Estimating Real and Nominal Term Structures using Treasury Yields, Inflation, Inflation Forecasts, and Inflation Swap Rates*

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November 21, 2008

*We thank seminar participants at Hong Kong University of Science and Technology, Nanyang Technical University, the National University of Singapore, Singapore Management University, the University of Hong Kong, and the University of Oklahoma for valuable comments.

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Abstract

This paper develops and estimates an equilibrium model of the term structures of nominal and real interest rates. The term structures are driven by state variables that include the short term real interest rate, expected inflation, a factor that models the changing level to which inflation is expected to revert, as well as four volatility factors that follow GARCH processes. We derive analytical solutions for the prices of nominal bonds, inflation-indexed bonds that have an indexation lag, the term structure of expected inflation, and inflation swap rates. The model parameters are estimated using data on nominal Treasury yields, survey forecasts of inflation, and inflation swap rates. We find that allowing for GARCH effects is particularly important for real interest rate and expected inflation processes, but that long-horizon real and inflation risk premia are relatively stable. Comparing our model prices of inflation-indexed bonds to those of Treasury Inflation Protected Securities (TIPS) suggests that TIPS were underpriced prior to 2004 but subsequently were valued fairly. We find that unexpected increases in both short run and longer run inflation implied by our model have a negative impact on stock market returns.

1 Introduction

The Treasury yield curve, by itself, provides a wealth of information. However, for many purposes it is important to know its key components: real rates, expected inflation, and real and inflation risk premia. These elements of the term structure are of interest to macroeconomists and to policymakers such as central bankers who wish to gauge investors' short- and long-run expectations of inflation. The characteristics of real and inflation-related components also are important to financial economists and practitioners interested in accurately pricing inflation-linked securities, such as inflation-indexed bonds and inflation derivatives. Issuance of inflation-related securities has grown in recent years, and the current turmoil in financial and commodity markets is likely to keep inflation volatility high and generate demand for securities that hedge inflation.

In this paper we develop a model of real and nominal yield curves and present an estimation technique that allows us to identify term structure components. The model we propose characterizes real rates and inflation by multifactor processes with stochastic volatilities. These volatilities also affect the risk premia associated with shocks to the factors. We are able to derive analytical solutions for the prices of inflation-indexed (real) bonds that include an indexation lag, a feature found in all inflation-linked securities. Similarly, our model can price inflation swaps, which we use along with Treasury yields and survey forecasts of inflation to estimate the model's parameters. Zero coupon inflation swaps are the most liquid of inflation derivatives traded in the over-the-counter (OTC) market. Employing data on inflation swaps helps us identify real interest rate risk premia and inflation risk premia.

Researchers have developed a multitude of models for the term structure of nominal interest rates.[1] Less numerous are models that can determine term structures of real and nominal interest rates together.[2] A satisfactory model of both real and nominal term structures requires at least two factors: one representing real interest rates and the other representing inflation. The necessity of several factors is supported by empirical evidence that finds multiple factors are required to explain the level, slope, and curvature of the nominal term structure (Litterman and Scheinkman (1991)). Moreover, empirical studies document that there is significant time variation in the volatility of interest rates.[3] The model we develop in this paper possesses these characteristics.

Specifically, our model of nominal and real yield curves is driven by state variables that include the short-term real interest rate, the short-term rate of expected inflation, and a third

[1] Recent surveys of nominal term structure models include Dai and Singleton (2003), Dai and Singleton (2004), Piazzesi (2005), and Rebonato (2004).

[2] A discussion of models of both real and nominal term structures can be found in Adrian and Wu (2008), Ang et al. (2008), Buraschi and Jiltsov (2005), and D'Amico et al. (2008).

[3] For example, see Ait-Sahalia (1996), Brenner et al. (1996), and Gallant and Tauchen (1998).

factor that models the changing level to which inflation is expected to revert, referred to as inflation's 'central tendency.' Moreover, there are four additional stochastic volatility factors related to these variables and to unanticipated inflation. These volatilities also affect the risk premia associated with the shocks in the underlying processes. When the stochastic volatilities are all turned off, our model reduces to a three factor constant volatility model that nests the two factor central tendency term structure models of Hull and White (1994), Jegadeesh and Pennacchi (1996), and Balduzzi et al. (1998) as well as the two factor real and nominal term structure models of Pennacchi (1991) and Jarrow and Yildirim (2003).[4]

Our modeling of stochastic volatilities extends the work of Heston and Nandi (2003) to a multivariate setting that can characterize both real and nominal term structures. The Heston and Nandi model is a discrete time term structure model where the nominal interest rate follows the nonlinear asymmetric GARCH process of Engle and Ng (1993).[5] Extensions of the Heston and Nandi model have been examined by Cvsa and Ritchken (2001) who allow conditional distributions of interest rates to be mixtures of normal and chi squared innovations, while maintaining GARCH volatility as a second state variable. Unlike these previous papers that focus on pricing nominal derivative contracts, we analyze the real and inflation-related components of the term structure and consider their implications for pricing inflation-indexed securities.

Modeling stochastic volatilities of term structure factors using multiple GARCH processes has some advantages relative to alternative methods. If volatilities are modeled as continuous time stochastic processes, the processes' parameters often are difficult to estimate because volatilities may not be directly observable. Frequently, estimation must rely on a cross section of security prices, rather than solely upon a time series. Using discrete time GARCH processes alleviates this problem because volatilities are *observable* functions of the history of the processes' innovations and can be exactly filtered from discrete observations. Using GARCH models, as opposed to continuous time stochastic volatility models, comes at almost no cost. Indeed, as shown by Foster and Nelson (1994), GARCH models can be made to converge to continuous time stochastic volatility processes as the time increment shrinks. Since our model has a multitude of different volatilities that play important roles not only for capturing the dynamics of real rates and expected inflation, but also for influencing fluctuations in risk premia, the use of discrete time GARCH models simplifies estimation with little sacrifice in realism. While the state variables in our model are conditionally normally distributed over a single period (which in our empirical work is taken to be a month), they are not normally distributed over multiple

[4] All of these models are generalizations of the one-factor Gaussian model of Vasicek (1977). A multivariate extension of the Vasicek model was analyzed by Langetieg (1980).

[5] Brenner et al. (1996) investigate alternative discrete time GARCH specifications for modeling the dynamics of the short term interest rate and find strong support for GARCH effects, especially when asymmetric responses to innovations are permitted.

periods. This permits multiperiod state variables and yields to have distributions that display skewness and kurtosis.

Another advantage of term structure modeling with GARCH processes relates to state variable correlations. Our model is an example of an affine term structure model.[6] The affine class is attractive due to the relative ease of computing solutions for bond yields which are affine (linear) functions of the state variables. However, one limitation of continuous time affine models is that a general correlation structure between state variables is possible only when they follow Gaussian (constant volatility) processes.[7] This is not the case in our discrete time model: it allows a general correlation structure between state variables yet permits them to display stochastic volatility. Moreover, correlations between our model's state variables can be time-varying, though in a more limited way than in models by Campbell et al. (2007) and Adrian and Wu (2008) which focus on stochastic covariation between real and nominal factors.

Regime switching is another approach to modeling stochastic volatility. For example, Ang et al. (2008) develop a term structure model that incorporates an observed inflation factor that switches regimes. An attractive feature of regime switching models is that they can be used to also change the state variables' conditional means. In our model, we permit changes in the conditional mean of inflation by introducing a stochastic central tendency. A stochastic central tendency for inflation, as well as GARCH volatilities for real and inflation factors, allows us to capture changing monetary and real economic environments. While regime switching models can provide valuable insights, it is our view that there may be advantages to a model that does not require discrete regimes. Even where distinct regimes are evident, the behavior of inflation and interest rates can differ markedly between regimes of the same type (Bordo and Haubrich (2004)). Furthermore, some variables of crucial interest to us, such as inflationary expectations, often show smooth transitions between regimes (Haubrich and Ritter (2000)).

We obtain several noteworthy empirical results. First, we find that short term real interest rates are the most volatile component of the yield curve, and it is especially important to allow their volatility to display GARCH behavior. Real rates were negative for much of the 2002 to 2005 period, which may have helped inflate a credit bubble. Second, we find that expected inflation is negatively correlated with real rates, and it also shows statistically significant changes in volatility. Both real rates and expected inflation display rather strong mean reversion. Third,

[6] Almost all research that models both real and nominal term structures fall within the affine class. An exception is Campbell et al. (2007) where nominal bond yields are assumed to be linear-quadratic functions of state variables but the short term real interest rate is an affine function of a constant volatility state variable. As will be discussed, our model's empirical results suggest that permitting stochastic (GARCH) volatility is especially important for describing the process followed by the real interest rate.

[7] For state variables to have stochastic volatilities in continuous time affine models, they must follow multivariate square root processes; that is, a multifactor extension of the model by Cox et al. (1985). However, this requires that the correlation between the state variables be nonnegative (Dai and Singleton (2000)). For example, D'Amico et al. (2008) justify their use of a multifactor Gaussian affine term structure model based on its flexible correlation structure.

over our sample period of 1982 to 2008, inflation's central tendency, which can be viewed as investors' expectation of longer term inflation, declined substantially. This is consistent with an increase in credibility regarding the Federal Reserve's desire to maintain low inflation.

Fourth, we find a real interest rate risk premium that is substantial and fairly stable, varying between 150 and 170 basis points for a ten-year maturity bond. The inflation risk premium on a ten-year bond varied between 38 and 60 basis points during our sample period. Fifth, by comparing our model's implied yields for inflation-indexed bonds to actual prices of U.S. Treasury Inflation-Protected Securities (TIPS), we document evidence that TIPS were underpriced prior to 2004 but subsequently appeared to be fairly valued. Lastly, we examine the relationship between our model's implied term structure components and stock market returns. We find that our model's implied shocks to both short run and longer run inflation have a negative impact on stock returns.

The paper proceeds as follows. Section 2 introduces a model of real interest rates and inflation that is used to derive the term structures of nominal bonds, inflation forecasts, inflation-indexed bonds, and inflation swap rates. Section 3 describes the data used to estimate the parameters of the model, and Section 4 explains the estimation technique. Section 5 describes the results and Section 6 concludes.

2 A Model of Nominal and Real Term Structures

2.1 Assumptions

Consider a discrete time environment with multiple periods, each of length Δt measured in years. Let $\frac{M_{t+\Delta t}}{M_t}$ be the nominal pricing kernel with dynamics:

$$\frac{M_{t+\Delta t}}{M_t} = e^{-i_t \Delta t - \frac{1}{2}\sum_{j=1}^{4} \phi_j^2 h_{j,t}^2 \Delta t - \sum_{j=1}^{4} \phi_j h_{j,t} \sqrt{\Delta t} \epsilon_{j,t+\Delta t}} \tag{1}$$

Here $\epsilon_{j,t+\Delta t}, j = 1, 2, .., 4$ are independent standard normal random variables and $\phi_j h_{j,t}$, $j = 1, 2, .., 4$ are market prices of risk associated with these four sources of uncertainty.[8] $h_{j,t}$, $j = 1, 2, .., 4$ represent four different volatility state variables whose dynamics will be specified later. Let $E_t[\cdot] = E[\cdot|\mathcal{F}_t]$ denote the expectations operator conditional on information at date t, $\mathcal{F}_t$. If we value a one period payoff of \$1, then

$$E_t\left[\frac{M_{t+\Delta t}}{M_t}\right] = e^{-i_t \Delta t} \tag{2}$$

so that i_t is the annualized, one period nominal interest rate.

[8] Note that $\sqrt{\Delta t}\epsilon_{j,t+\Delta t}$, $j = 1, ..., 4$ are discrete time analogues to Brownian motion processes.

Let the price index at date t be I_t. For example, I_t can denote the date t Consumer Price Index (CPI). Its dynamics are assumed to satisfy:

$$\frac{I_{t+\Delta t}}{I_t} = e^{\pi_t \Delta t - \frac{1}{2} h_{1,t}^2 \Delta t + h_{1,t} \sqrt{\Delta t} \epsilon_{1,t+\Delta t}} \tag{3}$$

where the variable $\pi_t = \frac{1}{\Delta t} \ln \left(E_t \left[I_{t+\Delta t} / I_t \right] \right)$ is rate of expected inflation for the period from t to $t + \Delta t$.

Given the processes for the nominal pricing kernel and the price index, we can compute the real pricing kernel, m_t. In particular:

$$\begin{aligned} \frac{m_{t+\Delta t}}{m_t} &= \frac{M_{t+\Delta t}}{M_t} \frac{I_{t+\Delta t}}{I_t} \\ &= e^{\left(\pi_t - i_t - \frac{1}{2} h_{1,t}^2\right)\Delta t - \frac{1}{2}\sum_{j=1}^4 \phi_j^2 h_{j,t}^2 \Delta t - \sum_{j=1}^4 \phi_j h_{j,t} \sqrt{\Delta t} \epsilon_{j,t+\Delta t} + h_{1,t}\sqrt{\Delta t}\epsilon_{1,t+\Delta t}} \end{aligned} \tag{4}$$

Taking expectations on the left-hand-side of (4) defines r_t, the one period real interest rate:

$$E_t \left[\frac{m_{t+\Delta t}}{m_t} \right] = e^{-r_t \Delta t} \tag{5}$$

Taking expectations on the right-hand-side of (4) and equating to (5), we obtain:

$$e^{-r_t \Delta t} = e^{\left(-i_t + \pi_t - \phi_1 h_{1,t}^2\right)\Delta t}, \tag{6}$$

from which

$$i_t = \pi_t + r_t - \phi_1 h_{1,t}^2 \tag{7}$$

All that remains to complete the model is to specify the dynamics of the state variables. It is assumed that

$$\begin{aligned} \pi_{t+\Delta t} - \pi_t &= \left[\alpha_t + a_1 r_t + a_2 \pi_t\right] \Delta t + \sqrt{\Delta t} \Sigma_{j=1}^2 \beta_j h_{j,t} \epsilon_{j,t+\Delta t} \\ r_{t+\Delta t} - r_t &= \left[b_0 + b_1 r_t + b_2 \pi_t\right] \Delta t + \sqrt{\Delta t} \Sigma_{j=1}^3 \gamma_j h_{j,t} \epsilon_{j,t+\Delta t} \\ \alpha_{t+\Delta t} - \alpha_t &= \left[c_0 + c_1 \alpha_t\right] \Delta t + \sqrt{\Delta t} \Sigma_{j=1}^4 \rho_j h_{j,t} \epsilon_{j,t+\Delta t} \\ h_{j,t+\Delta t}^2 - h_{j,t}^2 &= \left[d_{j0} + d_{j1} h_{j,t}^2 + d_{j2} \left(\epsilon_{j,t+\Delta t} - d_{j3} h_{j,t}\right)^2\right] \Delta t, \;\; j = 1, ..., 4 \end{aligned} \tag{8}$$

where α_t is an additional state variable that shifts the future path of the expected inflation rate. The first three equations in (8) satisfy a first-order vector autoregression. Subject to parameter stationarity conditions, the constants in these equations can be related to the unconditional mean (steady-state level) of expected inflation, $\overline{\pi}$, and the unconditional mean of the real rate,

$\overline{r}$. These relationships are

$$\overline{\pi} = -\frac{a_1 b_0 c_1 + b_1 c_0}{(a_1 b_2 - a_2 b_1)\, c_1} \tag{9}$$

$$\overline{r} = \frac{a_2 b_0 c_1 + b_2 c_0}{(a_1 b_2 - a_2 b_1)\, c_1} \tag{10}$$

The unconditional mean of α_t is $-c_0/c_1 = -(a_1\overline{r} + a_2\overline{\pi})$. If a constant is added to α_t such that $\widehat{\alpha}_t \equiv \alpha_t + a_1\overline{r} + (1 + a_2)\,\overline{\pi}$, then the unconditional mean of $\widehat{\alpha}_t$ equals $\overline{\pi}$, and $\widehat{\alpha}_t$ is commonly referred to as the 'central tendency' of the expected inflation rate.[9] It equals the current mean reversion level or target level to which the inflation rate is expected to tend. For simplicity, we shall refer to α_t as the central tendency, but it should be understood that it differs from the true central tendency, $\widehat{\alpha}_t$, by a constant.

The equations in (3) and (8) specify that actual inflation, expected inflation, the real interest rate, and inflation's central tendency follow imperfectly correlated processes having stochastic volatilities. These correlations depend on the β_j, γ_j, and ρ_j coefficients multiplying the four orthogonal shocks, $h_{j,t}\epsilon_{j,t+\Delta t}$, $j = 1, ..., 4$, but without loss of generality, we can restrict $\beta_2 = \gamma_3 = \rho_4 = 1$.[10] From (3), the one-period inflation rate, $\ln\left[I_{t+\Delta t}/I_t\right]$, has an annualized standard deviation, $h_{1,t}$, that follows a GARCH process driven by the inflation innovation, $\epsilon_{1,t}$. The first equation in (8) permits the rate of expected inflation, π_t, to follow a mean reverting process that tends toward a central tendency, α_t, which itself follows a mean reverting process. The change in expected inflation, $\pi_{t+\Delta t} - \pi_t$, depends on the surprise to actual inflation, $h_{1,t}\epsilon_{1,t+\Delta t}$, as well as an orthogonal shock, $h_{2,t}\epsilon_{2,t+\Delta t}$, where $h_{2,t}$ also follows a second GARCH process driven by $\epsilon_{2,t}$.

