Optimal Monetary Policy in a Heterogeneous Monetary Union*

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Abstract

We study optimal monetary policy in a monetary union with heterogeneous net foreign asset positions and uninsurable country-specific shocks. Member countries trade nominal, non-contingent claims, which allows for redistributive effects from inflation. Under discretion, an inflationary bias arises from the central bank's attempt to redistribute wealth towards debtor countries, which under incomplete markets have a higher marginal utility of net wealth. Under commitment, this redistributive motive to inflate is counteracted over time by the incentive to prevent expectations of future inflation from being priced into new bond issuances; under certain conditions, steady-state inflation is zero, as both effects cancel out. We calibrate our model to the euro area and find that the optimal commiment features first-order initial inflation followed by a gradual decline towards its (near zero) long-run value. Welfare losses from discretionary policy are first-order in magnitude, affecting both debtor and creditor countries.

Keywords: optimal monetary policy, commitment and discretion, incomplete markets, nominal debt, inflation, redistributive effects, continuous-time

JEL codes: E5, E62,F34.

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1 Introduction

A long-standing issue in monetary economics concerns the redistributive effects of inflation in the presence of nominal assets, whereby inflation benefits debtors and hurts creditors. This question has become quite relevant in the European Monetary Union (EMU), in light of significant heterogeneity in net nominal claims at the country level.[1] As shown in Figure 1, the distribution of net foreign asset positions across EMU member states was already quite dispersed in the early stages of the crisis, with some countries being heavily indebted in relation to their GDP and some other having relatively large net creditor positions.[2] The Great Recession and the subsequent European debt crisis have only increased such dispersion. This development has intensified the debate as to what should be the appropriate conduct of monetary policy in the context of a monetary union which such unequal net foreign asset positions.[3]

This paper addresses the above question by analyzing the optimal monetary policy of a benevolent central bank, both under commitment and discretion, in a continuous-time model of a monetary union where member countries have heterogeneous net asset positions and face uninsurable country-specific shocks. In the model, the individual union members trade nominal, non-contingent, long-term, domestic-currency-denominated financial claims with each other and with risk-neutral foreign investors, subject to an exogenous borrowing limit. As a result, inflation redistributes wealth from creditor member countries (and foreign investors) to debtor ones, all else equal. At the same time, expected future inflation raises the nominal costs of new debt issuances through higher inflation premia. Also, inflation entails direct welfare costs that can be rationalized on the basis of costly price adjustment. The monetary union as a whole is assumed to be a net debtor *vis-à-vis* foreign investors, such that domestic-currency-denominated nominal bonds are in positive net supply.

On the analytical front, we show that discretionary optimal policy features an 'inflationary bias', whereby the central bank tries to use inflation so as to redistribute wealth and hence consumption. In particular, we show that optimal discretionary inflation increases with the average cross-country net liability position weighted by each country's marginal utility of net wealth. The redistributive motive therefore has both an extra-monetary union and a within-monetary union dimension. On the one hand, inflation allows to redistribute from foreign investors to indebted union members. On the other hand, and somewhat more subtly, under market incompleteness and standard concave

[1] See e.g. Adam and Zhu (2016) for an in-depth analysis of the redistributional consequences of unexpected price level changes in the EMU.

[2] The left panel of Figure 1 displays a smoothed approximation to the actual distribution, constructed by means of a normal kernel. The latter is given by $\frac{1}{N}\sum_{n=1}^{N}\phi\left(\left(y-x_n\right)/\sigma\right)$, where ϕ is standard normal pdf and $\{x_n\}_{n=1}^{N}$ are the actual data points, i.e. the ratio of net financial asset position over GDP for the EMU country members. The smoothing parameter σ is set to 20 percent. Data on GDP and total financial assets/liabilities are from the ECB Statistical Data Warehouse.

[3] See e.g. The Economist (2014) or Eijffinger (2016).

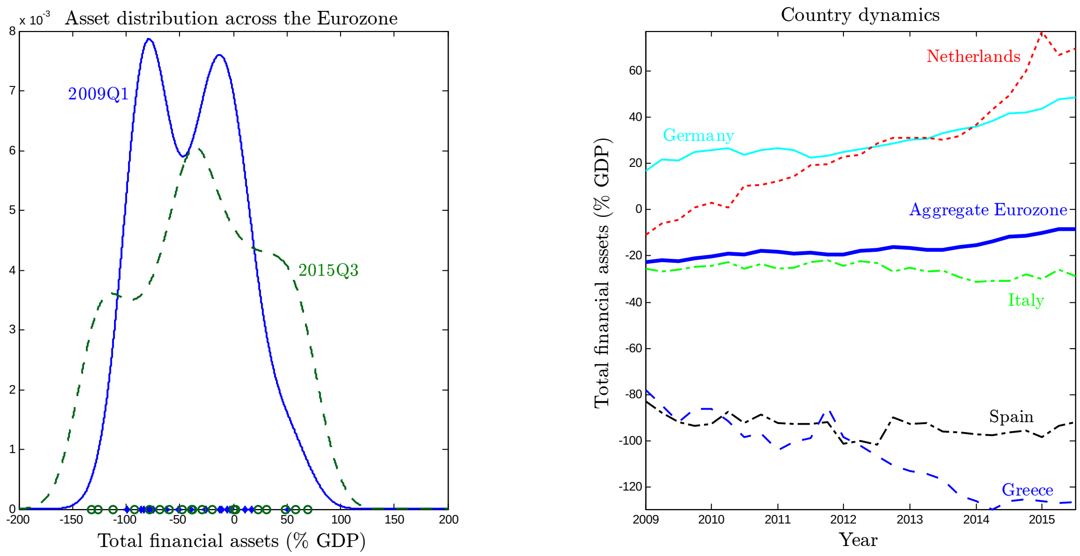

Figure 1: Asset distribution and dynamics in the Eurozone. Blue stars and green circles mark the exact financial position in 2009Q1 and 2015Q3 respectively. Source: ECB Statistical Data Warehouse.

preferences for consumption, debtor countries have a *higher* marginal utility of net wealth than creditor ones. As a result, they receive a higher effective weight in the optimal inflation decision, giving the central bank an incentive to redistribute wealth from creditor to debtor union members.

Under commitment, the same redistributive motive to inflate exists, but it is counteracted by an opposing force: the central bank internalizes how investors' expectations of future inflation affect their pricing of long-term nominal bonds from the time the optimal commitment plan is formulated ('time zero') onwards. At time zero, inflation is exactly the same as under discretion, as no prior commitments about inflation exist. However, the prices of bonds issued from time zero onwards do incorporate the central bank's promises about the future inflation path. This gives the central bank an incentive to commit to reduce inflation over time. In fact, we show that under certain conditions on preferences and parameter values, the steady state inflation rate under the optimal commitment is zero;[4] that is, in the long run the redistributive motive to inflate exactly cancels out with the incentive to reduce inflation expectations and nominal yields for a monetary union that is a net debtor.

We then solve numerically for the full transition path under commitment and discretion. We calibrate our model to match a number of features of the EMU, such as observed output fluctuations in its member states, as well as its consolidated net asset position *vis-à-vis* the rest of

[4]In particular, assuming separable preferences, then in the limiting case in which the central bank's discount rate converges to that of foreign investors optimal steady-state inflation under commiment converges to zero.

the World.[5] The optimal commitment policy depends on the time-zero net wealth distribution across the monetary union members, which is an infinite-dimensional object, whereas the optimal discretionary policy depends on such distribution at each point in time. To discipline our choice of initial net wealth distribution in our simulations, we construct it using information from the actual distribution of net foreign asset positions in the EMU. We find that optimal time-zero inflation, which as mentioned before is the same both under commitment and discretion, is first-order in magnitude. We show moreover that such initial inflation reflects mostly an incentive to redistribute from foreign investors to the monetary union as a whole, with the within-union redistributive motive contributing comparatively little. Under discretion, inflation remains high due to the inflationary bias discussed before, converging asymptotically to a level of about 1.5%. Under commitment, by contrast, inflation falls gradually towards its long-run level (essentially zero, under our calibration), reflecting the central bank's concern with preventing expectations of future inflation from being priced into new bond issuances. In summary, under commitment the central bank front-loads inflation so as to transitorily redistribute existing wealth among the monetary union's members and (especially) from foreign investors to the union's debtor members, but commits to gradually undo such initial inflation.

In welfare terms, the discretionary policy implies sizable (first-order) losses relative to the optimal commitment. Such losses are suffered by creditor member countries, but *also* by debtor ones. The reason is that, under discretion, expectations of permanent future positive inflation are fully priced into current nominal yields. This impairs the very redistributive effects of inflation that the central bank is trying to bring about, and leaves only the direct welfare costs of permanent inflation, which are born by creditor and debtor countries alike.

Overall, our findings shed some light on current policy and academic debates regarding the appropriate conduct of monetary policy in a monetary union with a cross-country distribution of net financial assets comparable to that of the Eurozone. In particular, our results suggest that an optimal plan that includes a commitment to price stability in the medium/long-run may also justify a relatively large (first-order) positive initial inflation rate, especially if the monetary union's consolidated position relative to the rest of the World is sufficiently negative.

Related literature. Our first main contribution is methodological. To the best of our knowledge, ours is the first paper to solve for a fully dynamic optimal policy problem, both under commitment and discretion, in a general equilibrium model with uninsurable idiosyncratic risk in which the cross-sectional net wealth distribution (an infinite-dimensional, endogenously evolving object) is a state in the planner's optimization problem. We exploit the fact that, in continuous time, the dynamics of the cross-sectional distribution are characterized by a partial differential

[5] In particular, the EMU's consolidated net asset position is used to inform the gap between the central bank's and foreign investors' discount rate, which as explained before is a key determinant of long-run inflation under commiment.

equation known as the *Kolmogorov forward* (KF) or *Fokker-Planck* equation, and therefore the problem can be solved by using calculus techniques in infinite-dimensional Hilbert spaces.[6]

Different papers have analyzed Ramsey problems with a continuum of heterogeneous agents. Dyrda and Pedroni (2014) study the optimal dynamic Ramsey taxation in a discrete-time Aiyagari economy. They assume that the paths for the optimal taxes follow splines with nodes set at a few exogenously selected periods, and perform a numerical search of the optimal node values. Acikgoz (2014), instead, follows the work of Davila et al. (2012) in employing calculus of variations to characterize the optimal Ramsey taxation in a similar setting. However, after having shown that the optimal long-run solution is independent of the initial conditions, he analyzes quantitatively the steady state but does not solve the full dynamic optimal path. Other papers, such as Gottardi, Kajii, and Nakajima (2011) or Itskhoki and Moll (2015), are able to find the optimal Ramsey policies in incomplete-market models under particular assumptions that allow for closed-form solutions. In contrast to these papers, here we introduce a methodology to compute the *full* dynamics under commitment in a general setting.[7] Regarding discretion, we are not aware of any previous paper that has quantitatively analyzed the *Markov Perfect Equilibrium* (MPE) with uninsurable idiosyncratic risk.[8]

Aside from the methodological contribution, our paper relates to several strands of the literature. First, our paper is related to a long-standing literature that analyzes the redistributive effects of monetary policy. This topic has received a considerable revival in recent years. In particular, a recent literature addresses this issue in the context of general equilibrium models with incomplete markets, (non-trivial) idiosyncratic uncertainty/heterogeneity, and distributional dynamics, with contributions from Meh, Ríos-Rull and Terajima (2010), Gornemann, Kuester and Nakajima (2012), McKay, Nakamura and Steinsson (2015), Challe, Matheron, Ragot and Rubio-Ramírez (2015), Luetticke (2015), Auclert (2015), and Kaplan, Moll and Violante (2016) among others. Contrary to the above papers, which focus on cross-household heterogeneity in closed-economy, discrete-time setups, we focus on cross-country heterogeneity in a continuous-time model of an open monetary union. More importantly, we analyze the *optimal* monetary policy, both under commit-

[6] These techniques were first introduced in Lucas and Moll (2014) and Nuño and Moll (2015) in order to find the first-best or the constrained-efficient allocation in heterogeneous-agent models.

[7] In addition, the numerical solution of the model is greatly improved in continuous-time, as discussed in Achdou, Lasry, Lions and Moll (2015) or Nuño and Thomas (2015). This is due to two properties of continuous-time models. First, the HJB equation is a deterministic partial differential equation which can be solved using efficient finite difference methods. Second, the dynamics of the distribution can be computed relatively quickly as they amount to calculating a matrix adjoint: the KF operator is the adjoint of the *infinitesimal generator* of the underlying stochastic process. This computational speed is essential as the computation of the optimal policies requires several iterations along the complete time-path of the distribution. In a home PC, the Ramsey problem presented here can be solved in less than 5 minutes.

[8] Amador, Aguiar, Farhi and Gopinath (2015), for example, consider a monetary union with a continuum of atomistic countries and define the MPE, but then they restrict their attention to a particular case with only two (groups of) countries, one with a high debt level and the other with a low debt level.

ment and discretion, in a model where the asset distribution is an (infinite-dimensional) state variable in the planner's problem. In this regard, the techniques developed here lend themselves naturally to the analysis of optimal policy (monetary, fiscal, etc.) in frameworks with non-trivial heterogeneity along alternative dimensions (e.g. households).

Starting from Clarida (1990), some authors have used multi-country models with idiosyncratic shocks and incomplete markets to study international capital flows. Examples are Castro (2005), Bai and Zhang (2010), Chang et al. (2013) and Fornaro (2014). We contribute to this literature by studying the optimal conduct of monetary policy in a monetary union with heterogenous nominal net debt positions and borrowing limits.

Our paper also relates to a long-standing and vast literature analyzing optimal monetary policy under commitment and discretion in a monetary union. Notable examples are Beetsma and Uhlig (1999), Dixit and Lambertini (2001, 2003), Chari and Kehoe (2007, 2008), Cooper, Kempf and Peled (2010, 2014), and Aguiar, Amador, Farhi and Gopinath (2015), among others. Most of this literature analyzes issues related to fiscal externalities on monetary policy, or to coordination between fiscal and monetary authorities, in a monetary union. We contribute to this literature by analyzing how, in a model of a monetary union with idiosyncratic shocks and incomplete markets, the endogenously-evolving distribution of net nominal assets determines the optimal inflationary policy under commitment and discretion.

Finally, our paper is related to the literature on *mean-field games* in Mathematics. The name, introduced by Lasry and Lions (2006a,b), is borrowed from the mean-field approximation in statistical physics, in which the effect on any given individual of all the other individuals is approximated by a single averaged effect. In particular, our paper is related to Bensoussan, Chau and Yam (2015), who analyze a model of a major player and a distribution of atomistic agents that shares some similarities with the Ramsey problem discussed here.[9]

2 A model of a heterogeneous monetary union

Let $(\Sigma, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space. Time is continuous: $t \in [0, \infty)$. There is a monetary union composed of a measure-one continuum of countries that are heterogeneous in their net financial wealth. There is a single, freely traded consumption good, the World price of which is normalized to 1. The price in the monetary union's currency (equivalently, the nominal exchange rate) at time t is denoted by P_t and evolves according to

$$dP_t = \pi_t P_t dt, \tag{1}$$

[9] Other papers analyzing mean-field games with a large non-atomistic player are Huang (2010), Nguyen and Huang (2012a,b) and Nourian and Caines (2013). A survey of mean-field games can be found in Bensoussan, Frehse and Yam (2013).

where π_t is the area-wide inflation rate (equivalently, the rate of nominal exchange rate depreciation).

2.1 Individual member countries

2.1.1 Output and net assets

Country $k \in [0,1]$ in the monetary union is endowed with y_{kt} units of the good at time t, where y_{kt} follows a two-state Poisson process: $y_{kt} \in \{y_1, y_2\}$, with $y_1 < y_2$. The process jumps from state 1 (which we may refer to as 'recession') to state 2 ('boom') with intensity λ_1 and vice versa with intensity λ_2.

In each country a representative household trades a nominal, non-contingent, long-term, domestic-currency-denominated bond with households in other union member states and with foreign investors. Let A_{kt} denote the net holdings of such bond in country k at time t; assuming that each bond has a nominal value of one unit of domestic currency, A_{kt} also represents the total nominal (face) value of net assets. In countries with a negative net position, $(-)\,A_{kt}$ represents the total nominal (face) value of outstanding net liabilities ('debt' for short). We assume that outstanding bonds are amortized at rate $\delta > 0$ per unit of time.[10] The nominal value of the country's net asset position thus evolves as follows,

$$dA_{kt} = \left(A_{kt}^{new} - \delta A_{kt}\right) dt,$$

where A_{kt}^{new} is the flow of new assets purchased at time t. The nominal market price of bonds at time t is Q_t. Country k incurs a nominal current account primary surplus $P_t\left(y_{kt} - c_{kt}\right)$, where c_{kt} is consumption. The aggregate budget constraint of country k is then

$$Q_t A_{kt}^{new} = P_t\left(y_{kt} - c_{kt}\right) + \delta A_{kt}.$$

Combining the last two equations, we obtain the following dynamics for net nominal wealth,

$$dA_{kt} = \left(\frac{\delta}{Q_t} - \delta\right) A_{kt} dt + \frac{P_t\left(y_{kt} - c_{kt}\right)}{Q_t} dt. \tag{2}$$

We define real net wealth as $a_{kt} \equiv A_{kt}/P_t$. Its dynamics are obtained by applying Itô's lemma to equations (1) and (2),

$$da_{kt} = \left[\left(\frac{\delta}{Q_t} - \delta - \pi_t\right) a_{kt} + \frac{y_{kt} - c_{kt}}{Q_t}\right] dt. \tag{3}$$

[10] See Woodford (2001) for an early use of nominal perpetual bonds with geometrically decaying coupons (which are isomorphic to the nominal perpetual bonds with a constant amortization rate used here) in a discrete-time macroeconomic framework. See also Hatchondo and Martinez (2009), and Chatterjee and Eyigungor (2012), for recent uses of such a modelling device in discrete-time open economy setups.

