Efficiency Improvement from Restricting the Liquidity of Nominal Bonds*

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Abstract

This paper provides a normative theory of partially illiquid bonds jointly with optimal monetary policy. The model has a centralized asset market and a decentralized goods market. Individuals face matching shocks which affect the marginal utility of consumption and which they cannot insure, borrow or trade assets against. The government imposes a legal restriction to prohibit nominal bonds from being used as a means of payments in a subset of trades. We show that this partial legal restriction can improve the society's welfare. In contrast to the literature, the efficiency role of the restriction exists in the steady state and it does not require the households to be able to trade assets after receiving the shocks. Moreover, even when lump-sum taxes are available, the efficiency role continues to exist under a condition that induces optimal money growth to be above the Friedman rule.

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1. Introduction

This paper provides a normative theory of partially illiquid bonds jointly with optimal monetary policy. I construct a model in which fiat money facilitates the exchange in the decentralized goods market. After buyers and sellers are matched in pairs, a shock determines the marginal utility of the goods. In a subset of trades, a legal restriction prevents the buyers from using nominal bonds as payments. I show first that the partial legal restriction can improve the society's welfare provided that money growth exceeds the so-called Friedman rule. Then I provide a condition under which optimal money growth is indeed above the Friedman rule.

The motivation for the analysis is straightforward. Money and nominal bonds have been co-existing for a long time, with bonds dominating money in the rate of return. For countries like the U.S. in the recent history, government bonds bear little default risk and they have all the intrinsic features that money has. However, they do not act as a medium of exchange to the same extent as money does and they dominate money in the rate of return. This return dominance is a classical issue in monetary economics (e.g. Hicks, 1939). To obtain return dominance, traditional models assume that the liquidity of bonds is reduced by legal restrictions, in the form of reserve requirements, cash in advance, or money in the utility function. However, those restrictions reduce the society's welfare.

An efficiency role of legal restrictions is needed for explaining why illiquid bonds are useful for a society. Moreover, monetary policy often relies on return dominance to achieve its effects. For example, open market operations explore the positive discount on bonds; the overnight market relies on collateral which has a higher rate of return than money. In the large literature that analyzes open market operations (e.g., Lucas, 1990), the restrictions that generate return dominance reduce efficiency, but eliminating these restrictions also eliminates the real effect of monetary policy. It is desirable to analyze the effects of monetary policy in a model where illiquid bonds have an efficiency role.

I introduce nominal bonds and a legal restriction into the search model of money in Shi (1997). The government sells bonds in a centralized (Walrasian) market, which is separated from the goods market. In the goods market where agents are matched in pairs, trading histories are private and so a medium of exchange facilitates the trade. After individuals are matched, a matching/taste shock determines whether the seller in the match can produce red or green goods. The two colors are equally costly to produce, but they yield different marginal utilities. The marginal utility of red goods relative to green goods is θ. Although green goods can be purchased with both money and bonds, a *legal restriction* prohibits the use of bonds as payments for red goods.

In the first version of model, the legal restriction is assumed to be enforced costlessly. The legal restriction can increase the steady-state welfare of the society provided that money growth is above the Friedman rule. The efficiency role arises when the relative taste for red goods, θ, is less than one but not too small. The reason for this result is simple: the legal restriction reduces the quantity of red goods and increases the quantity of green goods traded in a match. When θ is less than one, this shift of consumption from red goods to green goods reduces the gap between the marginal utilities of the two goods. As a result, the household's expected utility increases. Put differently, the illiquidity of bonds induced by the legal restriction serves as partial insurance against the matching shocks.

I then examine the joint determination of optimal money growth and the efficiency role of the legal restriction. Optimal money growth exceeds the Friedman rule if such higher growth can improve the efficiency of the extensive margin of trade, i.e., if it can reduce the matching inefficiency according to the principle described by Hosios (1990). Under the assumption that the buyer in a match makes a take-it-or-leave-it offer, this improvement in efficiency arises when money growth reduces the number of buyers in the goods market. I specify the condition for this effect to occur. Under this condition, the optimal structure of government liabilities contains money and partially illiquid bonds, with bonds dominating money in the rate of return.

In the second version of the model, I address the issue of enforcing the legal restriction. To do so, I introduce government sellers who have the technology to refuse to accept bonds as payments. Assuming that red goods are produced exclusively by government sellers, the setup ensures the legal restriction to be enforced in the trades of red goods. The main results in the first version of the model continue to hold.

Bryant and Wallace (1984) are among the first ones who examine the efficiency role of the legal restriction on nominal bonds. They model the legal restriction differently, as a prohibition against issuing bonds with small denominations. I will discuss the main similarities and differences between my model and the Bryant-Wallace model in section 5.

Another paper on the efficiency role of illiquid bonds is Kocherlakota (2003). Although illiquid bonds serve as partial insurance against shocks in both Kocherlakota's model and mine, there are important differences. First, I emphasize a different mechanism of welfare-improving illiquid bonds by deliberately shutting down the one in Kocherlakota. In Kocherlakota's model, a necessary condition for illiquid bonds to improve welfare is that individuals can first observe the taste shocks and then trade between money and bonds before going to the goods market. Such trading is not possible here because the shocks occur within the matches, at which time individuals are separated from each other. As a result, a universal legal restriction, like the one in Kocherlakota's model, cannot improve welfare here. Instead, a legal restriction imposed in only a subset of the trades can improve welfare. This is the mechanism I focus on. Second, the efficiency-improving role of the legal restriction sustains in the steady state in my model, while it lasts for only one period in Kocherlakota's model. Third, the efficiency role in my model can arise even when lump-sum taxes are available to implement the Friedman rule, while the role disappears in Kocherlakota's model if lump-sum taxes are introduced.

Let me relate this paper more broadly to the literature. Wallace (1983) argues explicitly that legal restrictions on bonds are inefficient in an overlapping generations model. Andolfatto (2006) extends Wallace's model to yield return dominance in an equilibrium, but he does not examine the efficiency role of legal restrictions. Aiyagari et al. (1996) examine the competition between money and bonds in a search model of money. They assume that money and bonds are indivisible and that individuals cannot always redeem matured bonds when they want to. These assumptions restrict the ability of bonds to compete against money and make the results difficult to interpret. I eliminate these assumptions using the construct of a large household in Shi (1997). A previous paper (Shi, 2005) also examines nominal bonds without these assumptions, but it does not focus on the efficiency role of illiquid bonds. Finally, Sun (2005) and Boel and Camera (2006) establish an efficiency role of illiquid bonds but, as Kocherlakota (2003), they assume that individuals can trade between bonds and money after observing the taste shocks.[1]

In section 2, I will describe the first version of the model where there is no government and where the legal restriction is assumed to be enforced. Section 3 will examine the efficiency role of the legal restriction: first with fixed money growth and then with optimal money growth. In section 4, I will describe the second version of the model that introduces government sellers to enforce the legal restriction. Section 5 will discuss several related issues. Section 6 will conclude and the Appendix will collect all the proofs.

2. A Search Economy with the Legal Restriction

2.1. Households, Matches and Assets

Consider a discrete-time economy with many types of households, all having the same discount factor $\beta \in (0,1)$. The number of households of each type is large and normalized to one. Households of the same type desire for a particular good, called the households' *consumption good*, which they cannot produce. Instead, they can produce a good that is desired by some other types of households. All goods are perishable between periods.

A household consists of a large number of members (normalized to one) who share consumption each period and regard the household's utility as the common objective. This assumption maintains analytical tractability. As I will describe soon, the trading in the goods market involves random matching which can generate a distribution of asset holdings across the individuals. These matching shocks are smoothed within each large household, and so the distribution of asset holdings across households is degenerated. As a result, I can select an arbitrary household as the representative household.[2]

[1] A related paper is Berentsen, et al. (2005), who allow individuals to borrow and lend after observing the matching shocks in the goods market. Such post-shock borrowing and lending performs a similar role as the trading between bonds and money in Kocherlakota's model.

[2] The assumption of large households is a modelling device extended from Lucas (1990), which is meant to capture an individual agent's allocation of time over different activities (see Shi, 1997). For an alternative way to make the distribution of asset holdings degenerate, see Lagos and Wright (2005).

The members of a household are divided into two groups: a fraction $n \in (0,1)$ of the members participate in the goods market and a fraction $(1-n)$ of the members are leisure seekers. This division is endogenous. Market participants are further divided into sellers and buyers. The measure of sellers is $\sigma \in (0,n)$, and the measure of buyers is $(n-\sigma)$. A seller can produce and sell goods, while a buyer is given assets to buy consumption goods for the household. To simplify the analysis, assume that σ is constant so that choosing n is equivalent to choosing the measure of buyers.[3]

The goods of each type have two colors, "red" and "green", which are indexed by $i \in \{R, G\}$. The cost of producing red and green goods is the same, specified by a disutility function, $\psi(.)$. However, the two colors generate different marginal utilities. The utility of consuming a consumption good of color i is $\theta^i u(c^i)$, where $\theta^G = 1$ and $\theta^R = \theta$ (> 0). The function ψ satisfies: $\psi(0) = 0$, $\psi'(0) = 0$, $\psi'(q) > 0$ and $\psi''(q) \geq 0$ for all $q > 0$. The function u satisfies: $u' > 0$, $u'' < 0$, $u'(0) = \infty$ and $u'(\infty) = 0$.[4] In addition, the utility of leisure in the household is $h(1-n)$, where $h(0) = 0$, $h'(0) = \infty$, $h' > 0$ and $h'' < 0$.

Let me describe the goods market first. In the goods market, buyers and sellers are randomly matched in pairs, and no match has a double coincidence of wants. A *trade match* is a match in which the seller produces the consumption good of the buyer's household. The total number of trade matches per household in a period is αN, where $\alpha > 0$ is a constant and N is the measure of market participants per household. A buyer encounters a trade match with probability $\alpha N/(N-\sigma)$, and a seller with probability $\alpha N/\sigma$. Assume that α is sufficiently small so that these expressions are bounded in $[0,1]$.