The real interest rate mean reverts to $\overline{r}$, and its change, $r_{t+\Delta t} - r_t$, is influenced by the innovations in actual inflation, expected inflation, as well as a third shock, $h_{3,t}\epsilon_{3,t+\Delta t}$. Here, $h_{3,t}$ follows a third GARCH process dependent on the innovation $\epsilon_{3,t}$. Finally, the process for inflation's central tendency, α_t, is correlated with actual inflation, expected inflation, and real rates, but also has its unique shock, $h_{4,t}\epsilon_{4,t+\Delta t}$, that satisfies a fourth GARCH process determined by $\epsilon_{4,t}$. Note from the pricing kernel equation (1), each of the four shocks $h_{1,t}\epsilon_{1,t+\Delta t}$, $h_{2,t}\epsilon_{2,t+\Delta t}$, $h_{3,t}\epsilon_{3,t+\Delta t}$, and $h_{4,t}\epsilon_{4,t+\Delta t}$ commands a risk premium of $\phi_1 h_{1,t}$, $\phi_2 h_{2,t}$, $\phi_3 h_{3,t}$, and $\phi_4 h_{4,t}$, respectively.

The dynamics of the $h_{j,t}$, $j = 1, ..., 4$, follow the Nonlinear Asymmetric GARCH model of Engle and Ng (1993). Subject to stationarity conditions, the steady-state levels of these

[9] Term structure models specifying a central tendency include Hull and White (1994), Jegadeesh and Pennacchi (1996), and Balduzzi et al. (1998).

[10] These three restrictions permit the identification of the levels of the stochastic volatilities $h_{2,t}$, $h_{3,t}$, and $h_{4,t}$.

processes are

$$\overline{h}_j^2 = -\frac{d_{j0} + d_{j2}}{d_{j1} + d_{j2} d_{j3}^2} \ , \quad j = 1, ..., 4 \tag{11}$$

This model extends a GARCH(1,1) to allow for asymmetric responses to the innovations when the parameters d_{j3}, $j = 1, ..., 4$ are non-zero. When d_{j3} is positive (*negative*), negative values of $\epsilon_{j,t}$ have a larger (*smaller*) impact on $h^2_{j,t+\Delta t}$ than do positive values. Collectively, the $h_{j,t}$ act as scaling factors that determine the local volatilities for inflation, expected inflation, real rates, and the central tendency. Of course, if all these GARCH effects are shut down, then there will be no stochastic volatility, and the model will reduce to a Markovian model with three state variables.[11] With stochastic volatility, our model has four stochastic drivers and seven state variables. While the one period distribution of the state variables π_t, r_t, and α_t are conditionally normal, over multiple periods, the distribution will not be normal. The parameters d_{j2} and d_{j3}, $j = 1, ..., 4$ heavily influence the skewness and kurtosis in the distribution of yields to maturity over multiple periods.

With these model assumptions, we can now derive the values of nominal bonds, inflation expectations, and inflation-linked securities.

2.2 Prices of Nominal Bonds

Let $P(t, t + n\Delta t)$ be the date t price of a nominal bond that pays \$1 at date $t + n\Delta t$, where n is a non-negative integer. We have:

$$P(t, t + n\Delta t) = E_t \left[\frac{M_{t+\Delta t}}{M_t} P(t + \Delta t, t + n\Delta t) \right] \tag{12}$$

Proposition 1 below provides the expressions for the term structure of nominal interest rates.

Proposition 1

Under the above dynamics, nominal bond prices are given by the following recursive equation:

$$P(t, t + n\Delta t) = e^{-K_n - A_n \pi_t - B_n r_t - C_n \alpha_t - \sum_{j=1}^{4} D_{j,n} h^2_{j,t}} \quad \textit{for } n \geq 1. \tag{13}$$

[11] The homoskedastic (constant volatility) case occurs when $d_{j1} = -1/\Delta t$ and $d_{j2} = d_{j3} = 0$. This would correspond to a multivariate Vasicek (1977) model as developed in Langetieg (1980). Pennacchi (1991) derives a special case of this model from a monetary production economy where expected inflation, π_t, and the real interest rate, r_t, are the only two state variables.

where $K_1 = 0,\ A_1 = \Delta t,\ B_1 = \Delta t,\ C_1 = 0,\ D_{1,1} = -\phi_1 \Delta t,\ D_{j,1} = 0$ *for* $j = 2, 3, 4,$ *and*

$$
\begin{aligned}
K_{n+1} &= K_n + (b_0 B_n + c_0 C_n + \Sigma_{j=1}^{4} d_{j0} D_{j,n})\Delta t + \frac{1}{2}\Sigma_{j=1}^{4} \ln(1 + 2d_{j2}\Delta t D_{j,n}) \\
A_{n+1} &= \Delta t + (1 + a_2 \Delta t)\, A_n + b_2 \Delta t B_n \\
B_{n+1} &= \Delta t + a_1 \Delta t A_n + (1 + b_1 \Delta t)\, B_n \\
C_{n+1} &= \Delta t A_n + (1 + c_1 \Delta t)\, C_n \\
D_{j,n+1} &= \left[1 + \left(d_{j1} + d_{j2} d_{j3}^2\right)\Delta t\right] D_{j,n} + \frac{1}{2}\phi_j^2 \Delta t - Q_j \Delta t,
\end{aligned}
$$

and

$$
\begin{aligned}
Q_1 &= \frac{(\phi_1 + A_n\beta_1 + B_n\gamma_1 + C_n\rho_1 - 2D_{1,n}d_{12}d_{13}\sqrt{\Delta t})^2}{2(1 + 2D_{1,n}d_{12}\Delta t)} + \phi_1 \\
Q_2 &= \frac{(\phi_2 + A_n\beta_2 + B_n\gamma_2 + C_n\rho_2 - 2D_{2,n}d_{22}d_{23}\sqrt{\Delta t})^2}{2(1 + 2D_{2,n}d_{22}\Delta t)} \\
Q_3 &= \frac{(\phi_3 + B_n\gamma_3 + C_n\rho_3 - 2D_{3,n}d_{32}d_{33}\sqrt{\Delta t})^2}{2(1 + 2D_{3,n}d_{32}\Delta t)} \\
Q_4 &= \frac{(\phi_4 + C_n\rho_4 - 2D_{4,n}d_{42}d_{43}\sqrt{\Delta t})^2}{2(1 + 2D_{4,n}d_{42}\Delta t)}.
\end{aligned}
$$

Proof: See the Appendix.

The proposition reveals that nominal bond prices reflect all of the model parameters. However, these parameters cannot be identified solely by data from a time series of bond prices of various maturities.[12] Intuitively, one needs other information that can separate nominal yields into their real and inflation-related components. This motivates our desire to use information from survey data on forecasts of inflation. Hence, we now compute expectations of multiperiod inflation implied by our model.

2.3 Expectations of Inflation

Define $I(t, t + n\Delta t) \equiv E_t\left[I_{t+n\Delta t}/I_t\right]$ to be the date t forecast of growth in the price level over the period from date t to date $t + n\Delta t$. This expectation is given by the following proposition.

Proposition 2

The date t expectation of inflation for a horizon of n periods is

$$I(t, t + n\Delta t) = e^{\bar{K}_n + \bar{A}_n \pi_t + \bar{B}_n r_t + \bar{C}_n \alpha_t + \sum_{j=1}^{4} \bar{D}_{j,n} h_{j,t}^2} \quad \text{for } n \geq 1. \tag{14}$$

[12] Dai and Singleton (2000) discus the identification restrictions for affine term structure models.

where $\bar{K}_1 = 0,\ \bar{A}_1 = \Delta t,\ \bar{B}_1 = 0,\ \bar{C}_1 = 0,\ \bar{D}_{j,1} = 0$ *for* $j = 1, 2, 3, 4,$ *and*

$$\begin{aligned}
\bar{K}_{n+1} &= \bar{K}_n + (b_0\bar{B}_n + c_0\bar{C}_n + \Sigma_{j=1}^4 d_{j0}\bar{D}_{j,n})\Delta t - \frac{1}{2}\Sigma_{j=1}^4 \ln(1 - 2d_{j2}\Delta t\bar{D}_{j,n}) \\
\bar{A}_{n+1} &= \Delta t + (1 + a_2\Delta t)\bar{A}_n + b_2\Delta t\bar{B}_n \\
\bar{B}_{n+1} &= a_1\Delta t\bar{A}_n + (1 + b_1\Delta t)\bar{B}_n \\
\bar{C}_{n+1} &= \Delta t\bar{A}_n + (1 + c_1\Delta t)\bar{C}_n \\
\bar{D}_{j,n+1} &= \bar{D}_{j,n}\left[1 + \left(d_{j1} + d_{j2}d_{j3}^2\right)\Delta t\right] + \bar{Q}_j\Delta t,
\end{aligned}$$

and

$$\begin{aligned}
\bar{Q}_1 &= \frac{(1 + \bar{A}_n\beta_1 + \bar{B}_n\gamma_1 + \bar{C}_n\rho_1 - 2\bar{D}_{1,n}d_{12}d_{13}\sqrt{\Delta t})^2}{2(1 - 2\bar{D}_{1,n}d_{12}\Delta t)} - \frac{1}{2} \\
\bar{Q}_2 &= \frac{(\bar{A}_n\beta_2 + \bar{B}_n\gamma_2 + \bar{C}_n\rho_2 - 2\bar{D}_{2,n}d_{22}d_{23}\sqrt{\Delta t})^2}{2(1 - 2\bar{D}_{2,n}d_{22}\Delta t)} \\
\bar{Q}_3 &= \frac{(\bar{B}_n\gamma_3 + \bar{C}_n\rho_3 - 2\bar{D}_{3,n}d_{32}d_{33}\sqrt{\Delta t})^2}{2(1 - 2\bar{D}_{3,n}d_{32}\Delta t)} \\
\bar{Q}_4 &= \frac{(\bar{C}_n\rho_4 - 2\bar{D}_{4,n}d_{42}d_{43}\sqrt{\Delta t})^2}{2(1 - 2\bar{D}_{4,n}d_{42}\Delta t)}.
\end{aligned}$$

Proof: See the Appendix

Proposition 2 provides the expectation of inflation starting from the current date t. Because our data also contains survey forecasts of an inflation rate that begins and ends at two future dates, it is useful to derive an expression for such a forecast. Let t be the current date, $t + n_1\Delta t$ be the date at which the inflation forecast starts, and $t + n_2\Delta t$ be the date at which the inflation forecast ends, where $n_2 > n_1$. Let $m \equiv n_2 - n_1$, for example, $m = 3$ periods (months) would occur if the forecast is of an inflation rate over a future quarter of a year. If survey participants forecast a continuously compounded rate, then their date t forecast is

$$E_t\left[\frac{1}{m\Delta t}\ln\left(\frac{I_{t+n_2\Delta t}}{I_{t+n_1\Delta t}}\right)\right] = \frac{1}{m\Delta t}\left(E_t\left[\ln\left(\frac{I_{t+n_2\Delta t}}{I_t}\right)\right] - E_t\left[\ln\left(\frac{I_{t+n_1\Delta t}}{I_t}\right)\right]\right) \tag{15}$$

Proposition 3

$$E_t\left[\ln\left(\frac{I_{t+n\Delta t}}{I_t}\right)\right] = K_n^* + A_n^*\pi_t + B_n^*r_t + C_n^*\alpha_t + \Sigma_{j=1}^4 D_{j,n}^* h_{j,t}^2 \tag{16}$$

where

$$
\begin{aligned}
K_{n+1}^* &= K_n^* + (b_0 B_n^* + c_0 C_n^* + \Sigma_{j=1}^4 (d_{j0} + d_{j2}) D_{j,n}^*)\Delta t \\
A_{n+1}^* &= \Delta t + (1 + a_2 \Delta t)\, A_n^* + b_2 \Delta t B_n^* \\
B_{n+1}^* &= a_1 \Delta t A_n^* + (1 + b_1 \Delta t) B_n^* \\
C_{n+1}^* &= \Delta t A_n^* + (1 + c_1 \Delta t) C_n^* \\
D_{j,n+1}^* &= D_{j,n}^* \left[1 + (d_{j1} + d_{j2} d_{j3}^2)\Delta t\right] - 1_{\{j=1\}} \frac{1}{2}\Delta t,
\end{aligned}
$$

where $1_{\{j=1\}} = 1$ *if* $j = 1$ *and* 0 *otherwise, and* $K_1^* = 0$, $A_1^* = \Delta t$, $B_1^* = 0$, $C_1^* = 0$, $D_{1,1}^* = -\frac{1}{2}\Delta t$, *and* $D_{j,1}^* = 0$, *for* $j = 2, 3, 4$.

Notice from Propositions 2 and 3 that not all of the parameters of the model enter into expectations of inflation or an inflation rate. In particular, the market prices of risk, ϕ_j, $j = 1, .., 4$, are absent. Augmenting nominal yield information with inflation forecasts is helpful in separating out expected inflation from nominal yields. However, inflation forecasts do little to distinguish between real and inflation-related risk premia. For this reason, our empirical work also uses information from securities having real (inflation-linked) payoffs to help identify risk premia.[13] The next subsections consider values for such securities.

2.4 Prices of Inflation-Indexed Bonds

Inflation-indexed (real) bonds are issued by many countries.[14] They make payments proportional to an inflation index, thereby protecting investors from the uncertainty of inflation. Inflation-indexed bonds issued by the U.S. Treasury are called Treasury Inflation-Protected Securities (TIPS), and these bonds pay semi-annual coupons. Since a coupon-bearing TIPS can be decomposed into a portfolio of zero-coupon TIPS contracts, it is sufficient to value a zero-coupon TIPS.[15] In practice, a TIPS contract does not provide full coverage against inflation. Rather, the inflation index for a TIPS payment is based on the Consumer Price Index (CPI) recorded at a date prior to the bond's date of payment. One reason for this is that the CPI is not revealed immediately at the date for which it is recorded, but is reported with a lag. In the

[13] Chernov and Mueller (2008) estimate a nominal and real term structure model based on Ang and Piazzesi (2003) that uses only Treasury yields and survey forecasts of inflation. However, in order to cope with the difficulty in estimating risk premia, they add to their likelihood function a 'penalization term' that is proportional to the variation in risk premia. This helps them avoid estimating very large values for risk premia. With our use of inflation swap rates, we are able to estimate reasonable risk premia without any modification of the likelihood function.

[14] These countries include the United States, the United Kingdom, France, Canada, Germany, Greece, Italy, Japan, and Sweden.

[15] Like nominal Treasury notes and bonds, TIPS can be stripped; that is, decomposed into separate coupon and principal payments and traded individually as zero-coupon bonds. However, the amount of TIPS that are in stripped form is relatively small.

U.S., for example, a payout for TIPS is based on the CPI recorded at a date three months ($\frac{1}{4}$ year) prior to the bond's payment date.[16]

Since this indexation lag feature can be important, we define $V^d(t; t_s, t_e)$ to be the date t value of a zero-coupon TIPS contract that pays an amount linked to the price index recorded at date t_e which is n periods of length Δt in the future. Thus, the bond payoff is based on the price index at date $t_e = t + n\Delta t$, i.e. $I_{t+n\Delta t}$, but the actual payment date, t_p, is d periods later at date $t_p = t_e + d\Delta t$. Therefore, d is the indexation lag. Following actual practice, the price index at the initiation date is also lagged by d periods. Let t_s represent the date at which the initial index is recorded. Then, if t is the initiation date, $t_s = t - d\Delta t$ and the TIPS payment at date t_p equals I_{t_e}/I_{t_s}. Now, note that at date $t_e = t + n\Delta t$ we can value the payment to be made d periods later as:

$$V^d(t_e; t_s, t_e) = \frac{I_{t_e}}{I_{t_s}} P(t_e, t_e + d\Delta t). \tag{17}$$

and at date t we have:

$$V^d(t; t_s, t_e) = E_t\left[\frac{M_{t+\Delta t}}{M_t} V^d(t + \Delta t; t_s, t_e)\right] \tag{18}$$

The following Proposition provides the recursive equation for pricing TIPS with an indexation lag of d periods.

Proposition 4

The date t value of a zero coupon TIPS that is indexed off the start date of t_s, has a payout determined by the index at the end date t_e, and pays out with a delay of $d\Delta t$ years at date $t_e + d\Delta t$, is given by

$$V^d(t = t_e - n\Delta t; t_s, t_e) = \frac{I_t}{I_{t_s}} e^{-\tilde{K}_n - \tilde{A}_n \pi_t - \tilde{B}_n r_t - \tilde{C}_n \alpha_t - \sum_{j=1}^4 \tilde{D}_n h_{j,t}^2} \quad \text{for } n \geq 0, \text{ and } t \geq t_s + d\Delta t \tag{19}$$

where $\tilde{K}_0 = K_d$, $\tilde{A}_0 = A_d$, $\tilde{B}_0 = B_d$, $\tilde{C}_0 = C_d$, $\tilde{D}_{j,0} = D_{j,d}$ *for* $j = 1, 2, 3, 4$, *and*

$$\begin{aligned}
\tilde{K}_{n+1} &= \tilde{K}_n + (b_0 \tilde{B}_n + c_0 \tilde{C}_n + \Sigma_{j=1}^4 d_{j,0} \tilde{D}_{j,n})\Delta t + \frac{1}{2}\Sigma_{j=1}^4 \ln(1 + 2d_{j2}\tilde{D}_{j,n}\Delta t) \\
\tilde{A}_{n+1} &= (1 + a_2\Delta t)\tilde{A}_n + b_2\Delta t \tilde{B}_n \\
\tilde{B}_{n+1} &= (1 + a_1 \tilde{A}_n)\Delta t + (1 + b_1\Delta t)\tilde{B}_n \\
\tilde{C}_{n+1} &= \Delta t \tilde{A}_n + (1 + c_1\Delta t)\tilde{C}_n \\
\tilde{D}_{j,n+1} &= \frac{1}{2}\phi_j^2 \Delta t + \tilde{D}_{j,n}(1 + d_{j1}\Delta t + d_{j2} d_{j3}^2 \Delta t) - \tilde{Q}_j \Delta t,
\end{aligned}$$

[16] Most valuation models of inflation-indexed bonds ignore this indexation lag feature. An exception is Risa (2001) which is a multifactor, essentially affine, Gaussian model.

and

$$\begin{aligned}
\tilde{Q}_1 &= \frac{(\phi_1 + \tilde{A}_n\beta_1 + \tilde{B}_n\gamma_1 + \tilde{C}_n\rho_1 - 1 - 2\tilde{D}_{1,n}d_{12}d_{13}\sqrt{\Delta t})^2}{2(1+2\tilde{D}_{1,n}d_{12}\Delta t)} + (\phi_1 - \frac{1}{2}) \\
\tilde{Q}_2 &= \frac{(\phi_2 + \tilde{A}_n\beta_2 + \tilde{B}_n\gamma_2 + \tilde{C}_n\rho_2 - 2\tilde{D}_{2,n}d_{22}d_{23}\sqrt{\Delta t})^2}{2(1+2\tilde{D}_{2,n}d_{22}\Delta t)} \\
\tilde{Q}_3 &= \frac{(\phi_3 + \tilde{B}_n\gamma_3 + \tilde{C}_n\rho_3 - 2\tilde{D}_{3,n}d_{32}d_{33}\sqrt{\Delta t})^2}{2(1+2\tilde{D}_{3,n}d_{32}\Delta t)} \\
\tilde{Q}_4 &= \frac{(\phi_4 + \tilde{C}_n\rho_4 - 2\tilde{D}_{4,n}d_{42}d_{43}\sqrt{\Delta t})^2}{2(1+2\tilde{D}_{4,n}d_{42}\Delta t)}.
\end{aligned}$$

Proof: See the Appendix

Like nominal bond prices, the prices of inflation-indexed bonds depend on all of the model's parameters. In principle, employing both types of bond prices can help distinguish between real and inflation-related risk premia. What we show next is that Proposition 4 is helpful not only for pricing indexed bonds, but also for determining inflation swap rates.

