternative assumptions, the estimations average over seven alternative restrictions across λ_0 and Λ_1 used in previous studies.

In addition, under the suspicion that term premium estimates may be sensitive to sample selection, particularly the addition of more recent data (e.g., Durham 2013b), the estimates use seven alternative end dates—namely the July 1993 through July 2007 period (i.e., sample end from Li and Wei 2012, which notably predates the global financial crisis), and subsequent annual extensions through the most recent data (i.e., July 2008, July 2009, July 2010, July 2011, July 2012, and July 2013). Finally, similar to Guimarães (2012) in general, the analyses use four alternative sets of yields in *Y*, spanning the tenors in Kim and Orphanides (2012) as well as Lemke and Werner (2009). Therefore in sum, the seven samples, four sets of bond maturities (three for Germany and France), six 5-factor specifications, and seven market price(s) of risk specifications, along with alternative use of direct proxies versus the first three principal components of the term structure, require 2352 sets of parameters or alternative models (1764 for Germany and France).[8]

To absorb the additional information, any average over the alternative specifications requires some arbitrary thresholds and requires a crucial balance—models that fit the data poorly should not inform the inferences, and yet the purpose of the sensitivity analyses is to avoid spurious reliance on a single (set of) estimate(s). Following the general notion in Granger and Uhlig (1990), the analyses consider an equally weighted average among selected models, which include the specification that best fits the yield curve, on the one hand, and the model that best fits stock returns, on the other. In addition, among the remaining specifications, the subset includes all specifications that fit bond yields (stock returns) more closely than the single model that fits stock returns (bond yields) the best. In this application, comparatively few models inform the estimates, indeed only four do in the case of the U.S., but other possible model-averaging techniques include the method described in Sala-i-Martin (1997) in the context of cross-country growth regressions.

For each of the three cases, estimated separately, the likelihood function corresponds to $L(\varphi) = \ln f(Y_1, \ldots, Y_T; \varphi)$, where *Y* is a vector of (monthly) data that include bond yields at *K* different maturities and (log) stock price returns as well as—in order to accommodate measurement error—the observed dividend yield, following Lemke and Werner (2009). Therefore, for a given observation *t* in the time series, *Y* and the measurement equation follows

[8] Guimarães (2012) conducts sensitivity analysis with respect to two alternative maturity sets as well as 3-, 4-, and 5-factor models and thereby considers six sets of parameters.

$$\begin{pmatrix} y_t^{N1} \\ \vdots \\ y_t^{NK} \\ \Delta v_t \\ \gamma_t \end{pmatrix} = \begin{pmatrix} A_{N1} \\ \vdots \\ A_{NK} \\ c \\ 0 \end{pmatrix} + \begin{pmatrix} B_{N1}^{'} & 0 \\ \vdots & 0 \\ B_{NK}^{'} & 0 \\ D^{'} & -D^{'} \\ \delta_{\gamma}^{'} & 0 \end{pmatrix} \begin{pmatrix} X_t \\ X_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \vdots \\ \varepsilon_{Kt} \\ 0 \\ \omega_t \end{pmatrix} \tag{25}$$

And,[9] the complete parameter set is

$$\varphi = vec\left(a, \kappa, \Sigma, \delta_o, \delta_1, \lambda_0, \Lambda_1, \varepsilon_1 \ldots \varepsilon_K, \omega\right) \tag{26}$$

where ε and ω correspond to estimated measurement errors for the *K* bond yields and the payout ratio, respectively. Estimation follows a common two-step procedure. The first entails rudimentary OLS of the underlying dynamics—*Σ*, *κ*, and *a*–following (1), and given these estimates, the second stage uses maximum likelihood estimation to determine the remaining parameters—$\delta_o, \delta_1, \lambda_0, \Lambda_1, \varepsilon_1 \ldots \varepsilon_K$, and *ω*. Note that in several GATSMs (e.g., Ang et al., 2006), the second-stage objective is to minimize the errors given a time-series of a selected cross-section of yields, as in (ignoring the calibration of errors)

$$L = \min \sum_{n=1}^{K} \sum_{t=1}^{T} \left(\hat{y}_t^n - y_t^n\right)^2 \tag{27}$$

However, given the necessary inclusion of (volatile) stock returns in (25), but in contrast to other GATSEMs, the objective function follows

$$L = \min_{\{\varphi\}} \left[\sum_{n=1}^{K} \frac{1}{\sigma_{y^n}^2} \sum_{t=1}^{T} \left(\hat{y}_t^n - y_t^n\right)^2 + \frac{1}{\sigma_{S\&P}^2} \sum_{t=1}^{T} \left(\hat{r}_t^{S\&P} - r_t^{S\&P}\right)^2 \right] \tag{28}$$

[9] To measure capital gains on stocks, the one-period lag of the factor is required, as $\Delta v_t = c + D^{'} \Delta X_t$. Also, following Lemke and Werner (2009), the assumption is that the dividend yield but not stock returns are measured with error. In addition, the proxies for level and slope are functions of the 3-month and 5-year yields, which are not assumed to be measured with error. Finally, following Ang et al. (2006), a constraint on the optimization is that the model fits the 3-month yield precisely, given that it is a perfect linear combination of the first two factors, again level and slope.

where $\sigma^2_{y^n}$ and $\sigma^2_{S\&P}$ refer to the sample variance of the (monthly) *N*-period bond yield and log changes in stock prices, respectively. The rationale for scaling each time series is to address any possible skew in the parameter selection toward minimizations of those elements in the cross section of *Y* that have greater (temporal) variance, namely stock returns at the expense of more inert government bond yields.[10]

5. Results

On balance, the general magnitudes and broad trends in the model-implied government bond term and equity risk premiums follow standard intuition. Exhibit 1 shows time-series of GATSEM-implied average 10-year expected short rate paths, 10-year zero-coupon term premiums, and the 10-year equity risk premium for the three cases. The top right panel indicates that, averaging across four specifications and similar to the trajectories in Kim and Wright (2005) as well as Lemke and Werner (2009), term premiums steadily decline through the sample and reach historical lows in the most recent period, which coincides with unconventional Federal Reserve policies. The series for Germany and France follow the same general pattern, although the levels of expected short rates and term premiums differ, as expected given overall yield differentials over the sample. Not unlike the results in Lemke and Werner (2009),[11] the level of the term premium for the U.S. is greater than the estimates in Kim and Wright (2005). However, the two series appear to track closely, and the correlation between the GATSEM-based average expected short rate path and the corresponding 10-year term premium is about 0.78, compared to approximately 0.74 with the Kim and Wright (2005) series. The correlations between estimated

[10] Lemke and Werner (2009) do not make such variance adjustment to the objective function.
[11] Lemke and Werner (2009) report a term premium that is 31 basis points greater (and 10 basis points less volatile) than in Kim and Wright (2005).

expected rates and term premiums are closer for Germany (0.87) and even moreso for France (0.96).