Once a buyer and a seller are matched, the seller receives a shock that determines whether he can produce the red good or the green good, with probability 1/2 for each realization. Let me call this shock a *matching shock*, because it occurs within each match.[5]

[3]We will assume that the buyer in each match makes a take-it-or-leave-it offer. This simplifying assumption is possible because the measure of sellers is fixed. If the household chooses the measures of both sellers and buyers, then the choice is non-trivial only if the seller in a trade obtains a positive surplus. Although this alternative modelling is feasible, as shown in Shi (2001), the algebra is more complicated.

[4]The analytical results hold for a more general specification $u(c^i, \theta^i)$, where the derivative of u with respect to c is increasing in θ.

[5]In the model described in this section, the shock can also be interpreted as a taste shock. However, labelling it a "matching shock" will be more accurate for the model in section 4.

A trade match in which red goods are produced is called a red trade, and a trade match in which green goods are produced is called a green trade. The matching shocks are identically and independently distributed across matches and over time. However, the number of trade matches of each color is deterministic at the household level, because each household consists of a large number of market participants.

In each trade, the buyer makes a take-it-or-leave-it offer. This assumption simplifies the determination of the trading quantities. For alternative assumptions that give both the buyer and the seller positive shares of the match surplus, see Shi (2001).

As is common in monetary models, the trading history of each household is private information, which prevents the use of credits in the trade. As a result, every trade entails a medium of exchange. There are two assets which can potentially perform this role, fiat money and nominal bonds. Both are issued by the government and can be stored without cost. The two assets are intrinsically worthless; i.e., they do not yield direct utility or facilitate production. Bonds are default-free, one-period bonds. At the maturity, each bond can be redeemed for one unit of money. Without loss of generality, I assume that all bonds are redeemed for money immediately at the maturity.[6]

The only difference between money and bonds is created by a partial legal restriction. While money can be used in both red and green trades, the legal restriction forbids the use of bonds as a means of payments in red trades. The enforcement of the legal restriction is an important issue, which I will examine explicitly in section 4. However, it is useful to analyze first a model where the enforcement is exogenously assumed.

In addition to the goods market, there is an asset market, where the government sells new bonds for money at an equilibrium price and redeems matured bonds. Let zM be the nominal amount of new bonds sold in each period, where $z \in (0, \infty)$ is a constant and M is the average stock of money per household. The government burns the receipts from selling new bonds and prints money to pay for the redemption of matured bonds. To

[6] In principle, a household can choose not to redeem the bonds at maturity and, instead, use them as a medium of exchange. However, such a choice is not optimal even if there is a slight chance that the bonds will be rejected in trade as a result of the legal restriction (see Shi, 2005).

focus on a stationary equilibrium, I assume that the government uses lump-sum monetary transfers to sterilize the effect of open market operations on the money supply. That is, the transfers keep money holdings per household growing at a constant (gross) rate $\gamma \geq \beta$. When $\gamma < 1$, the transfers are negative and, hence, are taxes.

2.2. Timing of Events and Capital Market Imperfections

To describe the timing of events, pick an arbitrary period t, suppress the time index t, and shorten the subscript $t \pm j$ as $\pm j$. Pick an arbitrary household as the representative household. Lower-case letters denote the decisions of this household and capital-case letters denote other households' decisions or aggregate variables. Normalize all nominal quantities and prices of goods by the aggregate money holdings per household, M.

t asset market	(m, b) measured		matching shocks		$t+1$
$\vert\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\vert\rightarrow$
redemption; transfers, T; new bonds, b	bonds mkt closed until next period	choices: n (q^i, x^i)	trades in goods:	consume	

Figure 1. Timing of events in a period

Figure 1 depicts the timing of events in a period. At the beginning of the period, the asset market opens. The household redeems matured bonds, receives lump-sum monetary transfers, T, and chooses to purchase new bonds. The (normalized) amount of new bonds sold by the government is z, which is exogenous. After the trade, the household's holdings consist of money, m, and bonds, b. Then, the asset market is closed and will remain closed for the rest of the period.

Next, the household chooses n, the fraction of members who will participate in the goods market. To the buyers, the household gives the assets and instructions on the quantities of trade. At this time, the matching shocks have not been realized yet, and so the household allocates the assets evenly among the buyers. Each buyer gets $m/(n - \sigma)$ units of money and $b/(n - \sigma)$ units of bonds. Moreover, the household gives the instructions to its buyers on the offer to make. Contingent on the realization of the matching shock, $i \in \{R, G\}$, the offer consists of the amount of goods to be purchased, q^i, and the amount of assets to be spent, x^i. Because of the legal restriction, x^R must be the amount of money alone.

Then, the traders of the household go to the goods market. Buyers and sellers are matched in pairs. In each match, the matching shock is realized to determine the color of the goods in the match. The buyers make the offers instructed earlier by the household. After the trade, the members bring the receipts of assets and goods back to the household. All members in the household share the same consumption, and the period ends.

Note that all the households are symmetric and face the same distribution of shocks. At the household level, there is no uncertainty about the amount and the composition of consumption. Thus, borrowing and lending between households is irrelevant in this model. A household would like to do is to redistribute assets *between matches* that have received different shocks, but cannot do so under the assumption of decentralized exchange. As capital market imperfections, this inability to trade assets between matches reflects the reality that the asset market is closed sometimes when individuals need the liquidity, albeit for a short time. As explained in the introduction, introducing these imperfections allows me to uncover a new channel through which illiquid bonds can improve efficiency, as opposed to the one in Kocherlakota (2003).[7]

2.3. Quantities of Trade in the Matches

Let m be the household's holdings of money and b the holdings of bonds immediately after the trading in the asset market. Both holdings are normalized by the aggregate stock of money holdings per household. Let $v(m,b) : R_+ \times R_+ \to R$ be the household's value function. Let ω^j be the shadow value of next period's asset j $(= m, b)$. That is,

$$\omega^m \equiv \frac{\beta}{\gamma} \frac{\partial}{\partial m_{+1}} v(m_{+1}, b_{+1}), \qquad \omega^b \equiv \frac{\beta}{\gamma} \frac{\partial}{\partial b_{+1}} v(m_{+1}, b_{+1}). \tag{2.1}$$

The marginal value of an asset in the future is discounted by the money growth rate γ, as well as β, because m_{+1} is normalized by next period's aggregate money stock. Other

[7] Because all members in a household enjoy the same consumption and utility, there is already insurance among the members. Insurance contracts between households are irrelevant here because all households have the same consumption and output. For dynamic contracts in a monetary model with private information, see Temzelides and Williamson (2001).

households' values of the assets are denoted similarly, with capital Ω.

In a trade where the buyer's matching shock is $i \in \{R, G\}$, the buyer makes a take-it-or-leave-it offer, (q^i, x^i). The offer must satisfy the following two constraints. The first is the seller's participation constraint: the offer must induce the seller to trade. Because the seller's surplus is equal to the value of the assets received in the trade, $\Omega^m x^i$, minus the cost of production, $\psi(q^i)$, the seller's participation constraint is:

$$x^i = \psi(q^i)/\Omega^m, \quad i = R, G. \tag{2.2}$$

The second constraint is the asset constraint, i.e., that the amount of assets offered cannot exceed the amount that the buyer can use. Because of the legal restriction, this constraint is different in a red trade and a green trade, as specified below:

$$x^R \leq \frac{m}{n - \sigma}, \tag{2.3}$$

$$x^G \leq \frac{m + b}{n - \sigma}. \tag{2.4}$$

It is unnecessary to specify how an offer in a green trade consists of money and bonds because the two assets have the same continuation value. Upon exiting from the trade, the only thing the household can do with the assets is to bring them to the next period, at which time the bonds will be redeemed for money at par. More precisely, the two assets have the same marginal value ω^m to the buyer and Ω^m to the seller.

If either (2.3) or (2.4) binds, money generates *liquidity services* in the goods market. These liquidity services are the *non-pecuniary return* to money in the goods market. In contrast, bonds yield liquidity services only if (2.4) binds. Bonds are *perfect substitutes* for money if they have the same value as money, i.e., if $\omega^b = \omega^m$.

2.4. A Household's Decision Problem

The household's choices in each period are the measure of market participants, n, the quantities of trade, (q^i, x^i), consumption, c^i, and future asset holdings, (m_{+1}, b_{+1}). The household takes other households' decisions (capital-case letters) and asset prices as given. The nominal price of bonds is denoted S, and the nominal interest rate is $r = 1/S - 1$.

The household's choices solve the following problem:

$$(PH) \quad v(m,b) = \max \left\{ \sum_{i=R,G} \left[\theta^i u\left(c^i\right) - \frac{\alpha N}{2} \psi\left(Q^i\right) \right] + h\left(1-n\right) + \beta v\left(m_{+1}, b_{+1}\right) \right\} \quad (2.5)$$

where

$$c^i = \frac{\alpha N(n-\sigma)}{2\left(N-\sigma\right)} q^i, \quad i \in \{R, G\},$$

and the constraints are as follows:

(i) the constraints in the goods market: (2.2), (2.3), and (2.4);

(ii) the law of motion for asset holdings:

$$m_{+1} + S_{+1} b_{+1} = \frac{1}{\gamma} \left[m + b + \frac{\alpha N}{2} \left(X^R + X^G \right) - \frac{\alpha N(n-\sigma)}{2(N-\sigma)} \left(x^R + x^G \right) \right] + T_{+1}. \quad (2.6)$$

In each period, consumption of color i goods is equal to the quantity of color i goods obtained in each color i trade, q^i, multiplied by the number of color i trades that the household's buyers experience. The latter number is equal to the probability with which each buyer obtains color i goods, which is $\alpha N / \left[2(N-\sigma)\right]$, multiplied by the number of buyers in the household, $(n-\sigma)$. Similarly, the number of color i trades experienced by the household's sellers is $\alpha N/2$ and total disutility of producing color i goods is $\psi(Q^i)\alpha N/2$, where Q^i is the quantity proposed by a buyer of other households.

