Enhanced Stress Testing and Financial Stability

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November 6, 2011

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Abstract

To date, regulatory stress tests have focused on ensuring the banking system is resilient to losses in one or a few stress scenarios that involve macro-economic weakness, but a theory of which stress-scenarios should be chosen to achieve systemic risk reduction objectives has not yet been developed. This paper proposes a framework for modeling systemic risk. Using the framework the paper analyzes current stress-testing practice and proposes a new approach to stress-testing and recapitalization policy that explicitly takes banks full set of risk exposures into account and is designed to ensure the banking system is robust to a wide set of shocks.

Keywords: Stress Testing, Financial Stability, Lending, Employment

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1 Introduction

The Great Recession underlined the role of the financial sector in real economic activity and highlighted the importance of assessing and monitoring financial stability. During the recession and the period that followed, the Federal Reserve conducted a series of stress tests, beginning with the Supervisory Capital Assessment Program (SCAP), and more recently the Comprehensive Capital Analysis and Review (CCAR). Both programs had the objective of improving financial stability through identifying vulnerabilities in banks' balance sheets, requiring more bank capital as needed, and through releasing information to the public on banks' performance. The stress tests proved to be useful in identifying capital shortfalls, and through increased capital and heightened transparency reduced investors uncertainty about the financial sector during turbulent times.[1] In the aftermath of the crisis, stress testing will be employed on a regular basis in the United States.[2] The European Banking Authority is also likely to continue stress testing European banks on a regular basis.

Broadly speaking there are two categories of stress-tests: micro-prudential stress tests, that are designed to assess the financial resilience of individual banks; and systemic tests that assess the resilience of the financial system. The focus of this paper is systemic tests. The financial system becomes systemically impaired when it cannot provide needed financial intermediation (FI) services to the real sector. Financial distress at a single bank will not necessarily cause systemic impairment if other institutions can step in and provide the services that cannot be provided by the distressed institution. However, if banks representing enough intermediation activity experience financial distress at the same time, then others may be not able to substitute for the lost FI services, and systemic impairment results.

This paper defines systemic risk as the risk that the financial system becomes systemically impaired. Systemic stress tests reduce systemic risk by reducing the likelihood that banks experience joint financial distress.[3] To do so, one or a few adverse macro-economic scenarios are applied to all banks. Based on the banks' losses, capital is increased to ensure that the banks can together adequately weather the adverse scenarios.[4,5]

[1]Supervisors requested 10 out of 19 banks participating in SCAP to raise $75 billion. All SCAP procedures and findings were made publically available for purposes of clarifying the SCAP process (Board of Governors of the Federal Reserve System, 2009a, 2009b). See Hirtle, Schuermann, and Stiroh (2009) for an overview of the SCAP. See Peristiani, Morgan, and Savino (2010) for an event study of analysis how the SCAP reduced uncertainty about banks.

[2]The Dodd-Frank Wall Street Reform and Consumer Protection Act mandates the Federal Reserve to perform annual stress tests of the major financial institutions. The institutions will also conduct their own stress tests. Institutions with total assets of at least $ 10 billion will stress their portfolios once a year, while institutions with at least $50 billion will conduct biannual tests.

[3]As noted above, the other way is through the provision of information.

[4]This paper does not take a stand on whether the capital should be privately raised or provided by the government

[5]The CCAR employed several stress-tests in addition to a single stress scenario applied to all banks. More specifically, in CCAR both supervisors and the banks forecasted banks' losses over a nine-quarter horizon under a baseline scenario, a stress scenario, and an adverse supervisory stress scenario. The baseline scenario was generated

The benchmark for the success of a program of systemic risk stress-testing and capital setting policy is its ability to make the provision of financial intermediation services robust to the severe but plausible shocks that can affect the financial system, and to accomplish this at low cost.

Relative to this benchmark, there are two important areas in which systemic risk stress-testing practices in the US can be refined. First, the systemic risk stress-tests are primarily focused on stresses emanating from the macro-economy to the banks; there is not a formal method for determining what other source of shocks should be considered. Second, the US approach to systemic risk stress-testing and capital policy make the banking system resilient (robust) to the stress-scenarios that are used in the test. But, because it is not part of the design, it is not clear how far this resilience extends against the other shocks that the banking system may face.

The first area in which stress-testing could be improved is that stress scenarios should be based on all of the risk exposures that banks face. During the financial crisis of 2008-2009 macro-economic weakness represented one of the most important threats to the banking system; and problems with the banking system would have made the macro-economic situation even worse. Systemic risk stress-tests based on a macroeconomic scenario were used then to shore-up the banking system and the macroeconomy against this threat.

During good times, the more relevant threats to the banking system and real activity may stem not from the US macroeconomy but rather from banks other risk exposures. The early stages of the recent financial crisis is a good example since the problems in the financial system preceded much of the weakness in the macroeconomy. Other examples are risks that emanate from financial institutions' overseas exposures. Historical examples include bank branches of Japanese banks cut back their US lending in response to the bursting of the real estate bubble in Japan in the late 80s; debt problems in Mexico, Brazil, and Argentina in the early '80s created large losses for some US banks; more recently there is a concern that direct or indirect exposures to European sovereign debt exposure could spillover to US markets and financial institutions. In all of these cases, important risk exposures and shocks could be missed if stress scenarios are chosen only based on U.S. macro-stresses without consideration of banks exposures to all sources of risk.

The second area where stress-testing and recapitalization practices should be enhanced is that they should be explicitly designed for robustness. A robust approach ensures that the financial system is not only well capitalized and resilient against the stress scenario under consideration, but also against a much broader set of plausible shocks to the financial system.