2.5 Inflation Swap Rates

Zero coupon inflation swaps are the most liquid of all inflation derivative contracts that trade in the over-the-counter (OTC) market. They are quoted with maturities ranging from 1 to 30 years. In addition, inflation swaps serve as the basic building blocks for the pricing of the majority of other inflation-related derivatives.

A zero coupon inflation swap is a contract whereby the inflation buyer pays a predetermined fixed nominal rate and in return receives from the seller an inflation linked payment. At the initiation date, t_0, the (consumer) price index is initialized to its value at the date $d\Delta t = \frac{1}{4}$ year earlier, say $t_s = t_0 - d\Delta t$. The ending date for the price index is denoted t_e, and the cash settlement or payment date is t_p where $t_p = t_e + d\Delta t$. At this final date a fixed payment is exchanged for I_{t_e}/I_{t_s}, which is the inflation over the period $[t_s, t_e]$. The fixed payment is denoted $(1+k)^{t_e - t_s}$ where k is the annually-compounded inflation swap rate. Thus, the net fixed for inflation swap payment is $(1+k)^{t_e-t_s} - I_{t_e}/I_{t_s}$.

Viewed from date t_0, the value of the fixed (nominal) leg is simply

$$V_{fix}(t_0) = P(t_0, t_p)(1+k)^{t_e - t_s}. \tag{20}$$

The payout of the inflation leg, $V_{inf}(t_0)$ say, equals the payout of a zero coupon TIPS, with payouts at date t_p linked to the index values at dates t_s and t_e:

$$V_{inf}(t_0) = V^d(t_0; t_s, t_e) \tag{21}$$

At the initiation date, t_0, the fair inflation swap rate is the value k that equates $V_{fix}(t_0)$ with $V_{inf}(t_0)$. The resulting value, $k^*(t_0; t_s, t_e)$, is given by:

$$k^*(t_0; t_s, t_e) = \left(\frac{V^d(t_0; t_s, t_e)}{P(t_0, t_p)}\right)^{1/(t_e - t_s)} - 1. \tag{22}$$

While the typical practice is to quote inflation swap rates as annually-compounded rates, we can convert this rate to a continosly compounded rate, say $k^c(t_0; t_s, t_e) = \ln[1 + k^*(t_0; t_s, t_e)]$. From (22) we can see that $k^c(t_0; t_s, t_e)$ is simply the difference between the continuously-compounded nominal bond yield and the equivalent maturity continosly compounded TIPS yield.

2.6 Prices of European Contingent Claims

In addition to inflation swaps, our model can be used to value other inflation-related derivatives, including those that have option-like payoffs, such as inflation caps and floors. Here we outline an approach to valuing a general European contingent claim whose payoff depends on one or more of our model's state variables. Let $C(t, T; \Psi_t)$ be the date t price of a European contingent claim that matures in τ periods at date $T = t + \tau\Delta t$, where $\Psi_t \equiv (I_t \;\; \pi_t \;\; r_t \;\; \alpha_t \;\; h^2_{j,t},\, j = 1, .., 4)$ is the vector of the state variables at date t. Then

$$\begin{aligned} C(t, T; \Psi_t) &= E_t\left[\frac{M_T}{M_t} C(T, T; \Psi_T)\right] \\ &= \widehat{E}_t\left[e^{-\Delta t \sum_{j=0}^{\tau-1} i_{t+j\Delta t}} C(T, T; \Psi_T)\right] \\ &= \widehat{E}_t\left[e^{-\Delta t \sum_{j=0}^{\tau-1} \pi_{t+j\Delta t} + r_{t+j\Delta t} - \phi_1 h^2_{1,t+j\Delta t}} C(T, T; \Psi_T)\right] \end{aligned} \tag{23}$$

where $\widehat{E}_t[\cdot]$ denotes the risk-neutral expectations operator. Define $\widehat{\epsilon}_{j,t+\Delta t} \equiv \epsilon_{j,t+\Delta t} + \sqrt{\Delta t}\phi_j h_{j,t}$, $j = 1, .., 4$. Then the state variables' risk-neutral dynamics are given by

$$
\begin{aligned}
\ln I_{t+\Delta t} - \ln I_t &= \pi_t \Delta t - \left(\frac{1}{2} + \phi_1\right) h_{1,t}^2 \Delta t + h_{1,t}\sqrt{\Delta t}\widehat{\epsilon}_{1,t+\Delta t} \\
\pi_{t+\Delta t} - \pi_t &= [\alpha_t + a_1 r_t + a_2 \pi_t - \Sigma_{j=1}^2 \phi_j \beta_j h_{j,t}]\Delta t + \sqrt{\Delta t}\Sigma_{j=1}^2 \beta_j h_{j,t}\widehat{\epsilon}_{j,t+\Delta t} \\
r_{t+\Delta t} - r_t &= [b_0 + b_1 r_t + b_2 \pi_t - \Sigma_{j=1}^3 \phi_j \gamma_j h_{j,t}]\Delta t + \sqrt{\Delta t}\Sigma_{j=1}^3 \gamma_j h_{j,t}\widehat{\epsilon}_{j,t+\Delta t} \\
\alpha_{t+\Delta t} - \alpha_t &= [c_0 + c_1 \alpha_t - \Sigma_{j=1}^4 \phi_j \rho_j h_{j,t}]\Delta t + \sqrt{\Delta t}\Sigma_{j=1}^4 \rho_j h_{j,t}\widehat{\epsilon}_{j,t+\Delta t} \\
h_{j,t+\Delta t}^2 - h_{j,t}^2 &= [d_{j0} + d_{j1} h_{j,t}^2 + d_{j2}\left(\widehat{\epsilon}_{j,t+\Delta t} - (d_{j3} + \sqrt{\Delta t}\phi_j) d_{j3} h_{j,t}\right)^2]\Delta t, \; j = 1, ..., 4
\end{aligned}
\tag{24}
$$

The risk-neutral expectation in (23) can be computed by Monte-Carlo simulation of the dynamics in (24) as in Boyle (1977). This simply involves generating multiple time series of four-element vectors of standard normal random variables ($\widehat{\epsilon}_{1,t}\ \widehat{\epsilon}_{2,t}\ \widehat{\epsilon}_{3,t}\ \widehat{\epsilon}_{4,t}$), which in turn generate a time series of the state variables and produce a risk-neutral distribution of the contingent claim's date T payoff.

3 Data Description

Estimation of our model uses monthly data on U.S. Treasury security yields, survey forecasts of inflation, rates of actual (realized) inflation, and inflation swap rates. Most data series are available over the period January 1982 to June 2008, though the data on inflation swap rates starts in only April 2003. Treasury security yields are obtained from three sources. First, we obtain zero coupon Treasury yields of 1, 2, 3, 5, 7, 10, and 15 years to maturity from daily off-the-run Treasury yield curves recently constructed by Gurkaynak et al. (2007).[17] Second, we collect daily secondary market yields for 3-month and 6-month Treasury bills from the Federal Reserve System's H.15 Release.[18] Third, we obtain a one-month (30-day) Treasury bill yield from the Center for Research in Security Prices (CRSP).[19] All of the Treasury yields are taken as of the first trading day of each month.

Survey forecasts of Consumer Price Index (CPI) inflation come from two different sources. First, a monthly series beginning in 1982 is obtained from Blue Chip Economic Indicators (BCEI) which surveys approximately 50 economists employed by financial institutions, non-financial corporations, and research organizations. At the beginning of each month, participants

[17] Their daily Treasury yield curves are available from 1961 to the present and can be downloaded from http://www.federalreserve.gov/econresdata/researchdata.htm. These zero-coupon yields were fitted by the method of Svensson (1994) using prices of off-the-run Treasury coupon notes and bonds.

[18] These data series can be obtained at http://research.stlouisfed.org/fred2/categories/116.

[19] CRSP provides a consistent time series for the one-month Treasury yield over the entire 1982 to 2008 period. A single time series was not available from Federal Reserve sources.

in this survey forecast future CPI inflation for quarterly time periods, starting from the current quarter and going out to at most 8 quarters (2 years) in the future. For January, February, and March, inflation rate forecasts for 8 future quarters are made. For April, May, and June, forecasts for 7 future quarters are made. For July, August, and September, forecasts for 6 future quarters are made, while for October, November, and December, forecasts for 5 future quarters are made. We use BCEI's reported 'consensus' forecast which is the average of the participants' forecasts.

Second, we use the median forecast of CPI inflation over the next ten years made by the approximately 40 participants of the Survey of Professional Forecasters (SPF), currently conducted by the Federal Reserve Bank of Philadelphia.[20] This 10-year forecast is at a quarterly frequency, and starts in December of 1991. Thus, we observe this forecast at the beginning of March, June, September, and December.[21] The analysis of Keane and Runkle (1990) suggests that SPF forecasts are rational expectations of inflation that incorporate all available public information. A recent study by Ang et al. (2007) finds that SPF forecasts significantly outperform a wide variety of other methods for predicting inflation. Since the participants in the BCEI survey have qualifications similar to those of the SPF participants, it is likely that the BCEI forecasts also possess these attractive features.

Our estimation method also uses a quarterly time series of actual (realized) inflation rates. We constructed this monthly series of actual CPI inflation to correspond with the monthly CPI inflation forecasts.[22]

In addition, we obtained inflation swap rates for annual maturities from 2 to 10 years, as well as 12-, 15-, 20-, and 30-year maturities. All inflation swap rates are for the first trading day of each month. The 2- to 10-year swap maturities start in April of 2003, the 12-, 15-, and 20-year inflation swap rates start in November 2003, and the 30-year inflation swap rates start in March 2004.

While not used in the estimation of our model, we will compare our estimated model's implied yields for inflation-indexed bonds to the yields on TIPS. Data on zero coupon TIPS

[20] This survey was originally performed by the American Statistical Association and the National Bureau of Economic Research. The data is available at http://www.philadelphiafed.org/econ/spf/. See Croushore (1993) and Stark (2004) for details of this survey.

[21] SPF participants make forecasts at approximately, the middle of February, May, August, and November of each year. To align this survey with our other data, we presume these forecasts come at the start of the next month.

[22] Since survey participants are asked to forecast the seasonally-adjusted CPI inflation rate, our monthly time series is also based on the seasonally-adjusted CPI. This data is available at http://research.stlouisfed.org/fred2/categories/9. However, it should be noted that TIPS and zero-coupon inflation swaps are indexed to the CPI that is not seasonally-adjusted. This difference is unlikely to have much impact on TIPS prices and swap rates, except perhaps for those with very short times to maturity. The variation in the CPI due to seasonal adjustments is likely to be small compared to other sources of CPI variation, particularly for medium term and longer term horizons.

yields are obtained from Gurkaynak et al. (2008) who have fit zero coupon TIPS yield curves based on the yields of actual coupon-paying TIPS.[23] We will also analyze how our model's implied term structure components relate to stock market returns. Monthly returns on the Standard & Poor's 500 (S&P500 returns including dividend distributions) are obtained from CRSP.

4 Estimation Technique

Our empirical technique imposes model restrictions on both the cross-sectional and time-series properties of bond yields, inflation, inflation forecasts, and inflation swap rates. Given that our data is observed at a monthly frequency, the model's period is taken to be $\Delta t = 1/12^{\text{th}}$ of a year. This implies that the model's nominal short rate, i_t, is the one-month Treasury bill rate. Similarly, π_t is the rate of inflation expected over the next month, and r_t is the one-month real interest rate. Note that while these rates correspond to a one-month horizon, we express them in annualized terms.

Our method is similar to that used for maximum likelihood estimation of GARCH models, except that we allow for measurement error in most of the Treasury yields, inflation rate forecasts, and inflation swap rates. Denote $y_t(n_i)$ as the annualized, continuously-compounded yield observed at date t on a nominal bond maturing in n_i months, where $i = 1, ..., b$. We assume

$$\begin{aligned} y_t(n_i) &= -\frac{1}{n_i \Delta t} \ln \left[P(t, t + n_i \Delta t) \right] + \omega_{t,i} \\ &= \frac{1}{n_i \Delta t} \left[K_n + A_n \pi_t + B_n r_t + C_n \alpha_t + \Sigma_{j=1}^4 D_{j,n} h_{j,t}^2 \right] + \omega_{t,i} \end{aligned} \quad (25)$$

where $\omega_{t,i}$ is an independent measurement error distributed $N\left(0, w^2\right)$.

Similarly, let $s_t(n_{t,i}^b, n_{t,i}^e)$ be the annualized, continuously-compounded expected inflation rate over the period from the beginning date $t + n_{t,i}^b \Delta t$ to the ending date $t + n_{t,i}^e \Delta t$ reported from the survey at date t by BCEI or SPF, where $i = 1, ..., f$.[24] It is assumed to take the form

$$\begin{aligned} s_t(n_{t,i}^b, n_{t,i}^e) &= \frac{1}{(n_{t,i}^e - n_{t,i}^b)\Delta t} \left(E_t \left[\ln \left(\frac{I_{t+n_{t,i}^e \Delta t}}{I_t} \right) \right] - E_t \left[\ln \left(\frac{I_{t+n_{t,i}^b \Delta t}}{I_t} \right) \right] \right) + \upsilon_{t,i} \\ &= \frac{1}{\Delta n_{t,i} \Delta t} \left[\Delta K_{t,i}^* + \Delta A_{t,i}^* \pi_t + \Delta B_{t,i}^* r_t + \Delta C_{t,i}^* \alpha_t + \Sigma_{j=1}^4 \Delta D_{t,j,i}^* h_{j,t}^2 \right] + \upsilon_{t,i} \end{aligned} \quad (26)$$

where $\upsilon_{t,i}$ is an independent measurement error distributed $N\left(0, v^2\right)$, $\Delta n_{t,i} = n_{t,i}^e - n_{t,i}^b$, $\Delta K_{t,i}^* \equiv$

[23] Their dataset is available at http://www.federalreserve.gov/econresdata/researchdata.htm.

[24] We convert the annualized, quarterly-compounded rates reported in the surveys to annualized, continuously-compounded rates.

$K^*_{n^e_{t,i}} - K^*_{n^b_{t,i}}$, $\Delta A^*_{t,i} \equiv A^*_{n^e_{t,i}} - A^*_{n^b_{t,i}}$, $\Delta B^*_{t,i} \equiv B^*_{n^e_{t,i}} - B^*_{n^b_{t,i}}$, $\Delta C^*_{t,i} \equiv C^*_{n^e_{t,i}} - C^*_{n^b_{t,i}}$, and $\Delta D^*_{t,j,i} \equiv D^*_{j,n^e_{t,i}} - D^*_{j,n^b_{t,i}}$. For the monthly forecasts taken from BCEI, $\Delta n_{t,i} = 3$ months since each inflation forecast is for a future quarter of a year. However, starting in December of 1991, our data includes quarterly-frequency predictions by the SPF for the inflation rate over the next 10 years. Thus, for the months of December, March, June, and September, we have an additional forecast where $\Delta n_{t,i} = 120$ - 0 =120 months.

Furthermore, let $k^c_t(n_i) \equiv \ln[1 + k^*(t; t - d\Delta t, t + (n_i - d)\Delta t)]$ be the continuously-compounded inflation swap rate whose payment date, t_p, is n_i periods in the future (at date $t_p = t + n_i\Delta t$), where $i = 1, ..., p$. Since $n_i\Delta t = t_e - t_s$, based on (22) it is assumed to take the form

$$\begin{aligned} k^c_t(n_i) &= \frac{1}{n_i\Delta t}\ln\left(\frac{V^d(t, t - d\Delta t, t + (n_i - d)\Delta t)}{P(t, t + n_i\Delta t)}\right) + \mu_{t,i} \\ &= y_t(n_i) - y^r_t(n_i) + \mu_{t,i} \end{aligned} \tag{27}$$

where $\mu_{t,i}$ is an independent measurement error distributed $N(0, u^2)$ and where $y_t(n_i)$ and $y^r_t(n_i)$ are the continuously-compounded yields on zero-coupon nominal bonds and real bonds (TIPS), respectively, that make their payments at date $t_p = t + n_i\Delta t$. Substituting in from (12) and (19), (27) can be re-written as

$$k^c_t(n_i) = \frac{1}{n_i\Delta t}\left[\Delta K^\dagger_i + \Delta A^\dagger_i\pi_t + \Delta B^\dagger_i r_t + \Delta C^\dagger_i\alpha_t + \Sigma^4_{j=1}\Delta D^\dagger_{j,i}h^2_{j,t}\right] + u_{t,i} \tag{28}$$

where $\Delta K^\dagger_i \equiv \ln(I_t/I_{t-d\Delta t}) + K_{n_i} - \tilde{K}_{n_i-d}$, $\Delta A^\dagger_i \equiv A_{n_i} - \tilde{A}_{n_i-d}$, $\Delta B^\dagger_i \equiv B_{n_i} - \tilde{B}_{n_i-d}$, $\Delta C^\dagger_i \equiv C_{n_i} - \tilde{C}_{n_i-d}$, and $\Delta D^\dagger_{j,i} \equiv D_{j,n_i} - \tilde{D}_{j,n_i-d}$.