We assume that each country faces a limit to the maximum amount of net debt that it may issue

$$a_{kt} \geq \phi. \tag{4}$$

where $\phi \leq 0$.

For future reference, we define the *nominal bond yield* r_t implicit in a nominal bond price Q_t as the discount rate for which the discounted future promised cash flows equal the bond price. The discounted future promised payments are $\int_0^\infty e^{-(r_t+\delta)s}\delta ds = \delta/(r_t+\delta)$. Therefore, the nominal bond yield is

$$r_t = \frac{\delta}{Q_t} - \delta. \tag{5}$$

2.1.2 Households

The representative household has preferences over paths for consumption c_{kt} and union-wide inflation π_t discounted at rate $\rho > 0$,

$$U_{k0} \equiv \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} u(c_{kt}, \pi_t) dt \right], \tag{6}$$

with $u_c > 0$, $u_\pi > 0$, $u_{cc} < 0$ and $u_{\pi\pi} < 0$. From now onwards we drop country subscripts k for ease of exposition. The household chooses consumption at each point in time in order to maximize its welfare. The *value function* of the household at time t can be expressed as

$$v(t,a,y) = \max_{\{c_s\}_{s=t}^\infty} \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} u(c_s, \pi_s) ds \right], \tag{7}$$

subject to the law of motion of net wealth (3) and the borrowing limit (4). We use the short-hand notation $v_i(t,a) \equiv v(t,a,y_i)$ for the value function in a recession ($i=1$) and in a boom ($i=2$). The Hamilton-Jacobi-Bellman (HJB) equation corresponding to the problem above is

$$\rho v_i(t,a) = \frac{\partial v_i}{\partial t} + \max_c \left\{ u(c, \pi(t)) + s_i(t,a,c) \frac{\partial v_i}{\partial a} \right\} + \lambda_i \left[v_j(t,a) - v_i(t,a) \right], \tag{8}$$

for $i,j = 1,2$, and $j \neq i$, where $s_i(t,a,c)$ is the *drift* function, given by

$$s_i(t,a,c) = \left(\frac{\delta}{Q(t)} - \delta - \pi(t) \right) a + \frac{y_i - c}{Q(t)}, \; i = 1,2. \tag{9}$$

The first order condition for consumption is

$$u_c(c_i(t,a), \pi(t)) = \frac{1}{Q(t)} \frac{\partial v_i(t,a)}{\partial a}. \tag{10}$$

Therefore, household consumption increases with nominal bond prices and falls with the slope of the value function. Intuitively, a higher bond price (equivalently, a lower yield) gives the household an incentive to save less and consume more. A steeper value function, on the contrary, makes it more attractive to save so as to increase net asset holdings.

2.2 Foreign investors

Households trade bonds with competitive risk-neutral foreign investors that can invest elsewhere at the risk-free real rate $\bar{r}$. In addition to the latter and the amortization rate δ, foreign investors discount future nominal payoffs with the accumulated union-wide inflation (i.e. exchange rate depreciation) between the time of the bond purchase and the time such payoffs accrue. Therefore, the nominal price of the bond at time t is given by

$$Q(t) = \int_t^\infty \delta e^{-(\bar{r}+\delta)(s-t)-\int_t^s \pi_u du} ds. \tag{11}$$

Taking the derivative with respect to time, we obtain

$$Q(t)\left(\bar{r} + \delta + \pi(t)\right) = \delta + Q'(t). \tag{12}$$

The partial differential equation (12) provides the risk-neutral pricing of the nominal sovereign bond. The boundary condition is

$$\lim_{t\to\infty} Q(t) = \frac{\delta}{\bar{r} + \delta + \pi(\infty)}, \tag{13}$$

where $\pi(\infty)$ is the inflation level in the steady state, which we assume exits.

2.3 Central Bank

There is a common central bank that chooses monetary policy for the union as a whole. We assume that there are no monetary frictions so that the only role of (outside) money is that of a unit of account (*cashless limit*). The monetary authority chooses the inflation rate π_t.[11] In Section 4, we will study in detail the optimal inflationary policy of the central bank.

[11] This could be done, for example, by setting the nominal interest rate on a lending (or deposit) short-term nominal facility with foreign investors.

2.4 Competitive equilibrium

The state of the economy at time t is the joint distribution of net wealth and output, $f(t,a,y_i) \equiv f_i(t,a)$, $i = 1,2$. The dynamics of this distribution are given by the *Kolmogorov Forward* (KF) equation

$$\frac{\partial f_i(t,a)}{\partial t} = -\frac{\partial}{\partial a}\left[s_i\left(t,a\right) f_i(t,a)\right] - \lambda_i f_i(t,a) + \lambda_j f_j(t,a), \tag{14}$$

$\forall a \in [\phi, \infty)$, $i,j = 1,2$, $j \neq i$. The distribution should satisfy the normalization

$$\sum_{i=1}^{2} \int_{\phi}^{\infty} f_i\left(t,a\right) da = 1. \tag{15}$$

We define a competitive equilibrium in this economy.

Definition 1 (Competitive equilibrium) *Given a sequence of inflation rates $\pi(t)$ and an initial wealth-output distribution $f(0,a,y)$, a competitive equilibrium is composed of a household value function $v(t,a,y)$, a consumption policy $c(t,a,y)$, a bond price function $Q(t)$ and a distribution $f(t,a,y)$ such that:*

1. *Given π, the price of bonds set by investors in (12) is Q.*
2. *Given Q and π, v is the solution of the households' problem (8) and c is the optimal consumption policy.*
3. *Given Q, π, and c, f is the solution of the KF equation (14).*

Notice that, given π, the problem of foreign investors can be solved independently of that of the household in each individual member country, which in turn only depends on π and Q but not on the aggregate distribution.

We can compute some aggregate (union-wide) variables of interest. The aggregate real net financial wealth in the monetary union is

$$\bar{a}_t \equiv \sum_{i=1}^{2} \int_{\phi}^{\infty} a f_i\left(t,a\right) da. \tag{16}$$

Aggregate consumption is

$$\bar{c}_t \equiv \sum_{i=1}^{2} \int_{\phi}^{\infty} c_i\left(a,t\right) f_i\left(t,a\right) da,$$

where $c_i\left(a,t\right) \equiv c(t,a,y_i)$, $i = 1,2$, and aggregate output is

$$\bar{y}_t \equiv \sum_{i=1}^{2} \int_{\phi}^{\infty} y_i f_i\left(t,a\right) da.$$

These quantities are linked by the union-wide current account identity,

$$\begin{aligned}
\frac{d\bar{a}_t}{dt} &= \sum_{i=1}^{2}\int_{\phi}^{\infty} a\frac{\partial f_i(t,a)}{\partial t}da = \sum_{i=1}^{2}\int_{\phi}^{\infty} a\left[-\frac{\partial}{\partial a}(s_i f_i)\,da - \lambda_i f_i(t,a) + \lambda_j f_j(t,a)\right]da \\
&= \sum_{i=1}^{2}\int_{\phi}^{\infty} -a\frac{\partial}{\partial a}(s_i f_i)\,da = -\sum_{i=1}^{2} a s_i f_i|_{\phi}^{\infty} + \sum_{i=1}^{2}\int_{\phi}^{\infty} s_i f_i da \\
&= \left(\frac{\delta}{Q_t} - \delta - \pi_t\right)\bar{a}_t + \frac{\bar{y}_t - \bar{c}_t}{Q_t}, \qquad (17)
\end{aligned}$$

where we have used (14) in the second equality, and we have applied the boundary conditions $s_1(t,\phi) f_1(t,\phi) = s_2(t,\phi) f_2(t,\phi) = 0$ in the last equality.[12]

Finally, we make the following assumption.

Assumption 1 *The value of parameters is such that in equilibrium the monetary union is a net debtor against the rest of the World:* $\bar{a}_t \leq 0\ \forall t$.

This condition is imposed for tractability. We have restricted households in the union to save only in bonds issued by other union member countries, and this would not be possible if the union as a whole was a net creditor *vis-à-vis* the rest of the World. In addition to this, we have assumed that the bonds issued by the union member countries are priced by foreign investors, which requires that there should be a positive net supply of bonds to the rest of the World to be priced. In any case, this assumption is consistent with the recent experience of the Euro area, which we take as a reference, as we discuss later in the calibration section.

3 Monetary transmission: the redistributive effect of inflation

Before analyzing the optimal monetary policy problem of the central bank, it is useful to gain some insight on the monetary transmission mechanism in our framework. In particular, we want to illustrate the redistributive effects of inflation, which will play an important role in our analysis of optimal monetary policy.

To this end, we consider that the monetary union rests at the steady-state implied by a zero inflation policy,[13] and that the central bank generates a surprise, transitory spike in inflation. Since

[12] This condition is related to the fact that the KF operator is the adjoint of the infinitesimal generator of the stochastic process (3). See Appendix A for more information.

[13] As we show in Section 5, optimal long-run inflation under commitment is essentially zero for our baseline calibration.

we are not able to solve the model analytically, we use numerical methods to solve continuous-time models with heterogeneous agents, as in Achdou et al. (2015) or Nuño and Moll (20015). Appendices B and C explains how to solve for the steady state and for the dynamics, respectively.

3.1 Calibration

Let the time unit be one year. We calibrate our monetary union model to the European Monetary Union (EMU). We set the world real interest rate $\bar{r}$ to 3 percent. We assume the following specification for preferences,

$$u(c, \pi) = \log(c) - \frac{\psi}{2}\pi^2. \tag{18}$$

As discussed in Appendix D, our quadratic specification for the inflation utility cost, $\frac{\psi}{2}\pi^2$, can be micro-founded by modelling firms explicitly and allowing them to set prices subject to standard quadratic price adjustment costs *à la* Rotemberg (1982). We set the scale parameter ψ such that the slope of the inflation equation in a Rotemberg pricing setup replicates that in a Calvo pricing setup for reasonable calibrations of price adjustment frequencies and demand curve elasticities.[14]

We set the discount rate of households in the monetary union ρ such that the long-run union-wide consolidated net foreign asset position *vis-à-vis* the rest of the world ($\bar{a}$) with zero inflation replicates the average one in the EMU in the period 2009:Q1-2015:Q3 (-17.3 percent of GDP); this yields $\rho = 3.01$ percent. We also set the borrowing limit ϕ to -2, or -200 percent of average GDP in the model (which as explained below is normalized to 1), such that the equilibrium range of net debt positions accommodates the largest country-specific ratios observed in the EMU (see Figure 1).[15]

Given $\bar{r}$, we choose the bond amortization rate δ such that the Macaulay bond duration $1/(\delta + \bar{r})$ equals 5 years, which is broadly consistent with international evidence (see e.g. Cruces et al. 2002).

The output process parameters are calibrated as follows. The transition rates between boom and recession (λ_1, λ_2) are chosen such that (i) the average duration of recessions equals $1/\lambda_1 = 2$

[14] The slope of the continuous-time New Keynesian Phillips curve in the Calvo model can be shown to be given by $\chi(\chi + \rho)$, where χ is the price adjustment rate (the proof is available upon request). As shown in Appendix D, in the Rotemberg model the slope is given by $\frac{\varepsilon-1}{\psi}$, where ε is the elasticity of firms' demand curves and ψ is the scale parameter in the quadratic price adjustment cost function in that model. It follows that, for the slope to be the same in both models, we need

$$\psi = \frac{\varepsilon - 1}{\chi(\chi + \rho)}.$$

Setting ε to 11 (such that the gross markup $\varepsilon/(\varepsilon - 1)$ equals 1.10) and χ to 4/3 (such that price last on average for 3 quarters), and given our calibration for ρ, we obtain $\psi = 5.5$.

[15] Quarterly data on GDP and total financial assets/liabilities for the period 2008Q1-2015Q3 are from the ECB Statistical Data Warehouse. The wealth dimension is discretized by using 1000 equally-spaced grid points from $a = -2$ to $a = 10$.

years and (ii) the average fraction of time spent in recession equals $\lambda_2/(\lambda_1+\lambda_2)=0.40$, which corresponds closely to the historical experience of current EMU member states.[16] We assume from now on that the output distribution is at its ergodic limit, such that the fraction of member countries in boom and recession are constant at $\lambda_1/(\lambda_1+\lambda_2)$ and $\lambda_2/(\lambda_1+\lambda_2)$, respectively. We normalize average output $\frac{\lambda_2}{\lambda_1+\lambda_2}y_1+\frac{\lambda_1}{\lambda_1+\lambda_2}y_2$ to 1, such that y_2 (the output level in booms) is a function of y_1 (output in recessions); we then set y_1 to replicate the average standard deviation of output fluctuations across current EMU member states.[17] Table 1 summarizes our baseline calibration.

Table 1. Baseline calibration

Parameter	Value	Description	Source/Target
$\bar{r}$	0.03	world real interest rate	standard
γ	1	relative risk aversion coef.	standard
ρ	0.0301	subjective discount rate	average NFA position
ϕ	-2	borrowing limit	max. debt = 200% of average GDP
δ	0.17	bond amortization rate	Macaulay duration = 5 years
λ_1	0.50	transition rate recession-to-boom	average recession duration
λ_2	0.33	transition rate boom-to-recession	average time spent in recession
y_1	0.930	output level in recession	standard deviation of cyclical output
y_2	1.047	output level in boom	$E(y)=1$
ψ	5.5	scale inflation disutility	slope NKPC in Calvo model

Figure 2 displays a number of objects in the zero-inflation steady state of the model, including the value functions $v_i(a,\infty)\equiv v_i(a)$, the consumption policies $c_i(a)$, the drifts $s_i(a)$ and the the long-run net asset distribution $f_i(a)$, for $i=1,2$. The gross debt is 64.3 percent and the current account surplus is 0.45 percent (of average GDP). Notice also that the value function is strictly concave in net wealth, as is typical of models with incomplete markets. While this is difficult to appreciate in the upper left plot of Figure 2, notice from equation (10) that $\partial v_i(\infty,a)/\partial a = Q(\infty)/c_i(\infty,a)$, and that consumption is strictly increasing in a, as clearly seen in the upper right plot.

[16] Using the Federal Reserve Bank of St. Louis' OECD-based Recession Indicators database, we find an average recession duration of 22.1 months (1.84 years) and an average fraction of time in recession of 0.42 in current EMU member states; we start the sample in 1960 (the first year in the database) so as to have as large a number of recessions as possible.

[17] We use annual real GDP data from Eurostat from 1995 to 2014. We first log and linearly detrend the series so as to obtain the cyclical component. We then take the standard deviation of cyclical output for each country. The average standard deviation across countries is 5.9%.

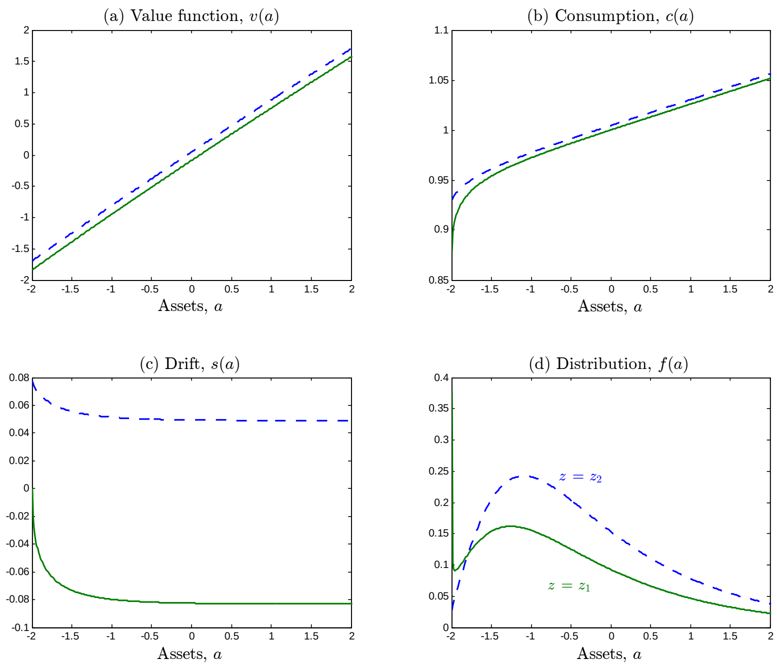

Figure 2: Steady state with zero inflation.

3.2 An inflation shock

Starting from the zero-inflation steady state just derived, assume that the central bank implements at time $t = 0$ a one-off unexpected increase in inflation of 10 percentage points that is reverted within a year. The responses of a number of union-wide variables are displayed in Figure 3.[18] The surprise increase in inflation produces a redistribution of wealth from creditor to debtor countries within the union, as shown by panels (e) and (f). This is due to a sharp decline in real yields, $r_t - \pi_t$, which reflects both the inflation spike and the comparatively tiny increase in nominal yields, as investors anticipate the short-lived nature of the inflation rise. The increase in nominal yields (equivalently, the fall in nominal bond prices) leads households in both creditor and debtor countries to reduce their consumption on impact, i.e. their primary deficit falls for given exogenous output. After the first year, however, debtor economies increase their consumption above steady state due to the reduction in their debt burden, whereas creditor ones keep theirs persistently below steady state for the opposite reason.