Ensuring robustness of the financial system does not mean that the financial system should

by the banks and reflected their expectations of the most likely path of the economy. The stress scenario was also generated by the banks and was targeted at stressing key sources of their revenue and loss. The adverse supervisory stress scenario was generated by the Federal Reserve and was intended to represent developments in the recession specified in terms of key macroeconomic indicators. Specifically, the Federal Reserve assumed a negative economic growth and a rise in unemployment for several quarters, among other developments.

be able to weather all shocks, but rather that it should be unlikely that distress spillovers from the financial system to affect the real economy. A physical analogy is useful: there will always be some floods that are large enough that they can overwhelm a dam of any size and spillover to the valley below; the robustness of the dam can be measured by the probability that it will not be overwhelmed. Similarly, the robustness of financial safeguards (such as capital buffers) can be measured by the probability or confidence that the financial sector is resilient in the sense that shocks to the financial sector will not spillover and harm the real economy. For example, a financial system is robust with confidence level 99 percent if its performance is resilient with probability of at least 99 percent.

The current US approach to systemic risk stress-testing, because it focuses on adverse scenarios, is likely to provide some amount of resilience of the financial system to other plausible shocks. But the extent of robustness is unclear. Geometrically, if a set of plausible shocks that could occur with 99 percent probability lie within a closed region such as the points in a multidimensional rectangle, then an approach to stress-testing that guarantees financial system resilience to all of the points in the region is robust with at least 99 percent confidence. By contrast the current US approach guarantees resilience to the scenarios used in the stress test, but the extent of resilience to other shocks in the region is unclear.

This paper's major contribution to stress-testing is that it analyzes systemic risk, stress testing, and capital policy within the framework of a formal model. Using the model, weakness in current stress testing practice are identified, and improvements to stress testing methodology are developed.

Stress testing practice is refined in two ways. . First, banks' risk exposures are used to construct scenarios. Choosing scenarios based on exposures helps to prevent important risk sources from being overlooked. Even if all potential risk sources are accounted for, it is important to use exposure information to choose the best direction of stress factor movements when forming a stress test. As a simple example of this principle, a downward move in stock prices is not stressful to an investor that is short the market, but an upward move will be.

Second, stress-scenarios are chosen using a methodology that is designed to make the financial system robust to a large set of shocks. To generate robustness, the scenario is a worst case for the financial sector among a set of scenarios that may occur with some chosen probability level such as 99 percent. If capital policy makes the financial system resilient against this worst-case shock, then the financial system should also be resilient to the set of all less severe shocks. Thus in the 99 percent region example, the new approach to stress-testing would make the financial sector resilient to the 99 percent of shocks that could occur within the region.

Implementing this new approach to systemic-risk stress-testing require new ways of thinking about how to use and collect supervisory information on banks and financial institutions. In particular, in the new approach, supervisory information on exposures will be used for constructing systemic risk stress tests. Therefore, exposure information should be collected and stored in ways that make it most useful for creating stress scenarios and conducting supervisory stress-tests.

In addition, tailoring stress-tests to achieve robustness with some degree of probabilistic confidence, requires models of the joint probability distribution of the risk factors that affect financial institutions. Therefore, models of the joint distribution of the factors should be further developed and refined for use in systemic-risk stress-testing.

The proposed stress-testing methodology is designed to achieve financial sector stability most of the time, where the financial sector is considered "stable" if it is able to withstand stressful events that result in losses among financial institutions without having a material effect on the real economy. This view of stability is adopted from Eric Rosengren (2011) who defines financial stability as "the ability of the financial system to consistently supply the credit intermediation and payment services that are needed in the real economy if it is to continue on its growth path."[6]

The remainder of the paper contains 4 sections. The next section illustrates the main ideas in the paper. Using a simple one-bank example, it illustrates shortcomings in current stress-testing practices, and illustrates our new approach for generating stress scenarios. Section 3 shows how a similar approach can be used to design stress scenarios and capital injections to control systemic risk in the banking system. Section 4 presents a concrete example of the stress-testing approach. A final section concludes.

2 Choosing a Stressful Scenario

This section discusses how to use information on banks' risk exposures to construct stress scenarios. The analysis starts from the assumption that banks portfolio risks can be represented by a potentially very large but common set of risk factors f. The risk factors represent innovations in the inputs that the banks use to value their assets and liabilities. Alternatively the risk factors may represent the factors that banks use to model the riskiness of their assets and liabilities as part of their risk management and measurement functions.

Let $V_{i,0}$ be the value of bank i today and $V_{i,T}$ be the value of bank i at date T, a future date of interest. The banks' value represents the market value of its assets minus the market value of its liabilities. Using Taylor series, the change in bank i's value, denoted ΔV_i $(= V_{i,t} - V_{i,0})$ can to

[6]There are many other notions of financial stability or instability. Mishkin (1999), for example, defines financial instability as misallocation of capital as a result of a disruption in information flows between potential borrowers and lenders. Capital misallocation results in inefficiency in the sense that capital does not necessarily flow to those with the most productive investment opportunities. Borio and Drehmann (2009) separate financial instability from financial distress. They define financial instability as a property of financial markets that may cause financial distress. Financial distress, in turn, is an event in which losses at financial institutions affect the real economy in terms of foregone output.

first and second order can be approximated as a function of f_T, the factor realizations at date T.

$$\Delta V^i = \approx \delta_i' f_T; \tag{1}$$

and,

$$\Delta V^i \approx \delta_i' f_T + .5 f_T' \Gamma_i f_T; \tag{2}$$

For simplicity, the analysis in the text will concentrate on the first-order approximation. Analysis for the second order approximation is contained in the appendix.