While most bond yields and inflation forecasts are assumed to be observed with error, we need to assume perfect observation of the short term (one-month maturity) nominal rate, $i_t = y_t(1)$ $= -\frac{1}{\Delta t}\ln[P(t, t + \Delta t)] = \pi_t + r_t - \phi_1 h^2_{1,t}$, and the survey inflation forecast at the one-month horizon, $I(t, t + \Delta t) = \frac{1}{\Delta t}E_t[I_{t+\Delta t}/I_t] = \exp(\pi_t\Delta t)$. These assumptions allow us to recover the exact one period real rate, $r_t = i_t - \pi_t + \phi_1 h^2_{1,t}$, given that $h^2_{1,t}$ is observed. Unfortunately, knowledge of the exact values of the i_t, π_t, and r_t by themselves, is not sufficient to update all of the volatility factors $h_{i,t}$, $i = 1, ..., 4$, because we also need to observe the central tendency, α_t. Therefore, we also assume that another particular maturity nominal bond yield is measured without error. For, example, if this particular yield has maturity n_x, then with $\omega_{t,x} = 0$ we have:

$$\begin{aligned} y_t(n_x) &= -\frac{1}{n_x\Delta t}\ln[P(t, t + n_x\Delta t)] \qquad (29) \\ &= \frac{1}{n_x\Delta t}\left(K_{n_x} + A_{n_x}\pi_t + B_{n_x}r_t + C_{n_x}\alpha_t + \Sigma^4_{j=1}D_{j,n_x}h^2_{j,t}\right) \end{aligned}$$

which implies:

$$\alpha_t = \frac{1}{C_{n_x}} \left(n_x \Delta t y_t \left(n_x\right) - K_{n_x} - A_{n_x} \pi_t - B_{n_x} r_t - \Sigma_{j=1}^4 D_{j,n_x} h_{j,t}^2 \right) \tag{30}$$

In principle, the perfectly observed yield $y_t\left(n_x\right)$ could be chosen from any one of the available yields in our data sample. However, because this yield is used to identify the central tendency, α_t, which largely determines the slope of the term structure, it would be reasonable to select a moderately long maturity bond yield. But since liquidity decreases as maturity expands, making the assumption of zero measurement error less plausible, as a compromise we select the five-year maturity ($n_x = 60$) as the bond maturity having no measurement error.

These assumptions allow us to observe π_t, r_t, and α_t and recover the $\epsilon_{j,t+\Delta t}$, $j = 1, ..., 4$ in equations (3) and (8). In turn, this allows us to update each of the volatility factors, $h_{j,t}$, $j = 1, ..., 4$. Given the state variables, $(\pi_t, r_t, \alpha_t, h_{j,t}^2$, $j = 1, .., 4)$ at date t, all of the theoretical zero coupon bond yields, inflation forecasts, and inflation swap rates can be computed. The difference between these theoretical quantities and their actual counterparts determine the measurement errors for bond yields, inflation forecasts, and inflation swap rates. The likelihood function can then be calculated recursively.

Let n_1, ..., n_b be the maturities of the b different bonds whose yields are assumed to be measured with error, let $(n_{t,1}^b, n_{t,1}^e)$,..., $(n_{t,f}^b, n_{t,f}^e)$ be the horizons of the f different inflation rate forecasts that are assumed to be measured with error at date t, and let n_1, ..., n_p be the maturities of the p different swap rates that are assumed to be measured with error. Note that at each monthly observation date, the bond yield maturities measured with error n_1, ..., n_b are the same, equal to 3, 6, 12, 24, 36, 84, 120, and 180 months. However, due to the nature of the inflation survey data, the number of inflation forecasts, f, and their horizons vary over different observation months. Similarly, the number of inflation swap rates, p, (but not their horizons) vary over different observation months. However, for a given observation month and number of inflation forecasts, f, and swap rates, p, define

$$
Y_t = \begin{pmatrix} ln\left(\frac{I_{t+\Delta t}}{I_t}\right) \\ \pi_{t+\Delta t} \\ r_{t+\Delta t} \\ \alpha_{t+\Delta t} \\ y_t(n_1) \\ \vdots \\ y_t(n_b) \\ s_t\left(n^b_{t,1}, n^e_{t,1}\right) \\ \vdots \\ s_t\left(n^b_{t,f}, n^e_{t,f}\right) \\ k^c_t\left(n_1\right) \\ \vdots \\ k^c_t\left(n_p\right) \end{pmatrix} \quad A_t = \begin{pmatrix} 0 \\ 0 \\ b_0\Delta t \\ c_0\Delta t \\ \frac{K_{n_1}}{n_1\Delta t} \\ \vdots \\ \frac{K_{n_b}}{n_b\Delta t} \\ \frac{\Delta K^*_{t,1}}{\Delta n_{t,1}\Delta t} \\ \vdots \\ \frac{\Delta K^*_{t,f}}{\Delta n_{t,f}\Delta t} \\ \frac{\Delta K^\dagger_{n_1}}{n_1\Delta t} \\ \vdots \\ \frac{\Delta K^\dagger_{n_p}}{n_p\Delta t} \end{pmatrix} \tag{31}
$$

$$
X_t = \begin{pmatrix}
\Delta t & 0 & 0 & -\frac{1}{2}\Delta t & 0 & 0 & 0 \\
1{+}a_2\Delta t & a_1\Delta t & \Delta t & 0 & 0 & 0 & 0 \\
b_2\Delta t & 1{+}b_1\Delta t & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1{+}c_1\Delta t & 0 & 0 & 0 & 0 \\
\frac{A_{n_1}}{n_1\Delta t} & \frac{B_{n_1}}{n_1\Delta t} & \frac{C_{n_1}}{n_1\Delta t} & \frac{D_{1,n_1}}{n_1\Delta t} & \frac{D_{2,n_1}}{n_1\Delta t} & \frac{D_{3,n_1}}{n_1\Delta t} & \frac{D_{4,n_1}}{n_1\Delta t} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{A_{n_b}}{n_b\Delta t} & \frac{B_{n_b}}{n_b\Delta t} & \frac{C_{n_b}}{n_b\Delta t} & \frac{D_{1,n_b}}{n_b\Delta t} & \frac{D_{2,n_b}}{n_b\Delta t} & \frac{D_{3,n_b}}{n_b\Delta t} & \frac{D_{4,n_b}}{n_b\Delta t} \\
\frac{\Delta A^*_{t,1}}{\Delta n_{t,1}\Delta t} & \frac{\Delta B^*_{t,1}}{\Delta n_{t,1}\Delta t} & \frac{\Delta C^*_{t,1}}{\Delta n_{t,1}\Delta t} & \frac{\Delta D^*_{t,1,1}}{\Delta n_{t,1}\Delta t} & \frac{\Delta D^*_{t,2,1}}{\Delta n_{t,1}\Delta t} & \frac{\Delta D^*_{t,3,1}}{\Delta n_{t,1}\Delta t} & \frac{\Delta D^*_{t,4,1}}{\Delta n_{t,1}\Delta t} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\Delta A^*_{t,f}}{\Delta n_{t,f}\Delta t} & \frac{\Delta B^*_{t,f}}{\Delta n_{t,f}\Delta t} & \frac{\Delta C^*_{t,f}}{\Delta n_{t,f}\Delta t} & \frac{\Delta D^*_{t,1,f}}{\Delta n_{t,f}\Delta t} & \frac{\Delta D^*_{t,2,f}}{\Delta n_{t,f}\Delta t} & \frac{\Delta D^*_{t,3,f}}{\Delta n_{t,f}\Delta t} & \frac{\Delta D^*_{t,4,f}}{\Delta n_{t,f}\Delta t} \\
\frac{\Delta A^\dagger_{n_1}}{n_1\Delta t} & \frac{\Delta B^\dagger_{n_1}}{n_1\Delta t} & \frac{\Delta C^\dagger_{n_1}}{n_1\Delta t} & \frac{\Delta D^\dagger_{1,n_1}}{n_1\Delta t} & \frac{\Delta D^\dagger_{2,n_1}}{n_1\Delta t} & \frac{\Delta D^\dagger_{3,n_1}}{n_1\Delta t} & \frac{\Delta D^\dagger_{4,n_1}}{n_1\Delta t} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\Delta A^\dagger_{n_p}}{n_p\Delta t} & \frac{\Delta B^\dagger_{n_p}}{n_p\Delta t} & \frac{\Delta C^\dagger_{n_p}}{n_p\Delta t} & \frac{\Delta D^\dagger_{1,n_p}}{n_p\Delta t} & \frac{\Delta D^\dagger_{2,n_p}}{n_p\Delta t} & \frac{\Delta D^\dagger_{3,n_p}}{n_p\Delta t} & \frac{\Delta D^\dagger_{4,n_p}}{n_p\Delta t}
\end{pmatrix} \tag{32}
$$

Then, our system of equations to be estimated can be written

$$
Y_t = A_t + X_t B_t + \Upsilon_t \tag{33}
$$

where

$$B_t = \begin{pmatrix} \pi_t \\ r_t \\ \alpha_t \\ h_{1,t}^2 \\ h_{2,t}^2 \\ h_{3,t}^2 \\ h_{4,t}^2 \end{pmatrix}, \Upsilon_t = \begin{pmatrix} \sqrt{\Delta t} h_{1,t}\epsilon_{1,t+\Delta t} \\ \sqrt{\Delta t}\sum_{j=1}^{2}\beta_j h_{j,t}\epsilon_{j,t+\Delta t} \\ \sqrt{\Delta t}\sum_{j=1}^{3}\gamma_j h_{j,t}\epsilon_{j,t+\Delta t} \\ \sqrt{\Delta t}\sum_{j=1}^{4}\rho_j h_{j,t}\epsilon_{j,t+\Delta t} \\ \omega_{t,1} \\ \vdots \\ \omega_{t,b} \\ \upsilon_{t,1} \\ \vdots \\ \upsilon_{t,f} \\ \mu_{t,1} \\ \vdots \\ \mu_{t,p} \end{pmatrix}. \tag{34}$$

The $\omega_{t,i}$ for $i = 1,..,b$, the $\upsilon_{t,i}$ for $i = 1,..,f$, and the $\mu_{t,i}$ for $i = 1,...,p$ are a sequence of independent normally distributed measurement errors.

Let Σ_t represent the variance covariance matrix of Υ_t. It has the block diagonal form:

$$\Sigma_t = \begin{pmatrix} \Delta t H_t & 0 & 0 & 0 \\ 0 & W & 0 & 0 \\ 0 & 0 & V & 0 \\ 0 & 0 & 0 & U \end{pmatrix} \tag{35}$$

where

$$H_t = \begin{pmatrix} h_{1,t}^2 & \beta_1 h_{1,t}^2 & \gamma_1 h_{1,t}^2 & \rho_1 h_{1,t}^2 \\ \beta_1 h_{1,t}^2 & \sum_{j=1}^{2}\beta_j^2 h_{j,t}^2 & \sum_{j=1}^{2}\beta_j\gamma_j h_{j,t}^2 & \sum_{j=1}^{2}\beta_j\rho_j h_{j,t}^2 \\ \gamma_1 h_{1,t}^2 & \sum_{j=1}^{2}\beta_j\gamma_j h_{j,t}^2 & \sum_{j=1}^{3}\gamma_j^2 h_{j,t}^2 & \sum_{j=1}^{3}\gamma_j\rho_j h_{j,t}^2 \\ \rho_1 h_{1,t}^2 & \sum_{j=1}^{2}\beta_j\rho_j h_{j,t}^2 & \sum_{j=1}^{3}\gamma_j\rho_j h_{j,t}^2 & \sum_{j=1}^{4}\rho_j^2 h_{j,t}^2 \end{pmatrix} \tag{36}$$

and where $W = w^2 I_b$, $V = v^2 I_f$, $U = u^2 I_p$, and I_b, I_f, and I_p are $b \times b$, $f \times f$, $p \times p$ identity matrices, respectively.

In principle, the model's 36 parameters can be estimated in one step using equation (33). However, note that the first element of Y_t is the process for the log of actual inflation, $\ln\left(I_{t+\Delta t}/I_t\right) = \pi_t\Delta t - \frac{1}{2}\Delta t h_{1,t}^2 + \sqrt{\Delta t} h_{1,t}\epsilon_{1,t+\Delta t}$. By estimating this equation alone using data only on I_t and π_t, we can recover estimates of the four parameters of the $h_{1,t}$ GARCH process, namely d_{10} (equivalently, $\overline{h}_1$), d_{11}, d_{12}, and d_{13}. Therefore, to make overall parameter estimation more manageable, we implement a two-step procedure where we first estimate the parameters of the $h_{1,t}$ process using data on only I_t and π_t. The 32 other parameters are estimated in a second step using equation (33) but with the parameters of the $h_{1,t}$ process fixed at those estimated in the first step. This two-step procedure would be equivalent to a one step weighted maximum likelihood procedure where the observations on the log of actual inflation, $\ln(I_{t+\Delta t}/I_t)$, are given much larger weights relative to those of the other observations.

5 Empirical Results

In this section, we first present estimates of the model's parameters and discuss their implications for state variable dynamics. Second, we consider the estimated model's implications for nominal and real yield curves as well as for real interest rate risk premia and inflation risk premia. Third, we compare our model's implied inflation-indexed bond yields to the yields on TIPS. Last, we explore movements in the model's term structure components to stock market returns.

5.1 Parameter Estimates and State Variable Dynamics

Table 1 presents results of the first step estimation of the parameters of the inflation volatility process, $h_{1,t}$, using data on the CPI (I_t) and the one-month forecast of inflation (π_t) derived from BCEI surveys over the period January 1982 to June 2008. The annualized, conditional standard deviation for inflation over a one-month horizon has a steady-state value of $\overline{h}_1 = 0.0083$; that is, 83 basis points.[25] The volatility of inflation displays significant GARCH effects: the estimated coefficient on the shock to inflation in the GARCH updating, d_{12}, is significantly positive. However, since d_{13} is insignificantly different from zero, there is no evidence that inflation's volatility responds asymmetrically to surprises.

Table 2 reports second step estimates of the model's other parameters. To gauge the statistical significance of permitting GARCH behavior, we estimated the unrestricted model as well as restricted models that assume some of the volatilities are constant; that is, $h_{j,t} = \overline{h}_j$. An assumption of constant volatility for a process $h_{j,t}$ entails the restrictions $d_{j1} = -1/\Delta t = -12$ and $d_{j2} = d_{j3} = 0$. The first column of Table 2 reports estimates assuming no GARCH behavior ($h_{j,t} = \overline{h}_j$, for $j = 2$, 3, and 4); the second column assumes GARCH behavior for only $h_{2,t}$; the third column assumes GARCH behavior only for $h_{3,t}$; and the fourth column assumes GARCH behavior for only $h_{4,t}$. Finally, the last column of Table 2 is the unrestricted model that permits GARCH behavior for $h_{2,t}$, $h_{3,t}$, and $h_{4,t}$.

Inspection of the log likelihood values for the different models at the bottom of Table 2 indicates that one can reject at the 1% level of significance the hypothesis of no GARCH behavior

[25] This is comparable to the annualized volatility of 87 basis points for inflation at a one-month horizon estimated by Jarrow and Yildirim (2003).

for all but one of the less restricted cases. Relative to the model with no GARCH behavior, the largest increase in likelihood value from permitting GARCH behavior for any single volatility process occurs with $h_{3,t}$, the volatility process for the independent component of the real interest rate, r_t. The second largest increase occurs when $h_{2,t}$ is able to display GARCH, which is the independent volatility component for expected inflation, π_t. The only instance where GARCH effects are not significant is for the independent volatility component of the central tendency. However, as indicated in the last column of Table 2, when $h_{2,t}$, $h_{3,t}$, and $h_{4,t}$ all are allowed to follow GARCH processes, one can reject the no GARCH restriction and each of the GARCH parameters (d_{22}, d_{32}, and d_{42}) are significantly positive. Based on these unrestricted model estimates and those for the inflation GARCH process in Table 1, measures of persistence for $h_{j,t}^2$, $j = 1, ..., 4$ can be computed. The half-life for a shock in $h_{j,t}^2$ to revert to its steady-state of $\overline{h}_j^2$ is 5.4 months, 3.8 months, 1.4 months, and 6.1 months for $j = 1$, 2, 3, and 4, respectively.[26]

It is noteworthy that we obtain reasonable estimates for the unconditional means of inflation and the real interest rate. For example, estimates using the unrestricted model give a value for $\overline{\pi}$ of 3.22% and for $\overline{r}$ of 1.57%. Allowing a central tendency for inflation also is important. In each of the estimations the estimate of the mean reversion parameter for c_1 is approximately -0.05 with a small standard error that makes it statistically different from zero. A model with no central tendency (α_t having a constant mean) would imply $c_1 = -1/\Delta t = -12$, so that a model lacking a central tendency is easily rejected by the data. In terms of the model's overall fit to the data, the estimated standard deviations of measurement errors for Treasury yields, survey forecasts of inflation rates, and inflation swap rates (w, v, and u) are 35 basis points, 39 basis points, and 27 basis points, respectively.