Compared to the rest of the World, the surprise bout in inflation also redistributes wealth from foreign investors to the monetary union as a whole, as shown by the reaction of the union's net asset position in panel (d). Union-wide consumption falls on impact due to the above-mentioned increase in nominal yields, but rises above its long-run value shortly afterwards, reflecting the

[18] We have simulated 200 years of data at monthly frequency.

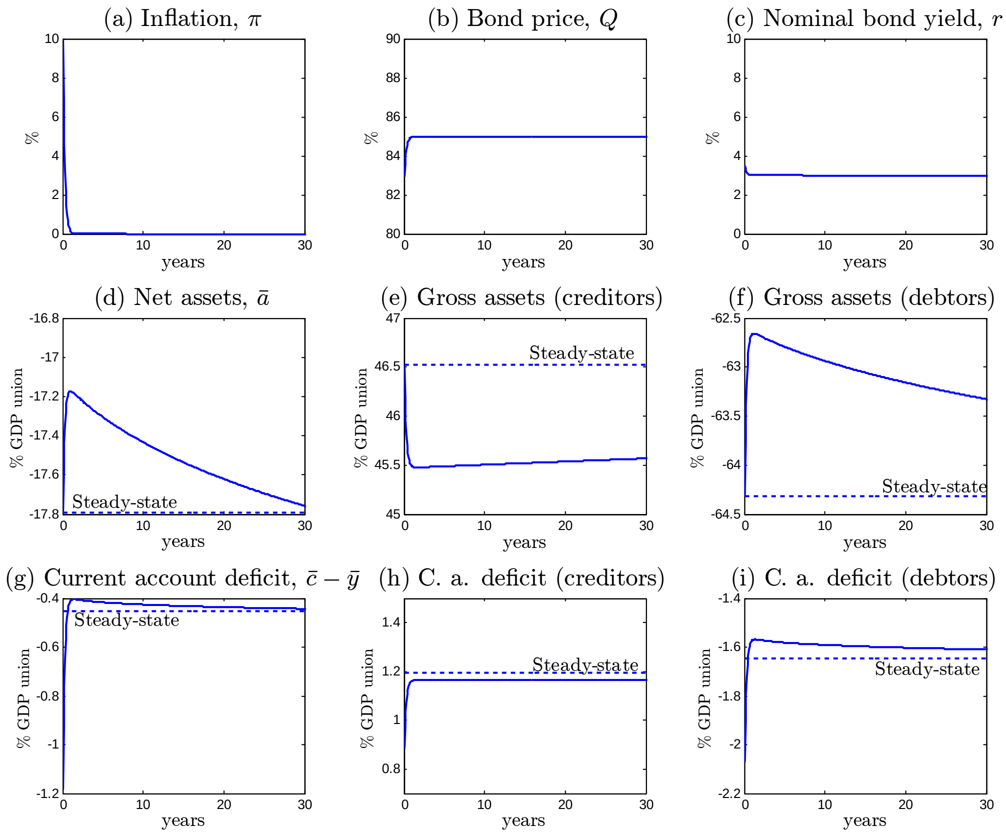

Figure 3: Transitional dynamics after an unexpected inflation increase.

relatively large consumption gains by creditor member countries.

4 Optimal monetary policy

We now turn to the design of the optimal monetary policy. We assume that the central bank is utilitarian, i.e. it gives the same Pareto weight to each household (and hence to each country) in the union. In order to illustrate the role of commitment vs. discretion in our framework, we will consider both the case in which the central can credibly commit to a future inflation path (the Ramsey problem), and the time-consistent case in which the central bank decides optimal current inflation given the current state of the economy (the Markov Perfect equilibrium).

4.1 Central bank's preferences

The central bank is assumed to be benevolent and hence maximizes union-wide aggregate welfare,

$$U_0^{CB} = \int_{\phi}^{\infty} \sum\nolimits_{i=1}^{2} v_i\left(0,a\right) f_i(0,a)da. \tag{19}$$

It will turn out to be useful to express the above welfare criterion as follows.

Lemma 1 *The welfare criterion (19) can alternatively be expressed as*

$$U_0^{CB} \equiv \int_0^{\infty} e^{-\rho s}\left[\int_{\phi}^{\infty} \sum\nolimits_{i=1}^{2} u\left(c_i\left(a,s\right),\pi\left(s\right)\right) f_i(s,a)da\right] ds. \tag{20}$$

4.2 Discretion (Markov Perfect Equilibrium)

Consider first the case in which the central bank cannot commit to any future policy. The inflation rate π then depends only on the current value of the aggregate state variable, the net wealth distribution $\{f_i\left(t,a\right)\}_{i=1,2} \equiv f\left(t,a\right)$; that is, $\pi\left(t\right) \equiv \pi^{MPE}\left[f\left(t,a\right)\right]$. This is a Markovian problem in a space of distributions. The value functional of the central bank is given by

$$J^{MPE}\left[f\left(t,\cdot\right)\right] = \max_{\{\pi_s\}_{s=t}^{\infty}} \int_t^{\infty} e^{-\rho(s-t)}\left[\sum_{i=1}^{2} \int_{\phi}^{\infty} u\left(c_{is}\left(a\right),\pi_s\right) f_i(s,a)da\right] ds, \tag{21}$$

subject to the law of motion of the distribution (14). Notice that the optimal value J^{MPE} and the optimal policy π^{MPE} are not ordinary functions, but *functionals*, as they map the infinite-dimensional state variable $f\left(t,a\right)$ into $\mathbb{R}$.

Let $f_0\left(\cdot\right) \equiv \{f_i\left(0,a\right)\}_{i=1,2}$ denote the initial distribution. We can define the equilibrium in this case.

Definition 2 (Markov Perfect Equilibrium) *Given an initial distribution* f_0, *a* symmetric Markov Perfect Equilibrium *is composed of a sequence of inflation rates* $\pi\left(t\right)$, *a household value function* $v(t,a,y)$, *a consumption policy* $c(t,a,y)$, *a bond price function* $Q\left(t\right)$ *and a distribution* $f(t,a,y)$ *such that:*

1. *Given* π, *then* v, c, Q *and* f *are a competitive equilibrium.*
2. *Given* c, Q *and* f, π *is the solution to the central bank problem (21).*

The fact that v, c, Q and f are part of a competitive equilibrium needs to be imposed in the definition of Markov Perfect Equilibrium, as it is not implicit in the central bank's problem (21).

Using standard dynamic programming arguments, the problem (39) can be expressed recursively as

$$J^{MPE}\left[f\left(t,\cdot\right)\right]=\max_{\{\pi_s\}_{s=t}^{\tau}}\int_t^{\tau}e^{-\rho(s-t)}\left[\int_{\phi}^{\infty}\sum\nolimits_{i=1}^{2}u\left(c_{is},\pi_s\right)f_i(s,a)da\right]ds+e^{-\rho(\tau-t)}J^{MPE}\left[f\left(\tau,\cdot\right)\right], \tag{22}$$

for any $\tau>t$ and subject to the law of motion of the distribution (14).

The solution to the central bank's problem is given in the following proposition.

Proposition 1 (Optimal inflation - MPE) *In addition to equations (14), (12), (8) and (10), if a solution to the MPE problem (21) exists, the inflation rate function $\pi\left(t\right)$ must satisfy*

$$\sum_{i=1}^{2}\int_{\phi}^{\infty}\left[a\frac{\partial v_i}{\partial a}-u_{\pi}\left(c_i\left(t,a\right),\pi\left(t\right)\right)\right]f_i\left(t,a\right)da=0. \tag{23}$$

In addition, the value functional must satisfy

$$J^{MPE}\left[f\left(t,\cdot\right)\right]=\sum_{i=1}^{2}\int_{\phi}^{\infty}v_i\left(t,a\right)f_i(t,a)da, \tag{24}$$

The proof is in Appendix A. Our approach combines the dynamic programming representation (22) with the *Riesz Representation Theorem*, which allows decomposing the central bank value functional J^{MPE} as an aggregation of individual values $v_i\left(t,a\right)$ across agents.

Equation (23) captures the basic static trade-off that the central bank faces when choosing inflation under discretion. The central bank balances the marginal utility cost of higher inflation across the monetary union (u_{π}) against the marginal welfare effects due to the impact of inflation on each country's net foreign asset position ($a\frac{\partial v_i}{\partial a}$). For debtor countries ($a<0$), the latter effect is positive as inflation erodes the real value of their debt burden, whereas the opposite is true for creditor countries ($a>0$). Under Assumption 1 (the monetary union as a whole is a net debtor), and provided the value function is concave in net wealth, then the central bank will have a double motive to use inflation for redistributive purposes. On the one hand, it will try to redistribute wealth from foreign investors to debtor member countries. On the other hand, and somewhat more subtly, if debtor countries have a higher marginal utility of net wealth than creditor ones, then the central bank will be led to redistribute from the latter to the former, as such course of action is understood to raise welfare in the union as a whole.

4.3 Commitment

Assume now that the central bank can credibly commit at time zero to an inflation path $\{\pi(t)\}_{t=0}^{\infty}$. The optimal inflation path is now a function of the initial distribution $f_0(a)$ and of time: $\pi(t) \equiv \pi^R[t, f_0(a)]$. The value functional of the central bank is now given by

$$J^R[f_0(\cdot)] = \max_{\{\pi_s, Q_s, v(s,\cdot), c(s,\cdot), f(s,\cdot)\}_{s=0}^{\infty}} \int_0^{\infty} e^{-\rho s} \left[\int_{\phi}^{\infty} \sum\nolimits_{i=1}^{2} u(c_{is}, \pi_s) f_i(s,a) da\right] ds, \tag{25}$$

subject to the law of motion of the distribution (14), the bond pricing equation (12), and each country's HJB equation (8) and optimal consumption choice (10). The optimal value J^R and the optimal policy π^R are again functionals, as in the discretionary case, only now they map the *initial* distribution $f_0(\cdot)$ into $\mathbb{R}$, as opposed to the distribution at each point in time. Notice that the central bank maximizes welfare taking into account not only the state dynamics (14), but *also* the HJB equation (8) and the bond pricing condition (12). That is, the central bank understands how it can steer the expectations of households in each member country and of foreign investors by committing to an inflation path. This is unlike in the discretionary case, where the central bank takes the expectations of other agents as given.

Definition 3 (Ramsey problem) *Given an initial distribution f_0, a* Ramsey problem *is composed of a sequence of inflation rates $\pi(t)$, a household value function $v(t,a,y)$, a consumption policy $c(t,a,y)$, a bond price function $Q(t)$ and a distribution $f(t,a,y)$ such that they solve the central bank problem (25).*

If v, f, c and Q are a solution to the problem (25), given π, they constitute a competitive equilibrium, as they satisfy equations (14), (12), (8) and (10). Therefore the Ramsey problem could be redefined as that of finding the π such that v, f, c and Q are a competitive equilibrium and the central bank's welfare criterion is maximized.

The Ramsey problem is an optimal control problem in a suitable function space. The solution is given by the following proposition.

Proposition 2 (Optimal inflation - Ramsey) *In addition to equations (14), (12), (8) and (10), if a solution to the Ramsey problem (25) exists, the inflation path $\pi(t)$ must satisfy*

$$\mu(t) Q(t) = \sum_{i=1}^{2} \int_{\phi}^{\infty} \left[a \frac{\partial v_{it}}{\partial a} - u_{\pi}(c_i(t,a), \pi(t))\right] f_i(t,a) da, \tag{26}$$

and a costate $\mu(t)$ with law of motion

$$\frac{d\mu(t)}{dt} = (\rho - \bar{r} - \pi(t) - \delta) \mu(t) + \sum_{i=1}^{2} \int_{\phi}^{\infty} \frac{\partial v_{it}}{\partial a} \frac{\delta a + y_i - c_i(t,a)}{Q(t)^2} f_i(t,a) da \tag{27}$$

and initial condition $\mu(0) = 0$.

The proof can also be found in Appendix A. Our approach is to solve the constrained optimization problem (25) in an infinite-dimensional Hilbert space. To this end, we need to employ a generalized version of the classical differential known as 'Gateaux differential'.[19]

The equation determining optimal inflation under commitment (26), is identical to that in the discretionary case (23), *except* for the presence on the left-hand side of the costate $\mu(t)$, which is the Lagrange multiplier associated to the bond pricing equation (12). Intuitively, $\mu(t)$ captures the value to the central bank of promises about time-t inflation made to foreign investors at time 0. Such value is zero only at the time of announcing the Ramsey plan ($t = 0$), because the central bank is not bound by previous commitments, but it will generally be different from zero at any time $t > 0$. By contrast, in the MPE case no promises are made at any point in time, hence the absence of such costate. Therefore, the static trade-off between the welfare cost of inflation and the welfare gains from inflating away net liabilities, explained above in the context of the MPE solution, is now modified by the central bank's need to respect past promises to investors about current inflation. If $\mu(t) < 0$, then the central bank's incentive to create inflation at time $t > 0$ so as to redistribute wealth will be tempered by the fact that it internalizes how expectations of higher inflation affect investors' bond pricing prior to time t.

Notice that the Ramsey problem is not time-consistent, due precisely to the presence of the (forward-looking) bond pricing condition in that problem.[20] If at some future time $t > 0$ the central bank decided to re-optimize given the current state $f(t, a, y)$, the new path for optimal inflation $\tilde{\pi}(t) \equiv \pi^R[t, f_t(\cdot)]$ would not need to coincide with the original path $\pi(t) \equiv \pi^R[t, f_0(\cdot)]$, as the value of the costate at that point would be $\tilde{\mu}(t) = 0$ (corresponding to a new commitment formulated at time t), whereas under the original commitment it is $\mu(t) \neq 0$.

4.4 Some analytical results

In order to provide some additional analytical insights on optimal policy, we make the following assumption on preferences.

Assumption 2 *Consider the class of separable utility functions*

$$u(c, \pi) = u^c(c) - u^\pi(\pi).$$

[19] The system composed of equations (8), (12), (14), (10), (26) and (27) is technically known as *forward-backward*, as both households and investors proceed backwards in order to compute their optimal values, policies and bond prices, whereas the distributional dynamics proceed forwards.

[20] As is well known, the MPE solution is time consistent, as it only depends on the current state.

The utility function also satisfies $u_c^c > 0$, $u_{cc}^c < 0$ *for* $c > 0$, $u_\pi^\pi > 0$ *for* $\pi > 0$, $u_\pi^\pi < 0$ *for* $\pi < 0$, $u_{\pi\pi}^\pi > 0$ *for all* π, *and* $u^\pi(0) = u_\pi^\pi(0) = 0$.

Our first result regards the existence of a positive *inflationary bias* under discretionary optimal monetary policy. This holds as long as the value function is concave in a.

Proposition 3 (Inflation bias under discretion) *Let preferences satisfy Assumption 2. Provided that* $\frac{\partial^2 v_i}{\partial a^2} < 0$, $i = 1, 2$, *the optimal inflation under discretion is positive:* $\pi(t) > 0$ *for all* $t \geq 0$.

The proof can be found in Appendix A. To gain intuition, we can use the above separable preferences in order to express the optimal inflation decision under discretion (equation 23) as

$$u_\pi^\pi(\pi(t)) = \sum_{i=1}^{2} \int_\phi^\infty (-a) \frac{\partial v_i}{\partial a} f_i(t,a)\, da. \tag{28}$$

That is, under discretion inflation increases with the average net liabilities across the union *weighted by each country's marginal utility of wealth*, $\partial v_i / \partial a$. Notice first that, from Assumption 1, the monetary union as a whole is a net debtor: $\sum_{i=1}^{2} \int_\phi^\infty (-a) f_i(t,a)\, da = (-)\, \bar{a}_t \geq 0$. This, combined with the strict concavity of each country's value function (such that debtor countries receive more weight than creditor ones), makes the right-hand side of (28) strictly positive. Since $u_\pi^\pi(\pi) > 0$ only for $\pi > 0$, it follows that inflation must be positive. Notice that, even if the monetary union as a whole is neither a creditor or a debtor ($\bar{a}_t = 0$), as long as there is within-union wealth dispersion and the individual value function is concave, the common monetary authority will have a reason to inflate.

The result in Proposition 3 is reminiscent of the classical inflationary bias of discretionary monetary policy originally emphasized by Kydland and Prescott (1977) and Barro and Gordon (1983). In those papers, the source of the inflation bias is a persistent attempt by the monetary authority to raise output above its natural level. Here, by contrast, it arises from the welfare gains that can be achieved for the monetary union as a whole by redistributing wealth towards debtor member countries.

We now turn to the commitment case. Under the above separable preferences, from equation (26) optimal inflation under commitment satisfies

$$u_\pi^\pi(\pi(t)) = \sum_{i=1}^{2} \int_\phi^\infty (-a) \frac{\partial v_i}{\partial a} f_i(t,a)\, da + \mu(t)\, Q(t). \tag{29}$$

In this case, the inflationary pressure coming from the redistributive incentives is counterbalanced by the value of time-0 promises about time-t inflation, as captured by the costate $\mu(t)$. Thus, a negative value of such costate leads the central bank to choose a *lower* inflation rate than the one it would set *ceteris paribus* under discretion.

Unfortunately, we cannot solve analytically for the optimal path of inflation. However, we are able to establish the following important result regarding the long-run level of inflation under commitment.

Proposition 4 (Optimal long-run inflation under commitment) *Let preferences satisfy Assumption 2. In the limit as $\rho \to \bar{r}$, the optimal steady-state inflation rate under commitment tends to zero:* $\lim_{\rho \to \bar{r}} \pi(\infty) = 0$.