It is useful to begin by highlighting the main deficiency of the macro-scenario based approach to systemic-risk stress-testing that is used in both the US and Europe. The macro-scenario based approach posits a macro-scenario, and then examines how well each bank performs under the scenario. To do so, a macro scenario is generated, and then its effects are projected onto the set of risk factors that affect the bank, producing a particular realization of f_T, call it $\tilde{f_T}$. The bank is evaluated based on $\delta' \tilde{f_T}$, the loss due to $\tilde{f_T}$. If this creates too large a loss, then the bank is deemed to be undercapitalized; but is viewed as well capitalized otherwise.

The following textbook example illustrate one of the potential problems with this approach. Consider a bank with, for expositional purposes, a very basic balance sheet. The bank funds itself with equity = \$1 million, and insured deposits which have value \$4 million. On the asset side, its portfolio has value \$ 5 million, which consists of cash holdings, a short position in one year zero coupon bonds, and a long position in 10 year zeros, as detailed in Table 1. The current zero coupon yield curve is modeled to be flat at 1 percent.

For the purposes of illustration, in this example, I assume stress-tests are being used with the objective of ensuring that the bank has enough equity capital (currently = \$1 million) to survive over the next three months with probability exceeding 99 percent. This objective will be achieved by evaluating the bank's capital under the stress scenario, and then requiring the bank to have more capital if it is needed. For illustrative purposes, in the example I assume the value of the bank's liabilities, which consist of insured deposits, are not affected by changes in interest rates.

Given the above objective, how should a stress scenario for the bank be chosen? One method that is often used, labelled the ES approach, chooses stresses that are extreme scenarios where an extreme stressful scenario specifies an extreme move for some variables and then sets other variables to their expected values given the extreme moves. In the yield curve example with two bonds, let X denote the yield change on one bond and let Y denote the yield change for the other. An example of an extreme move corresponds to setting X at its 1st or 99th percentile $[X \in \{X_1, X_{99}\}]$, and then choosing the yield for the other bond, Y so that $Y = E[Y|X]$. Depending on which bonds are Y and X, this generates four possible extreme stressful scenarios.

An alternative top-down approach, labelled the PS approach, shifts all risk factors in a class by the same amount; i.e. a parallel shift. In the context of the bond example, the stress scenario in the PS approach would shift both bonds yields by the same amount.

The effects that both approaches to stress-testing have on setting capital requirements, as well as as a third alternative, the max-loss approach (the approach advocated in this paper) are illustrated in Figure 1.

It is useful to begin with the maximal loss approach. To illustrate that approach the figure has a parallelogram shaped region that contains 99 percent of the probability mass from the joint distribution of the 1- and 10-year yield changes over a 3-month period. Of the yield changes inside the region the one that generates the largest losses results in a loss of approximately 2.03 million dollars. Because this loss is the maximal loss inside a region that contains 99% probability, it immediately follows that if the bank was required to hold sufficient economic capital to perform its financial intermediation activities after absorbing the 2.03 million dollars in losses associated with the worst scenario for the bank inside the region, then with probability of at least 99 percent the bank will be able to continue its financial intermediation activities given its risk exposures. Put differently, setting the bank's capital based on this worst-case shock makes the bank robust to a wide-set of shocks that in this case have probability of at least 99%.

It is useful to instead consider what would happen if the bank chose its capital based on the other ways of generating extreme stressful scenarios. Of the four extreme scenarios, the one that generates the most extreme losses generates a loss of $ .85 million (labeled -.85 in Figure 1) over a 3 month period. The other 3 scenarios generate a loss of $ 0.21 million or gains of $ 0.21 million and $ 0.85 million.

It turns out that the true 99th percentile of the portfolio's losses is about 1.3 million dollars of loss. Therefore, all four extreme scenarios turn out to greatly understate the economic capital that is required. If instead the stress-scenario that is used to set capital charges was based on parallel shifts in the yield curve, then the amount of capital held would also be inadequate. To illustrate, note that the bank's position is hedged against parallel yield curve changes. As a result, parallel yield curve changes, which occur along the blue 45 degree line in the figure generate almost no changes in the value of the bank's portfolio. This means if capital was naively set based on a parallel yield curve shift stress-test, then the amount of capital that was held would be small and would clearly be insufficient to achieve 99 percent confidence.

There are two reasons why the ES and PS approaches to generating stress scenarios and setting capital perform poorly in this example. The first is that they don't choose the correct direction for the stress. This is particularly clear in the case of the PS approach, which only considers shifts of the yield curve in a direction in which the bank is hedged.

The ES approach fails for a similar reason since the stress scenarios it chooses are also not too different from a parallel shift. The ES approach fails for another reason: in the ES approach capital is set based on X and $\mathrm{E}(Y|X)$. This approach to setting capital fails to consider the variation of Y around its conditional mean as a determinant of how much capital the bank should hold even though this variation should affect capital holdings in most settings.

This analysis shows that for stress-testing in the case of one bank, some of the very simple ways of setting up stress-tests don't require the bank to hold enough capital for its own survival, while an alternative approach based on maximal losses in a region is conservative, and ensures that the bank has sufficient capital against a wide-variety of shocks.

It is of course straightforward to find a stress that will create very large losses for a portfolio and generate enough capital holdings. For example, extremely high 10-year rates is one stress that will force this bank to be become insolvent, and holding capital against this stress will make insolvency extremely unlikely. However, stresses of this magnitude are implausible. A better set of questions are what types of plausible stresses is the bank vulnerable to, and for the set of plausible stresses how big are the losses that the bank can experience?