Given the unrestricted model's parameter estimates, we can also calculate the model's implied standard deviations and correlations for inflation, expected inflation, the real rate, and the central tendency. The values for these standard deviations and correlations come from the variables' covariance matrix, H_t, given in equation (36). Statistics for these values are given in Table 3.

The first column in Table 3 calculates the state variables' annualized standard deviations and correlations over a one-month horizon assuming that each of the GARCH processes are equal to their steady state values; that is, $h_{j,t} = \overline{h}_j$, $j = 1, ..., 4$. The real interest rate, r_t, and expected inflation, π_t, have the highest unconditional standard deviations of 3.18% and 2.66%, respectively. Conditional on its mean of π_t, the steady state one-month standard deviation of log inflation is 0.83% while the steady state standard deviation of the central tendency is 1.04%. One also sees that an innovation in actual inflation ($I_{t+\Delta t}$) has a 0.33 correlation with an innovation in expected inflation ($\pi_{t+\Delta t}$) and a 0.16 correlation with an innovation in

[26] Note from (8) that $E_t\left[h_{j,t+\Delta t}^2\right] = \varrho h_{j,t}^2 + (d_{j0} + d_{j2})\,\Delta t$, where $\varrho \equiv 1 + \left(d_{j1} + d_{j2}d_{j3}^2\right)\Delta t$. Thus, the half-life in periods (months) of length Δt is $\ln\left(\frac{1}{2}\right)/\ln(\varrho)$.

the central tendency (α_t). This suggests that when investors experience a positive inflation surprise, their one-month expectation of inflation is partially updated and, to a lesser degree, so is their longer-horizon expectation of inflation via the central tendency.

We also see that when starting from a steady-state, the one-month expected inflation and real rate are strongly negatively correlated at -0.844. This finding is consistent with Benninga and Protopapadakis (1983), Summers (1983), and Pennacchi (1991) who find that short run nominal interest rates do not adjust one-for-one with changes in real interest rates or expected inflation. Given that the Federal Reserve tends to keep short-maturity nominal interest rates stable by pegging the federal funds rate, this result might be expected. Controlling the short run nominal interest rate implies that any change in short run inflation expectations must result in an offsetting change in the short run real interest rate. Corroborating evidence that short term real interest rates are quite variable is found by Ang et al. (2008).

Of course, due to GARCH behavior, the standard deviations and correlations of the state variables are not constant. Columns two, three, and four of Table 3 calculate the model-implied average, minimum, and maximum of the state variables' standard deviations and correlations over the 1982 to 2008 sample period. As one might expect, the sample averages for standard deviations and correlations tend to be relatively close to their steady-state values. However, based on the minimum and maximum values, we see that standard deviations and correlations can vary significantly for most variables.

To illustrate this variation, Figure 1 displays the time series of the standard deviations of $\ln(I_{t+\Delta t}/I_t)$, π_t, r_t, and α_t. As one would expect, the standard deviations of expected inflation and the real interest rate were especially high during the early 1980s, a time when the Federal Reserve was battling to lower inflation expectations. Confirming the estimation results in Table 2, this figure also shows that there is little evidence of GARCH behavior for the central tendency.

Rather than their standard deviations, Figure 2 plots the model-implied levels of the state variables over the 1982 to 2008 period. The first panel in the figure indicates that the Federal Reserve was successful in lowering expected inflation. It shows that early in the period the central tendency for inflation was above short run expected inflation as investors apparently thought longer term inflation was likely to remain high. However, the Federal Reserve appears to have built credibility in lowering inflation, since the central tendency later declined to equal approximately the average of expected inflation. Early in 2008, the model is predicting that both expected inflation and its central tendency are on the rise.

The second panel in Figure 2 displays the one-month real interest rate, r_t, implied by our model estimates. Note that there was an unusually long period from 2002 to 2005 when it was negative. This finding supports the belief that a credit bubble may have been inflated by a policy of maintaining interest rates too low for too long. The figure also shows that at the beginning of 2008, the short run real interest rate is quite negative.

Figures 3 and 4 characterize the speeds of mean reversion for the state variables π_t, r_t, and α_t. Figure 3 presents impulse response functions that assume when there is a positive one standard deviation shock to a state variable, there is no instantaneous shock to the other state variables. It shows that there is somewhat stronger mean reversion for expected inflation compared to the real interest rate. A shock to the central tendency displays very weak reversion to its mean.

Figure 4 differs from Figure 3 in that when there is a positive one standard deviation shock to a state variable, the other state variables also suffer a shock commensurate with the estimated correlations given in Table 3. Under this scenario, mean reversion for expected inflation and the real interest rate becomes somewhat stronger than before. However, allowing for contemporaneous state variable shocks has little effect on the weak mean reversion of the central tendency.

5.2 Estimated Term Structures and Risk Premia

A basic question is whether our estimated model produces sensible-looking nominal and real (inflation-indexed) yield curves. The top panel in Figure 5 shows the unrestricted model's implied yield curves and expected inflation when each of the state variables is initially at its steady state level ($\pi_t = \widehat{\alpha}_t = \overline{\pi}$, $r_t = \overline{r}$, $h_{j,t} = \overline{h}_j$, $j = 1, ..., 4$). Indeed, the term structures do appear reasonable, even for maturities out to 30 years, a horizon where little data was used in the model's estimation. The slopes of the steady-state nominal yield curve (difference between yields and the one-month nominal rate i_t) equal 137, 192, 236, and 245 basis points at the 5-, 10-, 20-, and 30-year maturities, respectively. Similarly, the slopes of the real yield curve (difference between the yields and the one-month real rate r_t) equal 109, 150, 192, and 213 basis points at the 5-, 10-, 20-, and 30-year maturities, respectively. The substantial slope of the real yield curve contrasts with some other studies that find it to be relatively flat.[27] However, our model's real yield curve slopes are not that much larger than the average TIPS yield curve slopes over the January 1999 to June 2008 period. If we calculate the difference between average zero-coupon TIPS yields computed by Gurkaynak et al. (2008) and our one-month steady state real rate of $\overline{r} = 1.57\%$, then the 5-, 10-, and 20-year TIPS slopes average 76, 111, and 125 basis points, respectively.

The model's real (inflation-indexed) yields curves also look reasonable when plotted for each month in our sample period. Figure 6 shows the time-series of real yield curves. It indicates that real yields were high during the early 1980s, consistent with a tighter monetary policy whose goal was to bring down inflation expectations. In contrast, real yields have recently been

[27] For example, Ang et al. (2008) find an unconditional real yield curve that is slightly humped, peaking at a one year maturity before converging to 1.3% at a five-year maturity.

much lower. Figure 6 also confirms the earlier reported evidence that the volatility of short term real rates is high, though real yield volatilities lessen as maturities increase.

A component of the nominal term structure that has interested policymakers and academics is the term structure of inflation risk premia. There are at least two motivations for wanting to know this quantity. First, saving the cost of an inflation risk premium has been used to justify a government's issuance of inflation-indexed bonds. Second, one needs to subtract an inflation risk premium from the "break-even inflation rate" (difference between equivalent maturity nominal and inflation-indexed bonds) in order to construct a measure of inflation expectations.

We quantify the term structure of inflation risk premia, as well as the term structure of real interest rate risk premia, in the following manner. First, we compute nominal and real yield curves under the assumption that all of the market prices of risk equal zero; that is, $\phi \equiv (\phi_1\ \phi_2\ \phi_3\ \phi_4) = \mathbf{0}$. Recall that the yields on nominal and inflation-indexed bonds maturing in n_i periods are denoted as $y_t(n_i)$ and $y_t^r(n_i)$, respectively, so let their zero-risk premium counterparts be $y_t(n_i; \phi = \mathbf{0})$ and $y_t^r(n_i; \phi = \mathbf{0})$. As an illustration, the zero-risk premia nominal and real yield curves when all of the state variables are initially at their steady states are plotted in the bottom panel of Figure 5.

Second, define the date t nominal risk premium, $\Phi_t^n(n_i)$, and the real risk premium, $\Phi_t^r(n_i)$, for bonds maturing in n_i periods as:

$$\begin{aligned} \Phi_t^n(n_i) &= y_t(n_i) - y_t(n_i; \phi = \mathbf{0}) \\ \Phi_t^r(n_i) &= y_t^r(n_i) - y_t^r(n_i; \phi = \mathbf{0}) \end{aligned} \tag{37}$$

Finally, the inflation risk premium, $\Phi_t^{inf}(n_i)$ is defined as the difference between the nominal risk premium and the real risk premium for the same maturity:

$$\Phi_t^{inf}(n_i) = \Phi_t^n(n_i) - \Phi_t^r(n_i) \tag{38}$$

The term structures of nominal, real, and inflation risk premia when all of the state variables are initially at their steady states are plotted in Figure 7. Here, the term structure of nominal risk premia is simply the difference between the nominal yield curves in the top and bottom panels of Figure 5, while the term structure of real risk premia is the difference between the real yield curves in these same panels. The term structure of inflation risk premia in Figure 7 is then the difference between the nominal and real term structures of risk premia. When each of the state variables are initially at their steady states, we see that the real risk premia equal 111, 156, 212, and 250 basis points at the 5-, 10-, 20-, and 30-year maturities, respectively. The inflation risk premia equal 27, 51, 82, and 101 basis points at the 5-, 10-, 20-, and 30-year maturities, respectively.

We can also examine how these risk premia varied over time during our sample period. Figure 8 plots expected inflation, the real risk premium, and the inflation risk premium for a 10-year maturity during the 1982 to 2008 period. Interestingly, while inflation expected over 10 years varied substantially, the levels of the real and inflation risk premia did not. The real risk premium for a 10-year maturity bond varied from 150 to 170 basis points, averaging 157 basis points. This real risk premium is consistent with the substantial slope of the real yield curve discussed earlier. The inflation risk premium for a 10-year maturity bond varied from 38 to 60 basis points and averaged 51 basis points. These estimates of the 10-year inflation risk premium fall within the range of those estimated by other studies.[28]

Figure 9 plots the model's implied entire term structure of inflation expectations for each month of our sample period. Consistent with the earlier evidence in Figure 2 of a falling central tendency, one sees that inflation expectations generally have declined at all maturities. However, the term structure was often upward sloping during the mid-1980s, indicating that investors were not yet convinced that inflation would remain lower in the longer run. The figure also illustrates that expected inflation can be volatile at short maturities, but changes more smoothly at longer horizons.

5.3 Comparison to TIPS Yields

In the spirit of an out-of-sample test, we relate our model's implied yields for zero coupon inflation-indexed bonds to yields of actual zero coupon TIPS. We use zero coupon TIPS yields from Gurkaynak et al. (2008), which are available for the period January 1999 to June 2008. Taking their 5- and 10-year zero-coupon TIPS yields, we compare them to our unrestricted model's implied 5- and 10-year zero coupon yields for inflation-indexed bonds. The results of this exercise are given in Figure 10. This figure shows that our model significantly overvalues both the 5- and 10-year TIPS until about 2004. However, during the last four years, our model's yields and the TIPS yields appear to be tightly linked. One interpretation of this comparison is that our model performs poorly in pricing inflation-indexed bonds during the 1999 to 2004 period.

However, based on prior studies such as Sack and Elsasser (2004), Shen (2006), and D'Amico et al. (2008), a more likely interpretation is that TIPS were significantly undervalued prior to 2004. For example, D'Amico et al. (2008) find a large "liquidity premium" during the early years of TIPS's existence, especially before 2004. They conclude that until more recently, TIPS yields were difficult to account for within a rational pricing framework. Shen (2006) also finds evidence of a drop in the liquidity premium on TIPS around 2004. He notes that this

[28] For example, a 10-year inflation risk premium averaging 70 basis points and ranging from 20 to 140 basis points is found by Buraschi and Jiltsov (2005). Using data on TIPS, Adrian and Wu (2008) find a smaller 10-year inflation risk premium varying between -20 and 20 basis points.

may have been due to the U.S. Treasury's greater issuance of TIPS around this time, as well as the beginning of exchange traded funds that purchased TIPS. Another contemporaneous development that may have led to more fairly priced TIPS was the establishment of the U.S. inflation swap market beginning around 2003. Investors may have arbitraged the underpriced TIPS by purchasing them while simultaneously selling inflation payments via inflation swap contracts. In summary, the overall evidence supports the notion that our model can generate fair prices for inflation-indexed bonds.

5.4 Term Structure Shocks and Stock Returns

This section explores how our model's estimated shocks to the components of nominal and real term structures are related to stock market returns. In order to adequately describe stock return dynamics, we need to model a source of uncertainty that affects only stock returns that is in addition to those that affect nominal and real term structures. Toward this end, we incorporate a fifth stochastic shock, $\epsilon_{5,t}$, in the nominal pricing kernel:

$$\frac{M_{t+\Delta t}}{M_t} = e^{-i_t \Delta t - \frac{1}{2}\sum_{j=1}^{5} \phi_j^2 h_{j,t}^2 \Delta t - \sum_{j=1}^{5} \phi_j h_{j,t} \sqrt{\Delta t} \epsilon_{j,t+\Delta t}} \tag{39}$$

where $\phi_5 h_{5,t}$ is the market price of risk for $\epsilon_{5,t}$ and where

$$h_{5,t+\Delta t}^2 - h_{5,t}^2 = \left[d_{50} + d_{51} h_{5,t}^2 + d_{52} \left(\epsilon_{5,t+\Delta t} - d_{53} h_{5,t} \right)^2 \right] \Delta t. \tag{40}$$

We maintain all previous assumptions regarding I_t, π_t, r_t, α_t, and $h_{j,t}^2$, $j = 1,..,4$; that is, their dynamics do not depend on $\epsilon_{5,t}$. Since this fifth shock, $\epsilon_{5,t}$, is orthogonal to the state variables' dynamics, using the nominal pricing kernel in equation (39) will lead to exactly the same nominal and inflation-indexed security formulas as were derived using the nominal pricing kernel in equation (1). Hence, equation (39) is fully consistent with our previous model and empirical results.

Let us now assume that a stock portfolio with date t value S_t follows the process:

$$\frac{S_{t+\Delta t}}{S_t} = e^{\mu_{s,t} \Delta t - \frac{1}{2}\sum_{j=1}^{5} h_{j,t}^2 \Delta t + \sum_{j=1}^{5} \delta_j h_{j,t} \sqrt{\Delta t} \epsilon_{j,t+\Delta t}} \tag{41}$$

where, without loss of generality, it is assumed that $\delta_5 = 1$. One can solve for the stock portfolio's equilibrium expected rate of return, $\mu_{s,t}$, since

$$E_t \left[\frac{M_{t+\Delta t}}{M_t} \frac{S_{t+\Delta t}}{S_t} \right] = 1 \tag{42}$$

Substituting equation (39) and equation (41) into equation (42) implies

$$\mu_{s,t} = i_t + \sum_{j=1}^{5} \delta_j \phi_j h_{j,t}^2 \tag{43}$$

so that $\sum_{j=1}^{5} \delta_j \phi_j h_{j,t}^2$ is the stock portfolio's equity premium. Thus, the (continuously-compounded) rate of return on the stock portfolio follows the process

$$\ln\left(S_{t+\Delta t}/S_t\right) = \left[i_t + \Sigma_{j=1}^{5} \delta_j \phi_j h_{j,t}^2\right] \Delta t + \Sigma_{j=1}^{5} \delta_j h_{j,t} \sqrt{\Delta t} \epsilon_{j,t+\Delta t} \tag{44}$$

Given our prior empirical estimates, we have recovered $\epsilon_{j,t+\Delta t}$, $h_{j,t}$, and ϕ_j, $j = 1,..,4$, so that these values as well as i_t (the one-month Treasury yield) are known in equation (44). We need to estimate only the parameters δ_j, $j = 1,..,5$, ϕ_5, and the parameters of the fifth GARCH process in equation (40). This is done using monthly returns data on the S&P500 portfolio of stocks (including dividends) over the 1982 to 2008 sample period. The maximum likelihood estimates are given in Table 4.

Table 4 indicates that all of the parameters of $h_{5,t}$, the GARCH process that is unique to stocks, are statistically significant. The steady state standard deviation of this volatility process, $\overline{h}_5 = \sqrt{\left(d_{50} + d_{52}\right) / \left(d_{51} + d_{52} d_{53}^2\right)}$, is 15.45%. It is noteworthy that since d_{53} is positive, negative values of $\epsilon_{5,t}$ have a larger impact on increasing stock market volatility compared to similar-sized positive values of $\epsilon_{5,t}$.

Inspecting the estimated parameters determining the volatility of stock returns, one sees from the (marginally significantly) negative value of δ_1 that stock returns react negatively to surprises in the one-month actual inflation rate. Based on the small and insignificant values for δ_2 and δ_3, there does not appear to be much reaction by stocks to shocks from the independent components driving the one-month rate of expected inflation and the one-month real interest rate. However, based on the statistically significant negative value for δ_4, one sees that stock returns tend to fall when there is an unexpected rise in the independent component affecting inflation's central tendency.

One interpretation of these findings is that unexpectedly higher inflation, both in the short run and in the longer run, hurts stock returns. The economic channel through which this works might be tax distortions that are exacerbated by inflation.[29] Stock market returns appear to react little to changes in the one-month real rate and rate of expected inflation. As noted earlier, short run real rates and short run expected inflation are the most volatile of the model's

[29] As discussed by Feldstein (1980b) and Feldstein (1980a), the real value of a corporation's income tax deduction for depreciation expenses, which are based on historical cost, declines as inflation rises, leading to a higher real value of corporate taxation. In addition, the real value of personal taxes on nominal capital gains also rises as inflation increases. Thus, if the real after-tax returns on corporate cashflows received by investors declines with inflation, stock valuations will decline with unexpected rises in inflation.

state variables. However, recalling the results from our impulse response analysis, they both display strong mean reversion (while the central tendency does not). Since stocks are long-dated securities, it is not surprising that their values react minimally to short run shocks to expected inflation and real rates that do not persist. In contrast, since inflation's central tendency affects longer run inflation, and shocks to it are very persistent, it makes sense that it would have a significant impact on stock returns.