The equity risk premium series at the 10-year horizon also follows intuition and resembles the broad contours of the trajectory described in Lemke and Werner (2009), again even though these estimates rest on exclusively nominal as well as “macro-finance” observable factors. Following a familiar narrative, the 10-year U.S. equity premium, which ranges from less than four percent to around 10 percent through the sample, was low in the mid-1990s amid the internet stock bubble, edged higher around the fall of 1998, increased as share prices fell as the “tech bubble” burst and through the aftermath of September 11, 2001, nudged lower during the “considerable period” and “measured pace” phase of Federal Reserve policy during the mid-2000s, and increased to sample highs through the recent global financial crisis. The equity risk premium estimate for Germany largely follows the same trajectory, notably within a very similar range of around three to approximately 10 percent for the most recent period. The estimate for France also follows the general pattern, but the range of the premium spans zero to about seven percent, a bit lower than for the U.S. or Germany.

Again, an advantage of joint estimation is that such a common framework affords some assessment of any relation between required compensation for duration and equity risk. Toward that end, these estimates largely suggest that so-called flights-to-quality have been more benign than pernicious during the sample period, on balance, as investors at times shed credit but not necessarily duration risk. Indeed, the equity risk and term premium series for the U.S. are negatively correlated, at around -0.89, and even moreso for Germany (-0.97) and France (-0.99). This finding is broadly consistent with the interpretation that investors tended to sell (buy) shares as attitudes toward equity risk soured (improved) and reallocated funds to (from) longer-dated

Treasuries. A positive correlation would imply a broader aversion to risk along both the credit and duration dimensions, which might be manifested by a sharp selloff in shares and increased demand for Treasury bills (as opposed to notes or bonds) or capital flight from U.S. assets altogether.

Turning to cross-sectional as opposed to time-series variation in premiums, Exhibit 2 shows the decompositions of yield curves, which are upward-sloping in each case as of July 11, 2013, into anticipated instantaneous short rates and term premiums. For the U.S., forward term premiums are comparatively greater at the short end than at intermediate maturities, particularly nearer the belly of the curve. Correspondingly, expected short rates are negative through the two-year horizon, an implausible result of course.[12] The decomposition for German bunds notably differs in that, although 10-year yields are at comparable levels, forward term premiums (expected short rates) are meaningfully greater at the back (front) end of the term structure—which arguably reflects responses across the ECB and the Federal Reserve given the respective outlooks for inflation and the real economy. Then again, the decomposition for France, again at least for the last sample date, is more consistent with the trajectories for the U.S. than for its EMU partner. In particular, given the approximate 70 basis point spread in observed 10-year zero-coupon yields, the anticipated 10-year-ahead instantaneous short rate in France is about 100 basis points greater than for Germany. At first blush this magnitude may seem implausible, but not in terms of broad inferences if indeed, following the arguments in Durham (2013c),[13] spreads

[12] Vasicek-based models do not incorporate the zero nominal lower bound, a notable shortcoming of the approach, and these simple econometric projections of course do not follow a shadow rate model.
[13] Briefly, expected short rates between two sovereign issuers that share a central bank, and therefore monetary policies, cannot meaningfully diverge if investors have no doubts about the longevity of the currency union—spreads must owe to credit as opposed to convertibility risk (Durham, 2013c).

among EMU issuers reflect to a significant degree convertibility premiums as opposed to default risks per se.[14]

Turning to the term structure of equity risk premiums, the fact that GATSEMs produce such a schedule is ultimately advantageous compared to other estimates, particularly ex post measures. The shape of the required equity return term structure should broadly embed investors' perceptions and attitudes toward risk over a given horizon, and some phenomena likely affect the equity premium at different sections of the term structure. For example, any demographic trends related to retirement savings patterns presumably have a larger impact on longer-horizon premiums, as opposed to cyclical factors that drive investors' perceptions and attitudes toward equity risk in the shorter run. Exhibit 3 shows these schedules for the three cases as of July 11, 2013. For the U.S., the nearest-horizon weighted-average estimate is about six percent, and beyond the next few months, the required return increases to a peak of about 15 percent around the 4-year horizon. With due regard to false precision, the general path seems consistent with a perspective on unconventional U.S. monetary policy. Investors might very well expect not only large scale asset purchases (LSAPs) but also perhaps other measures to support risky asset prices through the portfolio rebalancing channel in the nearer-term, presumably before risk perceptions and attitudes asymptote toward greater levels when highly accommodative policy eventually unwinds. In contrast, the corresponding term structures of required returns are sharply downward sloping in Germany and France, starting from around 19 to 17 percent, respectively. Although the potential impact from the EMU crisis is global, this configuration perhaps reflects particularly acute near-term perceptions and aversion to risks within the single currency zone.

[14] Note that the fit of these models—with average mean squared errors of about 10 to 11 basis points across the term structure at best—is less precise than standard GATSMs, owning to the additional requirement to fit simultaneously stock returns.

Of course, spinning a plausible story behind the observed term structure on a single day hardly comprises broad confirmation, and Exhibit 4 shows the complete surfaces of schedules from July 5, 1993 through July 11, 2013. Clearly unlike the yield curve, the slope of the equity risk premium term structure is not so predominantly positive in either case. Instead, the schedule is notably downward sloping at times, most notably around the height of the global financial crisis, and at least in the U.S. (rather than in France or Germany), the early part of the sample. The former result seems quite plausible, as investors' perceptions and attitudes toward equity risk likely reflected pronounced near-term aversion to uncertainty.[15] However, although only applicable to the U.S., the latter episode amid the prolonged run-up in U.S. share prices and notably before the fall of 1998 seems somewhat curious.

In short, there is a general dearth of equity premium schedules in the literature. Nonetheless, the implied trajectory in this application at key sample points, and notably across cases, seems sensible. And by comparison, indeed the slope of the forward term premiums derived from standard GATSMs still pose challenging questions for interpretation. Shorter-horizon term premiums imprecisely reflect uncertainty around the cyclical monetary policy path, whereas given some evidence of "excess sensitivity" of distant-horizon forward rates to cyclical economic news (e.g., Gürkaynak et al., 2005), longer-run variation does not always tidily relate to uncertainty around the central bankers' targets or goals (or usual supply effects and unconventional policy measures). In the same way, investors' required forward compensation for equity risk might correlate surprisingly robustly with near-term conditions as opposed to potential longer-run factors, such as demographics, that might affect demand for the asset class.

[15] At first blush the magnitude of the equity premium at short horizons in these cases might seem excessive at some points in the sample. However, note that the average annual return on the S&P over the past century or so is roughly 10 percent with a percent standard deviation of roughly 20 percent. At least through some horizons, investors conceivably indeed require compensation for perceptions and attitudes toward risk commensurate with, say, the upper end of a one-standard-error band in historical realized returns, i.e., around 30 percent.

6. Premiums and the Global Financial Crisis

The recent global financial crisis and its aftermath, particularly unconventional monetary policy responses and the proximity of the nominal zero bound, motivates closer examination of parameter stability beyond the preceding sensitivity analyses of sample selection. Robustness to the most recent data could reflect underlying regime shifts along disparate dimensions (e.g., so-called financial repression, the end of the Great Moderation, an increase in the long-run NAIRU, etc.). Arbitrage-free models that nest bond and stock prices might afford more comprehensive insights on recent changes in perceptions and attitudes toward uncertainty across the risk-free and benchmark risky-assets.