Provided the discount factor of households in the monetary union (and of its benevolent central bank) is arbitrarily close to that of foreign investors, then optimal long-run inflation under commitment will be arbitrarily close to zero. The intuition is the following. The inflation path under commitment converges over time to a level that optimally balances the marginal welfare costs and benefits of trend inflation. On the one hand, the welfare costs include the direct utility costs, but also the increase in nominal bond yields that comes about with higher expected inflation; indeed, from the definition of the yield (5) and the expression for the long-run nominal bond price (13), the long-run nominal bond yield is given by the following long-run Fisher equation,

$$r(\infty) = \frac{\delta}{Q(\infty)} - \delta = \bar{r} + \pi(\infty), \tag{30}$$

such that nominal yields increase one-for-one with (expected) inflation in the long run. On the other hand, the welfare benefits of inflation are given by its redistributive effect (for given nominal yields). As $\rho \to \bar{r}$, these effects tend to exactly cancel out precisely at zero inflation.

Proposition 4 is reminiscent of a well-known result from the New Keynesian literature, namely that optimal long-run inflation in the standard New Keynesian framework is exactly zero (see e.g. Benigno and Woodford, 2005). In that framework, the optimality of zero long-run inflation arises from the fact that, at that level, the welfare gains from trying to exploit the short-run output-inflation trade-off (i.e. raising output towards its socially efficient level) exactly cancel out with the welfare losses from permanently worsening that trade-off (through higher inflation expectations). Key to that result is the fact that, in that model, price-setters and the (benevolent) central bank have the same (steady-state) discount factor. Here, the optimality of zero long-run inflation reflects instead the fact that, at zero trend inflation, the welfare gains from trying to redistribute wealth from creditors to debtors becomes arbitrarily close to the welfare losses from lower nominal bond prices when the discount rate of the investors pricing such bonds is arbitrarily close to that of the central bank.

Assumption 1 restricts us to have $\rho > \bar{r}$, as otherwise households would we able to accumulate enough wealth so that the monetary union would stop being a net debtor to the rest of the World.

However, Proposition 4 provides a useful benchmark to understand the long-run properties of optimal policy in our model when ρ is very close to $\bar{r}$. This will indeed be the case in our subsequent numerical analysis.

5 Quantitative analysis of optimal monetary policy

In the previous section we have characterized the optimal monetary policy in our model. In this section we solve numerically for the dynamic equilibrium under optimal policy. Before analyzing the dynamic path of this economy under the optimal policy, we first analyze the steady state towards which such path converges asymptotically. The numerical algorithms that we use are described in Appendices B (steady-state) and C (transitional dynamics).

5.1 Steady-state

We consider the same calibration of section 3. The steady-state values in the two monetary regimes (commitment and discretion) are displayed in Table 2. Under commitment, the optimal long-run inflation is close to zero (-0.02 percent), consistently with Proposition 4 and the fact ρ and $\bar{r}$ are essentially the same in our calibration.[21] The net total asset position is -17.9 percent, the gross debt is 64.4 percent and the current account surplus is 0.46 percent. From now on, we use $x \equiv x(\infty)$ to denote the steady state value of any variable x. As shown in the previous section, the long-run nominal yield is $r = \bar{r} + \pi$, where the World real interest rate $\bar{r}$ equals 3 percent in our calibration.

Table 2. Steady-state values

	units	Ramsey	MPE
Inflation, π	%	-0.02	1.47
Nominal yield, r	%	2.98	4.47
Net assets, $\bar{a}$	% union GDP	-17.94	-5.72
Gross assets (creditors)	% union GDP	46.43	54.37
Gross debt (debtors)	% union GDP	-64.37	-60.09
Current acc. deficit, $\bar{c} - \bar{y}$	% union GDP	-0.46	-0.14

Under discretion, by contrast, long run inflation is 1.47 percent, which reflects the inflationary bias discussed in the previous section. Net total assets amount to 5.7 percent, the gross debt is

[21] As explained in section 3, in our baseline calibration we have $\bar{r} = 0.03$ and $\rho = 0.0301$.

60.1 percent and the current account surplus is 0.14 percent. The presence of an inflationary bias makes nominal interest rates higher through the Fisher equation (30).

5.2 Optimal transitional dynamics

As explained in Section 4, the optimal policy paths depend on the initial (time-0) net wealth distribution across union member countries, $\{f_i(0,a)\}_{i=1,2}$, which is an (infinite-dimensional) primitive in our model.[22] In order to discipline our choice of initial distribution, and given our choice of the EMU as our target currency area for calibration, we consider an initial distribution that is close to that observed nowadays across EMU member states. In particular, we construct a smoothed approximation to the actual (19-point) distribution in the year 2009 using a normal kernel similar to that in Figure 1 but modified to take account of the different country sizes.[23] Our choice of 2009 as a reference year is mainly for the purpose of illustration, and is not intended to suggest that the ECB should have reassessed its monetary policy commitment based on the asset distribution in that year; using instead an initial distribution based on the actual one in 1999, i.e. the year the EMU started operating, produces similar results.[24]

Consider first the case under commitment (Ramsey policy). The optimal paths are shown by the green solid lines in Figure 4. Under our assumed functional form for preferences in (18), we have from equation (29) that initial optimal inflation is given by

$$\pi(0) = \frac{1}{\psi} \sum_{i=1}^{2} \int_{\phi}^{\infty} (-a) \frac{\partial v_i(0,a)}{\partial a} f_i(0,a)\, da,$$

where we have used the fact that $\mu(0) = 0$, as there are no pre-commitments at time zero. Therefore, the initial inflation rate, of about 2.5 percent, reflects exclusively the redistributive motive discussed in section 4.

As time goes by, optimal inflation under commitment gradually declines towards its (near) zero long-run level. The intuition is straightforward. At the time of formulating its commitment, the central bank exploits the existence of a stock of nominal bonds issued in the past. This means that the inflation created by the central bank has no effect on the prices at which those bonds

[22] As explained in section 3.1, in our numerical exercises we assume that the output distribution starts at its ergodic limit: $f_y(y_i) = \lambda_{j\neq i}/(\lambda_1+\lambda_2)$, $i = 1,2$. Also, in all our subsequent exercises we assume that the time-0 net asset distribution conditional on being in a boom is identical to that conditional on recession: $f_{a|y}(0, a \mid y_2) = f_{a|y}(0, a \mid y_1) \equiv f_0(a)$. Therefore, the initial joint density is simply $f(0,a,y_i) = f_0(a)\,\lambda_{j\neq i}/(\lambda_1+\lambda_2)$, $i = 1,2$.

[23] The normal kernel shown in Figure 1 uses the actual net foreign asset-to-GDP ratio of each EMU country member, implicitly giving the same weight to all 19 data points, both of small countries (e.g. Luxembourg) and large ones (e.g. Germany). Since all monetary union members are equally sized in our model, the resulting normal kernel overrepresents small countries and vice versa. Here, by contrast, we reweight each country's net foreign asset ratio with its weight in 2009 EMU GDP.

[24] These results are available upon request.

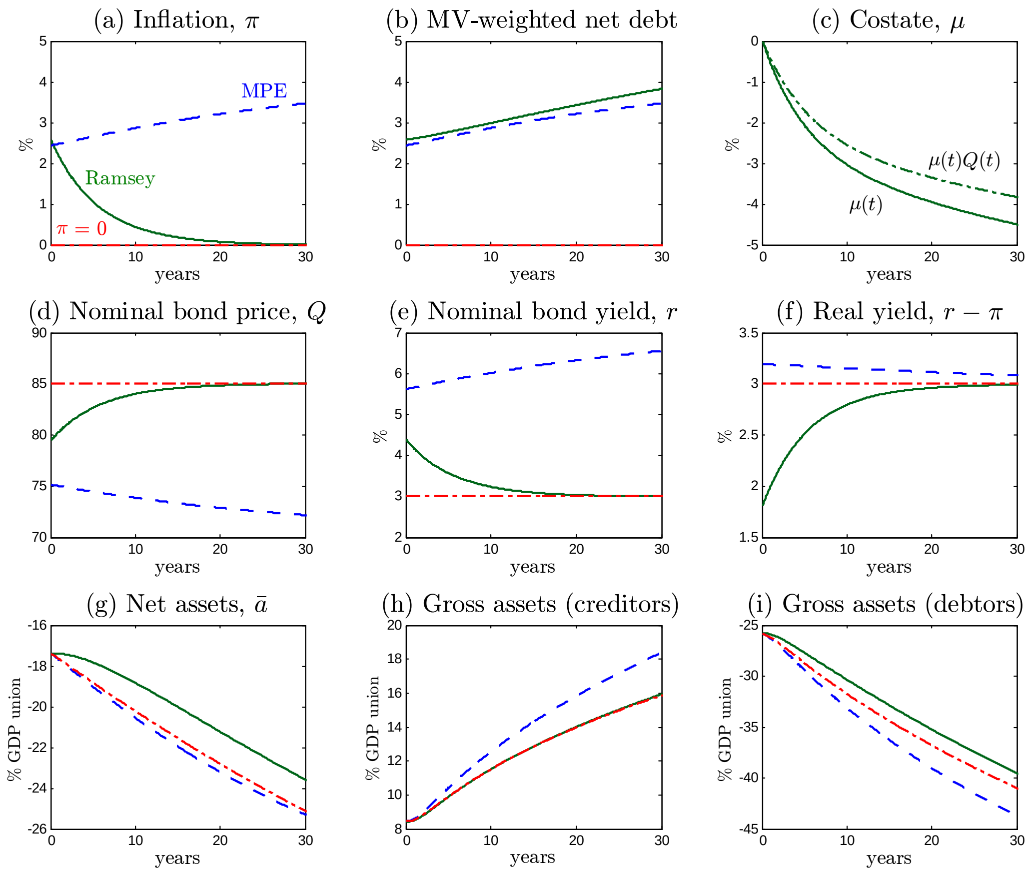

Figure 4: Dynamics under optimal monetary policy and zero inflation.

were issued. However, the price of nominal bonds issued from time 0 onwards *does* incorporate the expected future inflation path. Under commitment, the central bank internalizes that higher future inflation reduces nominal bond prices, i.e. it raises nominal bond yields, which hurts net bond issuers. This effect becomes stronger and stronger over time, as the fraction of total nominal bonds that were issued before the time-0 commitment becomes smaller and smaller. This gives the central bank the right incentive to gradually reduce inflation over time. Formally, in the equation that determines optimal inflation at $t \geq 0$,

$$\pi(t) = \frac{1}{\psi} \sum_{i=1}^{2} \int_{\phi}^{\infty} (-a) \frac{\partial v_i}{\partial a} f_i(t, a)\, da + \frac{1}{\psi} \mu(t) Q(t), \tag{31}$$

the (absolute) value of the costate $\mu(t)$, which captures the effect of time-t inflation on the price of bonds issued during the period $[0, t)$, becomes larger and larger over time. As shown in Figure 4, the increase in $|\mu(t) Q(t)|$ dominates that of the marginal-value-weighted average net liabilities, $\sum_i \int_{\phi}^{\infty} (-a) \frac{\partial v_i}{\partial a} f_i(t, a)\, da$, which from equation (31) produces the gradual fall in inflation.[25] In summary, under the optimal commitment the central bank *front-loads* inflation in order to redistribute existing wealth, committing to gradually reduce inflation towards zero in order to prevent inflation expectations from permanently raising nominal yields.

Under discretion (dashed blue lines in Figure 4), time-zero inflation is the same as under commitment, given the absence of prior commitments in the latter case. In contrast with the commitment case, however, from time zero onwards optimal discretionary inflation remains relatively high, with no tendency to fall. The reason is the inflationary bias that stems from the central bank's attempt to redistribute wealth from creditors (including foreign investors) to debtors. This inflationary bias is not counteracted by any concern about the effect of inflation expectations on nominal bond yields; that is, the costate $\mu(t)$ in equation (31) is zero at all times under discretion. This inflationary bias produces permanently lower nominal bond prices (higher nominal yields) than under commitment.

In order to further illustrate the effects of the optimal inflation path under commitment, we compare the latter policy with a strict policy rule of zero inflation at all times: $\pi(t) = 0$ for all $t \geq 0$. The implied equilibrium dynamics are shown by the red dashed lines in Figure 4. Under such a policy, bond prices and yields are constant at the levels $Q(t) = \frac{\delta}{\bar{r}+\delta} = 0.85$ and $\bar{r} = 3\%$, respectively, and no redistribution takes place between creditors and debtors. Compared to this zero-inflation equilibrium, the Ramsey allocation does achieve a certain degree of redistribution, as debtors' liabilities increase more slowly. The reason is that the optimal commitment plan manages

[25] Panels (b) and (c) in Figure 4 display the two terms on the right-hand side of (31), i.e. the marginal-value-weighted average net liabilities and $\mu(t) Q(t)$ both *rescaled* by the inflation disutility parameter ψ. Therefore, the sum of both terms equals optimal inflation under commitment.

to reduce real yields temporarily relative to $\bar{r}$, thanks both to the temporary inflation and the relatively mild increase in nominal yields, as investors anticipate the transitory nature of such inflation.

5.3 Welfare analysis

We now turn to the welfare analysis of alternative policy regimes. Aggregate union-wide welfare is defined as

$$\int_{\phi}^{\infty} \sum_{i=1}^{2} v_i(0,a) f_i(t,a)da = \int_0^{\infty} e^{-\rho t} \int_{\phi}^{\infty} \sum_{i=1}^{2} u(c_i(t,a), \pi(t)) f_i(t,a)dadt \equiv W[c],$$

Table 3 displays the welfare losses of suboptimal policies *vis-à-vis* the Ramsey optimal equilibrium. We express welfare losses as a permanent consumption equivalent, i.e. the number Θ (in %) that satisfies in each case $W^R[c] = W[(1+\Theta)c]$, where R denotes the Ramsey equilibrium.[26] The table also displays the welfare losses incurred respectively by creditor and debtor countries.[27] The welfare losses from discretionary policy versus commitment are of first order: 0.26% of permanent consumption. This welfare loss is suffered by creditor countries (0.12%), *but also by debtor ones* (0.15%), despite the fact that the discretionary policy is aimed precisely at redistributing wealth towards debtor countries. As shown in Figure 4, debtors' liabilities actually *increase* under discretion relative to the Ramsey equilibrium. The reason is that the lower bond prices force these countries to sell more bonds and thus increase their indebtedness. Moreover, the higher inflation under discretion does *not* help erode such indebtedness: nominal yields are much higher too, implying higher *real* yields and hence faster debt accumulation; in fact, real yields under discretion are even higher than under the zero inflation policy (for which $r_t - \pi_t = \bar{r} = 3\%$), as investors price in the rising path of future inflation. By contrast, and as explained in the previous subsection, the commitment policy does reduce real yields relative to $\bar{r}$, by creating some inflation initially while avoiding high nominal yields.

In summary, discretionary policy fails at producing the very redistribution towards debtor countries that it intends to achieve in the first place, while leaving both creditor and debtor members to bear the direct welfare costs of permanent positive inflation.

[26] Under our assumed separable preferences with log consumption utility, it is possible to show that $\Theta = \exp\left\{\rho\left(W^R[c] - W[c]\right)\right\} - 1$.

[27] That is, we report $\Theta^{a>0}$ and $\Theta^{a<0}$, where

$$\Theta^{a>0} = \exp\left[\rho\left(W^{R,a>0} - W^{MPE,a>0}\right)\right] - 1,$$

with $\Theta^{a<0}$ defined analogously, and where for each policy regime we have defined $W^{a>0} \equiv \int_0^{\infty} \sum_{i=1}^{2} v_i(0,a) f_i(t,a)da$, $W^{a<0} \equiv \int_{\phi}^{0} \sum_{i=1}^{2} v_i(0,a) f_i(t,a)da$. Notice that $\Theta^{a>0}$ and $\Theta^{a>0}$ do not exactly add up to Θ, as the expontential function is not a linear operator. However, Θ is sufficiently small that $\Theta \approx \Theta^{a>0} + \Theta^{a>0}$.

Table 3. Welfare losses relative to the optimal commitment

	Union-wide	Creditor countries	Debtor countries
Discretion	0.26	0.12	0.15
Zero inflation	0.01	-0.02	0.03

Note: welfare losses are expressed as a % of permanent consumption

We also compute the welfare losses from a policy of zero inflation, $\pi(t) = 0$ for all $t \geq 0$. As the table shows, the latter policy approximates the welfare outcome under commitment very closely, by balancing the welfare gains for creditor countries and the losses for debtor ones, both of which are in turn relatively small. Such small welfare losses reflects the transitory nature of optimal inflation under commitment and the limited extent of the resulting redistribution, as well as the fact that both policies avoid the first-order welfare costs from a redistribution-driven inflationary bias.

5.4 Robustness

Steady state inflation. In Proposition 4, we established that the Ramsey optimal long-run inflation rate converges to zero as the central bank's discount rate ρ converges to that of foreign investors, $\bar{r}$. In our baseline calibration, both discount rates are indeed very close to each other, implying that Ramsey optimal long-run inflation is essentially zero. We now evaluate the sensitivity of Ramsey optimal steady state inflation to the difference between both discount rates. From equation (31), Ramsey optimal steady state inflation is

$$\pi = \frac{1}{\psi} \sum_{i=1}^{2} \int_{\phi}^{\infty} (-a) \frac{\partial v_i}{\partial a} f_i(a)\, da + \frac{1}{\psi} \mu Q, \tag{32}$$

where the first term on the right hand side captures the redistributive motive to inflate in the long run, and the second one reflects the effect of central bank's commitments about long-run inflation. Figure 5 displays π (left axis), as well as its two determinants (right axis) on the right-hand side of equation (32). Optimal decreases approximately linearly with the gap $\rho - \bar{r}$. As the latter increases, two counteracting effects take place. On the one hand, it can be shown that as the monetary union's households become more impatient relative to foreign investors, the net asset distribution shifts towards the left, i.e. more and more member countries become net borrowers and come close to the borrowing limit, where the marginal utility of wealth is highest.[28] As shown in the figure, this increases the central bank's incentive to inflate for the purpose of redistributing

[28] The evolution of the long-run wealth distribution as $\rho - \bar{r}$ increases is available upon request.