The net effect of term structure shocks on overall stock market volatility is relatively small, at least when volatilities are at their steady states. The total steady state stock return volatility equals $\sqrt{\sum_{j=1}^{5} \delta_j^2 \overline{h}_j^2} = 15.59\%$, which is fourteen basis points higher than the volatility deriving from the independent component of stocks. The net effect of term structure variables on the steady state equity premium is slightly negative. This steady state equity premium equals $\sum_{j=1}^{5} \delta_j \phi_j \overline{h}_j^2 = 6.97\%$, which is nineteen basis points lower than the component of the equity premium deriving from the orthogonal shock, $\phi_5 \overline{h}_5^2 = 7.16\%$. Stock returns' negative exposure to inflation shocks ($\epsilon_{1,t}$), which carry a positive risk premium, and their positive exposure to the real rate innovation ($\epsilon_{3,t}$), which has a negative risk premium, enables term structure uncertainty to reduce equity's risk premium.

6 Conclusion

This paper presents an equilibrium model of the term structures of nominal and real interest rates. Its factors include the short term real interest rate, the short term expected inflation rate, and the inflation rate's central tendency. Along with actual inflation, these factors are assumed to be driven by four volatility processes that follow the nonlinear asymmetric GARCH model of Engle and Ng (1993). By allowing for a changing central tendency for inflation and for changing volatilities for real rates and inflation, our model is able to account for changing monetary and real economic conditions.

Although our model permits state variables to have a general correlation structure with stochastic volatilities, it still leads to analytical solutions for the prices of nominal bonds and inflation-indexed bonds that have an indexation lag, such as TIPS. Closed-form solutions for expected inflation rates and equilibrium rates on inflation swaps also can be derived.

The model's parameters were estimated using data on nominal Treasury yields, survey forecasts of inflation, and inflation swap rates. We found that allowing for GARCH effects is particularly important for real interest rate and expected inflation processes, but that long-horizon real and inflation risk premia are relatively stable. Our estimate for the 10-year inflation risk premium averaged 51 basis points and varied between 38 and 60 basis points during the 1982 to 2008 sample period. Somewhat different from prior studies, we find a sizeable real interest rate risk premium at the 10-year maturity, averaging 157 basis points and varying between 150 and 170 basis points.

Comparing our model's implied yields for inflation-indexed bonds to those of TIPS suggests that TIPS were underpriced prior to 2004 but more recently are fairly priced. Hence, the 'liquidity premium' in TIPS yields appears to have dissipated. The recent introduction of inflation derivatives, such as zero coupon inflation swaps, may have eliminated this mispricing by creating a more complete market for inflation-linked securities.

Our estimated model also suggests that shocks to both short run and longer run inflation coincide with negative stock returns. An implication is that stocks are, at best, an imperfect hedge against inflation. This underscores the importance of inflation-linked securities as a means for safeguarding the real value of investments.

Appendix

Lemma 1

Let X be a standard normal random variable. Then for $Q_2 > -\frac{1}{2}$,

$$E\left[e^{Q_1 X - Q_2 X^2}\right] = e^{\frac{Q_1^2}{2(1+2Q_2)} - \frac{1}{2}\ln(1+2Q_2)} \tag{A.1}$$

Proof:

The expectation can be written as:

$$E\left[e^{Q_1 X - Q_2 X^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{Q_1 x - Q_2 x^2 - \frac{1}{2}x^2} dx \tag{A.2}$$

The result follows after completing the square and using properties of the normal density function.

Proof of Proposition 1

We begin by substituting the nominal pricing kernel into the bond pricing equation (12):

$$P(t, t+n\Delta t) = E_t\left[e^{-(\pi_t + r_t - \phi_1 h_t^2)\Delta t - \frac{1}{2}\sum_{j=1}^4 \phi_j^2 h_{j,t}^2 \Delta t - \sum_{j=1}^4 \phi_j h_{j,t}\sqrt{\Delta t}\epsilon_{j,t+\Delta t}} P(t+\Delta t, t+n\Delta t)\right] \tag{A.3}$$

Now assume the bond price has the form

$$P(t, t+n\Delta t) = e^{-(K_n + A_n \pi_t + B_n r_t + C_n \alpha_t + \sum_{j=1}^4 D_{j,n} h_{j,t}^2)} \tag{A.4}$$

and substitute equation (A.4) into the left- and right-hand sides of equation (A.3). Substituting in for the state variables at date $t+\Delta t$ using equation (8), collecting all coefficients of the random variables of the same type together, and then taking expectations using Lemma 1 leads to the resulting recursive equations for the coefficients. The initial boundary conditions come from considering the case when $n = 1$.

Proof of Proposition 2

Assume the structure for growth in the price level, $I(t, t+n\Delta t)$ has the following form:

$$I(t, t+n\Delta t) = e^{\bar{K}_n + \bar{A}_n \pi_t + \bar{B}_n r_t + \bar{C}_n \alpha_t + \sum_{j=1}^4 \bar{D}_{j,n} h_{j,t}^2} \tag{A.5}$$

Then:

$$I(t, t+n\Delta t) = E_t\left[\frac{I_{t+\Delta t}}{I_t} I(t+\Delta t, t+n\Delta t)\right] \tag{A.6}$$

Substituting in the assumed form for $I(t+\Delta t, t+n\Delta t)$, we obtain

$$I(t, t+n\Delta t) = E_t \left[e^{\pi_t \Delta t - \frac{1}{2} h_t^2 \Delta t + h_t \sqrt{\Delta t} \epsilon_{1,t+\Delta t} + \bar{F}_{n-1} + \bar{A}_{n-1} \pi_{t+\Delta t} + \bar{B}_{n-1} r_{t+\Delta t} + \bar{C}_{n-1} \alpha_{t+\Delta t} + \sum_{j=1}^{4} \bar{D}_{j,n-1} h_{j,t+\Delta t}^2} \right]. \tag{A.7}$$

Substituting in for the state variables at date $t+\Delta t$ using equation (8), collecting all coefficients of the random variables of the same type together, and then taking expectations using Lemma 1 leads to the resulting recursive equations for the coefficients. The initial boundary conditions come from considering the case when $n = 1$.

Proof of Proposition 3

Let t be the current date, $t_e = t + n\Delta t$ and $t_p = t_e + d\Delta t$. Suppressing t_s, we need to compute $V\left(t; t_e = t + n\Delta t\right)$. Then assume the following structure:

$$V^d(t; t+n\Delta t) = \frac{I_t}{I_{t_s}} e^{-\tilde{K}_n - \tilde{A}_n \pi_t - \tilde{B}_n r_t - \tilde{C}_n \alpha_t - \sum_{j=1}^{4} \tilde{D}_{j,n} h_{j,t}^2} \tag{A.8}$$

Now

$$V^d(t; t+n\Delta t) = E_t \left[\frac{M_{t+\Delta t}}{M_t} V^d(t+\Delta t; t+n\Delta t) \right] \tag{A.9}$$

Substituting in the structure for $V^d(t+\Delta t; t+n\Delta t)$ leads to:

$$V^d(t; t+n\Delta t) = E_t \left[\frac{I_{t+\Delta t}}{I_{t_s}} \frac{M_{t+\Delta t}}{M_t} e^{-\tilde{K}_{n-1} - \tilde{A}_{n-1} \pi_{t+\Delta t} - \tilde{B}_{n-1} r_{t+\Delta t} - \tilde{C}_{n-1} \alpha_{t+\Delta t} - \sum_{j=1}^{4} \tilde{D}_{j,n-1} h_{j,t+\Delta t}^2} \right] \tag{A.10}$$

This can be rewritten as:

$$V^d(t; t+n\Delta t) = \frac{I_t}{I_{t_s}} E_t \left[\frac{I_{t+\Delta t}}{I_t} \frac{M_{t+\Delta t}}{M_t} e^{-\tilde{K}_{n-1} - \tilde{A}_{n-1} \pi_{t+\Delta t} - \tilde{B}_{n-1} r_{t+\Delta t} - \tilde{C}_{n-1} \alpha_{t+\Delta t} - \sum_{j=1}^{4} \tilde{D}_{j,n-1} h_{j,t+\Delta t}^2} \right] \tag{A.11}$$

Substituting in for the nominal pricing kernel and inflation process using equations (1) and (3), as well as for the state variables at date $t + \Delta t$ using equation (8), collecting all coefficients of the random variables of the same type together, and then taking expectations using Lemma 1 leads to the resulting recursive equations for the coefficients.

The boundary conditions are obtained by recognizing that at date $t+n\Delta t$, the final payment is known, but is deferred by d periods. So the boundary conditions with no periods to go are given by the known payment multiplied by the d-period discount bond price, the formula for which is given in Proposition 1.

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Table 1: Inflation Process Parameter Estimation

The table shows the estimates for the following inflation process:

$$\frac{I_{t+\Delta t}}{I_t} = e^{\pi_t \Delta t - \frac{1}{2} h_{1,t}^2 \Delta t + h_{1,t}\sqrt{\Delta t}\epsilon_{1,t+\Delta t}}$$

$$h_{1,t+\Delta t}^2 - h_t^2 = [d_{10} + d_{11} h_{1,t}^2 + d_{12}(\epsilon_{1,t+\Delta t} - d_{13} h_{1,t})^2]\Delta t$$

$$\bar{h}_1^2 = -\frac{d_{10} + d_{12}}{d_{11} + d_{12} d_{13}^2}$$

Maximum likelihood estimates were obtained using 1982 to 2008 monthly data on the Consumer Price Index and one-month expected inflation derived from Blue Chip Economic Indicator consensus forecasts of CPI inflation.

Parameter	**Estimate**	***t*-Statistic**	***p*-Value**
$\bar{h}_1$	0.0083	3.98	0
d_{11}	-1.446	-1.86	0.063
d_{12}	1.23×10^{-4}	2.34	0.019
d_{13}	-5.86	-0.19	0.849
Observations		318	

Table 2: Real and Nominal Term Structure Parameter Estimates

The estimates of the parameters of the process below are reported for five different specifications that differ according to which volatility dynamics are permitted to have GARCH specifications. *w*, *v*, and *u* are the standard deviations of the measurement errors for nominal Treasury yields, survey inflation rate forecasts, and inflation swap rates, respectively. For each set of estimates, the parameters of the GARCH process for inflation (*h*1) are fixed at the point estimates reported in Table 1. Note: ***, **, and * denotes statistical significance at the 1%, 5%, and 10% level.

$$\begin{aligned}
\pi_{t+\Delta t} - \pi_t &= \left[\alpha_t + a_1 r_t + a_2 \pi_t\right] \Delta t + \sqrt{\Delta t}\Sigma_{j=1}^{2}\beta_j h_{j,t}\epsilon_{j,t+\Delta t} \\
r_{t+\Delta t} - r_t &= \left[b_0 + b_1 r_t + b_2 \pi_t\right] \Delta t + \sqrt{\Delta t}\Sigma_{j=1}^{3}\gamma_j h_{j,t}\epsilon_{j,t+\Delta t} \\
\alpha_{t+\Delta t} - \alpha_t &= \left[c_0 + c_1 \alpha_t\right] \Delta t + \sqrt{\Delta t}\Sigma_{j=1}^{4}\rho_j h_{j,t}\epsilon_{j,t+\Delta t} \\
h_{j,t+\Delta t}^2 - h_{j,t}^2 &= \left[d_{j0} + d_{j1} h_{j,t}^2 + d_{j2}\left(\epsilon_{j,t+\Delta t} - d_{j3} h_{j,t}\right)^2\right] \Delta t,\ j = 1, \ldots, 4
\end{aligned}$$

Parameter	No GARCH	h_2 Only	h_3 Only	h_4 Only	*All GARCH*
$\overline{\pi}$	0.0258***	0.0241***	0.0416***	0.0258***	0.0322***
a_1	0.6929***	0.5625***	0.4949***	0.6928***	0.4916***
a_2	-2.762***	-2.7236***	-2.3683***	-2.762***	-2.4224***
β_1	0.7405***	0.8285***	0.8323***	0.7415***	1.0669***
b_1	-1.8036***	-1.4668***	-1.2476***	-1.8027***	-1.2543***
b_2	2.7787***	2.2795***	1.848***	2.7773***	1.8482***
γ_1	-0.1311	-0.597***	-0.6659***	-0.131	-0.9093***
γ_2	-0.9822***	-1.3765***	-1.0281***	-0.9826***	-1.0295***
$\overline{r}$	0.0126**	0.0119**	0.0252***	0.0127**	0.0157**
c_1	-0.0541***	-0.051***	-0.0482***	-0.0541***	-0.0483***
ρ_1	-0.0427	-0.0188	0.1372**	-0.0418	0.1975***
ρ_2	-0.0242	-0.0365***	0.017	-0.0278	0.0346*
ρ_3	-0.0752***	-0.073***	-0.1408***	-0.0759***	-0.1586***
$\overline{h}_2$	0.02494***	0.02504***	0.024576***	0.02494***	0.02508***
d_{21}	-	-1.9082***	-	-	-1.9891***
d_{22}	-	0.001388***	-	-	0.001419***
d_{23}	-	-25.48***	-	-	0.08
$\overline{h}_3$	0.02571***	0.026851***	0.016703***	0.02571***	0.017029***
d_{31}	-	-	-6.033***	-	-6.0448***
d_{32}	-	-	0.001331***	-	0.001386***
d_{33}	-	-	33.33***	-	32.61***
$\overline{h}_4$	0.010536***	0.011916***	0.010296***	0.010488***	0.009849***
d_{41}	-	-	-	-11.9984	-2.2841***
d_{42}	-	-	-	0.000022	0.000031***
d_{43}	-	-	-	-0.51	-179.51***
ϕ_1	-60.06***	-58.2***	16.00***	-59.96***	19.93***
ϕ_2	39.54***	39.35***	4.06**	39.49***	-6.88***
ϕ_3	7.41	13.17***	-50.18***	7.44	-50.25***
ϕ_4	-19.27***	-13.25***	-27.54***	-18.40***	-36.27***
w	0.0038	0.0037	0.0035	0.0038	0.0035
v	0.0039	0.0038	0.004	0.0039	0.0039
u	0.0027	0.0026	0.0027	0.0027	0.0027
Ln Likelihood	32443.314	32515.5	32632.206	32443.314	32686.266
Reject No GARCH		Yes	Yes	No	Yes

Table 3: Estimated Standard Deviations and Correlations

The table reports the annualized standard deviations and correlations among the state variables for one-month horizons and are based on parameter estimates from the unrestricted model.

	1982 – 2008 Sample Period			
	Steady State	**Average**	**Minimum**	**Maximum**
Standard Deviations				
$\ln(I_{t+\Delta t}/I_t)$	0.0083	0.0077	0.0039	0.0154
$\pi_{t+\Delta t}$	0.0266	0.0258	0.012	0.046
$r_{t+\Delta t}$	0.0318	0.0335	0.0158	0.0718
$\alpha_{t+\Delta t}$	0.0104	0.0108	0.0085	0.015
Correlations				
$\ln(I_{t+\Delta t}/I_t)$, $\pi_{t+\Delta t}$	0.333	0.326	0.191	0.671
$\ln(I_{t+\Delta t}/I_t)$, $r_{t+\Delta t}$	-0.237	-0.216	-0.465	-0.071
$\ln(I_{t+\Delta t}/I_t)$, $\alpha_{t+\Delta t}$	0.158	0.143	0.065	0.3
$\pi_{t+\Delta t}$, $r_{t+\Delta t}$	-0.844	-0.785	-0.992	-0.306
$\pi_{t+\Delta t}$, $\alpha_{t+\Delta t}$	0.131	0.122	0.059	0.235
$r_{t+\Delta t}$, $\alpha_{t+\Delta t}$	-0.244	-0.274	-0.641	-0.123

Table 4: Stock Return Process Estimates

Maximum likelihood estimates were obtained using monthly returns on the S&P500 (including dividends) over the 1982 to 2008 sample period. Term structure variables and parameters were set to their values estimated from the unrestricted model. The total number of monthly observations is 312. The dynamics of the stock return are:

$$\ln\left(S_{t+\Delta t}/S_t\right) = \left[i_t + \sum_{j=1}^{5} \delta_j \phi_j h_{j,t}^2\right] \Delta t + \sum_{j=1}^{5} \delta_j h_{j,t} \sqrt{\Delta t} \epsilon_{j,t+\Delta t}$$

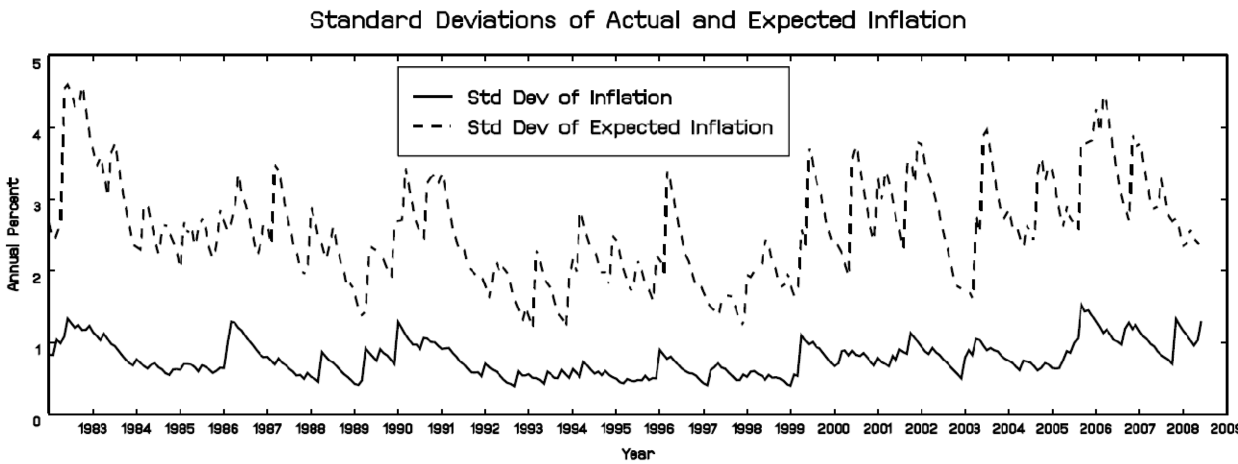
$$h_{5,t+\Delta t}^2 - h_{5,t}^2 = \left[d_{50} + d_{51} h_{5,t}^2 + d_{52}\left(\epsilon_{5,t+\Delta t} - d_{53} h_{5,t}\right)^2\right] \Delta t.$$

$$\bar{h}_5^2 = -\frac{d_{50} + d_{52}}{d_{51} + d_{52} d_{53}^2}$$

Parameter	Estimate	t-Statistic	p-Value
$\bar{h}_5$	0.1545	7.03	0.000
d_{51}	-8.5723	-3.8	0.000
d_{52}	0.05549	2.98	0.003
d_{53}	7.62	3.43	0.001
δ_1	-1.5414	-1.8	0.072
δ_2	-0.3543	-1.31	0.191
δ_3	0.3868	1.41	0.159
δ_4	-1.211	-2.32	0.021
ϕ_5	3.0011	2.2	0.028

Figure 1: Time Series of Volatilities of State variables

Using the parameter estimates of the full model with all GARCH effects we compute the time series of standard deviations of the state variables from 1982 to 2008.