To explain the law of motion for asset holdings, (2.6), start at the time in a period immediately after trading in the asset market (see Figure 1). The household's asset holdings at this time are (m, b). In the goods market, the total amount of assets obtained by selling goods is $\frac{\alpha N}{2}\left(X^R + X^G\right)$, and the total amount of assets spent by the buyers is $\frac{\alpha N(n-\sigma)}{2(N-\sigma)}\left(x^R + x^G\right)$. Thus, the amount inside the brackets [.] on the right-hand side of (2.6) is the amount of money that the household will have after redeeming the bonds in the *next period*. This amount is divided by γ to normalize it by next period's aggregate money stock per household. Adding the transfers in the next period to this amount, the household has the amount of money that can be used to acquire assets in the next period. The left-hand side of (2.6) gives such acquisitions.

2.5. Optimal Choices

Let λ^R be the shadow price of (2.3), and λ^G of (2.4). To simplify the formulas, multiply λ^i by the expected number of color i trades that the household's buyers experience, $\frac{\alpha N(n-\sigma)}{2(N-\sigma)}$, before incorporating the constraint into the maximization problem. The household's optimal decisions are characterized by the following conditions.

(i) For q^i:

$$\theta^i u'(c^i) = (\omega^m + \lambda^i)\frac{\psi'(q^i)}{\Omega^m}, \quad i = R, G. \tag{2.7}$$

(ii) For b_{+1}:

$$S_{+1} = \omega^b / \omega^m. \tag{2.8}$$

(iii) For (m, b) (envelope conditions):

$$\frac{\gamma}{\beta}\omega^m_{-1} = \omega^m + \frac{\alpha N}{2(N-\sigma)}\left(\lambda^G + \lambda^R\right), \tag{2.9}$$

$$\frac{\gamma}{\beta}\omega^b_{-1} = \omega^m + \frac{\alpha N}{2(N-\sigma)}\lambda^G. \tag{2.10}$$

(iv) For n:

$$h' = \frac{\alpha N}{2(N-\sigma)} \sum_{i=R,G} \left\{\theta^i u'\left(c^i\right)\left[q^i - \frac{\psi\left(q^i\right)}{\psi'\left(q^i\right)}\right]\right\}. \tag{2.11}$$

The condition (2.7) requires that a buyer's net gain from asking for an additional unit of good be zero. By getting an additional unit of good in a color i trade, the household's utility increases by $\theta^i u'(c^i)$. The cost is the additional amount $\psi'(q^i)/\Omega^m$ of assets that is needed to induce the seller to trade (see (2.2)). By giving one additional unit of asset, the buyer foregoes the discounted future value of the asset, ω^m, and causes the asset constraint in the trade to be more binding. Thus, $(\omega^m + \lambda^i)$ is the shadow cost of each additional unit of asset to the buyer's household, and the right-hand side of (2.7) is the cost of getting an additional unit of color i goods.

The condition (2.8) states the fact that the nominal price of bonds is equal to the relative value of bonds to money before the goods market opens. Thus, bonds are discounted only if they are not perfect substitutes for money in the goods market.

The envelope conditions require the current value of each asset to be equal to the future value of the asset plus the expected liquidity services generated by the asset in the goods market. Take money for example. The current value of money is given by the left-hand side of (2.9), where ω_{-1}^m is multiplied by γ/β because ω_{-1}^m is defined as the current value of money discounted to one period earlier. The right-hand side of (2.9) consists of the (discounted) future value of money, ω^m, and expected liquidity services generated by money in the two types of trades in the goods market.

In contrast to money, bonds can only generate liquidity services in green trades. Thus, $\omega^b < \omega^m$ if and only if $\lambda^R = 0$. That is, bonds are discounted if and only if the legal restriction binds. If $\lambda^R = 0$, then bonds are perfect substitutes for money.

Finally, (2.11) requires that the marginal disutility of allocating a member to the goods market (as a buyer) is equal to the expected gain. In the goods market, a buyer encounters a color i trade with probability $\alpha N/[2(N-\sigma)]$. The net gain from a color i trade to the buyer's household is $[\theta^i u'(c^i)q^i - (\omega + \lambda^i)x^i]$. After substituting x^i from (2.2) and λ^i from (2.7), the net gain becomes the expression inside the summation in (2.11).

2.6. Stationary Equilibrium

A symmetric equilibrium consists of a sequence of the representative household's choices, $(n, q, x, c, m_{+1}, b_{+1})$, the value function v, the shadow values of assets (ω^m, ω^b), and other households' choices such that the following requirements are met. (i) Optimality: given other households' choices, the household's choices solve (PH) and the value function satisfies (2.5); (ii) symmetry: the choices and shadow prices are the same across the households; (iii) clearing of the bonds market: $b = z$, with $0 < z < \infty$; (iv) positive and finite values of assets: $0 < \omega_{-1}^m m < \infty$ and $0 < \omega_{-1}^b b < \infty$; (v) stationarity: all real variables and the values $(\omega_{-1}^m m, \omega_{-1}^b b)$ are constant.

The total value of each asset is restricted to be positive and finite, in order to examine the coexistence of money and bonds.[8] This implies that ω^m falls over time at the money

[8]The value of each asset must be bounded in order to ensure that the household's optimal decisions are indeed characterized by the first-order conditions.

growth rate. Note that (iii) requires the choice of b to be interior, while stationarity implies $\omega^m_{-1} = \omega^m$ and $\omega^b_{-1} = \omega^b$. Symmetry implies $m = M = 1$. In the following analysis, I will equate the capital-case variables to the corresponding lower-case variables.

Let me establish existence of the equilibrium with $\gamma > \beta$. The real allocation under the Friedman rule ($\gamma = \beta$) can be obtained by taking the limit $\gamma \downarrow \beta$.

With $\gamma > \beta$, money must generate liquidity services in some trades: If both $\lambda^R = 0$ and $\lambda^G = 0$, a stationary equilibrium would exist only if $\gamma = \beta$. Thus, there are three cases of the equilibrium, depending on whether one or two of the asset constraints bind. To characterize the cases, define $\mu(n)$ and $f(k, n)$ as follows:

$$\mu(n) \equiv \frac{n - \sigma}{\alpha n}\left(\frac{\gamma}{\beta} - 1\right), \tag{2.12}$$

$$\frac{u'(\frac{\alpha n}{2} f(k,n))}{\psi'(f(k,n))} = k, \text{ for } k > 0. \tag{2.13}$$

The function $f(k/\theta^i, n)$ specifies the quantity of goods traded in a color i match that delivers k as the ratio of the marginal utility to the marginal cost. $f(k, n)$ decreases in (k, n) for all $k \in (0, \infty)$. For all $n \in (\sigma, 1)$, $\gamma > \beta$ implies $\mu > 0$.

Consider first the case where $\lambda^R = 0 < \lambda^G$. Refer to this case as Case PS (for perfect substitutability) where q^i_1 is the quantity of goods exchanged in a color i trade. Because the legal restriction does not bind in this case, bonds are perfect substitutes for money in the goods market. Precisely, (2.9) and (2.10) imply $\omega^b = \omega^m$, and (2.8) yields $S = 1$. To obtain q^G_1, substitute λ^G from (2.7) and $\lambda^R = 0$ into (2.9). This yields an equation for q^G_1. Setting $\lambda^R = 0$ in (2.7), I obtain an equation for q^R_1. Using the function f defined in (2.13), these quantities are as follows:

$$q^G_1(n) \equiv f\left(1 + 2\mu(n), n\right), \quad q^R_1(n) \equiv f\left(\frac{1}{\theta}, n\right). \tag{2.14}$$

Now consider the case where $\lambda^R > 0 = \lambda^G$. Refer to this case as Case BS (for bad substitutability) where q^i_3 is the quantity of goods exchanged in a color i trade. Because the legal restriction binds in this case, $\omega^b < \omega^m$. (2.10) implies $\omega^b_{-1} = \omega^m \beta/\gamma$. Because

$\omega^b = \omega^b_{-1}$, (2.8) yields $S = \beta/\gamma$. As in the above approach, I obtain:

$$q_3^G(n) \equiv f(1, n), \quad q_3^R(n) \equiv f\left(\frac{1 + 2\mu(n)}{\theta}, n\right). \tag{2.15}$$

Finally, consider the case where $\lambda^R > 0$ and $\lambda^G > 0$. Refer to this case as Case IS (for imperfect substitutability) where q_2^i is the quantity of goods in a color i trade. This case lies between Case PS and Case BS. As in Case BS, bonds are not perfect substitutes for money in the goods market, because $\lambda^R > 0$. However, since bonds yield liquidity services in green trades, they are not discounted by as much as in Case BS. Substituting $\omega^b_{-1} = S\omega^m_{-1}$ into (2.10), I obtain an equation for q_2^G, given S. Subtracting (2.10) from (2.9), I obtain an equation for q_2^R. Define

$$k^G(S, n) = 1 + \frac{2(n - \sigma)}{\alpha n}\left(\frac{\gamma}{\beta}S - 1\right), \quad k^R(S, n) = \frac{1}{\theta}\left[1 + \frac{2(n - \sigma)\gamma}{\alpha n \beta}(1 - S)\right]. \tag{2.16}$$

Express the quantities of goods traded as

$$q_2^i = Q_2^i(S, n) \equiv f\left(k^i(S, n), n\right), \text{ for } i = G, R. \tag{2.17}$$

Because the two asset constraints bind in this case, Q_2^G and Q_2^R satisfy:

$$\frac{\psi(Q_2^G(S, n))}{\psi(Q_2^R(S, n))} - (1 + z) = 0. \tag{2.18}$$

This equation determines $S = S(n)$, for any given n. Then, $q_2^i(n) = Q_2^i(S(n), n)$.

In each case, n solves (2.11). Let the equilibrium solution for n be n_1 in Case PS, n_2 in Case IS, and n_3 is BS. The following proposition describes existence and uniqueness of the equilibrium (see Appendix A for a proof):

Proposition 2.1. *Define $\gamma_0 = \beta\left[1 + \frac{\alpha}{2}(1 + \theta)\right]$. Assume that $\gamma > \beta$ and that z is sufficiently close to zero. If $\gamma < \gamma_0$, then a unique equilibrium exists and is characterized as in*

Table 1, where θ_1 and θ_3 are specified in Appendix A.