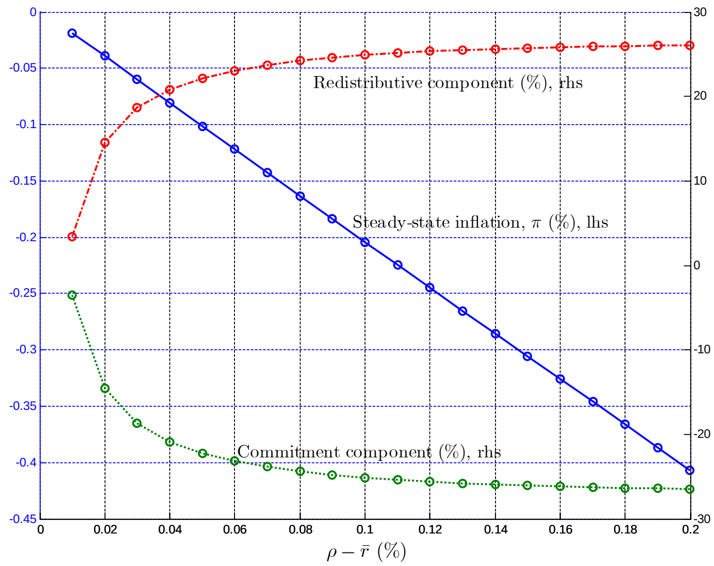

Figure 5: Sensitivity analysis to changes in $\rho - \bar{r}$.

wealth towards debtor countries. On the other hand, the more impatient households become relative to foreign investors, the more the central bank internalizes in present-discounted value terms the welfare consequences of creating expectations of higher inflation in the long run. This provides the central bank an incentive to committing to *lower* long run inflation. As shown by Figure 5, this second 'commitment' effect dominates the 'redistributive' effect, such that in net terms optimal long-run inflation becomes more negative as the discount rates gap widens.

Initial inflation. As explained before, time-0 optimal inflation and its subsequent path depend on the initial net wealth distribution across monetary union members, which is an infinite-dimensional object. In our baseline numerical analysis, we disciplined our choice of initial distribution by setting it approximately equal to the actual cross-country distribution in the EMU in 2009. We now investigate how initial inflation depends on such initial distribution. To make the analysis operational, we restrict our attention to the class of Normal distributions truncated at the borrowing limit ϕ. That is,

$$f(0, a) = \begin{cases} \phi(a; \mu, \sigma) / [1 - \Phi(\phi; \mu, \sigma)], & a \geq \phi \\ 0, & a < \phi \end{cases}, \tag{33}$$

where $\phi(\cdot; \mu, \sigma)$ and $\Phi(\cdot; \mu, \sigma)$ are the Normal pdf and cdf, respectively.[29] The parameters μ and σ allow us to control both (i) the initial net foreign asset position for the monetary union as a

[29] As explained in Section 5.2, in all our simulations we assume that the initial net asset distribution conditional on being in a boom or in a recession is the same: $f_{a|y}(0, a \mid y_2) = f_{a|y}(0, a \mid y_1) \equiv f_0(a)$. This implies that the marginal asset density coincides with its conditional density: $f(0, a) = \sum_{i=1,2} f_{a|y}(0, a \mid y_i) f_y(y_i) = f_0(a)$.

whole and (ii) the within-union cross-country dispersion in net foreign asset positions, and hence to isolate the effect of each factor on the optimal inflation path. Notice also that optimal long-run inflation rates do *not* depend on $f(0, a)$ and are therefore exactly the same as in our baseline numerical analysis regardless of μ and σ.[30] This allows us to focus here on inflation at time 0, which is the same under commitment and discretion, while noting that the transition paths towards the respective long-run levels are isomorphic to those displayed in Figure 4.[31]

Figure 6 displays optimal initial inflation rates for alternative initial net foreign asset distributions. In the first row of panels, we show the effect of increasing *within-union* net wealth dispersion while restricting the monetary union to have a zero net position *vis-à-vis* the rest of the World, i.e. we increase σ and simultaneously adjust μ to ensure that $\bar{a}(0) = 0$. In the extreme case of a (quasi) degenerate initial distribution at zero net assets (solid blue line in the upper left panel), the central bank has no incentive to create inflation, and thus optimal initial inflation is zero. As the degree of initial wealth dispersion increases, so does optimal initial inflation, although the latter remains within first-order magnitude.

The bottom row of panels in Figure 6 isolates instead the effect of increasing the monetary union's liabilities with the rest of the World, while assuming at the same time $\sigma \simeq 0$, i.e. eliminating any within-union wealth dispersion.[32] As shown by the lower right panel, optimal inflation increases fairly quickly with the monetary union's external indebtedness; for instance, an external debt-to-GDP ratio of 50% justifies an initial inflation of over 6%.

As we saw in Section 5.2, setting the initial asset distribution equal to (a smoothed approximation) of the actual cross-country distribution in the EMU in 2009 delivered an initial optimal inflation rate of $\pi(0) = 2.55\%$. Such distribution implies a consolidated net asset position for the monetary union of $\bar{a}(0) = -17.34\%$ of GDP. Using as initial condition a degenerate distribution at exactly that ratio (i.e. $\sigma \simeq 0$, $\mu = -17.34\%$) delivers $\pi(0) = 2.49\%$. This suggests that initial optimal inflation is mostly the outcome of the central bank's attempt to redistribute wealth from foreign investors to debtor union members, rather than from creditor to debtor countries within the union.

5.5 Political economy considerations

Although in our model the Ramsey optimal monetary policy is the best one for the monetary union as a whole, not all member countries prefer it to other alternative policies. And if all members

[30] As shown in Table 2, long-run inflation is -0.02% under commitment, and 1.47% under discretion.

[31] The full dynamic optimal paths under any of the alternative calibrations considered in this section are available upon request.

[32] That is, we approximate 'Dirac delta' distributions centered at different values of μ. Since such distributions are not affected by the truncation at $a = \phi$, we have $\bar{a}(0) \equiv \mu$, i.e. the monetary union's net foreign asset position coincides with μ.

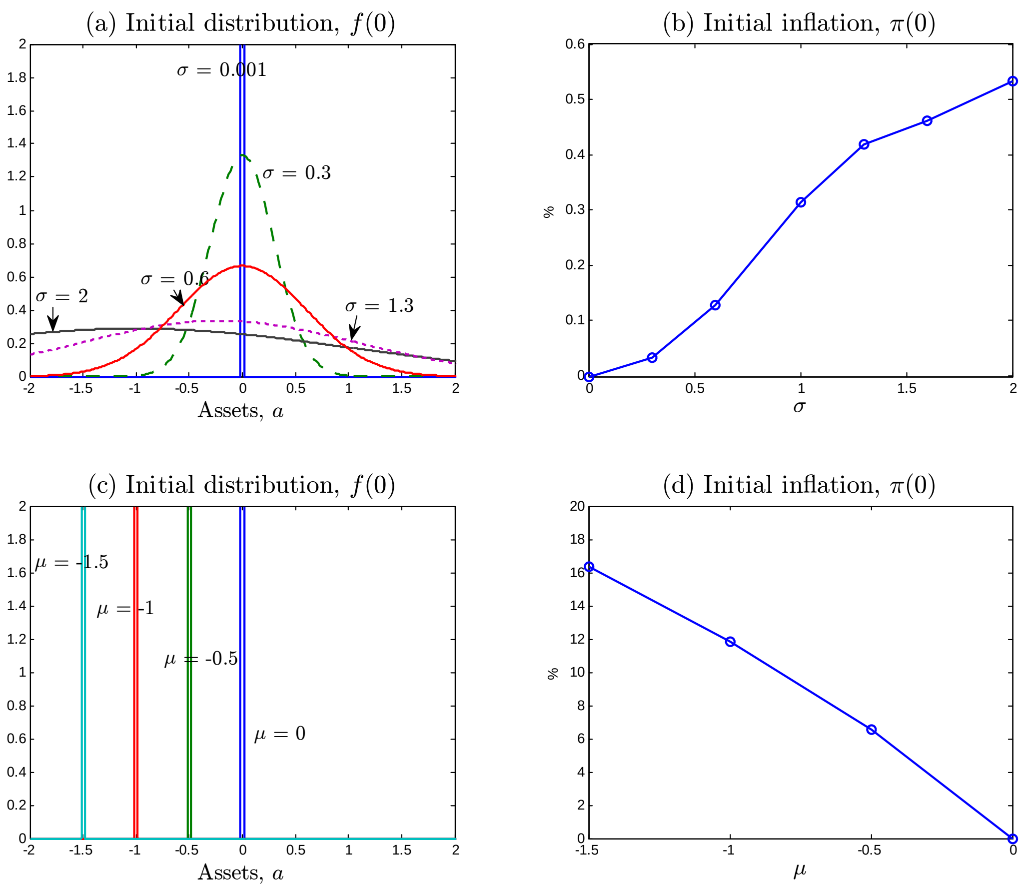

Figure 6: Ramsey optimal initial inflation for different initial net asset distributions.

have a say in the monetary policy decisions, then it is not guaranteed that the Ramsey policy would actually be implemented. For instance, we may consider an institutional setup in which all monetary policy decisions are taken by simple majority by a 'governing council' in the central bank where all countries have equal voting rights.[33] We now analyze whether the Ramsey optimal policy would actually be approved with such an institutional setup. This requires specifying an alternative monetary policy against which member countries can compare the Ramsey optimal one. For the purpose of illustration, we consider as benchmark the zero inflation policy, $\pi(t) = 0$ for all $t \geq 0$, which can be interpreted as 'strict inflation targeting'.

The upper panel in Figure 7 shows the time-0 value function for an individual country as a function of its initial net wealth under the Ramsey policy net of the same value function under strict inflation targeting.[34] Notice first that the level of net wealth that separates countries in favor and against the Ramsey policy is $a = -8.5\%$ of average GDP, such that all countries below that threshold would vote in favor. As it turns out, the fraction of countries below that threshold at time 0 is strictly higher than one half. This can also be seen by noticing that the *median* net wealth, $a = -11.5\%$ of average GDP, (marked by a star in the figure), is *below* the yes/no voting threshold. We conclude that the Ramsey policy would be approved by this hypothetical 'governing council', although by a relatively small margin.

5.6 Aggregate shocks

So far we have restricted our analysis to the transitional dynamics, given the economy's initial state, while abstracting from shocks that affected the monetary union as a whole. We now extend our analysis to allow for aggregate shocks in the case of the Ramsey optimal policy. In particular, we allow the World real interest rate $\bar{r}$ to vary over time and simulate a one-off, unanticipated increase at time 0 followed by a gradual return to its baseline value of 3%.

Figure 8 displays the exogenous path of $\bar{r}_t$ for two shock sizes (1pp and 2pp) and their impact on a number of variables in each case, measured as the difference between the equilibrium paths with and without the shock (the latter being those displayed in Figure 4). The shock raises nominal (and real) bond yields, which leads households in the monetary union to reduce their consumption. Notice however that these shocks barely affect the optimal inflation path, the reaction of which is an order of magnitude smaller than that of nominal yields or consumption. Therefore, we conclude

[33] This setup broadly resembles the actual one in the EMU, where monetary policy decisions are made by its Governing Council, in which all country members have one vote. Nonetheless, it should be emphasized that the ECB has traditionally aimed at reaching a full consensus among the members of the Governing Council when making its monetary policy decisions.

[34] The figure displays the difference in value functions both for countries in recession ($i = 1$) and in boom ($i = 2$). Both lines are virtually indistinguishable, i.e. at each asset level the welfare difference between the Ramsey and the zero inflation policy is independent of the country's output level.

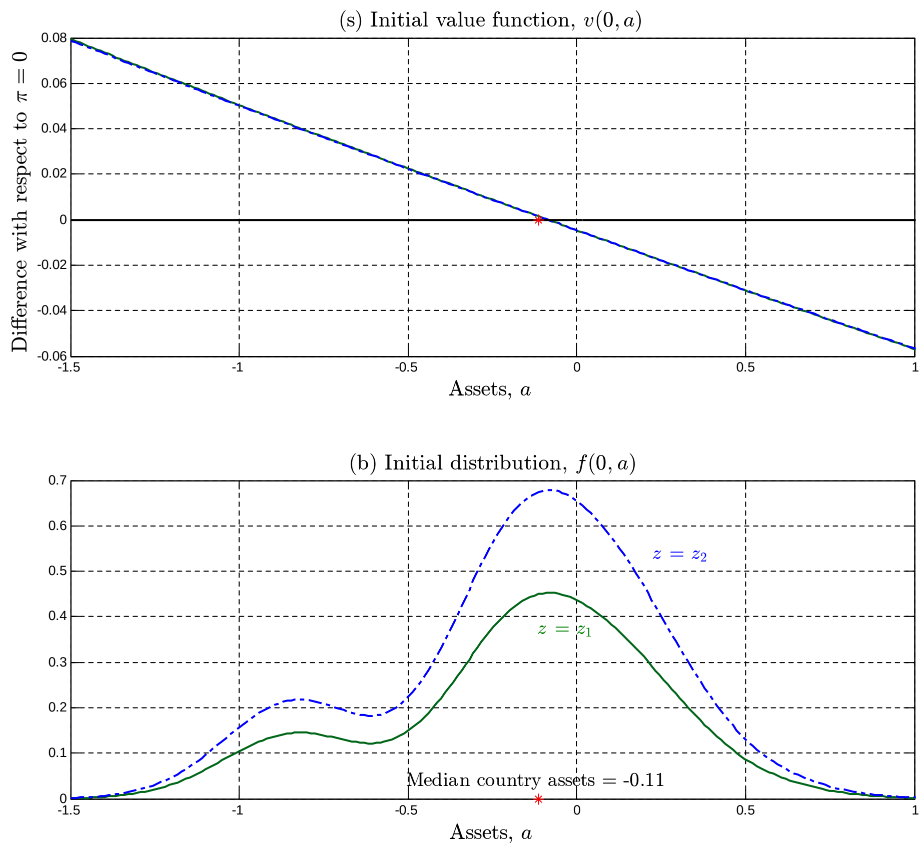

Figure 7: Winners and losers from the optimal commitment policy.

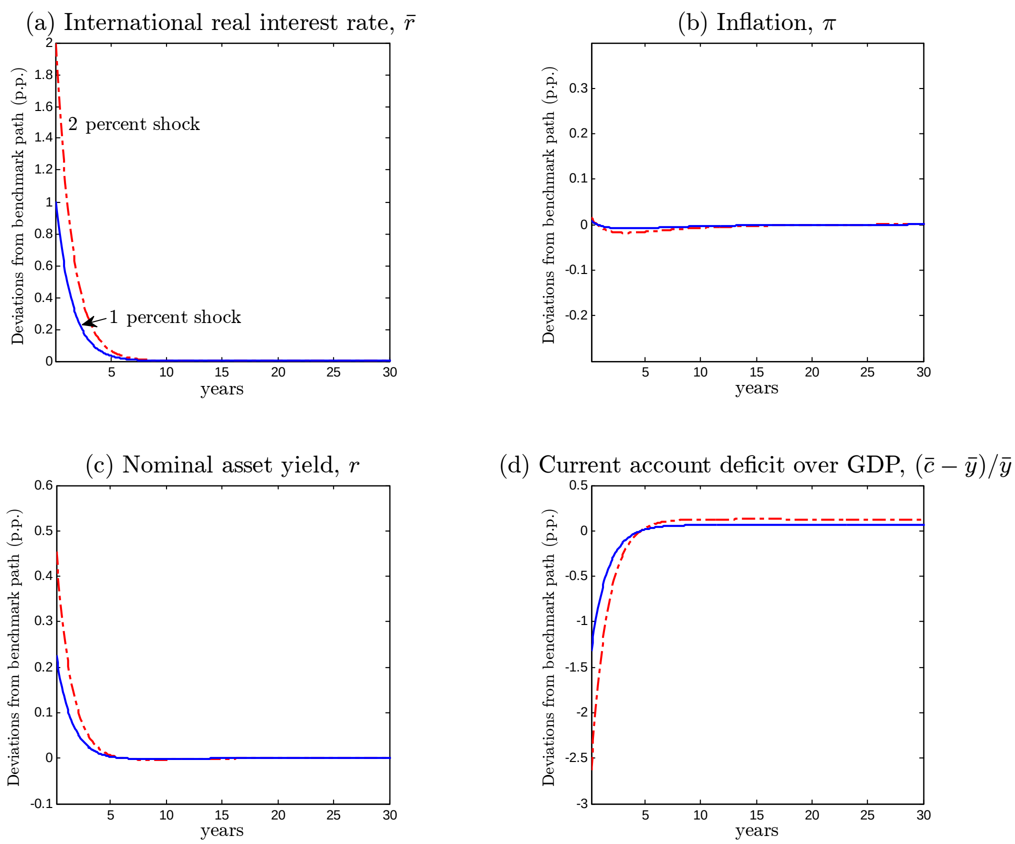

Figure 8: Impact of an interest rate shock under commitment.

that aggregate shocks such as those considered here barely affect the Ramsey optimal inflation path beyond the transition path analyzed before, which in turn arises from an incentive to (transitorily) redistribute wealth within the monetary union and also away from foreign investors.