Exhibit 5 includes the same information as Exhibit 1 for the U.S. but exclusively given parameter estimates from samples that end no earlier than July 2010. The implied equity risk premium series is somewhat lower but follows a similar trajectory as in Exhibit 1, and therefore the observation that the long-run equity risk premium remains elevated near the end of the sample is generally robust to this (arbitrary) sample break. Also, by crude visual inspection, changes in 10-year zero-coupon term premium estimates appear to track calculations from Kim and Wright (2005), but clearly the level diverges profoundly. Indeed, the decline in 10-year yields during the sample owes largely entirely to a decline in average expected short rates, which appear unreasonably low, rather than term premiums as the consensus suggests. Also, the correlation between the equity risk and Treasury term premium is 0.06, compared to about -0.89 as noted in Exhibit 1, which suggests a much more orthogonal relation between the two risk dimensions. Recent data might therefore imply more severe flights-to-quality, as investors shed not only credit but also duration risk at times, although the statistic of course implies no relation on balance.[16]

Thus, the results from this approach, similar to the results from some term structure models, question whether lower U.S. interest rates post-crisis reflect extraordinarily low term premiums as opposed to anticipated short rates.[17] At the same time, GATSEM-based equity risk premium estimates appear robust to the most recent data. This result is somewhat problematic for the view that unconventional Federal Reserve policy measures have worked through the portfolio-rebalancing channel per se, on net—i.e., if anything, policies near the zero bound correspond with very low expected short rates rather than term premiums using longer samples, amid an elevated equity risk premium. In other words, these estimates are arguably inconsistent with the view that unconventional policy removed duration from the market and lowered either investors' perceptions or attitudes toward risky assets. At the very least, although the equity risk premium series are comparatively robust, whether the post-crisis data produce a structural break in the series merits serious consideration from both policymakers, who might assume the term premium has grinded to remarkable lows amid unconventional policy, and investors, who could infer that the upward slope of the yield curve embeds significant anticipated capital depreciation on government bonds without meaningful compensation for bearing duration risk.

7. Discussion

The preceding analyses attempt to extend earlier work (e.g., Mamaysky, 2002; Lemke and Werner, 2009) that prices government bonds and stocks in a common affine arbitrage-free

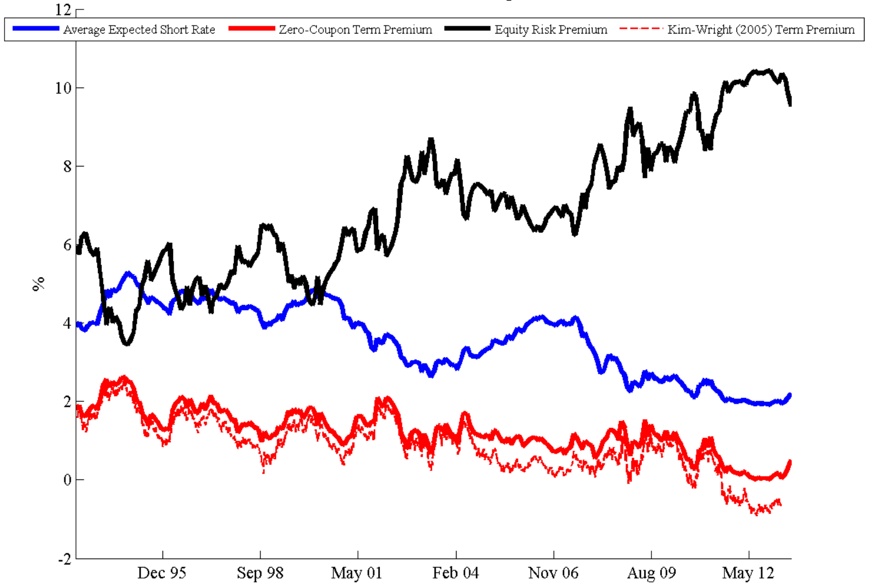
[16] The negative, as opposed to positive correlation between expected rates and term premiums is also noteworthy, and without a finer decomposition between real and nominal components, is not inconsistent with lower inflation risk premiums upon more restrictive anticipated monetary policy, or a counter-cyclical term premium with respect to the expected path of real rates.

[17] See Durham (2013c) or Bauer and Rudebusch (2012). Also, this broad signal is not wholly inconsistent with Guimarães (2012), who finds that the recent post-crisis net decline in nominal Gilt yields amid experience with unconventional Bank of England policy comprises lower expected real rates as opposed to real term premiums, inflation risk premiums, or anticipated inflation.

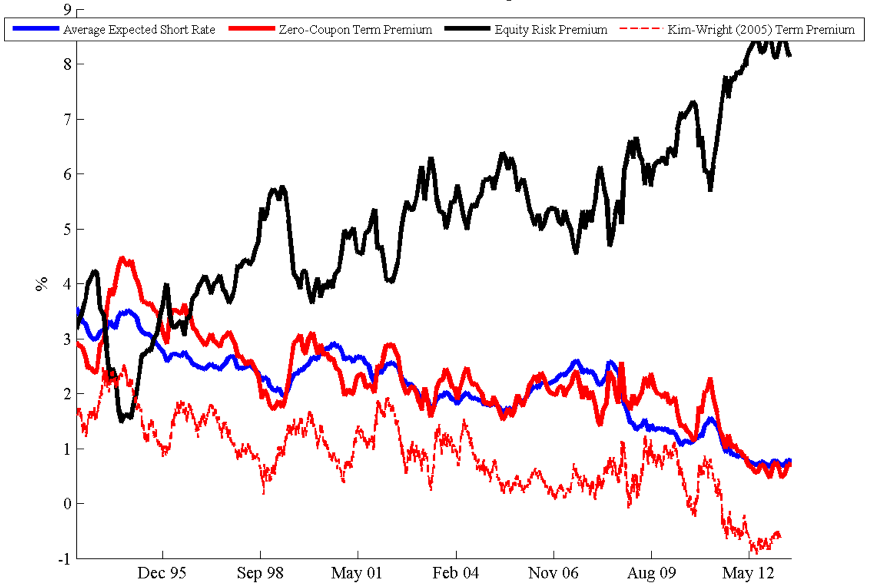
framework. A theoretical addendum is to identify the conditions under which such a framework is consistent with the final phase of a DDM, and an empirical objective is to introduce additional sensitivity analyses into the calibration of not only GATSEMs but also more limited (observable-factor) GATSMs. Findings as always should not be oversold, particularly given that the additional task of calibrating equity premiums is even more taxing on common GATSM parameter estimation problems, namely estimating the parameters for persistent yet mean-reverting yield series over short samples. Nonetheless, the variation in the implied Treasury term and equity risk premiums over the sample is broadly consistent with common priors, and although key episodes follow economic and financial logic (even in the context of contemporary policy), the results uncover room for additional theory behind (unidentified) factors that shape the term structure of the equity premium. In addition, germane to general GATSMs, the analyses raise questions regarding information through the aftermath of the global financial crisis. The equity premium estimates are largely robust, yet recent data challenge common assumptions about the trajectory and the level of U.S. Treasury term premiums, a critical input for monetary policymaking as well as investors' assessment of market-based anticipated returns on the "risk-free" asset class.