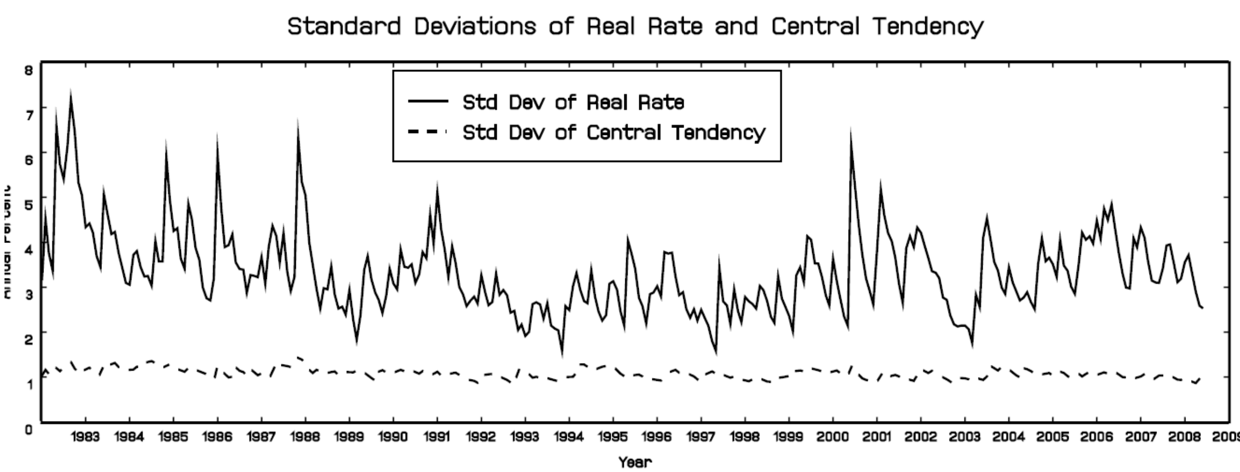
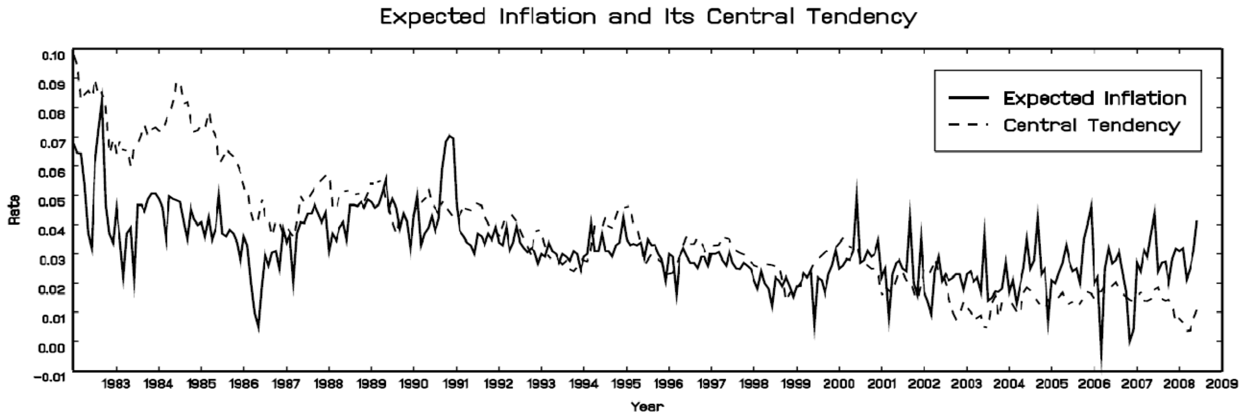


Figure 2: Time Series of State Variables

The figures below show the time series of Expected Inflation, its Central Tendency and Real Rates over the time span from 1982 to 2008. The estimated parameters for these time series come from using the full model with all GARCH effects.



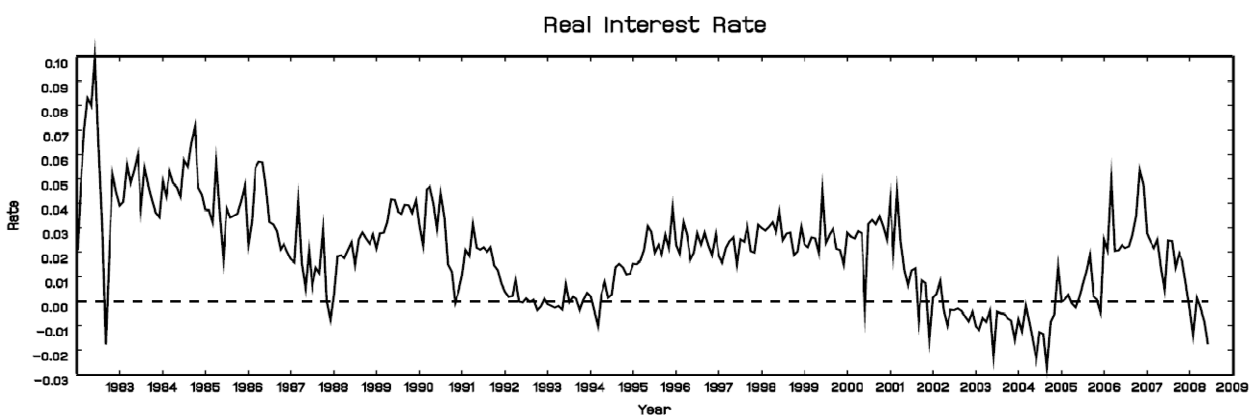


Figure 3: Impact of One Standard Deviation Shocks on State Variables

The figures show the expected path of Inflation, Real rates and Central Tendency following a one standard deviation move in each of these variables. The top panel shows the impact of shocks to Expected Inflation, the middle panel shows the sensitivity to shocks to Real Rates, and the bottom panel shows the sensitivity to shocks to the Central Tendency.

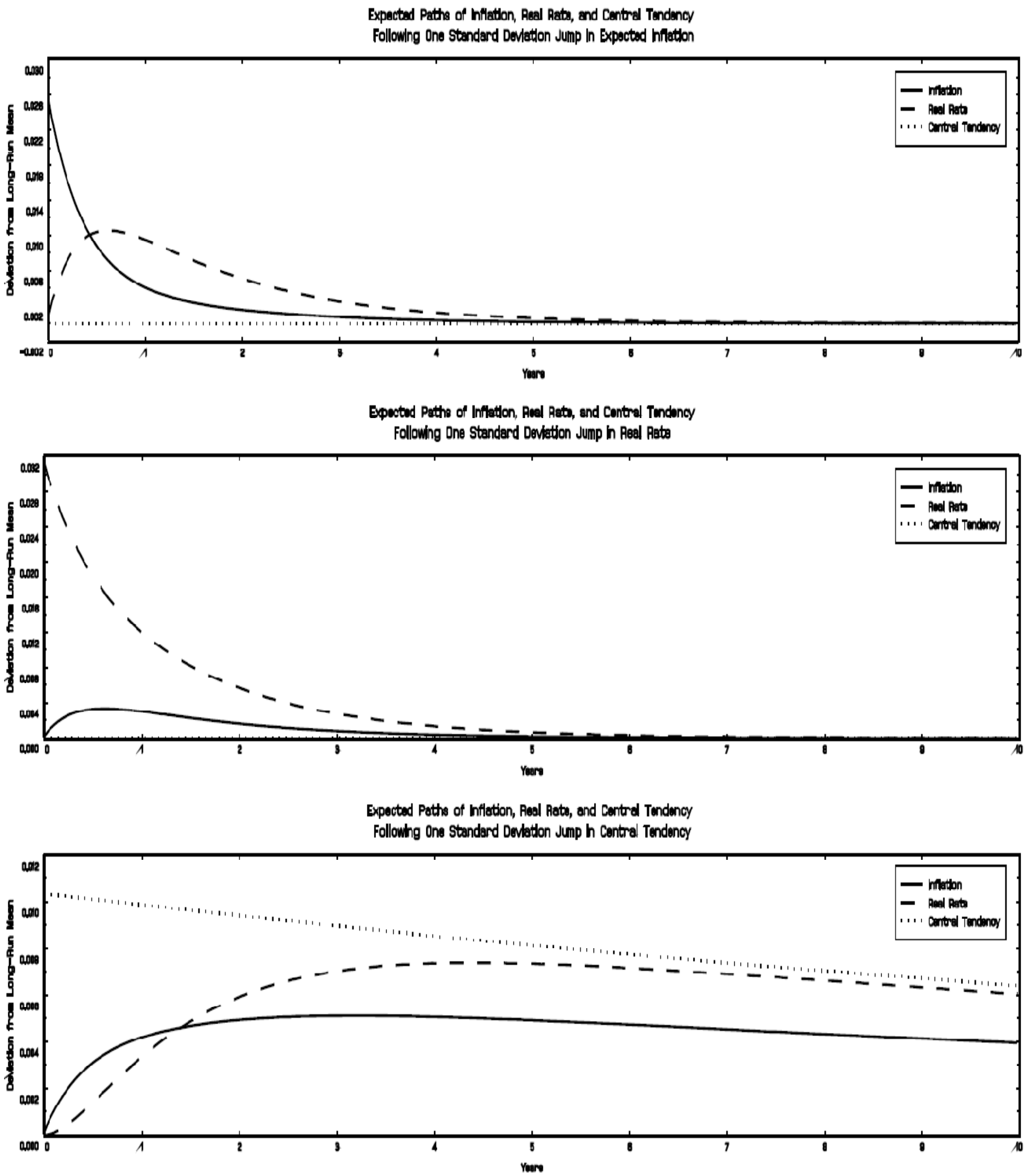


Figure 4: Impact of One Standard Deviation Shocks on State Variables

The figures show the expected path of Inflation, Real rates and Central Tendency following a one standard deviation move in each of these variables. The top panel shows the impact of shocks to Expected Inflation, the middle panel shows the sensitivity to shocks to Real Rates, and the bottom panel shows the sensitivity to shocks to the Central Tendency. Figure 4 differs from Figure 3 in that when there is a positive one standard deviation shock to a state variable, the other state variables also suffer a shock commensurate with the estimated correlations given in Table 3.

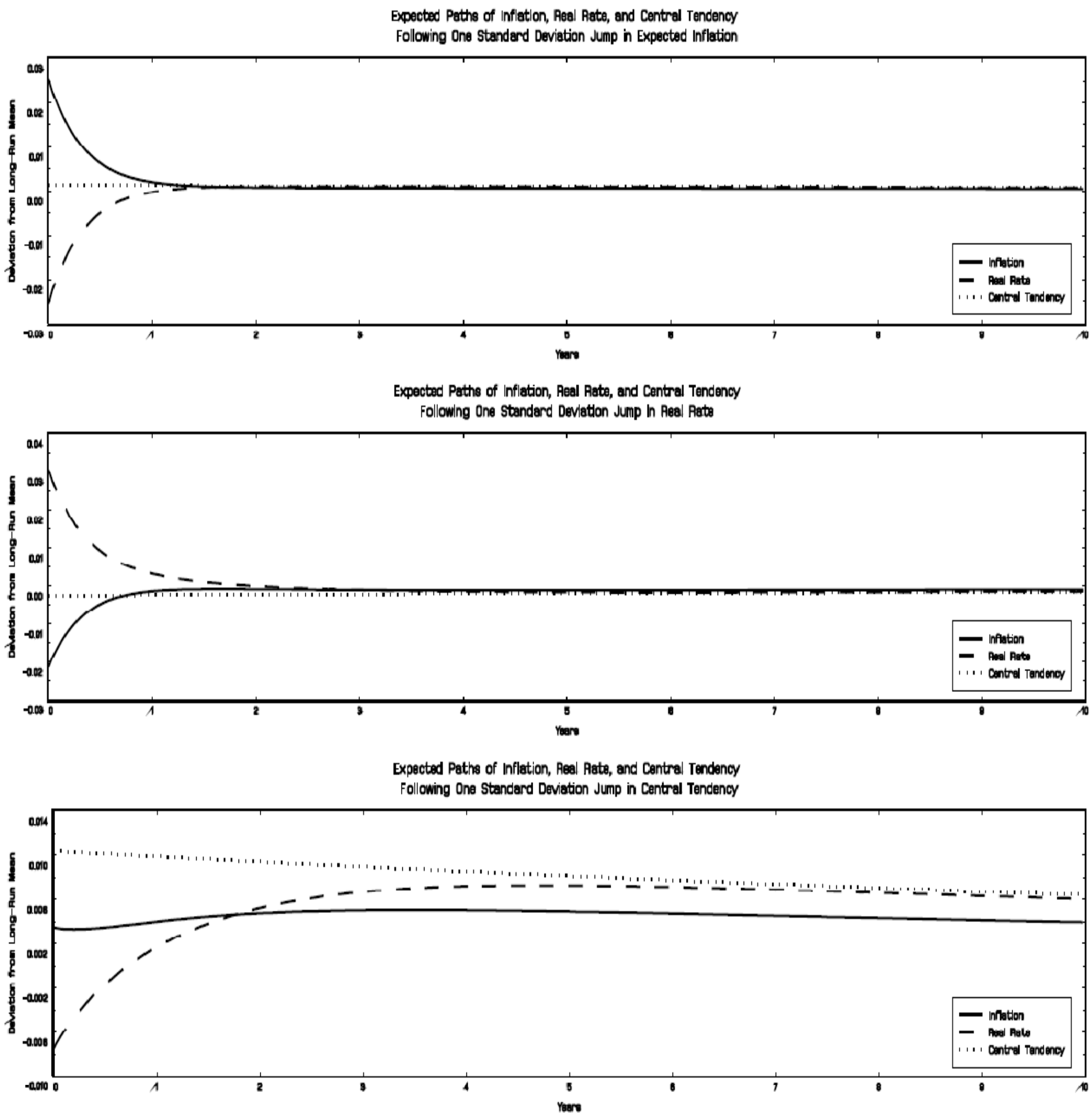


Figure 5: Real and Nominal Yield Curves with Inflation Expectations

The figure shows the nominal yield curve, the real yield curve and the term structure of expected inflation when the state variables are set equal to their steady state values.

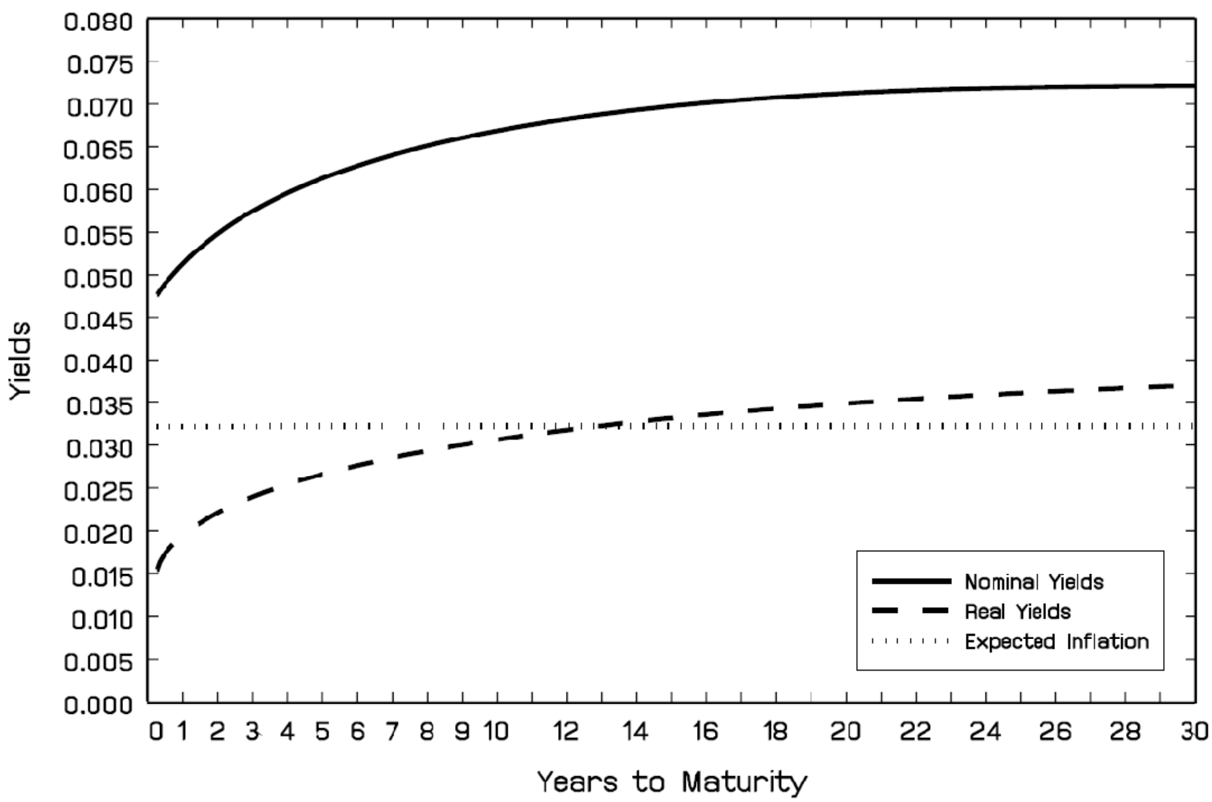


Figure 6: Time Series of Inflation Indexed Yield Curves

The figure shows the inflation indexed yield curves from 1982 to 2008. The yield curves were constructed using the parameter estimates from the full GARCH model as reported in Table 2.