Table 1. Three cases of the equilibrium

	Case PS	*Case IS*	*Case BS*
existence	$0 < \theta \leq \theta_1 \ (< 1)$	$\theta_1 < \theta < \theta_3$	$\theta \geq \theta_3$
asset constraints	$\lambda^R = 0 < \lambda^G$	$\lambda^R > 0,\ \lambda^G > 0$	$\lambda^R > 0 = \lambda^G$
bond price	$S = 1$	$S \in \left(\frac{\beta}{\gamma}, 1\right)$	$S = \frac{\beta}{\gamma}$
# of traders	$n_1 \in (\sigma, 1)$	$n_2 \in (n_1, n_3)$	$n_3 \in (\sigma, 1)$
red goods	$q_1^R(n_1)$	$q_2^R(n_2) \in \left(q_1^R(n_1), q_3^R(n_3)\right)$	$q_3^R(n_3)$
green goods	$q_1^G(n_1)$	$q_2^G(n_2)$	$q_3^G(n_3)$

The above proposition states intuitive properties of the equilibrium. When the tastes for red goods are very low, in the sense that $\theta < \theta_1$, a buyer in a red trade does not spend the entire amount of his money. As a result, the legal restriction in the goods market does not bind, and bonds are perfect substitutes for money. On the other hand, when the tastes for red goods are very high, in the sense that $\theta > \theta_3$, a buyer in a red trade is constrained by the amount of his money, but a buyer in a green trade is not constrained. Bonds are bad substitutes for money in this case. When the tastes for red goods are intermediate, in the sense that $\theta_1 < \theta < \theta_3$, the asset constraints in both a red and a green trade bind. Bonds are not perfect substitutes for money, but its substitutability for money is not as bad as in Case BS. The price of bonds differs in the three cases, which reflects the difference in the substitutability of bonds for money. The Fisher equation holds only in Case BS, i.e., only if bonds do not generate any liquidity service in the goods market.

To conclude this section, let me make two remarks on the above proposition. First, the two conditions in the above proposition, that $\gamma < \gamma_0$ and that z is small, are sufficient for existence. They are imposed here to ensure that the solution for n is unique in each case. If n were exogenous, then neither condition would be needed for existence and uniqueness of the equilibrium. Second, although there is a clear ranking across the three cases on the quantity of goods traded in a red match, it is difficult to obtain a ranking on the quantity of goods traded in a green match.

3. Efficiency-Improving Role of the Legal Restriction

This section provides a condition under which the legal restriction improves the society's welfare. This is done in two steps. First, for any fixed $\gamma \in (\beta, \gamma_0)$, where γ_0 is specified in Proposition 2.1, I show that the legal restriction can improve welfare. Second, I find a condition under which a deviation slightly above the Friedman rule is optimal. In this case, the optimal joint policy requires money growth that is higher than the Friedman rule and a legal restriction that distinguishes bonds from money in government liabilities. All proofs for this section appear in Appendix B.

3.1. Welfare Measure and the Way to Compare Economies

Social welfare is measured by the following steady-state utility per period:

$$(1-\beta)v = \sum_{i=R,G} \left[\theta^i u\left(c^i\right) - \frac{\alpha N}{2}\psi\left(Q^i\right)\right] + h\left(1-n\right). \tag{3.1}$$

This measure is standard, because all households are the same and there is no intrinsic dynamic adjustment toward the steady state. When the legal restriction increases this welfare measure, I say that the legal restriction improves efficiency.

To compare welfare, let me first examine an economy without the legal restriction. In such an economy, bonds are perfect substitutes for money, and so adding bonds increases the stock of assets in all trades uniformly. If bonds are eliminated in such an economy, the only change is a fall in the nominal price of goods. That is, the real allocation in such an economy is the same as the allocation with $z = 0$, provided that the money growth rate is fixed. With $z = 0$, however, whether the legal restriction exists is irrelevant to the allocation. For this reason, I will refer to the case $z = 0$ as an economy without the legal restriction and to the effects of an increase in z as the effects of the legal restriction.

Taking the limit $z \to 0$ in Proposition 2.1, I obtain the following allocation of an economy without the legal restriction:

Case A: $\theta \leq (1+2\mu)^{-1}$. In this case $q^G = q_1^G$ and $q^R = q_1^R$.

Case B: $(1+2\mu)^{-1} < \theta < 1+2\mu$. In this case, $q^G = q^R = q_2 \equiv f\left(\frac{2(1+\mu)}{1+\theta}, n\right)$.

Case C: $\theta \geq 1 + 2\mu$. In this case, $q^G = q_3^G$ and $q^R = q_3^R$.

Note that the real allocation is the same in Case A as in Case PS, and the same in Case C as in Case BS. Thus, the legal restriction does not affect the real allocation when the tastes for the two types of goods are far from symmetric. However, the restriction does affect the allocation when θ has intermediate values, as the allocation in Case IS is different from that in Case B. Moreover, the legal restriction reduces both the lower bound, θ_1, and the upper bound, θ_3, of the region in which Case IS occurs.

3.2. Effects of the Legal Restriction with Fixed Money Growth

Let me first isolate the effects of the legal restriction by fixing money growth at $\gamma \in (\beta, \gamma_0)$. A legal restriction may affect the equilibrium allocation and welfare on two margins. One is the extensive margin, n, which determines the number of trades. The other is the intensive margin which works through the quantities of goods traded in matches, q^R and q^G. The following lemma documents these effects:

Lemma 3.1. *For any fixed $\gamma \in (\beta, \gamma_0)$, a marginal increase in z from $z = 0$ increases q_2^G, reduces q_2^R, and has no effect on n.*

The legal restriction increases the quantity of goods traded in a green match and reduces the quantity of goods traded in a red match. These intensive effects are intuitive: the presence of bonds increases the total amount of assets and hence depresses the real value of both money and bonds. However, because the legal restriction forbids the use of bonds in red trades, not all bonds are used in the exchange. Thus, the value of assets does not fall one for one with the amount of bonds; instead, it falls by less than the increase in the amount of bonds. As a result, the real value of assets in a green trade increases, which increases the quantity of goods traded in a green trade. At the same time, the fall in the value of money reduces the quantity of goods traded in a red trade. Therefore, the legal restriction shifts consumption from red goods to green goods by shifting the purchasing power from red trades to green trades.

However, prices do adjust to the increased amount of assets in the goods market. Express prices of goods in terms of utility, i.e., by multiplying prices by the value of money, ω^m. Then, with an increase in the amount of bonds, the price of green goods increases and the price of red goods falls. However, these responses of prices do not fully offset the shift of the purchasing power between the two types of trades.

On the extensive margin of trade, n, the legal restriction has no first-order effect, provided that money growth is held constant. One way to explain this result is to note that the optimal choice of n is determined by the expected marginal gain to a buyer. Because buyers are the individuals who carry money into trades, the expected marginal gain to a buyer is the non-pecuniary return to money, which must be equal to the opportunity cost of holding money. When money growth is fixed, the opportunity cost of holding money is unchanged, and so the optimal choice of n does not change.

Note that the legal restriction reduces the marginal gain to a buyer in a green trade as q_2^G increases, and increases the marginal gain in a red trade as q_2^R decreases. The constancy of n means that these changes in the marginal gains exactly cancel each other out; that is, the expected marginal utility of a household does not change with the legal restriction. However, the expected *level* of utility does change with the legal restriction. The following proposition states this welfare effect:

Proposition 3.2. *Fix $\gamma \in (\beta, \gamma_0)$ and assume that z is sufficiently small. The legal restriction improves efficiency if and only if $(1 + 2\mu)^{-1} < \theta < 1$. This region of θ is non-empty, provided $\gamma > \beta$.*

It is easy to explain this welfare effect. When $\theta < 1$, the household has a stronger desire for green goods than red goods. By shifting the purchasing power between the two types of trades, the legal restriction shifts consumption from the good with a lower marginal utility to the good with a higher marginal utility. Thus, expected utility increases. Put differently, the legal restriction allows bonds to serve as partial insurance against the matching shocks in this case.[9] If $\theta > 1$, on the other hand, the legal restriction reduces expected utility.

[9]The condition $\theta > (1 + 2\mu)^{-1}$ in the proposition comes from the existence condition for Case B. It is needed because the legal restriction affects the allocation only when both Case IS and Case B exist.

For any $\gamma > \beta$, the insurance role of the legal restriction exists in the specified region of θ. However, the role disappears when $\gamma = \beta$. Under the Friedman rule, a household is indifferent in spending a marginal unit of money or holding it to the next period. In this case, the constraints (2.3) and (2.4) do not bind, and the quantity of goods traded in a match equates the marginal utility of consumption to the marginal cost of production. In this sense, money provides perfect insurance against the matching shocks, and so it renders the legal restriction useless as a device of indirect insurance.

As partial insurance against the matching shocks, the efficiency role of the legal restriction has a similarity to the role of illiquid bonds in Kocherlakota (2003).[10] However, the above analysis has illustrated two main differences between the efficiency result here and that in Kocherlakota (2003). First, the legal restriction can improve efficiency in the steady state here, but it improves efficiency only for one period in Kocherlakota's model. Second, the efficiency role is new here because it does not rely on the mechanism in Kocherlakota.

To explain the second difference, it is important to emphasize that Kocherlakota assumes that households are able to trade between bonds and money after the matching shocks and before they go to the goods market. This asset trade enables households with high shocks to bring more money to the goods market than households with low shocks, thus achieving the desired effect of reducing the gap in the marginal utility between different households. In the model here, Kocherlakota's mechanism amounts to allowing the buyers with different shocks to trade assets between matches, which would violate the assumption of decentralized exchange in the goods market.

Despite the absence of asset trades after the shocks are realized, the legal restriction can still improve efficiency in my model because it is imposed only in a fraction of trades. As explained before, the partiality shifts consumption between the two types of trades by preventing the real value of assets from falling one for one with the increase in the amount of bonds. If the legal restriction were imposed universally, then illiquid bonds would have no real effect. I will illustrate this point more precisely at the end of section 4.