Exhibit 1

US: 5-factor GATSEM (07/05/1993-07/11/2013): Weighted Average of Specifications (4/2352 models)
Correlations: Equity Risk/Term Premium (-0.89446), Short Rate/Term Premium: (0.78604),
Short Rate/Term Premium (Kim-Wright): (0.73809)



Germany: 5-factor GATSEM (07/05/1993-07/11/2013): Weighted Average of Specifications (8/1764 models)
Correlations: Equity Risk/Term Premium (-0.97557), Short Rate/Term Premium: (0.87385),
Short Rate/Term Premium (Kim-Wright): (0.73809)



France: 5-factor GATSEM (07/05/1993-07/11/2013): Weighted Average of Specifications (22/1764 models)
Correlations: Equity Risk/Term Premium (-0.99271), Short Rate/Term Premium: (0.96162),
Short Rate/Term Premium (Kim-Wright): (0.73809)

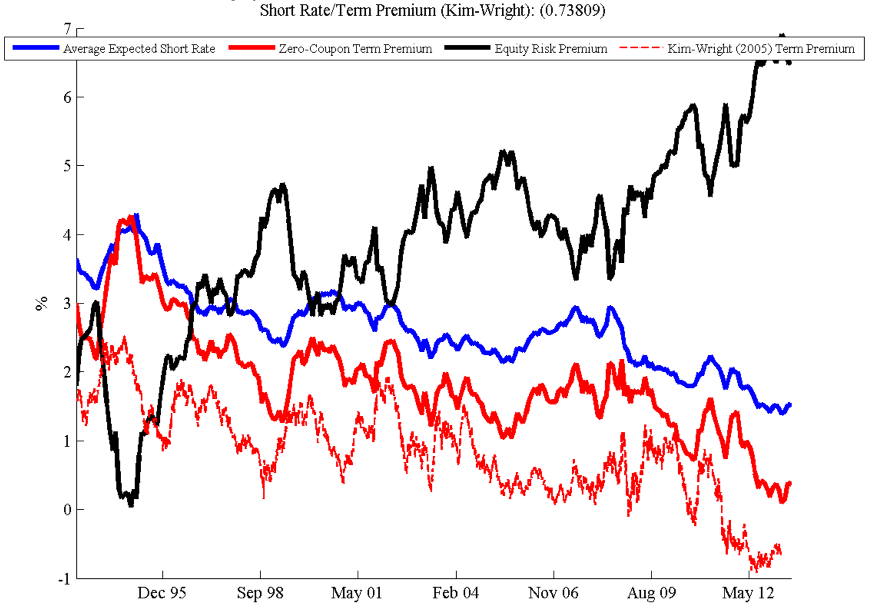
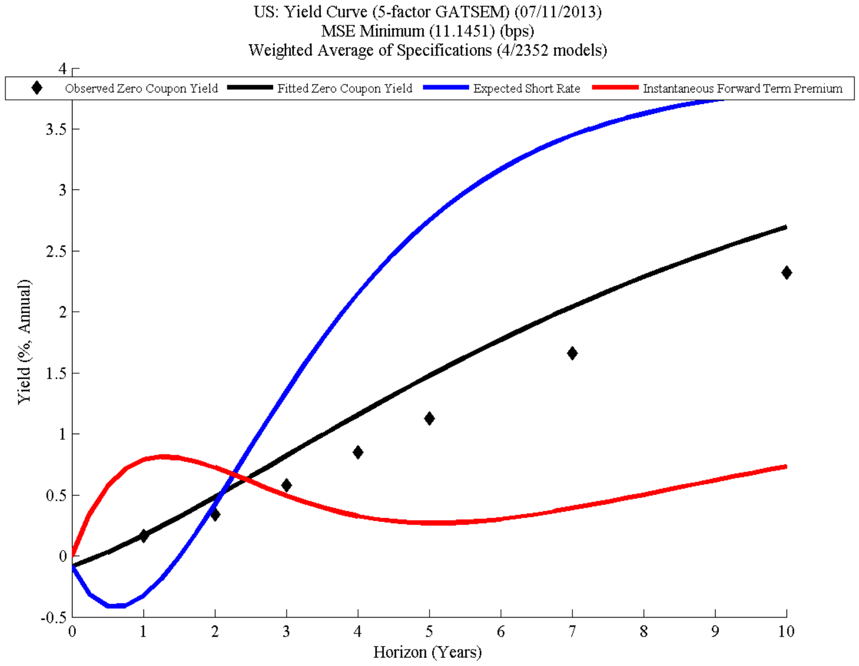
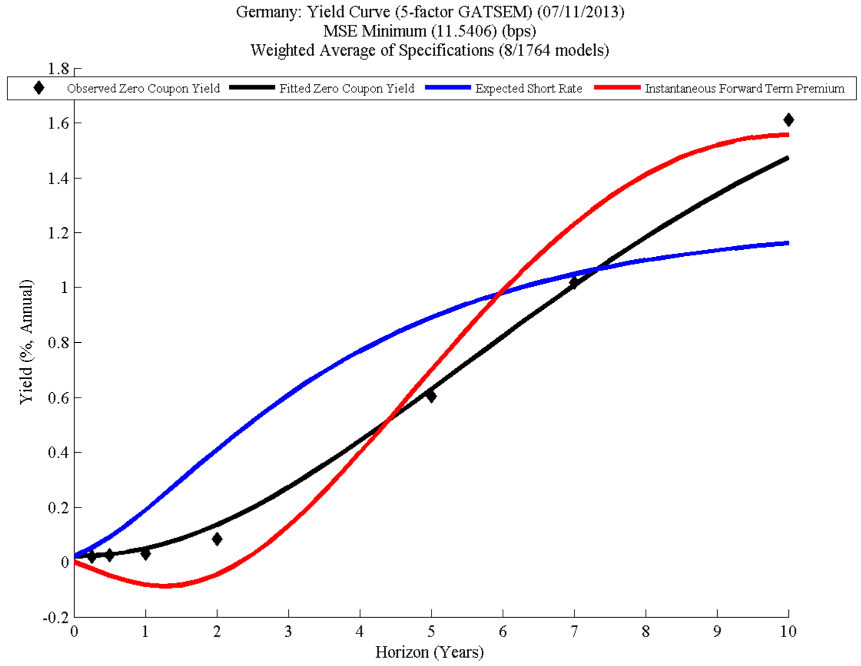


Exhibit 2


US: Yield Curve (5-factor GATSEM) (07/11/2013)
MSE Minimum (11.1451) (bps)
Weighted Average of Specifications (4/2352 models)
Observed Zero Coupon Yield
Fitted Zero Coupon Yield
Expected Short Rate
Instantaneous Forward Term Premium
Yield (%, Annual)
Horizon (Years)


Germany: Yield Curve (5-factor GATSEM) (07/11/2013)
MSE Minimum (11.5406) (bps)
Weighted Average of Specifications (8/1764 models)
Observed Zero Coupon Yield
Fitted Zero Coupon Yield
Expected Short Rate
Instantaneous Forward Term Premium
Yield (%, Annual)
Horizon (Years)