To provide answers, plausible scenarios are modeled under the assumption that f_T has an elliptical distribution that for simplicity is multivariate normal.[7]

$$f_T \sim \mathcal{N}(0, \Omega). \tag{3}$$

A scenario is defined to be plausible if the realization of f_T in the scenario is not too far out in the tails of a multivariate normal. Formally, a scenario for f_T is considered plausible if

$$\Omega^{-.5} f_T \in A, \tag{4}$$

where in the case of two factors

$$A = \{(x, y)|x \in [-a, a] \text{ and } y \in [-a, a]\}.$$

For notational purposes I will refer to the set A as a *trust set*, which represents a set of factor outcomes against which the bank is sufficiently capitalized. The plausibility condition places restrictions on the shape of the trust sets. The plausibility condition can best be understood as restrictions on the transformed risk factors $u = \Omega^{-.5} f_T$. The transformation expresses the bank's

[7] With some additional difficulty the factors could instead be modeled with a fat-tailed distribution such as a multivariate student-t.

risk factors f_T in terms of a set of independent risk factors u that are distributed $\mathcal{N}(0, I)$.[8] The restriction of the independent risk factors to the set A means the realizations of each element of u lie beween $-a, a$; hence each independent risk factor in the system cannot lie too far out in the tail of its marginal distribution, where "too far out" is determined by the choice of a (which then determines the set A). Because the bank is only being stressed for scenarios in A, it is also important that the probability of the set A is large enough to insure that the bank is well capitalized against a large set of plausible scenarios.[9] To ensure that this is the case, a can be chosen so that the probability of the set A, given by,

$$Prob(A) = [\Phi(a) - \Phi(-a)]^2,$$

is sufficiently large.[10] For example, to ensure that the bank can weather bad scenarios with probability of at least 99%, a should be chosen so that

$$[\Phi(a) - \Phi(-a)]^2 = 0.99;$$

and the stress scenario f_{min} should be chosen so that

$$f(min) = \min_f \delta' f \tag{5}$$

such that,

$$\Omega^{-.5} f \in A.$$

This minimization problem can be rewritten as:

$$\begin{aligned} & \min_{f|\Omega^{-.5}f\in A} [\delta'\Omega^{.5}] \times [\Omega^{-.5}f] \\ = & \min_{u\in A} \delta^{*\prime} u \\ = & \min_{u_i\in[-a,a]} \sum_i \delta_i^* \times u_i \end{aligned}$$

The solution for u and f (denoted $u(min)$ and $f(min)$, and the worst linear loss in portfolio

[8]The risk factors are independent in this case because they are normally distributed and uncorrelated. If the risk factors are instead multi-variate student-t, then the transformed risk factors will be uncorrelated but not independent.

[9]The set A is often defined by the condition $f_T'\Omega^{-1}f_T \leq k^2$. As discussed in the appendix, specifying the set A in this way can lead to extreme and very unrealistic worst-case scenarios in some cases. In addition, there is typically not a closed form solution for the worst-case loss and portfolio when using quadratic approximation, although it can be found fairly easily numerically [Studer and Luthi (1997)]. By contrast, defining the set A as we do here places more realistic bounds on f_T. In addition, the computation of the worst-case scenarios is very straightforward.

[10]If f_T is multivariate student-t, with covariance matrix Ω, then the elements of $u = \Omega^{-.5}f_T$ are not independent. In that case a solves

$$\int_{-a}^{a}\int_{-a}^{a} g(s_1, s_2)ds_1 ds_2 = (1-\alpha),$$

where $g(.,.)$ is the density function of a multivariate student-t.

value are

$$u(min)_i = -a \times \text{sign}(\delta_i^*) \quad for all i;$$
$$f(min) = \Omega^{.5} u(min).$$

$$\text{Worst Linear Loss} = \delta' f(min)$$

For Example 1, the trust set is a box in terms of the transformed variables u (not shown) and is a parallelogram in terms of f_T. Figure 1 illustrates the boundary of the set of plausible scenarios for f. The approximate worst-case linear loss in A is at one of the corners of the parallelogram. Because a first-order approximation is used to find this point, the actual worst case loss over the set A differs from the amount identified, but the difference is small in this case. Using quadratic approximation of the change in portfolio value, with the same trust set A the difference will typically be even smaller. It is important to add however that when using quadratic approximation of the change in portfolio value, it is computationally convenient to choose a slightly different trust set that will also contain 99% of the probability mass of changes in the bond yields.[11] To understand how the choice of trust sets affect the amount of capital that is held, note that if the approximations for the change in portfolio value as a function of the factors are exact, then if capital is set aside against the worst-loss in either trust set then that amount of capital is more than sufficient to cover 99% of the losses that will occur for the bank because with no approximation error both loss estimates are upper bounds for the amount for the amount of capital that the bank needs. If there is no approximation error, it then is best to choose the trust set whose maximal lossess are lowest because then less economic capital will be needed for the bank to meet its risk objective. If there is approximation error, then this error also needs to be accounted for when setting economic capital.

Before continuing, it is important to summarize the main lessons from this example.

1. Stress tests need to use information on banks risk exposures in order to ensure that they choose directions of stress that are meaningful.
2. If stress tests are used to set economic capital, then they should appropriately use information on the joint distribution of the risk factors when setting capital.
3. If a goal of stress testing and capital policy is to ensure that banks are robust to other plausible scenarios, then stress tests and capital policies should be explacitly designed to achieve this objective.

[11] There are two ways to implement the maximization with quadratic approximation. The more difficult method imposes the constraint $\Omega^{-1/2} f_T \in A$. The more straightforward method imposes the constraint $P'\Omega^{-1/2} f_T \in A$ where P is a rotation matrix that depends on Ω and Γ. See the appendix for details.

The next section applies the insights from designing stress scenarios for a single bank to the problem of designing systemic risk stress tests for the banking system.