[10] Also similar to Kocherlakota's model, the legal restriction reduces the price of bonds and, hence, increases the nominal interest rate.

3.3. The Efficiency Role of the Legal Restriction under Optimal Money Growth

The analysis so far has fixed the money growth rate. Arguably, the most interesting issue is the joint determination of optimal monetary policy and the optimal structure of government liabilities. In the literature on the efficiency of return dominance, it is common to assume that the government (or the social planner) is not able to collect lump-sum taxes, e.g., Bryant and Wallace (1984) and Kocherlakota (2003). Under this assumption, Proposition 3.2 has already contained the essential result. That is, the optimal structure of government liabilities should consist of money and illiquid bonds, with bonds dominating money in the rate of return. In fact, this result only needs a weaker assumption that the government cannot collect lump-sum taxes to the extent that implements the Friedman rule. Thus, one way to establish the efficiency role of the legal restriction under optimal money growth is to specify the environment in detail to rationalize this weak assumption.

I take up an alternative and more difficult task here: assuming that the government is able to implement the Friedman rule with lump-sum taxes, I show that optimal money growth can exceed the Friedman rule under a certain condition. This will be shown first for the case $z = 0$. Then, continuity implies that there exists a neighborhood of $z \geq 0$ in which a small deviation of money growth above the Friedman rule is optimal. Because Proposition 3.2 holds for all $\gamma \in (\beta, \gamma_0)$, then there exists a neighborhood of $\theta < 1$ such that the legal restriction improves efficiency under optimal money growth.

Again, focus on Case IS, which becomes Case B when $z = 0$. In Case B, $q_2^G = q_2^R = q_2$ and $(1+\theta)\, u'(c_2) = 2(1+\mu)\psi'(q_2)$. Differentiating (2.11) with respect to γ and evaluating the derivative at $z = 0$ and $\gamma = \beta$, it can be verified that $[dn/d\gamma]_{\gamma=\beta} < 0$ iff

$$\frac{q_0\psi'(q_0)}{\psi(q_0)} - 1 + k_0 f_1(k_0, n)\frac{q_0\psi''(q_0)}{\psi(q_0)} < 0, \tag{3.2}$$

where $k_0 \equiv 2/(1+\theta)$ and $q_0 \equiv f(k_0, n)$. Moreover,

$$\frac{1-\beta}{\alpha}\left[\frac{dv}{d\gamma}\right]_{\gamma=\beta} = -\frac{\sigma}{n-\sigma}\left[q_0\psi'\left(q_0\right) - \psi\left(q_0\right)\right]\left[\frac{dn}{d\gamma}\right]_{\gamma=\beta}.$$

Because $q\psi' > \psi$ for all $q > 0$, welfare increases with γ near $\gamma = \beta$ if and only if n decreases with γ. This result, together with the argument in the second paragraph of this subsection, leads to the following proposition:

Proposition 3.3. *A deviation slightly above the Friedman rule improves welfare if and only if it reduces the number of buyers in the goods market and, hence, if and only if (3.2) is satisfied. Therefore, under (3.2), there exists a neighborhood of $\theta < 1$ where optimal monetary policy is $\gamma > \beta$ and where the legal restriction improves efficiency.*

In contrast to the legal restriction, money growth affects the extensive margin of trade by affecting the gain from trade to a buyer. In turn, this effect can be decomposed into two effects. The first effect is negative: money growth reduces the value of a match by reducing the quantity of goods traded in a match. The second effect is positive: by reducing consumption, money growth increases the value of each unit of good that a buyer receives from a trade. These two effects work through the two terms in the summation of (2.11). The first effect dominates if and only if (3.2) is satisfied. Under this condition, the gain from trade and, hence, the number of buyers decreases with money growth.

Money growth also affects the intensive margin of trade, because it reduces the quantity of goods traded in a match. However, when γ is close to β and θ is close to 1, this intensive margin only has a second-order effect on welfare because the quantities of goods are close to the efficient ones that equate the marginal utility of consumption to the marginal cost of production. In this case, the extensive margin is the dominating margin of welfare.

An increase in money growth slightly above the Friedman rule improves welfare if and only if money growth reduces the number of buyers in the goods market. To explain why this is the case, compare buyers' contribution to matches with their bargaining power in trade. By assumption, a buyer in a trade takes the entire surplus of the match. However, buyers contribute to only a part of the formulation of matches. Because the matching function is αN, the share of buyers' contribution to matches is:

$$\frac{d\ln(\alpha N)}{d\ln(N-\sigma)} = \frac{N-\sigma}{N} < 1.$$

Buyers are over-compensated in trade for their contribution to matches, according to the efficiency condition specified by Hosios (1990) for a matching market. Thus, inflation increases efficiency by reducing the number of buyers.

To see whether (3.2) can be satisfied, consider the functional forms $u(c) = \frac{c^{1-\eta}-1}{1-\eta}$ and $\psi(q) = \psi_0 q^{\xi}$, where $\eta > 0$, $\xi > 1$ and $\psi_0 > 0$. Then, (3.2) is satisfied if and only if $\eta < 1$.

Let me conclude this section with a clarification about the robustness of the result in Proposition 3.3. The result that reducing the number of buyers in the goods market improves efficiency is specific to the assumption that the buyer in a match has all the bargaining power. If sellers have sufficiently high bargaining power, then the measure of buyers can be inefficiently low in equilibrium, in which case efficiency entails an increase in the number of buyers. Despite this variation, deviations above the Friedman rule can still improve efficiency by affecting the extensive margin of trade, as shown by Shi (1997) and Berentsen et al. (2007). Furthermore, this efficiency role remains even when direct (distortionary) taxes are introduced and set optimally (e.g., Ritter, 2007). With these qualifications, the above proposition reflects the general possibility that the legal restriction can improve efficiency even under optimal monetary policy.

4. Enforcement of the Legal Restriction

I now examine the enforcement of the legal restriction by introducing government sellers. The proofs for this section are all collected in Appendix C.

Let me introduce a measure $g > 0$ of government sellers per household, whose disutility function of producing goods is the same as that of private sellers. Government sellers enforce the legal restriction by refusing to accept bonds as payments for goods. For example, the government gives each of its sellers a machine that is programmed to accept only money as payments. To simplify the analysis, assume that government sellers produce only red goods while private sellers produce only green goods. In this case, the legal restriction is enforced only on red goods. The fraction of trades in which the legal restriction is enforced is equal to the fraction of sellers who are government agents. Denote this fraction as $p = g/(\sigma + g)$. Also, denote $p^R = p$ and $p^G = 1 - p$. Assume $0 < p < 1$.

Whether a buyer faces the legal restriction is determined by the matching outcome. In the presence of government sellers, the total measure of individuals in the goods market is $(N + g)$. Similar to the specification in previous sections, the total number of trade matches is $\alpha(N + g)$. For a (private) buyer, the probability of getting a trade match is $\alpha(N + g)/(N - \sigma)$. Conditional on a trade match, the match is a color i trade with probability p^i, where $i \in \{R, G\}$. For a seller, government or private, the probability of getting a trade match is $\alpha(N + g)/(\sigma + g)$. This match is a green trade if the seller is a private seller and a red trade if the seller is a government seller. Note that the total number of green trades encountered by all of the household's sellers is $p^G \alpha(N + g)$.

With the above modifications, a household's consumption of color i goods is:

$$c^i = p^i \frac{\alpha(N + g)}{N - \sigma}(n - \sigma)q^i, \quad i \in \{R, G\}.$$

The maximization problem of a private household becomes:

$$(PH') \quad v(m, b) = \max\left\{u(c^G) + \theta u\left(c^R\right) - p^G \alpha(N + g)\psi\left(Q^G\right) + h\left(1 - n\right) + \beta v\left(m_{+1}, b_{+1}\right)\right\}$$

subject to (2.2), (2.3), (2.4), and the following law of motion of assets:

$$m_{+1} + S_{+1}b_{+1} = \frac{1}{\gamma}\left[m + b + p^G \alpha\left(N + g\right)X^G - \frac{\alpha(N + g)}{N - \sigma}(n - \sigma)\left(p^R x^R + p^G x^G\right)\right] + T_{+1}.$$

Here, I have assumed that government sellers value money with the same marginal value Ω^m as private households do.

To examine efficiency, let me redefine the society's welfare to include the disutility of production incurred by government sellers, as well as the utility of private households.[11] Expected disutility incurred by a government seller in a period is $\psi(q^R)\alpha(n + g)/(\sigma + g)$. Let v^s be the average value for an individual in the economy, which is obtained by giving the same weight to all individuals. The weight given to all private households together is $1/(1 + g)$, and the weight given to all government sellers together is $g/(1 + g)$. Thus,

$$(1 - \beta)v^s = \frac{(1 - \beta)v}{1 + g} - \frac{g}{1 + g}\frac{\alpha(n + g)}{\sigma + g}\psi(q^R).$$

[11] The value of money received from trade by government sellers is not counted in the social welfare function because it is a transfer from the private sector to the government.

Substituting v and simplifying yields:

$$(1-\beta)(1+g)v^s = h + \sum_{i=R,G} \left[\theta^i u\left(c^i\right) - \alpha(n+g)p^i\psi(q^i)\right]. \quad (4.1)$$

The legal restriction improves efficiency if it increases v^s.