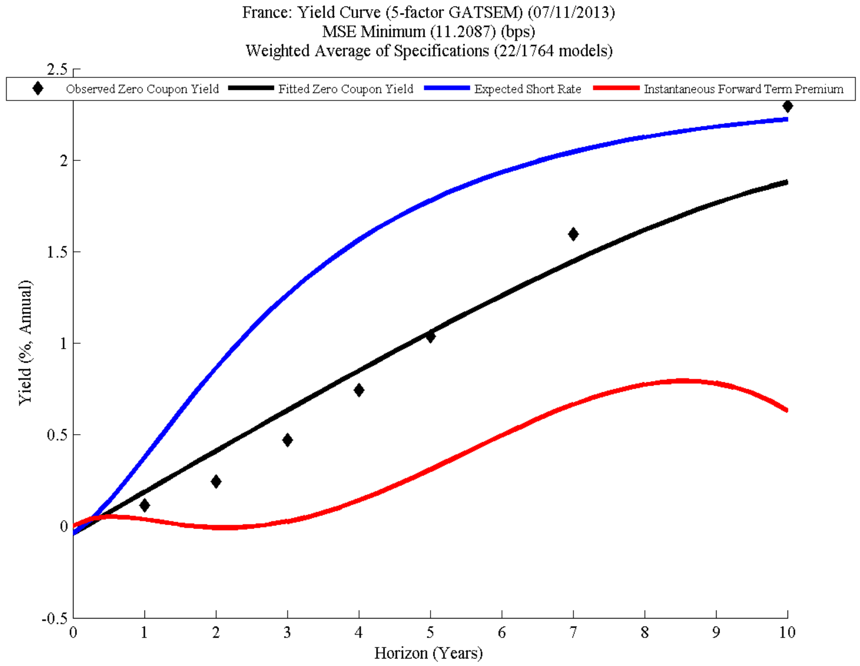
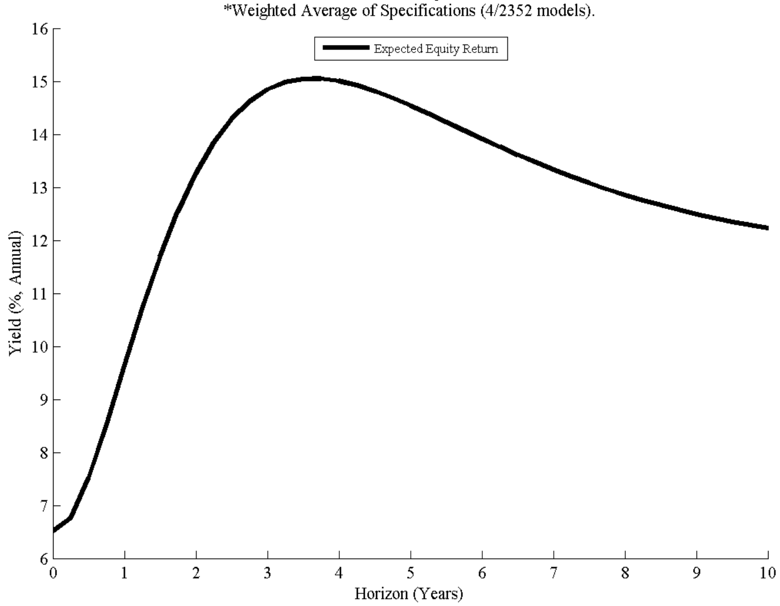
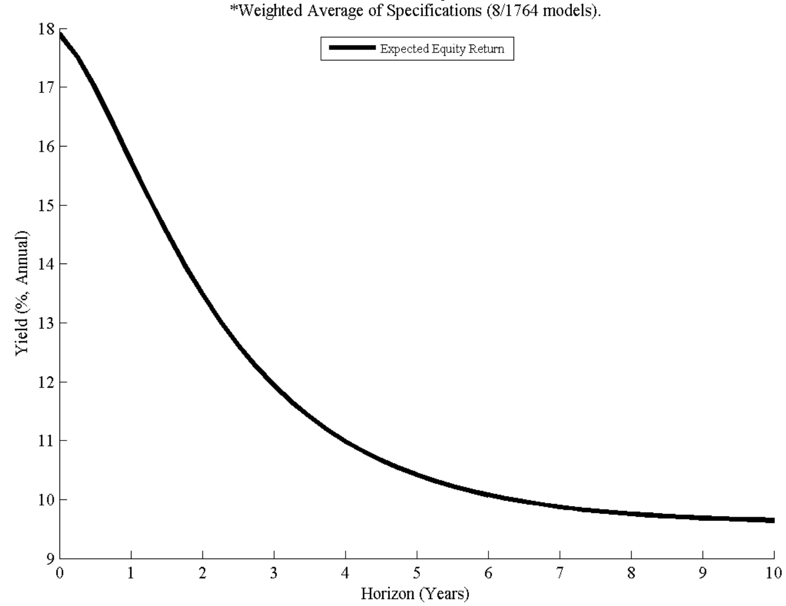

France: Yield Curve (5-factor GATSEM) (07/11/2013)
MSE Minimum (11.2087) (bps)
Weighted Average of Specifications (22/1764 models)
Observed Zero Coupon Yield
Fitted Zero Coupon Yield
Expected Short Rate
Instantaneous Forward Term Premium
Yield (%, Annual)
Horizon (Years)

Exhibit 3

US: Equity Expected Return Term Structure:5-factor GATSEM (07/11/2013)
Parameter Estimation Sample for Closest Fit: (06/07/1993-07/14/2008)
MSE Minimum (1.4489%), Sample Std. Dev. (5.2145%)
*Weighted Average of Specifications (4/2352 models).



Germany: Equity Expected Return Term Structure:5-factor GATSEM (07/11/2013)
Parameter Estimation Sample for Closest Fit: (06/07/1993-07/09/2007)
MSE Minimum (2.3323%), Sample Std. Dev. (5.8491%)
*Weighted Average of Specifications (8/1764 models).



France: Equity Expected Return Term Structure:5-factor GATSEM (07/11/2013)
Parameter Estimation Sample for Closest Fit: (06/07/1993-07/09/2007)
MSE Minimum (2.3006%), Sample Std. Dev. (5.7075%)
*Weighted Average of Specifications (22/1764 models).