6 Conclusion

We have analyzed optimal monetary policy, under commitment and discretion, in a continuous-time model of a monetary union where member countries are heterogenous in their net foreign asset positions and receive country-specific shocks. Markets are incomplete: each country can only trade nominal, noncontingent bonds with other member countries and with (risk neutral) foreign investors, subject to an exogenous borrowing limit. Our analysis sheds light on a recent policy and academic debate on the consequences that large differences in net foreign asset positions across member countries in a monetary union (such as those characterizing the European Monetary Union since its inception) should have for the conduct of monetary policy. On a methodological level, to the best of our knowledge our paper is the first to compute the fully dynamic optimal policy, both under commitment and discretion, in a continuous-time model with uninsurable idiosyncratic risk where the wealth distribution (an infinite-dimensional, endogenously time-varying object) is a state in the planner's problem.

We show analytically that, whether under discretion or commitment, the central bank has an incentive to create inflation in order to redistribute wealth both within the monetary union from creditor to debtor countries, to the extent that the latter have a higher marginal utility of net wealth, and away from foreign investors, to the extent that these are net creditors *vis-à-vis* the monetary union as a whole. Under commitment, however, such an inflationary force is counteracted by the central bank's understanding of how expectations of future inflation affect current nominal bond prices. We show moreover that, in the limiting case in which the central bank's discount factor converges to that of foreign investors, the long-run inflation rate under commitment converges to zero.

We calibrate the model to the EMU, including its cross-country net foreign asset distribution. We show that the optimal policy under commitment features first-order positive initial inflation, followed by a gradual decline towards its (near zero) long-run level. That is, the central bank front-loads inflation so as to transitorily redistribute existing wealth both within the union and away from international investors, committing to gradually abandon such redistributive stance. By contrast, discretionary monetary policy keeps inflation permanently high; such a policy is shown to reduce welfare substantially, both for creditor and for debtor union members.

Our analysis thus suggest that, in a monetary union with heterogenous net foreign positions in nominal assets, inflationary redistribution should only be used temporarily, avoiding any temptation to prolong positive inflation rates over time.

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Appendix

A. Proofs

Mathematical preliminaries

First we need to introduce some mathematical concepts. An *operator* T is a mapping from one vector space to another. Given the stochastic process a_t defined in (3), its *infinitesimal generator* is an operator $\mathcal{A}$ defined by

$$\mathcal{A}v \equiv \begin{pmatrix} s_1(t,a)\frac{\partial v_1(t,a)}{\partial a} + \lambda_1 \left[v_2(t,a) - v_1(t,a)\right] \\ s_2(t,a)\frac{\partial v_2(t,a)}{\partial a} + \lambda_2 \left[v_1(t,a) - v_2(t,a)\right] \end{pmatrix}, \tag{34}$$

so that the HJB equation (8) can be expressed as

$$\rho v = \frac{\partial v}{\partial t} + \max_c \left\{ u\left(c, \pi\right) + \mathcal{A}v \right\},$$

where $v \equiv \binom{v_1(t,a)}{v_2(t,a)}$ and $u\left(c,\pi\right) \equiv \binom{u(c_1,\pi)}{u(c_2,\pi)}$.

From now on, we assume that there is an upper bound arbitrarily large $\varkappa$ such that $f(t,a,y) = 0$ for all $a > \varkappa$. In steady state this can be proved in general following the same reasoning as in Proposition 2 of Achdou et al. (2015). Alternatively, we may assume that there is a maximum constraint in asset holding such that $a \leq \varkappa$, and that this constraint is so large that it does not affect to the results. In any case, let $\Phi \equiv [\phi, \varkappa]$ be the valid domain. The space of Lebesgue-integrable functions $L^2\left(\Phi\right)$ with the inner product

$$\langle v, f\rangle_\Phi = \sum_{i=1}^{2} \int_\Phi v_i f_i da = \int_\Phi v^{\mathbf{T}} f da, \ \forall v, f \in L^2\left(\Phi\right),$$

is a Hilbert space.[35]

Given an operator $\mathcal{A}$, its *adjoint* is an operator $\mathcal{A}^*$ such that $\langle f, \mathcal{A}v\rangle_\Phi = \langle \mathcal{A}^* f, v\rangle_\Phi$. In the case of the operator defined by (34) its adjoint is the operator

$$\mathcal{A}^* f \equiv \begin{pmatrix} -\frac{\partial(s_1 f_1)}{\partial a} - \lambda_1 f_1 + \lambda_2 j_2 \\ -\frac{\partial(s_2 f_2)}{\partial a} - \lambda_i f_i + \lambda_1 j_1 \end{pmatrix},$$

with boundary conditions

$$s_i\left(t,\phi\right) f_i\left(t,\phi\right) = s_i\left(t,\varkappa\right) f_i\left(t,\varkappa\right) = 0, \quad i = 1, 2, \tag{35}$$

[35] See Luenberger (1969) or Brezis (2011) for references.

such that the KF equation (14) results in

$$\frac{\partial f}{\partial t} = \mathcal{A}^* f, \tag{36}$$

for $f \equiv \binom{f_1(t,a)}{f_2(t,a)}$. We can see that $\mathcal{A}$ and $\mathcal{A}^*$ are adjoints as

$$\begin{aligned}
\langle \mathcal{A}v, f \rangle_\Phi &= \int_\Phi (\mathcal{A}v)^{\mathbf{T}} f da = \sum_{i=1}^{2} \int_\Phi \left[s_i \frac{\partial v_i}{\partial a} + \lambda_i \left[v_j - v_i \right] \right] f_i da \\
&= \sum_{i=1}^{2} v_i s_i f_i |_\phi^\varkappa + \sum_{i=1}^{2} \int_\Phi v_i \left[-\frac{\partial}{\partial a} \left(s_i f_i \right) - \lambda_i f_i + \lambda_j j_j \right] da \\
&= \int_\Phi v^{\mathbf{T}} \mathcal{A}^* f da = \langle v, \mathcal{A}^* f \rangle_\Phi \,.
\end{aligned}$$

We introduce the concept of Gateaux and Frechet differentials as generalizations of the standard concept of derivative to infinite-dimensional spaces.[36]

Definition 4 (Gateaux differential) *Let $J[f]$ be a linear continuous functional and let h be arbitrary in $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$. If the limit*

$$\delta J[f;h] = \lim_{\alpha \to 0} \frac{J[f + \alpha h] - J[f]}{\alpha} \tag{37}$$

exists, it is called the Gateaux differential *of J at f with increment h. If the limit (37) exists for each $h \in L^2(\Omega)$, the functional J is said to be* Gateaux differentiable *at f.*

If the limit exists, it can be expressed as $\delta J[f;h] = \frac{d}{d\alpha} J[f + \alpha h]|_{\alpha=0}$. A more restricted concept is that of the Fréchet differential.

Definition 5 (Fréchet differential) *Let h be arbitrary in $L^2(\Omega)$. If for fixed $f \in L^2(\Omega)$ there exists $\delta J[f;h]$ which is linear and continuous with respect to h such that*

$$\lim_{\|h\|_{L^2(\Omega)} \to 0} \frac{|J[f+h] - J[f] - \delta J[f;h]|}{\|h\|_{L^2(\Omega)}} = 0,$$

then J is said to be Fréchet differentiable *at f and $\delta[Jf;h]$ is the* Fréchet differential *of J at f with increment h.*

The following proposition links both concepts.

[36] See Luenberger (1969), Gelfand and Fomin (1991) or Sagan (1992).

Theorem 2 *If the Fréchet differential of J exists at f, then the Gateaux differential exists at f and they are equal.*

Proof. See Luenberger (1969, p. 173). ■

The familiar technique of maximizing a function of a single variable by ordinary calculus can be extended in infinite dimensional spaces to a similar technique based on more general differentials. We use the term *extremum* to refer to a maximum or a minimum over any set. A a function $f \in L^2(\Omega)$ is a maximum of $J[f]$if for all functions h, $\|h - f\|_{L^2(\Omega)} < \varepsilon$ then $J[f] \geq J[h]$. The following theorem is the Fundamental Theorem of Calculus.

Theorem 3 *Let J have a Gateaux differential, a necessary condition for J to have an extremum at f is that $\delta J[f;h] = 0$ for all $h \in L^2(\Omega)$.*

Proof. Luenberger (1969, p. 173), Gelfand and Fomin (1991, pp. 13-14) or Sagan (1992, p. 34). ■

In the case of constrained optimization in an infinite-dimensional Hilbert space, we have the following Theorem.

Theorem 4 (Lagrange multipliers) *Let H be a mapping from $L^2(\Omega)$ into $\mathbb{R}^p$. If J has a continuous Fréchet differential, a necessary condition for J to have an extremum at f under the constraint $H[f] = 0$ at the function f is that there exists a function $\eta \in L^2(\Omega)$ such that the Lagrangian functional*

$$\mathcal{L}[f] = J[f] + \langle \eta, H[f] \rangle_{\Omega} \tag{38}$$

is stationary in f, i.e., $\delta\mathcal{L}[f;h] = 0$.

Proof. Luenberger (1969, p. 243). ■

Proof of Lemma 1

Given the welfare criterion defined as in (19), we have

$$\begin{aligned}
U_0^{CB} &= \int_{\phi}^{\infty} \sum_{i=1}^{2} v_i(0,a) f_i(0,a) da = \int_{\phi}^{\infty} \sum_{i=1}^{2} \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho t} u(c_t, \pi_t) dt | a(0) = a,\ y(0) = y_i \right] f_i(0,a) da \\
&= \int_{\phi}^{\infty} \sum_{i=1}^{2} \left[\sum_{j=1}^{2} \int_{\phi}^{\infty} \int_0^{\infty} e^{-\rho t} u(c,\pi) f(t, \tilde{a}, \tilde{y}_j; a, y_i) dt d\tilde{a} \right] f_i(0,a) da \\
&= \int_0^{\infty} \sum_{j=1}^{2} e^{-\rho t} \int_{\phi}^{\infty} u(c,\pi) \left[\sum_{i=1}^{2} \int_{\phi}^{\infty} f(t, \tilde{a}, \tilde{y}_j; a, y_i) f_i(0,a) da \right] d\tilde{a} dt \\
&= \int_0^{\infty} \sum_{i=1}^{2} e^{-\rho t} \int_{\phi}^{\infty} u(c,\pi) f_j(t, \tilde{a}) d\tilde{a} dt,
\end{aligned}$$

where $f(t, \tilde{a}, \tilde{y}_j; a, y)$ is the transition probability from $a_0 = a$, $y_0 = y_i$ to $a_t = \tilde{a}$, $y_t = \tilde{y}_j$ and

$$f_j(t, \tilde{a}) = \sum_{j=1}^{2} \int_{\phi}^{\infty} f(t, \tilde{a}, \tilde{y}_j; a, y_i) f_i(0, a) da,$$

is the Chapman–Kolmogorov equation.

Proposition 1. Solution to the MPE

The idea of the proof is to employ dynamic programming in order to transform the problem of the central bank in a family -indexed by time- of static calculus of variations problems. Then we solve each of these problems using differenciation techniques in infinite-dimensional Hilbert spaces.

Step 1: Representation The first step is to show how the central bank functional $J[f]$ is a linear continuous functional in the Hilbert space $L^2(\Phi)$. The fact that the functional

$$J[f(t, \cdot)] = \int_t^{\infty} e^{-\rho(s-t)} \sum_{i=1}^{2} \int_{\Phi} [u(c_s, \pi_s)] f_i(s, a) da ds, \tag{39}$$

is linear in $f(t, \cdot)$ is trivial. The functional is *bounded* (or *continuous*) if there is a constant M such that

$$\|J\| \equiv \sup_{f \neq 0} \frac{|J(f)|}{\|f\|_{L^2(\Phi)}} \leq M.$$

We can check that as long as the instantaneous utility is bounded in Φ :

$$|u(c, \pi)| \leq M_0, \quad \forall a \in \Phi, \; y \in \{y_1, y_2\}$$

the functional J is continuous:

$$\begin{aligned} \sup_{f \neq 0} \frac{|J[f]|}{\|f\|_{L^2(\Phi)}} &< \sup_{f \neq 0} \frac{1}{\|f\|_{L^2(\Phi)}} \sum_{i=1}^{2} \int_{\Phi} \int_t^{\infty} e^{-\rho(s-t)} |u(c_s)| f_i(s, a) da ds \\ &\leq \sup_{f \neq 0} \frac{M_0}{\|f\|_{L^2(\Phi)}} \int_t^{\infty} e^{-\rho(s-t)} ds \leq \frac{M_0 \sqrt{\varkappa - \phi}}{\rho} = M, \end{aligned}$$

where the last inequality follows from the Cauchy-Schwarz inequality and the normalization condition (15):

$$1 = \sum_{i=1}^{2} \int_{\Phi} f_i da = \langle 1, f \rangle_{\Phi} \leq \|f\|_{L^2(\Phi)} \|1\|_{L^2(\Phi)} = \|f\|_{L^2(\Phi)} \sqrt{\int_{\Phi} 1^2 da} = \|f\|_{L^2(\Phi)} \sqrt{\varkappa - \phi}.$$

We may then apply the Riesz representation theorem.

Theorem 5 (Riesz representation theorem) *Let $J[f] : L^2(\Phi) \to \mathbb{R}$ be a linear continuous functional. Then there exists a unique function $j \in L^2(\Phi)$ such that*

$$J[f] = \langle j, f \rangle_{\Phi} = \sum_{i=1}^{2} \int_{\Phi} j_i f_i da.$$

Proof. See Brezis (2011, pp. 97-98). ■

Therefore, the central banks functional (39) can be represented as

$$J[f(t,\cdot)] = \sum_{i=1}^{2} \int_{\Phi} j_i(t,a) f_i(t,a) da,$$

where $j(t,\cdot) \in L^2(\Phi)$ is the *central bank's value* at time t of a country with debt a.[37]

Step 2: Dynamic programming Second, for any initial condition $f(t_0,\cdot)$ we have an optimal control path $\{\pi(t)\}_{t=t_0}^{\infty}$ and we may apply the *Bellman's Principle of Optimality*

$$J[f(t_0,\cdot)] = \int_{t_0}^{t} e^{-\rho(s-t_0)} \sum_{i=1}^{2} \int_{\Phi} u(c_s, \pi_s) f_i(s,a) da ds + e^{-\rho(t-t_0)} J[f(t,\cdot)]. \tag{40}$$

Let $\Xi[f]$ be defined as

$$\Xi[f] \equiv \sum_{i=1}^{2} \int_{\Phi} u(c_t, \pi_t) f_i(t,a) da = \langle u, f \rangle_{\Phi}.$$

Taking derivatives with respect to time in equation (40) and the limit as $t \to t_0$:

$$\begin{aligned} 0 &= \Xi[f] - \rho J[f(t,\cdot)] + \frac{\partial}{\partial t} J[f(t,\cdot)] = \Xi[f] - \rho J[f(t,\cdot)] + \frac{\partial}{\partial t} \sum_{i=1}^{2} \int_{\Phi} j_i(t,a) f_i(t,a) da \quad (41) \\ &= \Xi[f] - \rho J[f(t,\cdot)] + \sum_{i=1}^{2} \int_{\Phi} \left[\frac{\partial j_i}{\partial t} f_i(t,a) + j_i(t,a) \frac{\partial f_i}{\partial t} \right] da. \end{aligned}$$

Equation (41) is the Bellman equation of the problem (39). This is equivalent to the HJB

[37] The functional is $J[f(t,\cdot), f(t' > t, \cdot)]$ and thus $j(t,a,y) = j[f(t' > t, \cdot)]$ is the *functional derivative* of J with respect to f :

$$j = \frac{\delta J}{\delta f}.$$

equation, but in an integral form. The last term of the expression is

$$\sum_{i=1}^{2}\int_{\Phi} j_i(t,a)\frac{\partial f_i}{\partial t}da = \left\langle j, \frac{\partial f}{\partial t}\right\rangle_{\Phi} = \langle j, \mathcal{A}^* f\rangle_{\Phi} = \langle \mathcal{A}j, f\rangle_{\Phi},$$

where in the last equality we have applied the KF equation (36).

We may express the Bellman equation as

$$\rho J[f] = \rho\int_{\Phi} jf da = \max_{\pi}\int_{\Phi}\left(u(c,\pi) + \frac{\partial j}{\partial t} + \mathcal{A}j\right)^{\mathbf{T}} f da. \tag{42}$$

Step 3: Optimal inflation The first order condition with respect to inflation in (42) is

$$\begin{aligned}
&\frac{\partial}{\partial\pi}\int_{\Phi}\left(u(c,\pi) + \frac{\partial j}{\partial t} + \mathcal{A}j\right)^{\mathbf{T}} f da \\
= &\sum_{i=1}^{2}\int_{\Phi}\left(u_\pi f_i + \frac{\partial s_i}{\partial\pi}\frac{\partial j_i}{\partial a}\right) f_i da \\
= &\sum_{i=1}^{2}\int_{\Phi}\left(u_\pi f_i - af_i\frac{\partial j_i}{\partial a}\right) da = 0,
\end{aligned} \tag{43}$$

so that the optimal inflation should satisfy

$$\sum_{i=1}^{2}\int_{\Phi}\left(af_i\frac{\partial j_i}{\partial a} - u_\pi f_i\right) da = 0. \tag{44}$$

Step 4: Central Bank's HJB In order to find the value of $j(t,\cdot)$, we compute the Gateaux differential of the Bellman equation (41). If we take the Gateaux differential at both sides of equation (42), we obtain

$$\begin{aligned}
&\frac{d}{d\alpha}\int_{\Phi}\left(u(c,\pi) + \frac{\partial j}{\partial t} + \mathcal{A}j - \rho j\right)^{\mathbf{T}}(f+\alpha h)\, da\Bigg|_{\alpha=0} \\
= &\int_{\Phi}\left(u(c,\pi) + \frac{\partial j}{\partial t} + \mathcal{A}j - \rho j\right)^{\mathbf{T}} h da = 0,
\end{aligned}$$

For any $h \in L^2(\Phi)$ we have $\int_{\Phi}\left(u(c,\pi) + \frac{\partial j}{\partial t} + \mathcal{A}j - \rho j\right)^{\mathbf{T}} h da = 0$ and hence $u(c,\pi) + \frac{\partial j}{\partial t} + \mathcal{A}j - \rho j = 0$, $\forall a \in \Phi$, $y \in \{y_1, y_2\}$:

$$\rho j_i(t,a) = u(c_i,\pi) + \frac{\partial j_i}{\partial t} + s_i(t,a)\frac{\partial j_i}{\partial a} + \lambda_i\left(j_k(t,a) - j_i(t,a)\right),\; i = 1,2,\; k \neq i. \tag{45}$$

Equation (45) is the same as the individual HJB equation (8). The boundary conditions are also the same (state constraints on the domain Φ) and therefore its solution should be the same: $j(t,a,y) = v(t,a,y)$, that is, the marginal social value to the central bank under discretion $j(\cdot)$ equals the individual value $v(\cdot)$.