Focus on Case IS where the asset constraints (2.3) and (2.4) both bind. Similar to the previous model, refer to the economy with $z = 0$ as an economy without the legal restriction and to the effects of increasing z as those of the legal restriction.[12]

As in the previous model, an increase in z tends to increase the quantity of goods traded in a green match and reduce the quantity of goods traded in a red match. In contrast to the previous model, an increase in z may now affect the extensive margin of trade as well. This new effect arises from the feature that the household does not consume the same amount of green goods as red goods, even in the case where the quantity of goods traded is the same in a green match as in a red match. The reason is that, when $p \neq 1/2$, the number of green trades is different from the number of red trades. With this new feature, an increase in z may affect the expected marginal utility of consumption when it shifts consumption from red goods to green goods. As a result, the expected non-pecuniary return to holding money may change with z. This effect must be offset by a change in n, because the opportunity cost of holding money is fixed when money growth is fixed. In order to keep the analysis as close as possible to the one in the previous model, I eliminate the extensive effect of z by the following assumption:

Assumption 1. *Either $u'(a_1c)/u'(a_2c)$ is independent of c for all (a_1, a_2, c), or $p = 1/2$.*

The previous model satisfies the assumption because it resembles the case $p = 1/2$. The assumption is also satisfied if $u(c)$ has constant relative risk aversion.[13] Define $\delta =$

[12] It is tempting to use the parameter g as an alternative measure of the legal restriction and conduct comparative statics. Doing so is not a good idea because a change in g changes a household's expected utility even when the legal restriction is absent. Moreover, note that eliminating the government in this economy, by setting $g = 0$, is not optimal under the assumption $u'(0) = \infty$.

[13] Assumption 1 is not needed in an alternative model where green and red goods are both produced by the government and private households, with a half of the sellers (private or government) producing each color. In such an economy, the number of green trades is always equal to the number of red goods. We do not present this alterntive model because it is more difficult to analyze – there are more types of matches and, hence, more pairs of quantities in trade to be determined.

$u'(1-p)/u'(p)$. Redefine $f(k,n)$ and (k_0, q_0) as follows:

$$\frac{u'\left(\alpha(n+g)(1-p)f(k,n)\right)}{\psi'(f(k,n))} = k, \tag{4.2}$$

$$k_0 = (1-p+p\theta/\delta)^{-1}, \quad q_0 = f(k_0, n). \tag{4.3}$$

The equilibrium can be characterized by adapting the analysis in section 2.6. In Appendix C, I sketch the characterization of Case IS, find the corresponding limit $z \downarrow 0$ (i.e., Case B), and prove the following proposition:

Proposition 4.1. *Under Assumption 1, the legal restriction increases q_2^G, reduces q_2^R, but has no first-order effect on n. If (3.2) is satisfied with (f, k_0, q_0) redefined above, then there exists a neighborhood of $\theta < \delta$ where optimal money growth is $\gamma > \beta$ and where the legal restriction improves efficiency.*

The results in the above proposition are similar to those in Propositions 3.2 and 3.3. In particular, the same condition, (3.2), is needed for optimal money growth to be above the Friedman rule. The main modification of the results is the neighborhood of θ in which the legal restriction improves efficiency: it is now around $\theta < \delta$ rather than $\theta < 1$. As said earlier, the previous model resembles the case $p = 1/2$. If I set $p = 1/2$ in the current model, then $\delta = 1$ indeed.

The case $p = 1$ is also informative. In this case, all sellers are government sellers, all trades are red, and so the legal restriction is enforced in all trades. In this case, it is easy to show that increasing z from $z = 0$ has no effect on welfare (see Appendix C). Thus, a partially enforced legal restriction can be better than a universally enforced legal restriction or no legal restriction at all.[14]

5. Discussions

In this section I discuss a few related issues. One obvious question is whether the legal restriction is "essential" in the sense that the social planner can achieve better allocations

[14]After I wrote the first version of the current paper in 2002, I became aware of a paper by Rocheteau (2002), who uses a search model to examine the legal restrictions in the goods market. His model is different from mine. Also, his result on the welfare-improving role of the legal restriction is largely numerical and he does not address the issue of how the legal restriction is enforced.

with the legal restriction than without the restriction. The answer is likely affirmative, provided that the social planner cannot observe the types of matches that individuals experience. Although a formal support for this answer requires a setup of mechanism design, which is out of the scope this paper, I provide an informal argument as follows. The social planner needs to keep two reports of individuals' matching histories in order to allocate consumption and production optimally. One is the report on whether an individual is a buyer or a seller in a match. To record this type of trading histories, the social planner can alter an individual's money holdings. Another report is on whether an individual has a red or green trade. Because the marginal utility of consumption depends on the color of the goods, it is efficient for the social planner to describe different quantities of goods to be traded in matches with different colors. To record this type of histories, the planner needs another asset – by altering an individual's holdings of this second asset according to the individual's report, the planner can induce the individual to trade the corresponding quantity of goods. The legal restriction in my model implements this dependence of an individual's holdings of the second asset on the reported color of trade.

Another issue is the relationship between my efficiency result and the general principle that introducing some distortions into an economy with imperfect markets can improve efficiency. Because the economy in my analysis has incomplete markets and bargaining, the efficiency role of the legal restriction conforms with this general principle. However, not every distortion can improve efficiency in an imperfect economy while many distortions can reduce efficiency. In particular, introducing return dominance has reduced efficiency in most monetary models. For this reason, it is an important result that return dominance generated by the legal restriction can improve efficiency in my model.

The next issue is how my model is related to Bryant and Wallace (1984). In an overlapping-generations model, Bryant and Wallace assume that nominal bonds are issued in large denominations and that a legal restriction prohibits intermediaries from issuing small-denomination bills. They show that this legal restriction can increase expected utility when lump-sum taxes are not possible. The similarity between this efficiency result and the one in my model is that both rely on market segmentation. In Bryant and Wallace, the indivisibility of large denomination bonds makes an individual's consumption set non-convex which, in turn, makes it optimal to price discriminate individuals who hold bonds from those who do not hold bonds. In my model, the discrimination is achieved by the legal restriction and the separation between matches.

The two models have significant differences. First, the legal restriction is modelled differently – in my model, the legal restriction directly prevents the traders in a subset of trades from using bonds as payments. Second, my model does not have non-convexity. In fact, it would be awkward to introduce the type of non-convexity in Bryant and Wallace into my model, given that each household experiences a large number of trades.[15] Third, the efficiency role of the legal restriction can exist in my model even when lump-sum taxes are available, which is not true in Bryant and Wallace.

Finally, one may ask why return dominance exists in reality, despite the apparent lack of legal restrictions on issuing private money. Introducing private money with small denominations will eliminate return dominance in the Bryant-Wallace model, but not in my model. For example, suppose that private money is introduced into the economy as described in section 4. If the legal restriction requires government sellers not to accept private money, then private money must dominate fiat money in the rate of return, just as nominal bonds must. Put differently, private money will unlikely be essential beyond fiat money and nominal bonds. Thus, the particular legal restriction in my model can provide a normative answer to the question why private money is not widespread.

6. Conclusion

In this paper I examined whether a legal restriction that reduces the liquidity of nominal bonds can improve welfare for the society. To do so, I introduced nominal bonds and an asset market into a microfounded model of money. While the asset market is Walrasian, the goods market is decentralized, where the government imposes a legal restriction in a

[15] To see this difficulty, note that the assumption of consumption sharing within each household convexifies the model even if indivisible bonds are introduced. Also, note that the non-convexity in Bryant and Wallace is eliminated if individuals can participate in lotteries that allocate the large-denomination bonds.

fraction of the trades. Individuals face matching shocks, which affect the marginal utility of consumption and which they cannot insure, borrow or trade assets against. I show that the partial legal restriction can improve efficiency of the society. In contrast to some previous models (see the introduction), the efficiency role of the legal restriction persists in the steady state and it does not require households to be able to trade bonds for money after receiving the shocks. Moreover, even when lump-sum taxes are available, the legal restriction can still improve welfare under a condition that induces optimal money growth to be above the Friedman rule.

As explained in the introduction, the current model will be useful for analyzing monetary policy, because it provides an efficiency role for return dominance that has been relied upon in all monetary policy analyses. In particular, the model can be extended to incorporate limited participation, which captures the liquidity effect of open market operations (see Lucas, 1990). By providing a microfoundation for money and open market operations, such an extension will provide a justification for the liquidity effect on the basis of efficiency. It may also uncover new propagation mechanisms for monetary shocks. This task is left for a sequel.[16] Another potential use of the current model is to examine whether there is an efficiency gain to restricting the circulation of foreign currency in a country. Such a restriction is similar to the legal restriction examined in the current paper.

[16] Williamson (2005) constructs a different model of limited participation to prolong the real effects of monetary injection. However, he does not examine the essentiality of illiquid bonds.

Appendix

A. Proof of Proposition 2.1

I divide the proof into three parts.

Part 1. Given the solution for $n \in (\sigma, 1)$, I characterize each of the three cases. For Case PS to occur, the quantities (q_1^R, q_1^G) given by (2.14) must induce $\lambda^R = 0 < \lambda^G$. That is, the asset constraints in trade must hold as follows: $\frac{1}{n-\sigma} \geq \frac{\psi(q_1^R)}{\omega^m}$ and $\frac{1+z}{n-\sigma} = \frac{\psi(q_1^G)}{\omega^m}$, where I have substituted the market clearing conditions $m = 1$ and $b = z$. Combining the two constraints to eliminate ω^m, I obtain the following condition for Case PS:

$$q_1^R(n_1) \leq Q_1(n_1) \equiv \psi^{-1}\left(\frac{\psi(q_1^G(n_1))}{1+z}\right). \tag{A.1}$$

Substituting (q_1^R, q_1^G) from (2.14) and f from (2.13), I rewrite (A.1) as $\theta \leq \theta_1$ where

$$\theta_1 = \Theta_1(n_1) \text{ and } \Theta_1(n) \equiv \frac{\psi'(Q_1(n))}{u'(\frac{\alpha n}{2}Q_1(n))}. \tag{A.2}$$

For all $z > 0$ and $n \in (\sigma, 1)$, $Q_1(n) < q_1^G(n)$ and so $\Theta_1(n) < [1 + 2\mu(n)]^{-1} < 1$.