3 A Framework for Systemic-Risk Stress Testing

The analysis of systemic-risk stress-testing proceeds in three parts. First systemic risk stress-testing is contrasted with stress-testing for individual banks; second, the analysis provides definitions of systemic stress and finanial stability; the analysis concludes by showing how to design stress-tests and capital injection policy in order to attain financial stability.

Systemic Risk Stress Tests vs Stress Tests for Individual Banks

The purpose of systemic risk stress tests is to ensure that the banking and other parts of the financial system are sufficiently capitalized as a whole to support normal levels of financial intermediation activity. In this respect, the purpose of systemic risk stress-tests is different from individual bank capital requirements.

Individual bank capital requirements are designed to prevent the insolvency of individual banks by for example requiring them to hold enough capital to survive for a year with probability exceeding 99.9 percent. However, even if a bank holds enough capital to survive for a year, if it is solvent but becomes poorly capitalized, it may be unable to lend for a while. If other lenders are financially healthy enough to step in and lend to that banks borrowers, it may not represent a problem for the financial system. But, if many lenders are solvent but become poorly capitalized at the same time, it could create problems since there may not be enough healthy lenders to step in and provide financial intermediation. The purpose of systemic risk stress tests is to avoid this type of impairment to financial intermediation.

Systemic Risk and Financial Stability Defined

The analysis of systemic risk proceeds under the assumption that banks that are too undercapitalized cannot perform needed financial intermediation activities. Additionally, if too many banks are undercapitalized, then others will not be able to step-in and fill the gap, and hence systemic risk ensues. Based on these ideas, the amount of systemic risk at date T is measured by the percentage of banking assets that are held by banks in financial distress at that date. This amount is denoted by SAD_T, which stands for Systemic Assets in Distress at T. SAD_T should in general depend on the economic state at date T, which is represented by the vector of risk-factor realizations f_T. To simplify notation, the dependence of SAD_T on f_T will typically be suppressed.

θ-systemic risk is defined as the event that $SAD_T = \theta$. Building on the dam analogy and robustness concepts discussed in section 2, a financial system is defined to be systemically stable if its probability of θ-systemic risk exceeding a pre-specified threshold is low. Specifically, a financial system is defined to be weakly alpha-theta stable (written as $\alpha - \theta$ stable hereafter) over the time horizon that begins today and ends at T if $\text{Prob}_0(SAD_T \geq \theta)$ is less than α. For example, if T is one year, θ is 10 percent, and α is one percent, then weak $\alpha - \theta$ stability at horizon T means the probability conditional on information today (time 0) that banks representing 10 percent or more financial assets will be in financial distress a year from today is less than 1 percent.

In this paper instead of focusing on weak $\alpha - \theta$ stability, for computational purposes the focus is on a strengthened concept, $\alpha - \theta$ stability, defined as follows:

Definition 1 *Let f_T be the vector of risk-factor realizations that affect the value of financial firms at time T. The financial system is $\alpha - \theta$ stable at horizon T if there is a set of possible factor realizations F_T such that*

$$Prob_0(f_T \in F_T) \quad = \quad 1 - \alpha,$$

and for all $f_T \in F_T$, $SAD_T(f_T) \leq \theta$.

The definition of $\alpha - \theta$ stability implies that for all realizations of f_T within a set F_T that has probability $1 - \alpha$ as of date 0, the amount of systemic assets in distress is less than or equal to θ. An immediate implication is that the set of f_T for which $SAD_T > \theta$ has probability which is less than or equal to α. Therefore, $\alpha - \theta$ stability implies weak $\alpha - \theta$ stability.

The definition of $\alpha - \theta$ stability relies on a set of factor realizationsF_T such that the probability that the factor realizations lie within the set is $1 - \alpha$. There are many possible sets that have this property. In the analysis that follows below, the sets are chosen to achieve two objectives. The first is to avoid overly conservative stress-scenarios that can occur when optimizing over worst stress scenarios for some choices of the set F_T. The second objective is to simplify computation of the stress scenarios that maximize losses. The set F_T that is chosen is the multidimensional analog of the constraint in equation 4. For whatever set is chosen, it is possible to design stress-tests and capital injections that achieve $\alpha - \theta$ stability.

The definitions of systemic risk and financial stability are based on a definition of financial distress. For the purposes of this paper bank i is defined to be in financial distress if its economic capital ratio falls below some threshold c_i^*.[12] This threshold is greater than the regulatory minimums; it instead represents the amount of capital that bank i must hold given its risk in order to perform its financial intermediation activities without any impairments due to low capital ratios.

[12] c_i^* has a subscript i because the appropriate threshold should depend on the businesses that the bank conducts. Generally, it may also depend on other factors such as the riskiness of the economic environment. For simplicity the other factors that affect c_i^* are suppressed from the analysis.

To formally model financial distress and systemic risk, let $A_{i,t}$ and $L_{i,t}$ denote bank i's assets and liabilities at time t, and let $C_{i,t} = (A_{i,t} - L_{i,t})/A_{i,t}$ denote its economic capital ratio. The current date is $t = 0$, and the focus is bank i's economic capital ratio at date T. $C_{i,T}$ is approximated using a first or second order Taylor series expansion in the risk factors f that affect the bank's value. The second order expansion has form

$$C_{i,T} \approx C_{i,0} + \delta_i' f_T + \frac{1}{2} f_T' \Gamma_i f_T. \tag{6}$$

As above, δ and Γ to represent the first and second derivative with respect to the risk factors, but unlike above, here they represent first and second derivatives of bank i's capital ratio with respect to the factors.