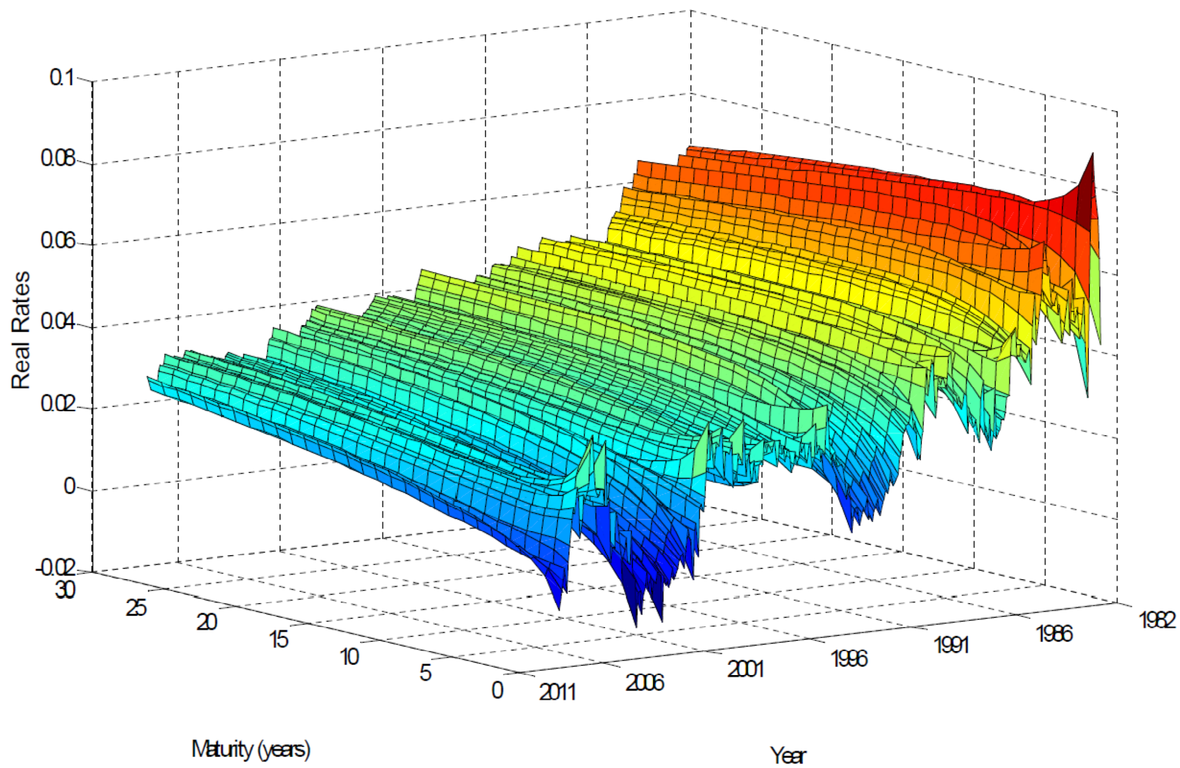


Figure 7: Real and Nominal Yield Curves with Inflation Expectations

The figure shows the nominal yield curve, the real yield curve and the term structure of expected inflation when the state variables are set equal to their steady state values but all market prices of risk are set equal to zero.

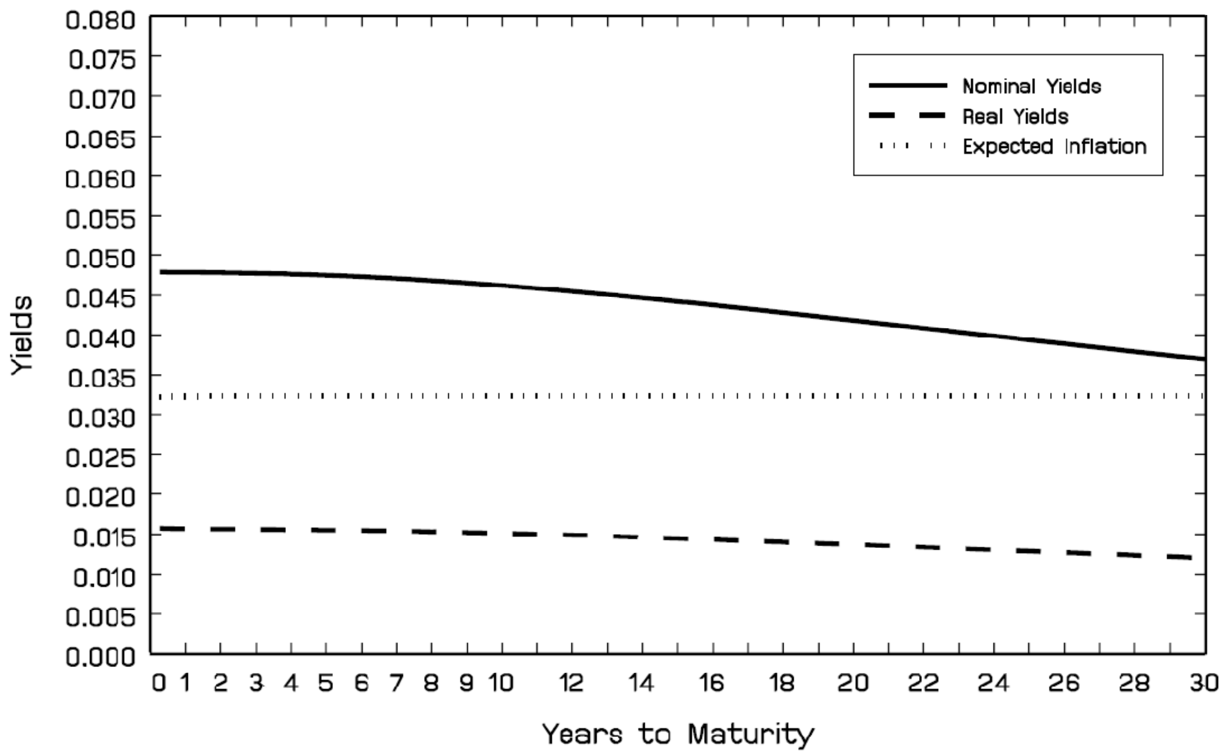


Figure 8: Nominal, Real and Inflation Risk Premia

The figure shows the term structure of risk premia for nominal and real yields as well as for expected inflation when all the state variables are initialized at their steady state levels.

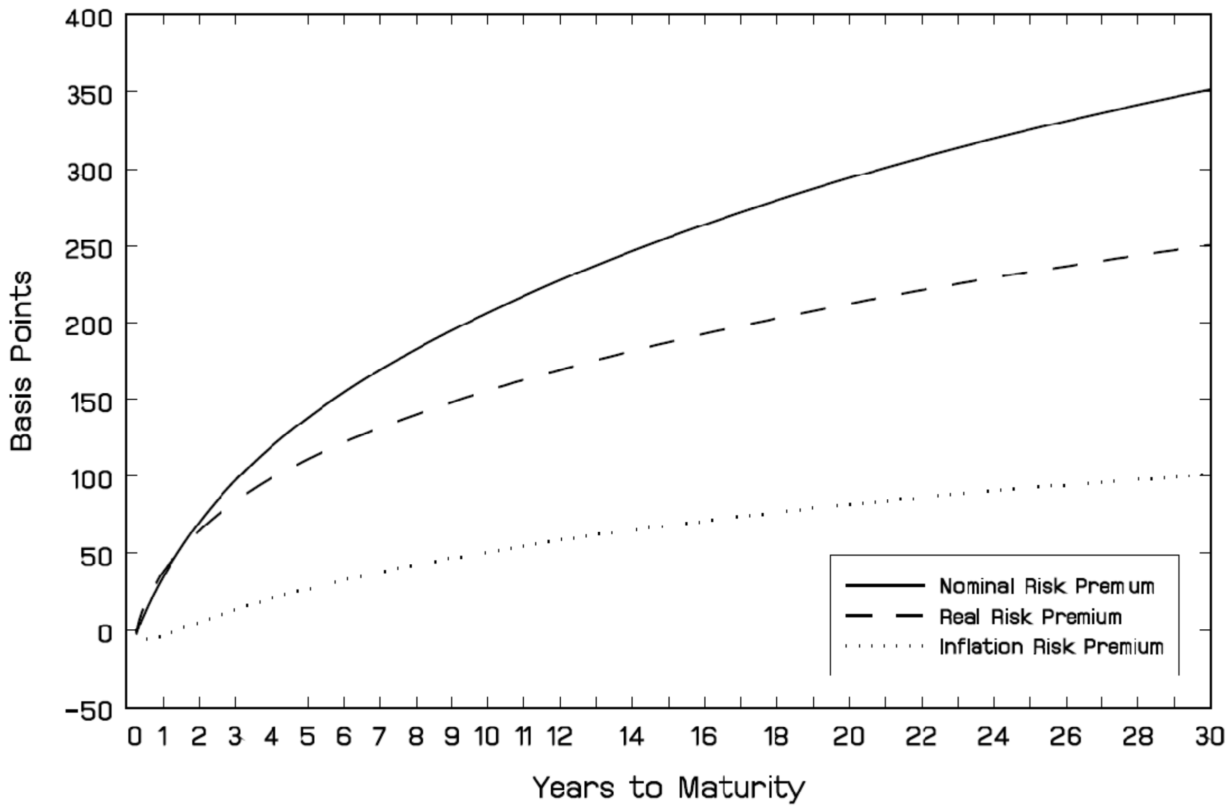


Figure 9: Ten Year Expected Inflation and real and Nominal Risk Premia

The Figure shows the time variation in the ten year risk premia. The state variables used for this analysis come from the full GARCH model, with parameter values provided in Table 2.

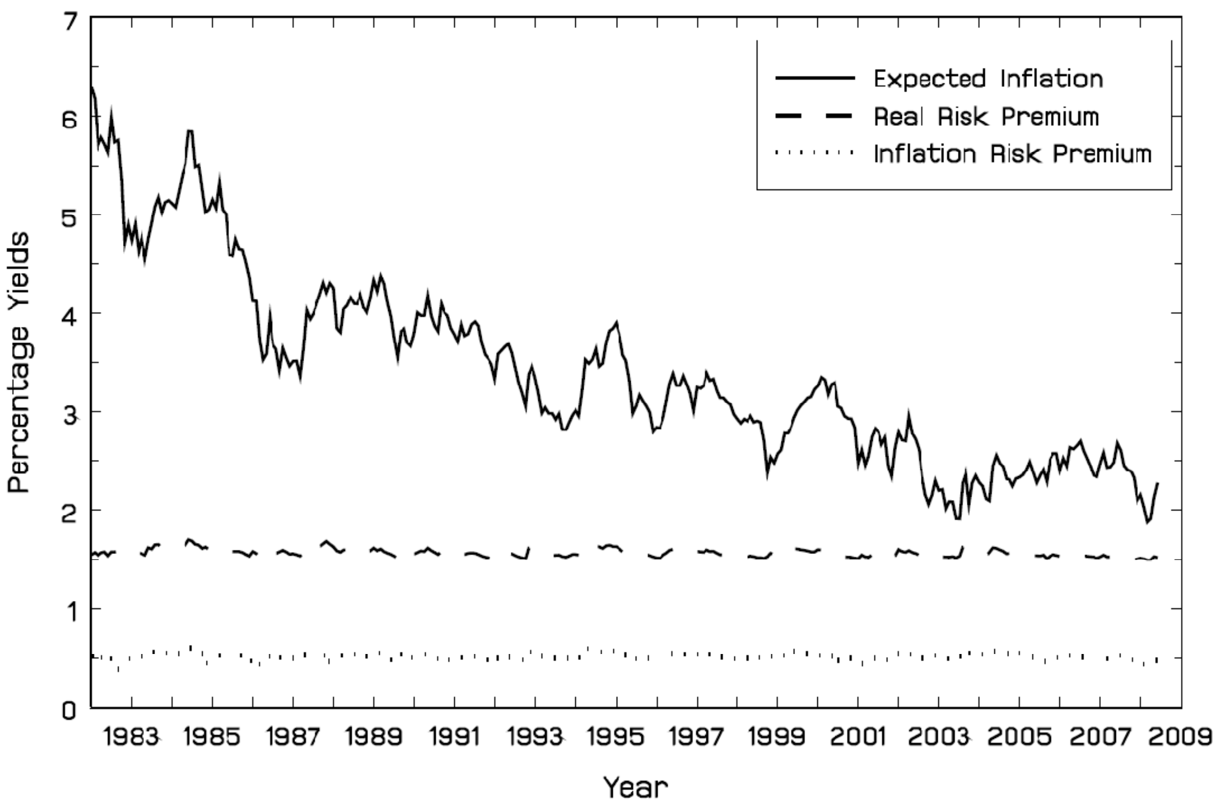


Figure 10: Term Structure of Expected Inflation

The Figure shows the time series of term structures of inflation expectations over the time period from 1982 to 2008. The parameter estimates for the figure correspond to the full GARCH model and are reported in Table 2.

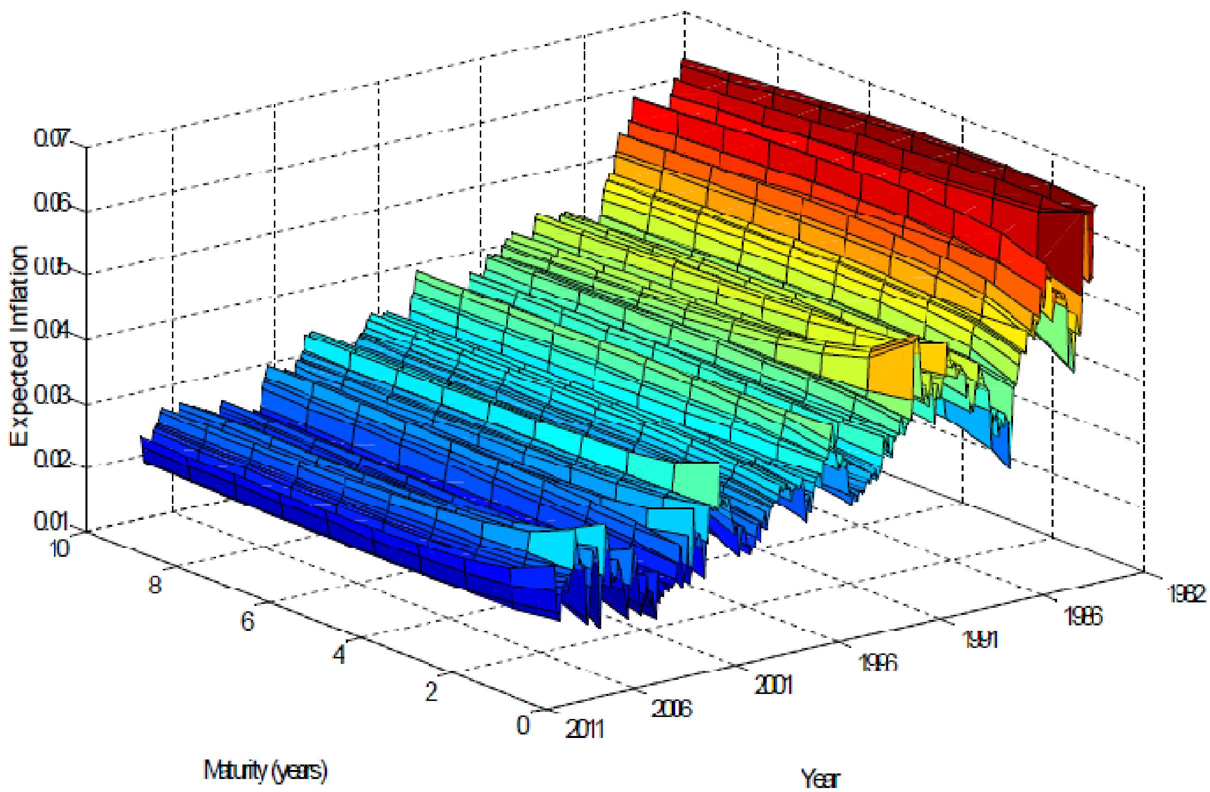


Figure 11: Five and Ten Year TIPS Yields versus Real Yields

The top panel compares the five year TIPS yields with the five year real yields produced by the full GARCH model. The bottom panel compares the ten year TIPS yields with the 10 year real yields. Data for the TIPS yields were obtained from Gurkaynak, Sack and Wright.