Similarly, examine Case BS, which requires $\lambda^R > 0 = \lambda^G$. That is, $\frac{1}{n-\sigma} = \frac{\psi(q_3^R)}{\omega^m}$ and $\frac{1+z}{n-\sigma} \geq \frac{\psi(q_3^G)}{\omega^m}$. Combining these two constraints yields:

$$q_3^R(n_3) \geq Q_3(n_3) \equiv \psi^{-1}\left(\frac{\psi(q_3^G(n_3))}{1+z}\right). \tag{A.3}$$

Substituting (q_3^R, q_3^G) from (2.15) and f from (2.13), I rewrite (A.3) as $\theta \geq \theta_3$ where

$$\theta_3 = \Theta_3(n_3) \text{ and } \Theta_3(n) \equiv \frac{[1 + 2\mu(n)]\,\psi'(Q_3(n))}{u'(\frac{\alpha n}{2}Q_3(n))}. \tag{A.4}$$

For all $z > 0$ and $n \in (\sigma, 1)$, $Q_3(n) < q_3^G(n)$ and so $\Theta_3(n) < 1 + 2\mu(n)$.

Now turn to Case IS. This case requires λ^R and λ^G to be positive. By (2.7), this requirement is equivalent to $\theta^i u'(c_2^i) > \psi'(q_2^i)$ for both $i = G$ and $i = R$. That is, $k^G > 1$ and $k^R > 1/\theta$. Using (2.17), I can express these requirements as $\beta/\gamma < S < 1$. Temporarily denote the left-hand side of (2.18) as $LHS(S, n)$. For any given $n \in (\sigma, 1)$, $k^G(S, n)$ increases in S and $k^R(S, n)$ decreases in S. Because $f(k, n)$ is decreasing in k, then $Q_2^G(S, n)$ decreases in S and $Q_2^R(S, n)$ increases in S. Thus, $LHS(S, n)$ decreases in S for any given $n \in (\sigma, 1)$. Moreover, because $Q_2^G(\beta/\gamma, n) = q_3^G(n)$ and $Q_2^R(\beta/\gamma, n) = q_3^R(n)$, then $LHS(\beta/\gamma, n) > 0$ iff $q_3^R(n) < Q_3(n)$, i.e., iff $\theta < \Theta_3(n)$. Similarly, because $Q_2^G(1, n) = q_1^G(n)$ and $Q_2^R(1, n) = q_1^R(n)$, then $LHS(1, n) < 0$ iff $q_1^R(n) > Q_1(n)$, i.e., iff $\theta > \Theta_1(n)$. As shown later, $n_2 \to n_3$ when $S \to \beta/\gamma$, and $n_2 \to n_1$ when $S \to 1$. Therefore, Case IS exists iff $\theta_1 = \Theta_1(n_1) < \theta < \Theta_3(n_3) = \theta_3$.

To show that $\theta_1 < \theta_3$ holds, note that, when $z = 0$, I have $\Theta_1(n) = [1 + 2\mu(n)]^{-1}$ and $\Theta_3(n) = 1 + 2\mu(n)$. Thus, for all $\gamma > \beta$ and all $n, n' \in (\sigma, 1)$, I have $\Theta_1(n) < \Theta_3(n')$. This implies $\theta_1 < \theta_3$ when z is sufficiently small.

Part 2. Determine n in each case. In Case PS, substitute $u'(c_1^G) = (1+2\mu)\psi'(q_1^G)$ and $\theta u'(c_1^R) = \psi'(q_1^R)$ to rewrite (2.11) as

$$\frac{2h'(1-n)}{\frac{\alpha n}{n-\sigma}[1+2\mu(n)]} = \left[q_1^G(n)\,\psi'(q_1^G(n)) - \psi(q_1^G(n))\right] + \frac{q_1^R(n)\,\psi'(q_1^R(n)) - \psi(q_1^R(n))}{1+2\mu(n)}. \quad \text{(A.5)}$$

Note that the expression $\frac{\alpha n}{n-\sigma}[1+2\mu(n)]$ is a decreasing function of n, and so the left-hand side of this equation increases in n. Because $\mu(n)$ decreases in n, and $q_1^G(n)$ and $q_1^R(n)$ both decrease in n, the right-hand side of the equation decreases in n. The solution for n to the equation is unique if it exists. Existence can be verified with the assumptions on (h, u, ψ). Denote the solution as $n_1 \in (\sigma, 1)$.

Similarly, in Case BS, I can rewrite (2.11) as

$$\frac{2h'(1-n)}{\frac{\alpha n}{n-\sigma}[1+2\mu(n)]} = \left[q_3^R(n)\,\psi'(q_3^R(n)) - \psi(q_3^R(n))\right] + \frac{q_3^G(n)\,\psi'(q_3^G(n)) - \psi(q_3^G(n))}{1+2\mu(n)}, \quad \text{(A.6)}$$

and show that that a unique solution exists. Denote the solution as $n_3 \in (\sigma, 1)$.

Denote the equilibrium value of n in Case IS as n_2, and the solution for S to (2.18) as $S(n)$. To determine n_2, denote:

$$k^{*i}(n) = k^i(S(n), n), \;\; q_2^i(n) = Q_2^i(S(n), n), \text{ where } i = G, R. \qquad \text{(A.7)}$$

Then, (2.11) in Case IS can be rewritten as follows:

$$h'(1-n) = \sum_{i=G,R} \frac{\alpha n k^{*i}(n)}{2(n-\sigma)} \theta^i \left[q_2^i(n)\,\psi'\left(q_2^i(n)\right) - \psi\left(q_2^i(n)\right)\right]. \qquad \text{(A.8)}$$

I establish first existence and uniqueness of the solution to this equation when $z = 0$. When $z = 0$, the solution $S(n)$ to (2.18) yields $q_2^G(n) = q_2^R(n)$ and hence $k^{*G}(n) = k^{*R}(n)$. Differentiating (2.18) with respect to n to obtain $S'(n)$, I can compute the following derivatives at $z = 0$ for both $i = G$ and $i = R$:

$$\frac{dk^{*i}(n)}{dn} = \frac{2(\gamma/\beta - 1)\sigma}{(1+\theta)\alpha n^2} > 0; \quad \frac{dq^i(n)}{dn} = f_1 \frac{dk^{*i}(n)}{dn} + f_2 < 0.$$

Using these results, I can compute further:

$$\frac{d}{dn}\left[\frac{\alpha nk^{*i}(n)}{2(n-\sigma)}\right]=\frac{\sigma}{(1+\theta)\,n\,(n-\sigma)}\left[(1-S(n))\frac{\gamma}{\beta}-\theta\left(\frac{\gamma}{\beta}S(n)-1\right)-\frac{\alpha n(1+\theta)}{2(n-\sigma)}\right].$$

Note that the expression inside [.] decreases in S. Since $S\geq\beta/\gamma$ and $n>n-\sigma$, I can show that the expression is less than $(\gamma-\gamma_0)/\beta$, where γ_0 is defined in Proposition 2.1. If $\gamma\in(\beta,\gamma_0)$, the above derivative is negative, in which case the right-hand side of (A.8) decreases in n. Because the left-hand side increases in n, the solution to (A.8) is unique if $\gamma\in(\beta,\gamma_0)$ and $z=0$. Existence of the solution follows from the assumptions on (h,u,ψ).

Since the solution is continuous in z in a neighborhood of $z=0$, then the solution to (A.8) exists and is unique if $\gamma\in(\beta,\gamma_0)$ and if z is sufficiently small.

Part 3. Compare the n'es and q's among the three cases. First, I show that $n_1<n_3$. For any fixed $n\in(\sigma,1)$, the following result holds:

$$q_1^R(n)=f\left(\frac{1}{\theta},n\right)\leq f\left(\frac{1}{\theta_1},n\right)<q_3^G(n)=f(1,n).$$

The first inequality comes from the facts that $\theta\leq\theta_1$ in Case PS and that $f(k,n)$ decreases in k. The second inequality comes from the fact that $\theta_1\leq 1/\left[1+2\mu(n_1)\right]<1$. Moreover, because $\Theta_3(n)=1+2\mu(n)$ when $z=0$, then $\theta_3>1$ when z is sufficiently small. For any fixed $n\in(\sigma,1)$ and for sufficiently small z, I have:

$$q_1^G(n)<f\left(\frac{1+2\mu(n)}{\theta_3},n\right)\leq q_3^R(n)=f\left(\frac{1+2\mu(n)}{\theta},n\right).$$

The first equality comes from $\theta_3>1$, and the second inequality from $\theta\geq\theta_3$ in Case BS. Thus, for sufficiently small z and for any fixed $n\in(\sigma,1)$, the right-hand side of (A.5) is strictly smaller than that of (A.6). As a result, $n_1<n_3$ when z is small.

Second, I show that $n_1<n_2<n_3$. With k^{*i} defined in (A.7), I have:

$$\frac{\alpha nk^{*G}(n)}{2(n-\sigma)}=\frac{\alpha n}{2(n-\sigma)}+\frac{\gamma}{\beta}S(n)-1,$$

$$\frac{\theta\alpha nk^{*R}(n)}{2(n-\sigma)}=\frac{\alpha n}{2(n-\sigma)}+\frac{\gamma}{\beta}\left[1-S(n)\right].$$

If I substitute these expressions and $q_2^i(n)=Q_2^i(S(n),n)$ into (A.8) and treat S as a separate variable, then the right-hand side of (A.8) becomes a decreasing function of S at $z=0$. When $S\to\beta/\gamma$, (A.8) becomes (A.6), and so $n_2\to n_3$. When $S\to 1$, (A.8) becomes (A.5), and so $n_2\to n_1$. This procedure also shows that $n_1<n_2<n_3$ when z is sufficiently small.