The above Taylor series implicitly assumes that bank i does not inject any capital. Because capital injections play a role in systemic risk stress-testing, it is important to allow for them. The above equation is modified to incorporate capital injections that may be required just after date 0. This modification is made because date 0 represents when stress-tests take place; and any required capital injections are assumed to be made soon thereafter. $CI_{i,0}$ denotes the increase in bank $i's$ capital ratio just after date 0 due to a capital injection. The equation for capital injections is accordingly modified to become

$$C_{i,T} \approx C_{i,0} + CI_{i,0} + \delta_i' f_T + \frac{1}{2} f_T' \Gamma_i f_T. \tag{7}$$

The size of the capital injection is scaled by the size of the bank's assets, thus to raise capital ratios by $CI_{i.0}$ just after date 0 requires a capital injection of $A_i \times CI_{i,0}$.

Using the definition of distress, the event of bank i's financial distress at date T can be denoted by the indicator function $d_{i,T}$ where

$$d_{i,T} = \begin{cases} 1 & C_{i,T} < C_i^* \\ 0 & C_{i,T} \geq C_i^* \end{cases}$$

While it is convenient to define financial distress as occurring when a bank's capital ratio is less than some threshold, banks that are slightly above or below the threshold are probably experiencing about the same amounts of financial distress. Therefore it makes more sense to model financial distress as a continuous function of a bank's capital ratio. To do so, the binary distress function $d_{i,T}$ is replaced with the continuous distress function $D_{i,T}$, given by

$$D_{i,T} = \frac{1}{1 + e^{-a_i - k_i(C_i^* - C_{i,T})}}, \tag{8}$$

which takes values between 0 and 1. It approaches 1 as $C_{i,T}$ becomes small and 0 as $C_{i,T}$ becomes large. The parameters a_i and k_i are tuning parameters that can be dependent on the characteristics of bank i and provide flexibility in modeling. Specifically, a_i determines the level of bank i's distress when $C_{i,T} = C_i^*$, and k_i determines the rate at which bank i's distress changes when $C_{i,T}$ moves away from C_i^*.

Assets in distress at date T are defined as the date 0 asset holdings of banks that experience distress at date T. Because distress is modeled as a continuous variable, assets in distress at T, denoted AD_T is given by

$$AD_T = \sum_i A_{i,0} D_{i,T}. \tag{9}$$

Similarly systemic risk, measured by the percent of assets held by intermediaries in distress, is denoted SAD_T (systemic assets in distress) is defined as:

$$SAD_T = \frac{\sum_i A_{i,0} D_{i,T}}{\sum_i A_{i,0}}. \tag{10}$$

Note that SAD_T is stochastic because it depends on how the risk factors that affect banks' capital ratios evolve. In addition, SAD_T depends on any capital that is injected into banks just after date 0.

The question answered below is how to use stress-tests and capital injections to achieve $\alpha - \theta$ stability, where recall the condition for $\alpha - \theta$ stability is $SAD_T[f_T] < \theta$ for all $f_T \in F_T$, where the set F_T has probability $1 - \alpha$.

Choosing Stress-Scenarios and Capital Injection Policy to Achieve Financial Stability

A 6 - step approach is proposed for choosing a stress-scenario and capital injection policy that attains financial stability at low cost. The steps are outlined below with details provided (eventually) in the appendix.

Step 1: Approximate SAD_T as a function of the risk factors f_T.

To do so, for each i, a first order Taylor expansion of $D_{i,T}$ is created terms of the expression for $C_{i,T}$ from equation 7, expanded around $\mathrm{E}\{C_{i,T}\}$.[13] The resulting expansions are plugged into the expression for SAD_T. The result is an approximation for SAD_T as a quadratic function of the

[13] Because $f_T \sim \mathcal{N}(0, \Omega)$,

$$\mathrm{E}\, C_{i,T} = C_{i,0} + CI_0 + (1/2) \times \mathrm{Trace}[\Omega \Gamma_i].$$

risk factors f. This approximation is denoted $\widehat{SAD}_T[f]$.

Step 2: Choose f_T to maximize $\widehat{SAD}_T[f]$ subject to a plausibility constraints that take the form given in equation 4.[14] Because this a quadratic maximization, its solution can be found using the same approach as in the analogous problem from section 2. Denote the solution as f_T^M, where M denotes the solution that maximizes systemic losses.

Step 3: Plug the solution for f_T^M into the expression for $SAD_T(f)$ and evaluate it.

If $SAD_T(f_T^M) < \theta$, then assuming that maximizing the Taylor series approximation to $SAD_T(f_T^M)]$ finds the true f_T^M, it follows that the financial system is $\alpha-\theta$ stable by construction, and is therefore also weakly $\alpha-\theta$ stable.

If the inequality is not satisfied, then capital injections will be needed to ensure $\alpha-\theta$ stability. Additionally, the excess amount of systemic risk in the system, denoted $ESAD_T$ is approximately:

$$ESAD_T \approx SAD_T(f_T^M) - \theta, \tag{11}$$

where the reason for approximation is because the analysis is based on Taylor series expansion.

If there is excess systemic risk, then steps 4−6 are required to find the required capital injections.

Step 4: Create a second order Taylor expansion of $SAD_T(f_T^M)$ in terms of capital injections for each bank i centered around when each bank's capital injection is equal to 0.

Denote this expansion as $\widehat{SAD}_T(f_T^M, CI)$, where CI represents the vector of capital injections that are chosen. In the expansion SAD_T will be decreasing in $CI_{i,0}$ for each bank. For plausible parameterizations the rate at which $SAD_T(f_T^M)$ goes down with each banks capital injection will be a diminishing function of each bank's capital injection. The diminishing benefits of each banks capital injections will be reflected in the quadratic terms of the expansion.