Proposition 2. Solution to the Ramsey problem

The problem of the central bank is given by

$$J\left[f_0\left(\cdot\right)\right] = \max_{\{\pi_s, Q_s, v(s,\cdot), c(s,\cdot), f(s,\cdot)\}_{s=0}^{\infty}} \sum_{i=1}^{2} \int_0^{\infty} e^{-\rho s}\left[\int_{\Phi} u\left(c_s, \pi_s\right) f_i(s,a) da\right] ds,$$

subject to the law of motion of the distribution (14), the bond pricing equation (12) and the individual HJB equation (8). This is a problem of constrained optimization in an infinite-dimensional Hilbert space $\hat{\Phi} = [0,\infty) \times \Phi$. Notice that we are working now in the Hilbert space $\hat{\Phi}$, including the time dimension, where the inner product is

$$\langle v, f\rangle_{\hat{\Phi}} = \sum_{i=1}^{2} \int_0^{\infty} \int_{\Phi} v_i f_i da dt = \int_0^{\infty} \langle v, f\rangle_{\Phi} dt, \ \forall v, f \in L^2\left(\hat{\Phi}\right).$$

In this case, the Lagrangian is

$$\begin{aligned}
\mathcal{L}\left[\pi, Q, f, v, c\right] &= \int_0^{\infty} e^{-\rho t} \langle u, f\rangle_{\Phi} dt + \int_0^{\infty} \left\langle e^{-\rho t}\zeta\left(t,a\right), \mathcal{A}^* f - \frac{\partial f}{\partial t}\right\rangle_{\Phi} dt \\
&+ \int_0^{\infty} e^{-\rho t}\mu\left(t\right)\left(Q\left(\bar{r} + \pi + \delta\right) - \delta - \dot{Q}\right) dt \\
&+ \int_0^{\infty} \left\langle e^{-\rho t}\theta\left(t,a\right), u + \mathcal{A}v + \frac{\partial v}{\partial t} - \rho v\right\rangle_{\Phi} dt \\
&+ \int_0^{\infty} \left\langle e^{-\rho t}\eta\left(t,a\right), u_c - \frac{1}{Q}\frac{\partial v}{\partial a}\right\rangle_{\Phi} dt
\end{aligned}$$

where $\zeta(t,a)$, $\eta(t,a)$, $\theta(t,a) \in L^2\left(\hat{\Phi}\right)$ and $\mu(t) \in L^2[0,\infty)$ are the Lagrange multipliers associated to equations (14), (10), (8) and (12), respectively. The Lagragian can be expressed as

$$\begin{aligned}
\mathcal{L}[\pi,Q,f,v,c] &= \int_0^\infty e^{-\rho t}\left\langle u+\frac{\partial\zeta}{\partial t}+\mathcal{A}\zeta-\rho\zeta+\mu\left(Q(\bar{r}+\pi+\delta)-\delta-\dot{Q}\right),f\right\rangle_\Phi dt \\
&+\int_0^\infty e^{-\rho t}\left(\langle\theta,u\rangle_\Phi+\left\langle\mathcal{A}^*\theta-\frac{\partial\theta}{\partial t},v\right\rangle_\Phi+\left\langle\eta,u_c-\frac{1}{Q}\frac{\partial v}{\partial a}\right\rangle_\Phi\right)dt \\
&+\langle\zeta(0,\cdot),f(0,\cdot)\rangle_\Phi-\lim_{T\to\infty}\left\langle e^{-\rho T}\zeta(T,\cdot),f(T,\cdot)\right\rangle_\Phi \\
&+\lim_{T\to\infty}\left\langle e^{-\rho T}\theta(T,\cdot),v(T,\cdot)\right\rangle_\Phi-\langle\theta(0,\cdot),v(0,\cdot)\rangle+\int_0^\infty e^{-\rho t}\sum_{i=1}^2 v_i s_i\theta_i\big|_\phi^\varkappa dt,
\end{aligned}$$

where we have applied $\langle\zeta,\mathcal{A}^*f\rangle=\langle\mathcal{A}\zeta,f\rangle$, $\langle\theta,\mathcal{A}v\rangle=\langle\mathcal{A}^*\theta,v\rangle_\Phi+\sum_{i=1}^2 v_i s_i\theta_i\big|_\phi^\varkappa$ and integrated by parts

$$\begin{aligned}
\int_0^\infty\left\langle e^{-\rho t}\zeta,-\frac{\partial f}{\partial t}\right\rangle dt &= -\sum_{i=1}^2\int_0^\infty\int_\Phi e^{-\rho t}\zeta_i\frac{\partial f_i}{\partial t}dadt \\
&= -\sum_{i=1}^2\int_\Phi f_i e^{-\rho t}\zeta_i\big|_0^\infty da+\sum_{i=1}^2\int_0^\infty\int_\Phi f_i\frac{\partial}{\partial t}\left(e^{-\rho t}\zeta_i\right)dadt \\
&= \sum_{i=1}^2\int_\Phi f_i(0,a)\zeta_i(0,a)\,da-\lim_{T\to\infty}\sum_{i=1}^2\int_\Phi e^{-\rho T}f_i(T,a)\zeta_i(T,a)\,da \\
&+\sum_{i=1}^2\int_0^\infty\int_\Phi e^{-\rho t}f_i\left(\frac{\partial\zeta_i}{\partial t}-\rho\zeta_i\right)dadt \\
&= \langle\zeta(0,\cdot),f(0,\cdot)\rangle_\Phi-\lim_{T\to\infty}\left\langle e^{-\rho T}\zeta(T,\cdot),f(T,\cdot)\right\rangle_\Phi \\
&+\int_0^\infty e^{-\rho t}\left\langle\frac{\partial\zeta}{\partial t}-\rho\zeta,f\right\rangle_\Phi dt,
\end{aligned}$$

and

$$
\begin{aligned}
\int_0^\infty \left\langle e^{-\rho t}\theta, \frac{\partial v}{\partial t} - \rho v \right\rangle dt &= \sum_{i=1}^{2} \int_0^\infty \int_\Phi e^{-\rho t}\theta_i \left(\frac{\partial v_i}{\partial t} - \rho v_i \right) dadt \\
&= \sum_{i=1}^{2} \int_\Phi \theta_i e_i^{-\rho t} v \Big|_0^\infty da - \sum_{i=1}^{2} \int_0^\infty \int_\Phi v_i \left[\frac{\partial}{\partial t}\left(e^{-\rho t}\theta_i\right) + \rho\theta_i \right] dadt \\
&= \lim_{T\to\infty} \sum_{i=1}^{2} \int_\Phi e^{-\rho T} v_i\left(T,a\right)\theta_i\left(T,a\right) da - \sum_{i=1}^{2} \int_\Phi v_i\left(0,a\right)\theta_i\left(0,a\right) da \\
&\quad - \sum_{i=1}^{2} \int_0^\infty \int_\Phi e^{-\rho t} v_i \left(\frac{\partial \theta_i}{\partial t} \right) dadt \\
&= \lim_{T\to\infty} \left\langle e^{-\rho T}\theta\left(T,\cdot\right), v\left(T,\cdot\right)\right\rangle_\Phi - \left\langle \theta\left(0,\cdot\right), v\left(0,\cdot\right)\right\rangle_\Phi \\
&\quad + \int_0^\infty e^{-\rho t} \left\langle -\frac{\partial \theta}{\partial t}, v \right\rangle_\Phi dt,
\end{aligned}
$$

In order to find the extrema, we need to take the Gateaux differentials with respect to the controls π, Q, v and c.

The Gateaux differential with respect to f is

$$
\begin{aligned}
\frac{d}{d\alpha}\mathcal{L}\left[\pi, Q, f + \alpha h\left(t,a\right), v, c\right]|_{\alpha=0} &= \left\langle \zeta\left(0,\cdot\right), h\left(0,\cdot\right)\right\rangle_\Phi - \lim_{T\to\infty} \left\langle e^{-\rho T}\zeta\left(T,\cdot\right), h\left(T,\cdot\right)\right\rangle_\Phi \\
&\quad - \int_0^\infty e^{-\rho t} \left\langle u + \frac{\partial \zeta}{\partial t} + \mathcal{A}\zeta - \rho\zeta, h \right\rangle_\Phi dt,
\end{aligned}
$$

which should equal zero for any function $h\left(t,a\right) \in L^2\left(\hat{\Phi}\right)$ such that $h\left(0,\cdot\right) = 0$, as the initial value of $f\left(0,\cdot\right)$ is given. If we consider functions with $h\left(T,\cdot\right) = 0$, we obtain

$$
\rho\zeta = u + \frac{\partial \zeta}{\partial t} + \mathcal{A}\zeta, \tag{46}
$$

and taking this into account and considering functions $h\left(T,\cdot\right) \neq 0$, we obtain the boundary condition

$$
\lim_{T\to\infty} e^{-\rho T}\zeta\left(T,a\right) = 0.
$$

We may apply the Feynman–Kac formula to (46) and express $\zeta\left(t,a\right)$ as

$$
\zeta\left(t,a\right) = \mathbb{E}_t\left[\int_t^\infty e^{-\rho(s-t)} u(c_s, \pi_s) ds \mid a_t = a\right],
$$

subject to the evolution of a_t given by equation (3). This is the expression of the individual value function (7). Therefore, we may conclude that $\zeta\left(\cdot\right) = v\left(\cdot\right)$.

In the case of $c(t,a)$:

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{L}\left[\pi,Q,f,v,c+\alpha h(t,a)\right]|_{\alpha=0} &= \int_0^{\infty} e^{-\rho t}\left\langle \left(u_c-\frac{1}{Q}\frac{\partial \zeta}{\partial a}\right)h,f\right\rangle_{\Phi} dt \\ +\int_0^{\infty} e^{-\rho t}\left(\left\langle \theta,\left(u_c-\frac{1}{Q}\frac{\partial v}{\partial a}\right)h\right\rangle_{\Phi}+\langle \eta,u_{cc}h\rangle_{\Phi}\right)dt &= 0,\end{aligned}$$

where $\frac{\partial}{\partial a}(\mathcal{A}\zeta)=-\frac{1}{Q}\frac{\partial \zeta}{\partial a}$. Due to the first order conditions (10) and to the fact that $\zeta(\cdot)=v(\cdot)$ this expression reduces to

$$\int_0^{\infty} e^{-\rho t}\langle \eta(t,a),u_{cc}(t,a)\,h(t,a)\rangle_{\Phi}\,dt=0,$$

so that $\eta(t,a)=0\ \forall (t,a)\in\hat{\Phi}$, that is, the first order condition (10) is not binding as its associated Lagrange multiplier is zero.

In the case of $v(t,a)$:

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{L}\left[\pi,Q,f,v+\alpha h(t,a),c\right]|_{\alpha=0} &= \int_0^{\infty} e^{-\rho t}\left(\left\langle \mathcal{A}^*\theta-\frac{\partial \theta}{\partial t},h\right\rangle_{\Phi}\right)dt \\ +\lim_{T\to\infty}\left\langle e^{-\rho T}\theta(T,\cdot),h(T,\cdot)\right\rangle_{\Phi}-\langle \theta(0,\cdot),h(0,\cdot)\rangle_{\Phi}+\sum_{i=1}^{2}h_i s_i\theta_i|_{\phi}^{\varkappa} &= 0,\end{aligned}$$

where we have already taken into account the fact that $\eta(t,a)=0$. Proceeding as in the case of f, we conclude that this yields a Kolmogorov forward equation

$$\frac{\partial \theta}{\partial t}=\mathcal{A}^*\theta, \tag{47}$$

with boundary conditions

$$\begin{aligned}s_i(t,\phi)\,\theta_i(t,\phi) &= s_i(t,\varkappa)\,\theta_i(t,\varkappa)=0,\quad i=1,2,\\ \lim_{T\to\infty} e^{-\rho T}\theta(T,\cdot) &= 0,\\ \theta(0,\cdot) &= 0.\end{aligned}$$

This is exactly the same KF equation than in the case of f, but the initial distribution is $\theta(0,\cdot)=0$. Therefore, the distribution at any point in time should be zero $\theta(\cdot,\cdot)=0$. Both the Lagrange multiplier of the HJB equation θ and that of the first-order condition η are zero, reflecting the fact that the HJB condition is not binding, that is, that the monetary authority would choose the same consumption as the individual economies.

In the case of $\pi(t)$:

$$\frac{d}{d\alpha}\mathcal{L}\left[\pi+\alpha h\left(t\right),Q,f,v,c\right]|_{\alpha=0}=\int_0^{\infty}e^{-\rho t}\left\langle u_{\pi}-a\left(\frac{\partial\zeta}{\partial a}\right)+\mu Q,f\right\rangle_{\Phi}hdt=0,$$

where we have already taken into account that so the fact that $\theta\left(t,a\right)=0$. Taking into account that $\zeta\left(t,a\right)=v\left(t,a\right)$:

$$\mu\left(t\right)Q\left(t\right)=\sum_{i=1}^{2}\int_{\Phi}\left(a\frac{\partial v_i}{\partial a}-u_{\pi}\right)f_i\left(t,a\right)da.$$

In the case of $Q\left(t\right)$:

$$\begin{aligned}&\frac{d}{d\alpha}\mathcal{L}\left[\pi,Q+\alpha h\left(t\right),f,v,c\right]|_{\alpha=0}\\ =&\int_0^{\infty}e^{-\rho t}\left\langle-\frac{\delta h}{Q^2}a\frac{\partial\zeta}{\partial a}-\frac{\left(y-c\right)h}{Q^2}\frac{\partial\zeta}{\partial a}+\mu\left[h\left(\bar{r}+\pi+\delta\right)-\dot{h}\right],f\right\rangle_{\Phi}dt=0\end{aligned}$$

Integrating by parts

$$\begin{aligned}\int_0^{\infty}e^{-\rho t}\left\langle-\mu\dot{h},f\right\rangle_{\Phi}dt&=-\int_0^{\infty}e^{-\rho t}\mu\dot{h}\left\langle 1,f\right\rangle_{\Phi}dt=-\int_0^{\infty}e^{-\rho t}\mu\dot{h}dt\\&=-e^{-\rho t}\mu h\Big|_0^{\infty}+\int_0^{\infty}e^{-\rho t}\left(\dot{\mu}-\rho\mu\right)hdt\\&=\mu\left(0\right)h\left(0\right)+\int_0^{\infty}e^{-\rho t}\left\langle\left(\dot{\mu}-\rho\mu\right)h,f\right\rangle_{\Phi}dt.\end{aligned}$$

Therefore,

$$\int_0^{\infty}e^{-\rho t}\left\langle-\frac{\delta}{Q^2}a\frac{\partial\zeta}{\partial a}-\frac{\left(y-c\right)}{Q^2}\frac{\partial\zeta}{\partial a}+\mu\left(\bar{r}+\pi+\delta-\rho\right)+\dot{\mu},f\right\rangle_{\Phi}hdt+\mu\left(0\right)h\left(0\right)=0,$$

so that, using similar arguments as in the case of θ above we can show that $\mu\left(0\right)=0$ and

$$\left\langle-\frac{\delta}{Q^2}a\frac{\partial v}{\partial a}-\frac{\left(y-c\right)}{Q^2}\frac{\partial v}{\partial a},f\right\rangle_{\Phi}+\mu\left(\bar{r}+\pi+\delta-\rho\right)+\dot{\mu}=0,\ t>0.$$

Finally,

$$\frac{d\mu}{dt}=\left(\rho-\bar{r}-\pi-\delta\right)\mu+\frac{1}{Q^2\left(t\right)}\sum_{i=1}^{2}\int_{\Phi}\frac{\partial v_i}{\partial a}\left[\delta a+\left(y-c\right)\right]f_i\left(t,a\right)da,$$

here we have applied the fact that $\zeta=v$. In steady-state, this results in

$$\mu=\frac{1}{\left(\bar{r}+\pi+\delta-\rho\right)Q^2}\sum_{i=1}^{2}\int_{\Phi}\frac{\partial v_i}{\partial a}\left[\delta a+\left(y-c\right)\right]f_i\left(a\right)da.$$

Proposition 3: Inflation bias in MPE

As we are assuming that the value function is concave in a, then it satisfies that

$$\frac{\partial v_i(t,\tilde{a})}{\partial a} < \frac{\partial v_i(t,0)}{\partial a} < \frac{\partial v_i(t,\hat{a})}{\partial a}, \ \forall \tilde{a} \in (0,\infty), \ \hat{a} \in (\phi, 0), \ t \geq 0, \ i = 1, 2. \tag{48}$$

In addition, the condition that the union as a whole is a net debtor ($\bar{a}_t < 0$) implies

$$\sum_{i=1}^{2} \int_{\phi}^{0} (-a)\, f_i(t,a)da \geq \sum_{i=1}^{2} \int_{0}^{\varkappa} (a)\, f_i(t,a)da, \ \forall t \geq 0. \tag{49}$$

Therefore

$$\sum_{i=1}^{2} \int_{0}^{\varkappa} a f_i \frac{\partial v_i(t,a)}{\partial a} da \ < \ \frac{\partial v_i(t,0)}{\partial a} \sum_{i=1}^{2} \int_{0}^{\varkappa} a f_i da \leq \frac{\partial v_i(t,0)}{\partial a} \sum_{i=1}^{2} \int_{\phi}^{0} (-a)\, f_i(t,a)da \tag{50}$$

$$< \ \sum_{i=1}^{2} \int_{\phi}^{0} (-a)\, f_i(t,a) \frac{\partial v_i(t,a)}{\partial a} da, \tag{51}$$

where we have applied (48) in the first and last stepts and (49) in the intermediate one. The optimal inflation in the MPE case (23) with separable utility $u = u^c - u^\pi$ is

$$\sum_{i=1}^{2} \int_{\phi}^{\varkappa} \left(a f_i \frac{\partial v_i}{\partial a} - u_\pi f_i \right) da = \sum_{i=1}^{2} \int_{\phi}^{\varkappa} a f_i \frac{\partial v_i}{\partial a} da + u_\pi^\pi = 0.$$

Combining this expression with (50) we obtain

$$u_\pi^\pi = \sum_{i=1}^{2} \int_{\phi}^{\varkappa} (-a)\, f_i \frac{\partial v_i}{\partial a} da > 0.$$

Finally, taking into account the fact that $u_\pi^\pi > 0$ only for $\pi > 0$ we have that $\pi(t) > 0$.