Third, I compare $q_1^R(n_1)$ with $q_3^R(n_3)$. Consider the case $z=0$. In this case, $\Theta_3(n) = 1+2\mu(n)$. For all $\theta \geq \theta_3$,

$$\frac{1+2\mu(n_3)}{\theta} \leq \frac{1+2\mu(n_3)}{\theta_3} = 1.$$

This implies $q_3^R(n_3) \geq q_3^G(n_3)$, which further implies:

$$q_3^R(n_3)\,\psi'(q_3^R(n_3)) - \psi(q_3^R(n_3)) \geq \frac{1+2\mu(n_3)}{2\left[1+\mu(n_3)\right]} RHS(\text{A.6})|_{n=n_3} = \frac{h(1-n_3)}{\frac{\alpha n_3}{n_3-\sigma}\left[1+\mu(n_3)\right]}.$$

The equality comes from (A.6). Similarly, when $z=0$, $q_1^R(n_1) \leq q_1^G(n_1)$ and so

$$q_1^R(n_1)\,\psi'(q_1^R(n_1)) - \psi(q_1^R(n_1)) \leq \frac{1+2\mu(n_1)}{2\left[1+\mu(n_1)\right]} RHS(\text{A.5})|_{n=n_1} = \frac{h(1-n_1)}{\frac{\alpha n_1}{n_1-\sigma}\left[1+\mu(n_1)\right]}.$$

Because the function $\frac{n}{n-\sigma}\left[1+\mu(n)\right]$ decreases in n, the last expression increases in n. Since $n_1 < n_3$ when $z=0$, then

$$q_3^R(n_3)\,\psi'(q_3^R(n_3)) - \psi(q_3^R(n_3)) \geq q_1^R(n_1)\,\psi'(q_1^R(n_1)) - \psi(q_1^R(n_1)).$$

This implies $q_3^R(n_3) > q_1^R(n_1)$ when $z=0$ and, hence, when z is sufficiently small.

A similar procedure, together with the facts that $S \in (\beta/\gamma, 1)$ and $n_2 \in (n_1, n_3)$, leads to the result that $q_1^R(n_1) < q_2^R(n_2) < q_3^R(n_3)$. **QED**

B. Proofs for Section 3

Let me start with the proof of Lemma 3.1. Denote the derivative of $f(k,n)$ to the jth argument as f_j, where f is defined in (2.13). Then, $f_1 < 0$ and $f_2 < 0$. All the derivatives below with respect to z are evaluated at $z=0$, and the notation for this evaluation is suppressed. Recall that, when $z=0$, $q_2^G = q_2^R = q_2 = f(k,n)$, where $k = 2(1+\mu)/(1+\theta)$. Differentiating (2.17) with respect to z and evaluating at $z=0$ yields:

$$\frac{dq_2^i}{dz} = f_1\frac{dk^i}{dz} + f_2\frac{dn}{dz}, \text{ for } i = G, R, \tag{B.1}$$

where k^i is defined in (2.16). Differentiating (2.18) with respect to z and evaluating at $z=0$, I get:

$$\frac{\psi}{f_1\psi'} = \frac{dq_2^G}{dz} - \frac{dq_2^R}{dz},$$

where the argument of ψ and ψ' is q_2. Use the two results above and substitute dk^i/dz to solve for dS/dz. Then,

$$\frac{dk^G}{dz} = \frac{1}{1+\theta}\left(\theta\frac{\psi}{f_1\psi'} + 2\frac{d\mu}{dz}\right), \quad \theta\frac{dk^R}{dz} + \frac{dk^G}{dz} = 2\frac{d\mu}{dz}.$$

Note that $f_1 < 0$ and $\frac{d\mu}{dz} = \frac{\sigma}{\alpha n^2}\left(\frac{\gamma}{\beta} - 1\right)\frac{dn}{dz}$. If $dn/dz = 0$, as stated in Lemma 3.1, then $dk^G/dz < 0$ and $dk^R/dz > 0$. In this case, (B.1) implies $dq_2^G/dz > 0$ and $dq_2^R/dz < 0$.

To show $dn/dz = 0$, substitute $u'(c_2^i) = k^i\psi'(q_2^i)$ to rewrite (2.11) as:

$$\frac{2(n-\sigma)}{\alpha n}h' = \theta k^R\left[q_2^R\psi'(q_2^R) - \psi(q_2^R)\right] + k^G\left[q_2^G\psi'(q_2^G) - \psi(q_2^G)\right].$$

Differentiating this equation with respect to z, evaluating at $z = 0$, and substituting dq/dz and dk^i/dz, I obtain $dn/dz = 0$. This completes the proof of Lemma 3.1.

To prove Proposition 3.2, differentiate the welfare measure in (3.1) with respect to z, evaluate at $z = 0$, and substitute the above derivatives of (q_2^R, q_2^G, n). Then,

$$\frac{1-\beta}{\alpha n}\frac{dv}{dz} = \frac{(1-\theta)\psi(q_2)}{2(1+\theta)}. \tag{B.2}$$

Thus, $dv/dz > 0$ if and only if $\theta < 1$. Since this effect exists only if Case IS and Case B both exist, $\theta > [1 + 2\mu(n_1)]^{-1}$ is required. Thus, when $[1 + 2\mu(n_1)]^{-1} < \theta < 1$ and when z is small, $dv/dz > 0$ for all $\gamma \in (\beta, \gamma_0)$. Finally, this region of θ is non-empty if and only if $\mu(n_1) > 0$, which is equivalent to $\gamma > \beta$. **QED**

C. Proofs for Section 4

Let me first characterize Case IS of the equilibrium. Redefine (μ, k^G, k^R) as follows:

$$\mu(n) = \frac{n-\sigma}{\alpha(n+g)}\left(\frac{\gamma}{\beta} - 1\right),$$

$$k^G(S,n) = 1 + \frac{(n-\sigma)(S\gamma/\beta - 1)}{(1-p)\alpha(n+g)}, \quad k^R(S,n) = \frac{\delta}{\theta}\left(1 + \frac{(n-\sigma)(1-S)\gamma/\beta}{p\alpha(n+g)}\right).$$

Note that Assumption 1 implies that

$$\frac{u'(\alpha(n+g)(1-p)q)}{u'(\alpha(n+g)pq)} = \frac{u'(1-p)}{u'(p)} \equiv \delta \text{ for all } q.$$

With this feature and the redefined (f, k^i), (q_2^G, q_2^R, S) in Case IS of the equilibrium continue to satisfy (2.17) and (2.18). Similar to (2.11), the optimal choice of n satisfies:

$$\frac{n-\sigma}{\alpha(n+g)}h' = (1-p)k^G\left[q_2^G\psi'(q_2^G) - \psi(q_2^G)\right] + p\frac{\theta}{\delta}k^R\left[q_2^R\psi'(q_2^R) - \psi(q_2^R)\right], \tag{C.1}$$

where I have substituted the relationships $u'(c_2^G) = k^G\psi'(q_2^G)$ and $u'(c_2^R) = (k^R/\delta)\psi'(q_2^R)$. The definitions of k^G and k^R yield:

$$(1-p)k^G(S,n) + \frac{p\theta}{\delta}k^R(S,n) = 1 + \mu. \tag{C.2}$$

Case IS requires the solution to (2.18) for S to lie in $(\beta/\gamma, 1)$. As in Appendix A, this requirement can be expressed as $\theta \in (\theta_1, \theta_3)$, where $\theta_1 = \Theta_1(n_1)$, $\theta_3 = \Theta_3(n_3)$ and

$$\Theta_1(n) \equiv \frac{\delta\psi'(Q_1(n))}{u'(\alpha(n+g)(1-p)Q_1(n))}, \ \Theta_3(n) \equiv \frac{\delta[1+\mu(n)/p]\,\psi'(Q_3(n))}{u'(\alpha(n+g)(1-p)Q_3(n))}.$$

Here, Q_1 and Q_3 are given in Appendix A, with q_1^G and q_3^G being redefined as $q_1^G(n) = f(1+\mu(n)/(1-p), n)$ and $q_3^G(n) = f(1, n)$.

When $z \downarrow 0$, Case IS of the equilibrium becomes Case B of the equilibrium without the legal restriction. The bounds of the region of existence approach the following limits:

$$\Theta_1(n) \uparrow \frac{\delta}{1+\mu(n)/(1-p)}, \quad \Theta_3(n) \uparrow \delta\left(1+\frac{\mu(n)}{p}\right).$$

Also, $q_2^G = q_2^R = q_2 = f(k,n)$ and $k^G = k^R = k$, where k satisfies (C.2). If $\gamma = \beta$, in addition, then $\mu = 0$, $k = k_0$ and $q = q_0 = f(k_0, n)$, where k_0 is defined in (4.3).

Now I prove Proposition 4.1. With the redefined (f, k^i), the effects of z can be verified with the same procedure as the one in the proof of Lemma 3.1. In particular, differentiating (C.1) with respect to z and evaluating at $z = 0$ yields $dn/dz = 0$. Moreover,

$$\frac{dq_2^i}{dz} = f_1\frac{dk^i}{dz}, \ \frac{dk^G}{dz} = \frac{\theta p\psi/(f_1\psi')}{\theta p + \delta(1-p)}, \ \frac{p\theta}{\delta}\frac{dk^R}{dz} = -(1-p)\frac{dk^G}{dz}.$$

Differentiating (4.1) with respect to z and evaluating at $z = 0$ yields:

$$\frac{(1+\beta)(1+g)}{\alpha(n+g)}\frac{dv^s}{dz} = (\delta-\theta)\frac{p(1-p)\psi}{\theta p+\delta(1-p)}.$$

Note that $dv^s/dz = 0$ if either $p = 1$ or $p = 0$. Because $0 < p < 1$ by assumption, $dv^s/dz > 0$ iff $\theta < \delta$. Because this result holds only in the parameter region where Case IS and Case B coexist, it requires $\delta/\left(1+\frac{\mu(n_1)}{1-p}\right) < \theta < \delta\left(1+\frac{\mu(n_3)}{p}\right)$. Thus, $dv^s/dz > 0$ iff $\delta/\left(1+\frac{\mu(n_1)}{1-p}\right) < \theta < \delta$. This region of parameters is non-empty for all $\gamma > \beta$.

Finally, one can adapt the analysis leading to Proposition 3.3 to show that $dv^s/d\gamma > 0$ at $z = 0$ and $\gamma = \beta$ if and only if $dn/d\gamma < 0$ and, hence, if and only if (3.2) is satisfied with the redefined (f, k_0, q_0). Thus, under (3.2), there exists a neighborhood of $\theta < \delta$ where optimal money growth is $\gamma > \beta$ and where the legal restriction improves efficiency. **QED**

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