Step 5: Solve for the least costly way to inject capital to achieve $\alpha-\theta$ stability.

The total amount of capital injected into the banking system just after date 0, denoted TCI is:

$$TCI = \sum_i A_i \times CI_{i,0} \tag{12}$$

where $CI_{i,0}$ is the increase in bank $i's$ capital ratio as a result of the injection, and A_i is the amount of capital that must be injected per unit increase in i's capital ration.

[14] Formally, f is the union of all the risk factors that banks use and the vectors $delta_i$ and Γ_i represent banks exposures to those factors.

Assuming that injecting capital is costly, the objective is to inject as little capital as needed in order to achieve $\alpha - \theta$ stability. This is accomplished by solving the problem:

$$\min_{CI} \sum_i A_i \times CI_{i,0} \tag{13}$$

such that

$$\widehat{SAD}_T(f_T^M, CI) < \theta.$$

The constraint in the maximization is equivalent to requiring the drop in systemic risk evaluated at f_T^M to exceed the amount of excess systemic risk.

$$SAD_T(f_T^M) - \widehat{SAD}_T(f_T^M, CI) > ESAD_T$$

Injecting capital by solving the problem in step 5 should achieve $alpha - \theta$ stability because it is designed to achieve it for f_T^M, which is the worst scenario for systemic stability among a set of scenarios that have probability $1 - \alpha$. Because the minimization problem has a linear-quadratic form, it is simple.

However, because the analysis is based on approximations, the solution for the capital injections are also only approximate. It therefore becomes necessary to check whether they achieve $\alpha - \theta$ stability, and fix them if they don't. The method for doing so is described in the next step.

Step 6: Check whether the capital injections are adequate and iterate as needed.

To verify whether the capital injections are adequate, one should first plug them into the expression for $SAD_T(f_T^M)$ and verify whether it is less than θ. If it is, then provided f_T^M is a worst case with probability α, then the financial system is $\alpha - \theta$ stable. However, because of the capital injections f_T^M is likely to no longer be the worst case. Therefore it is important to verify that the system with the capital injections is $\alpha - \theta$ stable. To do so, it suffices to repeat steps 1 - 3 above, and then verify if at the new f_T^M with the capital injections, the system appears to be $\alpha - \theta$ stable.

If it is stable, then stress-scenarios and capital injections have been found that together produce $\alpha - \theta$ stability. If the new system is still not stable, it may be necessary to repeat steps 4 - 5, and then 1 - 3, iteratively until stability is achieved.

4 An Example of Stress-Tests and Capital Injections for Systemic Risk.

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5 Conclusion

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Appendix

A Quadratic Minimization

The quadratic approximation of the change in value of an individual firm is given by:

$$\Delta V \approx \delta' f + .5 f' \Gamma f$$

where

$$f \sim \mathcal{N}(0, \Omega).$$

After the change in variable,

$$u = \Omega^{-.5} f,$$

the change in value can be written as

$$\Delta V \approx \delta' \Omega^{.5} u + .5 u' \Omega^{.5} \Gamma \Omega^{.5} u, \tag{14}$$

where $u \sim \mathcal{N}(0, I)$.

Because $\Omega^{.5}\Gamma\Omega^{.5}$ is symmetric, it has representation

$$\Omega^{.5}\Gamma\Omega^{.5} = PDP' \tag{15}$$

where D is a real diagonal matrix and P is a matrix of orthonormal eigenvectors.

Applying the change of variables $x = P'u$ then tranforms ΔV to

$$\begin{aligned} \Delta V &\approx \delta' \Omega^{.5} P x + .5 x' D x \\ &\approx \tilde{\delta} x + .5 x' D x, \end{aligned} \tag{16}$$

where $x \sim \mathcal{N}(0, I)$.

To solve find the worst case stress-scenario over over a set of f that has probability $1-\alpha$, we solve

$$\Delta v_{min} = \min_x \tilde{\delta}' x + .5 x' D x \tag{17}$$

such that

$$x_i \in [-a, a] \text{for all i.},$$

and

$$[\Phi(a) - \Phi(-a)]^N = (1-\alpha).$$

Specifying the constraints on x in this way guarantees that mapping from x to A, the set of possible f realizations has probability $1-\alpha$.

The reason for doing a change of variables to x is that the minimization problem has the very simple form

$$\min_{x_i \in [-a,a]} \sum_i \tilde{\delta}_i x_i + .5 x_i^2 D_{ii},$$

which is just N trivial constrainted quadratic minimizations. Denote the solution for x as x_{min}.