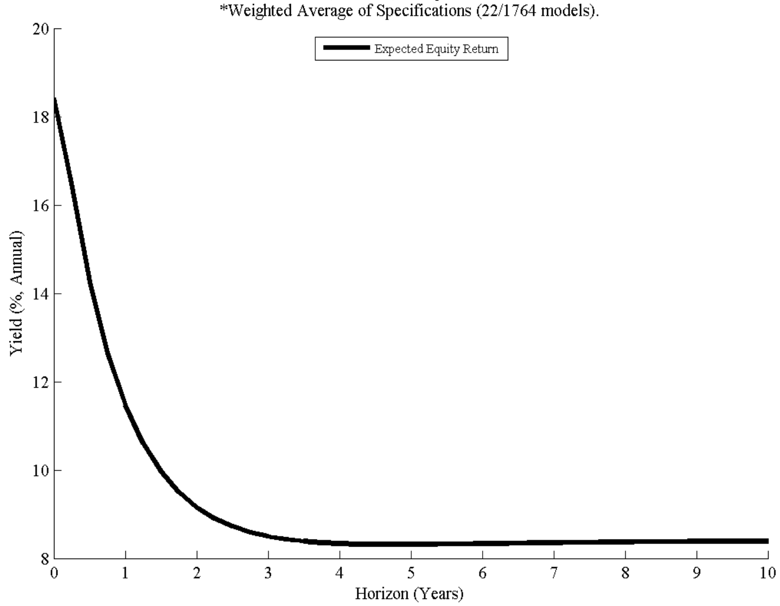
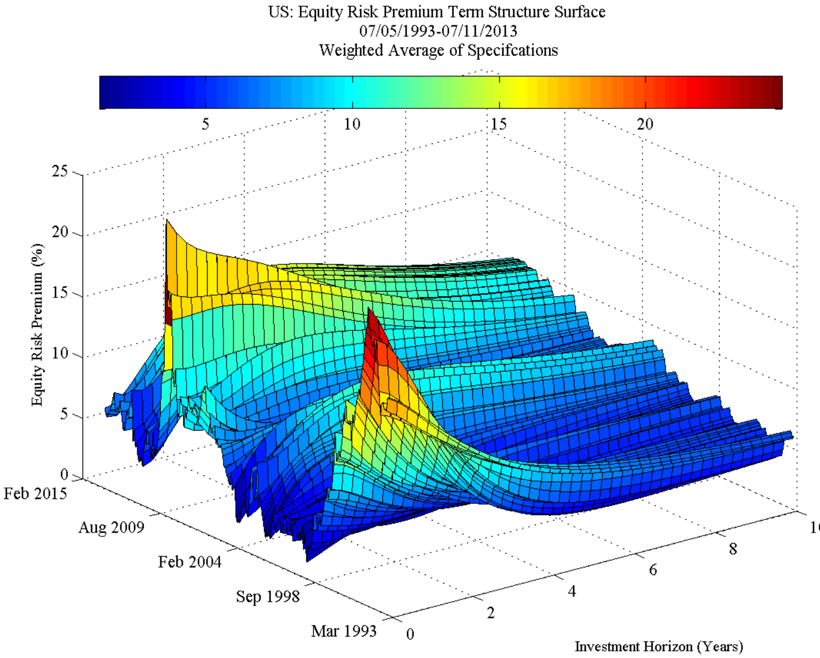
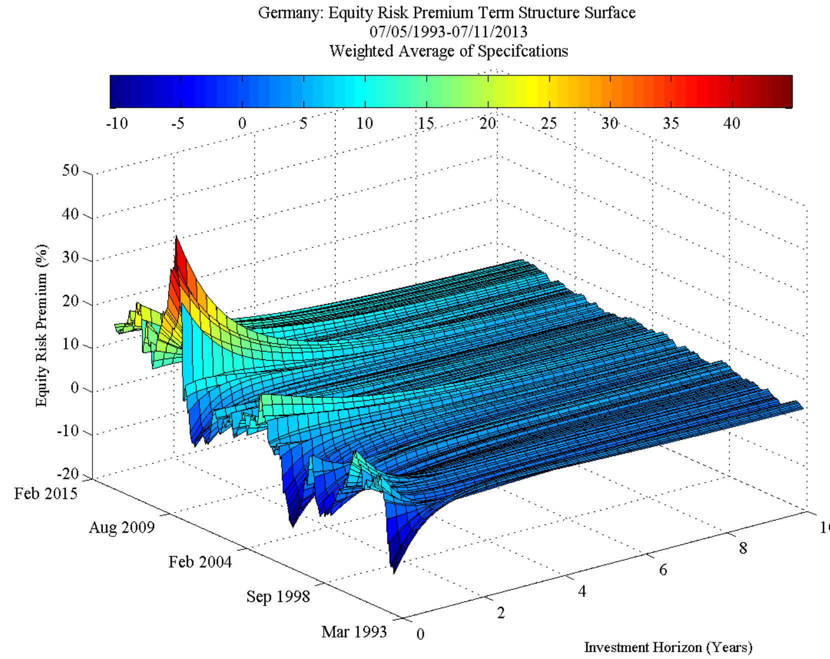


Exhibit 4


US: Equity Risk Premium Term Structure Surface
07/05/1993-07/11/2013
Weighted Average of Specifcations
5
10
15
20
25
0
Equity Risk Premium (%)
Feb 2015
Aug 2009
Feb 2004
Sep 1998
Mar 1993
0
2
4
6
8
10
Investment Horizon (Years)


Germany: Equity Risk Premium Term Structure Surface
07/05/1993-07/11/2013
Weighted Average of Specifcations
-10
-5
0
5
10
15
20
25
30
35
40
50
-20
Equity Risk Premium (%)
Feb 2015
Aug 2009
Feb 2004
Sep 1998
Mar 1993
0
2
4
6
8
10
Investment Horizon (Years)

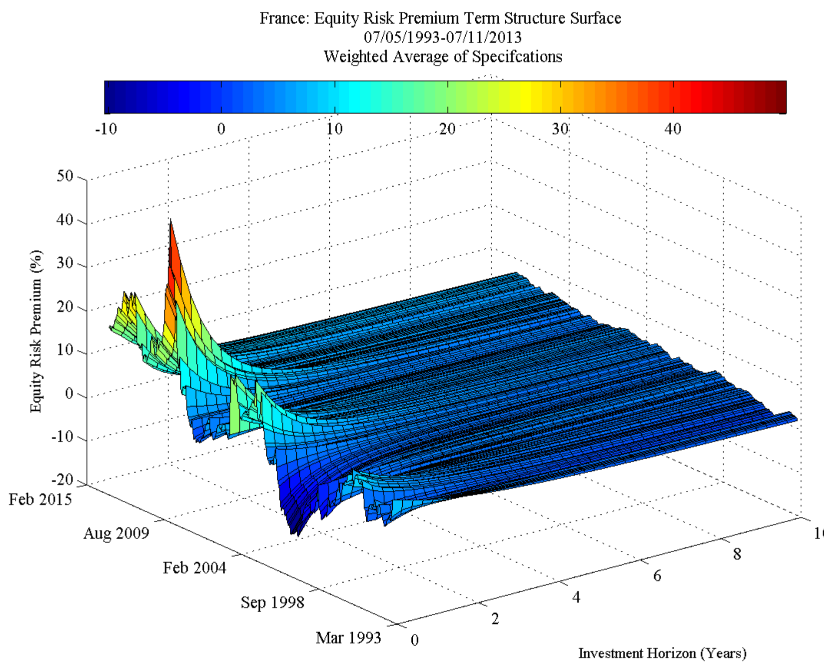
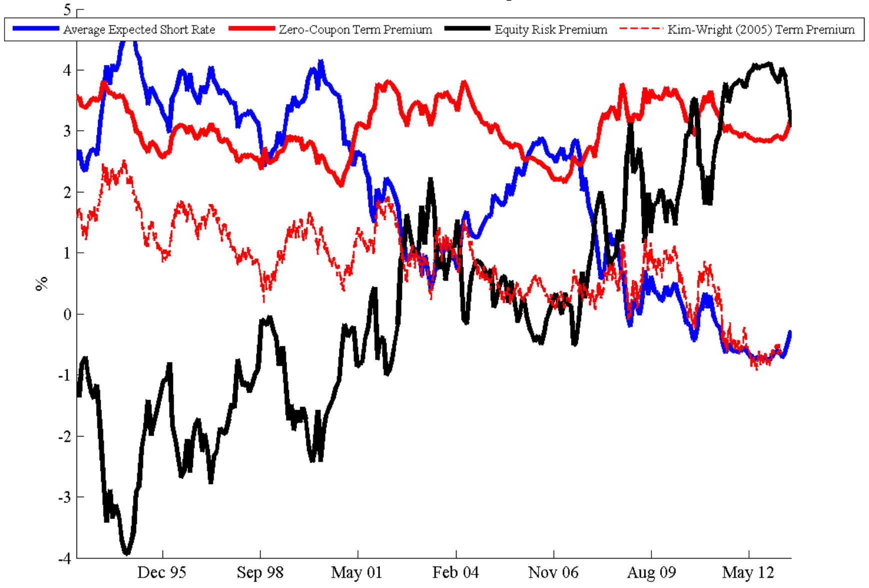