After the minimum has been found, transforming back produces

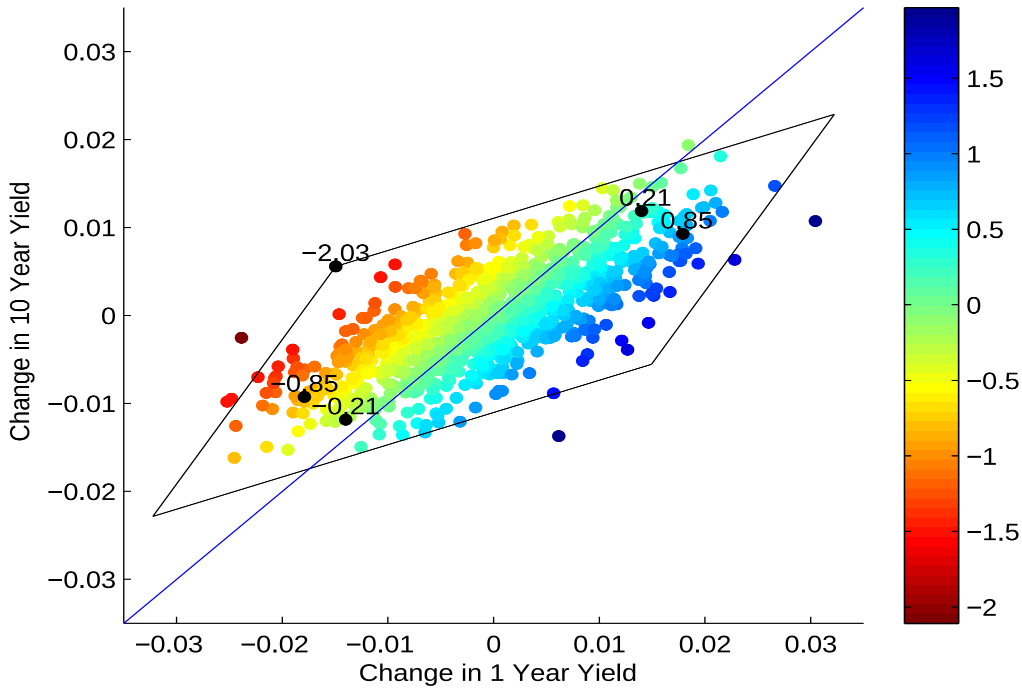
$$f_{min} = \Omega^{.5} P x_{min} \tag{18}$$

Table 1: Assets for Textbook Example

Asset	Yield	Amount Invested
Cash	0	95 million
1 Yr ZCB	1 %	-100 million
10 Yr ZCB	1 %	10 million

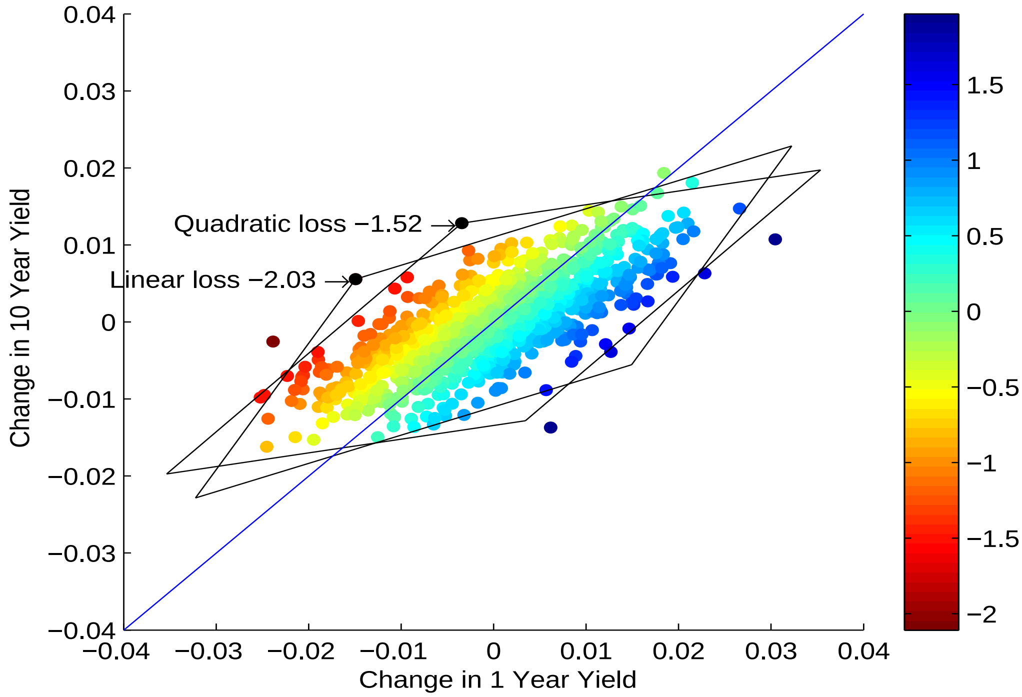
Notes: For the example in section 2, the Table provides information on the banks asset portfolio, which consists of positions in cash, and one- and ten- year zero coupon bonds (ZCB).

Figure 1: Setting Capital Based on Stress Scenarios: An Example Using Interest-Rate Shocks



Notes: For the stylized bank whose balance sheet is described in Table 1, the figure presents a plot of changes in portfolio value that can occur for a set of ten-year and one-year yield changes. The figure is used to examine the efficacy of setting capital by using stress-scenarios. Scenarios are generated three ways. First, a scenario is generated by finding yield shocks that generate the worst portfolio loss for shocks that lie within a parallelogram that contains 99% of the probability mass of shocks (loss = -2.03 million). Second, 4 extreme scenarios are generated that shock one yield up or down by two standard deviations, and the other by its expected change conditional on the first shock (losses = -.85, -.21; gains = .85, .21). Third, scenarios are generated through parallel shifts to the yield curve, which corresponds to movements along the blue 45 degree line in the figure.
The true 99th percentile of loss for the portfolio is a loss of 1.3 million dollars. Therefore, if capital is set based on the first alternative, it will be more than sufficient to cover this loss. If capital is set instead based on the second alternative (extreme scenarios) or the third alternative (parallel yield curve shifts), then the capital holdings will be inadequate to absorb up to 99th percentile of the loss distribution of the bank's portfolio.

Figure 2: Linear and Quadratic 99% Trust Sets



Notes: For the stylized bank in Figure 1, the figure shows the 99% trust set that was used in Figure 1, as well as an alternative trust set that is more convenient to maximize over when the bank's value is approximated using a quadratic function of the risk factors. For both trust sets, the worst case loss over the trust set is presented in the figure. Both worst case losses exceed the true 99th percentile of loss. Therefore, setting capital for this bank using the maximal loss criterion and either trust set would result in adequate economic capital for the example considered.