France: Equity Risk Premium Term Structure Surface
07/05/1993-07/11/2013
Weighted Average of Specifcations
-10
0
10
20
30
40
50
-20
Equity Risk Premium (%)
Feb 2015
Aug 2009
Feb 2004
Sep 1998
Mar 1993
0
2
4
6
8
10
Investment Horizon (Years)

Exhibit 5

US: 5-factor GATSEM (07/05/1993-07/11/2013): Weighted Average of Specifications (3/1344 models)
Correlations: Equity Risk/Term Premium (0.11892), Short Rate/Term Premium: (-0.32655),
Short Rate/Term Premium (Kim-Wright): (0.73809)



Appendix 1:
The Affine Solution for Equity Prices

Following (5) in general as well as (2), (6), and (8), the equity index price is a function of the pricing kernel, the expected payout, and anticipated price (de)appreciation, as in

$$V_t = E_t\left\{\exp\left[\ln M_{t+1} + \ln\Gamma_{t+1} + \ln V_{t+1}\right]\right\}$$
$$= E_t\left\{\exp\left[\left(-\delta_0 - \delta_1^{'}X_t - \frac{1}{2}\lambda_t^{'}\lambda_t - \lambda_t^{'}\eta_{t+1}\right) + \rho_0 + \rho_1^{'}X_{t+1} + c\left(t+1-t_0\right) + D^{'}X_{t+1}\right]\right\} \tag{29}$$

With substitution for X_{t+1} given the dynamics of the factors, (1), the expression simplifies with respect to X_t and η_{t+1} following

$$V_t = E_t\left\{\exp\left[-\delta_0 - \delta_1^{'}X_t - \frac{1}{2}\lambda_t^{'}\lambda_t - \lambda_t^{'}\eta_{t+1} + \left(\rho_1^{'} + D^{'}\right)\left(a + \kappa X_t + \Sigma\eta_{t+1}\right) + \rho_0 + c\left(t+1-t_0\right)\right]\right\}$$
$$= E_t\left\{\exp\left[-\frac{1}{2}\lambda_t^{'}\lambda_t - \delta_0 + \left(\delta_\gamma^{'} + D^{'}\right)a + \rho_0 + c\left(t+1-t_0\right) + \left(\left(\rho_1^{'} + D^{'}\right)\kappa - \delta_1^{'}\right)X_t + \left(\left(\rho_1 + D\right)^{'}\Sigma - \lambda_t^{'}\right)\eta_{t+1}\right]\right\} \tag{30}$$

In general, recall that for a normally distributed random variable (such as stock returns), *W*, and the moment generating function for a Gaussian,

$$E_t\left\{\exp\left(W_{t+1}\right)\right\} = \exp\left(E_t\left\{W_{t+1}\right\} + \frac{1}{2}\mathrm{var}_t\left\{W_{t+1}\right\}\right) \tag{31}$$

Therefore, the mean of (30) follows

$$-\frac{1}{2}\lambda_t^{'}\lambda_t - \delta_0 + \left(\delta_\gamma^{'} + D^{'}\right)a + \rho_0 + c\left(t+1-t_0\right) + \left(\rho_1^{'} + D^{'}\right)\kappa X_t - \delta_1^{'}X_t \tag{32}$$

And, the conditional variance is

$$= \left[\left(\rho_1^{'} + D\right)^{'}\Sigma - \lambda_t^{'}\right]\left[\left(\rho_1^{'} + D\right)^{'}\Sigma - \lambda_t^{'}\right]^{'}$$
$$= \left(\rho_1^{'} + D\right)^{'}\Sigma\Sigma^{'}\left(\rho_1^{'} + D\right) + \lambda_t^{'}\lambda_t - 2\left(\rho_1^{'} + D\right)^{'}\Sigma\lambda_t \tag{33}$$

Substituting (31) and (32) into (30), the price then, gathering terms with respect to X_t, follows

$$
\begin{aligned}
V_t &= \exp\Big[-\frac{1}{2}\lambda_t^{'}\lambda_t - \delta_0 + \left(\rho_1^{'} + D^{'}\right) a + \rho_0 + c\left(t+1-t_0\right) + \left(\rho_1^{'} + D^{'}\right)\kappa X_t - \delta_1^{'} X_t \ldots \\
&\quad + \frac{1}{2}\Big[\left(\rho_1 + D\right)^{'} \Sigma\Sigma^{'} \left(\rho_1 + D\right) + \lambda_t^{'}\lambda_t - 2\left(\rho_1 + D\right)^{'} \Sigma\lambda_t \Big]\Big] \\
&= \exp\Big[-\delta_0 + \left(\rho_1 + D\right)^{'} a + \frac{1}{2}\left(\rho_1 + D\right)^{'} \Sigma\Sigma^{'} \left(\rho_1 + D\right) - \left(\rho_1 + D\right)^{'} \Sigma\lambda_t + \rho_0 + c\left(t+1-t_0\right) \ldots \\
&\quad + \left(\rho_1 + D\right)^{'} \kappa X_t - \delta_1^{'} X_t \Big] \\
&= \exp\Big[-\delta_0 + \left(\rho_1 + D\right)^{'} a + \frac{1}{2}\left(\rho_1 + D\right)^{'} \Sigma\Sigma^{'} \left(\rho_1 + D\right) - \left(\rho_1 + D\right)^{'} \Sigma\left(\lambda_0 + \Lambda_1 X_t\right) + \rho_0 + c\left(t+1-t_0\right) \ldots \\
&\quad + \left(\rho_1 + D\right)^{'} \kappa X_t - \delta_1^{'} X_t \Big] \\
&= \exp\Big[-\delta_0 + \left(\rho_1 + D\right)^{'} a + \frac{1}{2}\left(\rho_1 + D\right)^{'} \Sigma\Sigma^{'} \left(\rho_1 + D\right) - \left(\rho_1 + D\right)^{'} \Sigma\lambda_0 + \rho_0 + c\left(t+1-t_0\right) \ldots \\
&\quad + \Big[\left(\rho_1 + D\right)^{'} \kappa - \delta_1^{'} - \left(\rho_1 + D\right)^{'} \Sigma\Lambda_1 \Big] X_t \Big]
\end{aligned}
\tag{34}
$$

Therefore, again noting the grouping of terms with respect to X_t, the two equations for *c* and *D* must follow

$$
\begin{aligned}
D^{'} &= \left(\rho_1 + D\right)^{'} \kappa - \delta_1^{'} - \left(\rho_1 + D\right)^{'} \Sigma\Lambda_1 \\
c\left(t - t_0\right) &= -\delta_0 + \left(\rho_1 + D\right)^{'} a + \frac{1}{2}\left(\rho_1 + D\right)^{'} \Sigma\Sigma^{'} \left(\rho_1 + D\right) - \left(\rho_1 + D\right)^{'} \Sigma\lambda_0 + \rho_0 + c\left(t+1-t_0\right)
\end{aligned}
\tag{35}
$$

with solution (9).

Appendix 2:
Expected Multi-period Stock Returns under the Affine Model

To derive the formula for multi-period-period expected stock returns, rewrite (14) as a function of the state variables, following

$$
\begin{aligned}
E\left\{r_{t+N}^{S\&P,(N)}\right\} &= E\left\{\frac{1}{N}\sum_{i=1}^{N}\left(c + D^{'}\Delta X_{t+i} + \rho_0 + \rho_1^{'} X_{t+i}\right)\right\} \\
&= \frac{1}{N}\left(Nc + N\rho_0 + D^{'}\sum_{i=1}^{N} E\left\{\Delta X_{t+i}\right\} + \rho_1^{'}\sum_{i=1}^{n} E\left\{X_{t+i}\right\}\right)
\end{aligned} \tag{36}
$$

Therefore, the sum of expectations for both the level and change in *X* are required. For the level, note by simple recursion that

$$
\begin{aligned}
E\left\{X_{t+1}\right\} &= a + \kappa X_t \\
E\left\{X_{t+2}\right\} &= a + \kappa E\left\{X_{t+1}\right\} \\
E\left\{X_{t+i}\right\} &= a + \kappa a + \kappa^2 a + \cdots + \kappa^{i-1} a + \kappa^i X_t \\
&= \left(\sum_{j=0}^{i-1}\kappa^j\right) a + \kappa^i X_t \\
&= \left(I-\kappa\right)^{-1}\left(I-\kappa^i\right) a + \kappa^i X_t \\
&= \left(I-\kappa\right)^{-1} a - \left(I-\kappa\right)^{-1}\kappa^i a + \kappa^i X_t
\end{aligned} \tag{37}
$$

Therefore,[18] the sum of expectations follows

$$
\begin{aligned}
\sum_{i=1}^{N} E\left\{X_{t+i}\right\} &= \sum_{i=1}^{N}\left[\left(I-\kappa\right)^{-1} a - \left(I-\kappa\right)^{-1}\kappa^i a + \kappa^i X_t\right] \\
&= \sum_{i=1}^{N}\left(I-\kappa\right)^{-1} a - \left(I-\kappa\right)^{-1}\left(\sum_{i=1}^{N}\kappa^i\right) a + \sum_{i=1}^{N}\kappa^i X_t \\
&= \left[\sum_{i=1}^{N}\left(I-\kappa\right)^{-1} - \left(I-\kappa\right)^{-1}\left(\sum_{i=1}^{N}\kappa^i\right)\right] a + \sum_{i=1}^{N}\kappa^i X_t \\
&= \left[\left(I-\kappa\right)^{-1} NI - \left(I-\kappa\right)^{-1}\left(\sum_{i=1}^{N}\kappa^i\right)\right] a + \sum_{i=1}^{N}\kappa^i X_t \\
&= \left(I-\kappa\right)^{-1}\left[NI - \left(\sum_{i=1}^{N}\kappa^i\right)\right] a + \sum_{i=1}^{N}\kappa^i X_t \\
&= \left(I-\kappa\right)^{-1}\left(NI - R\left(\kappa,N\right)\right) a + R\left(\kappa,N\right) X_t
\end{aligned} \tag{38}
$$

where $R\left(\kappa,N\right) = \left(I-\kappa\right)^{-1}\kappa\left(I-\kappa^N\right)$. With respect to changes in *X*, and given (37),

[18] Note that (37) makes use of the following

$$\sum_{j=1}^{i} A^j = \left(I-A\right)^{-1}\left(A - A^{i+1}\right)$$

$$
\begin{aligned}
E\{\Delta X_{t+1}\} &= a + (\kappa - I) X_t \\
E\{\Delta X_{t+2}\} &= a + (\kappa - I) E\{X_{t+1}\} \\
E\{\Delta X_{t+i}\} &= a + (\kappa - I) E\{X_{t+i-1}\} \\
&= a + (\kappa - I)\left[(I-\kappa)^{-1} a - (I-\kappa)^{-1} \kappa^{i-1} a + \kappa^{i-1} X_t\right] \\
&= a - a + \kappa^{i-1} a - (I - K)\kappa^{i-1} X_t \\
&= \kappa^{i-1} a - (I-\kappa)\kappa^{i-1} X_t \\
&= \kappa^{i-1} a - (\kappa^{-1} - I)\kappa\kappa^{i-1} X_t \\
&= \kappa^{i-1} a + (I - \kappa^{-1})\kappa^{i} X_t
\end{aligned}
\tag{39}
$$

And, the sum of expected changes therefore follows

$$
\begin{aligned}
\sum_{i=1}^{n} E\{\Delta X_{t+i}\} &= \sum_{i=1}^{n}\left[\kappa^{i-1} a + (I-\kappa^{-1})\kappa^{i} X_t\right] \\
&= \sum_{i=1}^{n}\kappa^{i-1} a + \sum_{i=1}^{n}\left[(I-\kappa^{-1})\kappa^{i} X_t\right] \\
&= \left(\sum_{i=1}^{n}\kappa^{i}\right)\kappa^{-1} a + (I-\kappa^{-1})\sum_{i=1}^{n}\kappa^{i} X_t \\
&= R(\kappa,n)\kappa^{-1} a + (I-\kappa^{-1}) R(\kappa,n) X_t
\end{aligned}
\tag{40}
$$

Finally, substituting (38) and (40) into (36) produces (14), following

$$
\begin{aligned}
E\left\{r_{t+N}^{(N)}\right\} &= \frac{1}{N}\Big(Nc + N\rho_0 + D'\left(R(\kappa,N)\kappa^{-1} a + (I-\kappa^{-1}) R(\kappa,N) X_t\right)\dots \\
&\quad + \rho_1'\left((I-\kappa)^{-1}\left(NI - R(\kappa,N)\right) a + R(\kappa,N) X_t\right)\Big) \\
&= c + \rho_0 + \frac{1}{N} D' R(\kappa,N)\kappa^{-1} a + \frac{1}{N}\rho_1'(I-\kappa)^{-1}\left(NI - R(\kappa,N)\right) a\dots \\
&\quad + \left[\frac{1}{N} D'(I-\kappa^{-1}) R(\kappa,N) + \frac{1}{N}\rho_1' R(\kappa,N)\right] X_t \\
&= f_N + F_N' X_t
\end{aligned}
\tag{41}
$$

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