Proposition 4: optimal long-run inflation under commitment in the limit as $\bar{r} \to \rho$

In the steady state, equations (27) and (29) in the main text become

$$(\rho - \bar{r} - \pi - \delta)\, \mu + \frac{1}{Q^2} \sum_{i=1}^{2} \int_{\phi}^{\infty} \frac{\partial v_i}{\partial a} \left[\delta a + (y_i - c_i) \right] f_i(a)\, da = 0,$$

$$\mu Q = u_\pi^\pi(\pi) + \sum_{i=1}^{2} \int_{\phi}^{\infty} a \frac{\partial v_i}{\partial a} f_i(a)\, da,$$

respectively. Consider now the limiting case $\rho \rightarrow \bar{r}$, and guess that $\pi \rightarrow 0$. The above two equations then become

$$\begin{aligned}
\mu Q &= \frac{1}{\delta Q}\sum_{i=1}^{2}\int_{\phi}^{\infty}\frac{\partial v_i}{\partial a}\left[\delta a + (y_i - c_i)\right] f_i(a)\, da, \\
\mu Q &= \sum_{i=1}^{2}\int_{\phi}^{\infty} a\frac{\partial v_i}{\partial a} f_i(a)\, da,
\end{aligned}$$

as $u_{\pi}^{\pi}(0) = 0$ under our assumed preferences in Section 4.4. Combining both equations, and using the fact that in the zero inflation steady state the bond price equals $Q = \frac{\delta}{\delta + \bar{r}}$, we obtain

$$\sum_{i=1}^{2}\int_{\phi}^{\infty}\frac{\partial v_i}{\partial a}\left(\bar{r}a + \frac{y_i - c_i}{Q}\right) f_i(a)\, da = 0. \tag{52}$$

In the zero inflation steady state, the value function v satisfies the HJB equation

$$\rho v_i(a) = u^c(c_i(a)) + \left(\bar{r}a + \frac{y_i - c_i(a)}{Q}\right)\frac{\partial v_i}{\partial a} + \lambda_i\left[v_j(a) - v_i(a)\right], \quad i = 1, 2, \; j \neq i, \tag{53}$$

where we have used $u^{\pi}(0) = 0$ under our assumed preferences. We also have the first-order condition

$$u_c^c(c_i(a)) = Q\frac{\partial v_i}{\partial a} \Rightarrow c_i(a) = u_c^{c,-1}\left(Q\frac{\partial v_i}{\partial a}\right).$$

We guess and verify a solution of the form $v_i(a) = \kappa_i a + \vartheta_i$, so that $u_c^c(c_i) = Q\kappa_i$. Using our guess in (53), and grouping terms that depend on and those that do not, we have that

$$\begin{aligned}
\rho\kappa_i &= \bar{r}\kappa_i + \lambda_i(\kappa_j - \kappa_i), && (54)\\
\rho\vartheta_i &= u_c\left(u_c^{c,-1}(Q\kappa_i)\right) + \frac{y_i - u_c^{c,-1}(Q\kappa_i)}{Q}\kappa_i + \lambda_i(\vartheta_j - \vartheta_i), && (55)
\end{aligned}$$

for $i, j = 1, 2$ and $j \neq i$. In the limit as $\bar{r} \rightarrow \rho$, equation (54) results in $\kappa_j = \kappa_i \equiv \kappa$, so that consumption is the same in both states. The value of the slope κ can be computed from the boundary conditions.[38] We can solve for $\{\vartheta_i\}_{i=1,2}$ from equations (55),

$$\vartheta_i = \frac{1}{\rho} u_c \left(u_c^{c,-1} \left(Q\kappa \right) \right) + \frac{y_i - u_c^{c,-1} \left(Q\kappa \right)}{\rho Q} \kappa + \frac{\lambda_i \left(y_j - y_i \right)}{\rho \left(\lambda_i + \lambda_j + \rho \right) Q} \kappa,$$

for $i, j = 1, 2$ and $j \neq i$. Substituting $\frac{\partial v_i}{\partial a} = \kappa$ in (52), we obtain

$$\sum_{i=1}^{2} \int_{\phi}^{\infty} \left(\bar{r} a + \frac{y_i - c_i}{Q} \right) f_i \left(a \right) da = 0. \tag{56}$$

Equation (56) is exactly the zero-inflation steady-state limit of equation (17) in the main text (once we use the definitions of $\bar{a}$, $\bar{y}$ and $\bar{c}$), and is therefore satisfied in equilibrium. We have thus verified our guess that $\pi \to 0$.

B. Computational method: the stationary case

B.1 Exogenous monetary policy

We describe the numerical algorithm used to jointly solve for the equilibrium value function, $v \left(a, y \right)$, and bond price, Q, given an exogenous inflation rate π. The algorithm proceeds in 3 steps. We describe each step in turn.

Step 1: Solution to the Hamilton-Jacobi-Bellman equation Given π, the bond pricing equation (12) is trivially solved in this case:

$$Q = \frac{\delta}{\bar{r} + \pi + \delta}. \tag{57}$$

The HJB equation is solved using an *upwind finite difference* scheme similar to Achdou et al. (2015). It approximates the value function $v(a)$ on a finite grid with step $\Delta a : a \in \{a_1, ..., a_J\}$, where $a_j = a_{j-1} + \Delta a = a_1 + \left(j - 1 \right) \Delta a$ for $2 \leq j \leq J$. The bounds are $a_1 = \phi$ and $a_I = \varkappa$, such that $\Delta a = \left(\varkappa - \phi \right) / \left(J - 1 \right)$. We use the notation $v_{i,j} \equiv v_i(a_j)$, $i = 1, 2$, and similarly for the policy function $c_{i,j}$.

[38] The condition that the drift should be positive at the borrowing constraint, $s_i \left(\phi \right) \geq 0$, $i = 1, 2$, implies that

$$s_1 \left(\phi \right) = \bar{r} \phi + \frac{y_1 - u_c^{c,-1} \left(Q\kappa \right)}{Q} = 0,$$

and

$$\kappa = \frac{u_c^c \left(\bar{r} \phi Q + y_1 \right)}{Q}.$$

In the case of state $i = 2$, this guarantees $s_2 \left(\phi \right) > 0$.

Notice first that the HJB equation involves first derivatives of the value function, $v_i'(a)$ and $v_i''(a)$. At each point of the grid, the first derivative can be approximated with a forward (F) or a backward (B) approximation,

$$v_i'(a_j) \approx \partial_F v_{i,j} \equiv \frac{v_{i,j+1} - v_{i,j}}{\Delta a}, \tag{58}$$

$$v_i'(a_j) \approx \partial_B v_{i,j} \equiv \frac{v_{i,j} - v_{i,j-1}}{\Delta a}. \tag{59}$$

In an upwind scheme, the choice of forward or backward derivative depends on the sign of the *drift function* for the state variable, given by

$$s_i(a) \equiv \left(\frac{\delta}{Q} - \delta - \pi\right) a + \frac{(y_i - c_i(a))}{Q}, \tag{60}$$

for $\phi \leq a \leq 0$, where

$$c_i(a) = \left[\frac{v_i'(a)}{Q}\right]^{-1/\gamma}. \tag{61}$$

Let superscript n denote the iteration counter. The HJB equation is approximated by the following upwind scheme,

$$\frac{v_{i,j}^{n+1} - v_{i,j}^n}{\Delta} + \rho v_{i,j}^{n+1} = \frac{(c_{i,j}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 + \partial_F v_{i,j}^{n+1} s_{i,j,F}^n \mathbf{1}_{s_{i,j,F}^n > 0} + \partial_B v_{i,j}^{n+1} s_{i,j,B}^n \mathbf{1}_{s_{i,j,B}^n < 0} + \lambda_i \left(v_{-i,j}^{n+1} - v_{i,j}^{n+1}\right),$$

for $i = 1, 2,\ j = 1, ..., J$, where $\mathbf{1}(\cdot)$ is the indicator function and

$$s_{i,,jF}^n = \left(\frac{\delta}{Q} - \delta - \pi\right) a + \frac{y_i - \left[\frac{Q}{\partial_F v_{i,j}^n}\right]^{1/\gamma}}{Q}, \tag{62}$$

$$s_{i,j,B}^n = \left(\frac{\delta}{Q} - \delta - \pi\right) a + \frac{y_i - \left[\frac{Q}{\partial_B v_{i,j}^n}\right]^{1/\gamma}}{Q}. \tag{63}$$

Therefore, when the drift is positive ($s_{i,,jF}^n > 0$) we employ a forward approximation of the derivative, $\partial_F v_{i,j}^{n+1}$; when it is negative ($s_{i,j,B}^n < 0$) we employ a backward approximation, $\partial_B v_{i,j}^{n+1}$. The term $\frac{v_{i,j}^{n+1} - v_{i,j}^n}{\Delta} \to 0$ as $v_{i,j}^{n+1} \to v_{i,j}^n$. Moving all terms involving v^{n+1} to the left hand side and the rest to the right hand side, we obtain

$$\frac{v_{i,j}^{n+1} - v_{i,j}^n}{\Delta} + \rho v_{i,j}^{n+1} = \frac{(c_{i,j}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 + v_{i,j-1}^{n+1}\alpha_{i,j}^n + v_{i,j}^{n+1}\beta_{i,j}^n + v_{i,j+1}^{n+1}\xi_{i,j}^n + \lambda_i v_{-i,j}^{n+1}, \tag{64}$$

where

$$\begin{aligned}
\alpha_{i,j}^n &\equiv -\frac{s_{i,j,B}^n \mathbf{1}_{s_{i,j,B}^n<0}}{\Delta a}, \\
\beta_{i,j}^n &\equiv -\frac{s_{i,j,F}^n \mathbf{1}_{s_{i,j,F}^n>0}}{\Delta a} + \frac{s_{i,j,B}^n \mathbf{1}_{s_{i,j,B}^n<0}}{\Delta a} - \lambda_i, \\
\xi_{i,j}^n &\equiv \frac{s_{i,j,F}^n \mathbf{1}_{s_{i,j,F}^n>0}}{\Delta a},
\end{aligned}$$

for $i = 1, 2,\ j = 1, ..., J$. Notice that the state constraints $\phi \leq a \leq 0$ mean that $s_{i,1,B}^n = s_{i,J,F}^n = 0$.

In equation (64), the optimal consumption is set to

$$c_{i,j}^n = \left(\frac{\partial v_{i,j}^n}{Q} \right)^{-1/\gamma}. \tag{65}$$

where

$$\partial v_{i,j}^n = \partial_F v_{i,j}^n \mathbf{1}_{s_{i,j,F}^n>0} + \partial_B v_{i,j}^n \mathbf{1}_{s_{i,j,B}^n<0} + \partial \bar{v}_{i,j}^n \mathbf{1}_{s_{i,F}^n \leq 0} \mathbf{1}_{s_{i,B}^n \geq 0}.$$

In the above expression, $\partial \bar{v}_{i,j}^n = Q(\bar{c}_{i,j}^n)^{-\gamma}$ where $\bar{c}_{i,j}^n$ is the consumption level such that $s\left(a_i\right) \equiv s_i^n = 0:$

$$\bar{c}_{i,j}^n = \left(\frac{\delta}{Q} - \delta - \pi \right) a_j Q + y_i.$$

Equation (64) is a system of $2 \times J$ linear equations which can be written in matrix notation as:

$$\frac{1}{\Delta}\left(\mathbf{v}^{n+1} - \mathbf{v}^n\right) + \rho \mathbf{v}^{n+1} = \mathbf{u}^n + \mathbf{A}^n \mathbf{v}^{n+1}$$

where the matrix $\mathbf{A}^n$ and the vectors v^{n+1} and $\mathbf{u}^n$ are defined by

$$\mathbf{A}^n = -\begin{bmatrix}
\beta_{1,1}^n & \xi_{1,1}^n & 0 & 0 & \cdots & 0 & \lambda_1 & 0 & \cdots & 0 \\
\alpha_{1,2}^n & \beta_{1,2}^n & \xi_{1,2}^n & 0 & \cdots & 0 & 0 & \lambda_1 & \ddots & 0 \\
0 & \alpha_{1,3}^n & \beta_{1,3}^n & \xi_{1,3}^n & \cdots & 0 & 0 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha_{1,J-1}^n & \beta_{1,J-1}^n & \xi_{1,J-1}^n & 0 & \cdots & \lambda_1 & 0 \\
0 & 0 & \cdots & 0 & \alpha_{1,J}^n & \beta_{1,J}^n & 0 & 0 & \cdots & \lambda_1 \\
\lambda_2 & 0 & \cdots & 0 & 0 & 0 & \beta_{2,1}^n & \xi_{2,1}^n & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \lambda_2 & 0 & \cdots & \alpha_{2,J}^n & \beta_{2,J}^n
\end{bmatrix}, \quad \mathbf{v}^{n+1} = \begin{bmatrix}
v_{1,1}^{n+1} \\ v_{1,2}^{n+1} \\ v_{1,3}^{n+1} \\ \vdots \\ v_{1,J-1}^{n+1} \\ v_{1,J}^{n+1} \\ v_{2,1}^{n+1} \\ \vdots \\ v_{2,J}^{n+1}
\end{bmatrix} \tag{66}$$

$$\mathbf{u}^n = \begin{bmatrix} \frac{(c_{1,1}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 \\ \frac{(c_{1,2}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 \\ \vdots \\ \frac{(c_{1,J}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 \\ \frac{(c_{2,1}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 \\ \vdots \\ \frac{(c_{2,J}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 \end{bmatrix}.$$

The system in turn can be written as

$$\mathbf{B}^n \mathbf{v}^{n+1} = \mathbf{d}^n \tag{67}$$

where , $\mathbf{B}^n = \left(\frac{1}{\Delta} + \rho\right) \mathbf{I} - \mathbf{A}^n$ and $\mathbf{d}^n = \mathbf{u}^n + \frac{1}{\Delta}\mathbf{v}^n$.

The algorithm to solve the HJB equation runs as follows. Begin with an initial guess $\{v_{i,j}^0\}_{j=1}^J$, $i = 1, 2$. Set $n = 0$. Then:

1. Compute $\{\partial_F v_{i,j}^n, \partial_B v_{i,j}^n\}_{j=1}^J$, $i = 1, 2$ using (58)-(59).

2. Compute $\{c_{i,j}^n\}_{j=1}^J$, $i = 1, 2$ using (61) as well as $\{s_{i,j,F}^n, s_{i,j,B}^n\}_{j=1}^J$, $i = 1, 2$ using (62) and (63).

3. Find $\{v_{i,j}^{n+1}\}_{j=1}^J$, $i = 1, 2$ solving the linear system of equations (67).

4. If $\{v_{i,j}^{n+1}\}$ is close enough to $\{v_{i,j}^{n+1}\}$, stop. If not set $n := n + 1$ and proceed to 1.

Most computer software packages, such as Matlab, include efficient routines to handle sparse matrices such as $\mathbf{A}^n$.

Step 2: Solution to the Kolmogorov Forward equation The stationary distribution of debt-to-GDP ratio, $f(a)$, satisfies the Kolmogorov Forward equation:

$$0 = -\frac{d}{da}\left[s_i(a) f_i(a)\right] - \lambda_i f_i(a) + \lambda_{-i} f_{-i}(a), \; i = 1, 2. \tag{68}$$

$$1 = \int_\phi^\infty f(a)da. \tag{69}$$

We also solve this equation using an finite difference scheme. We use the notation $f_{i,j} \equiv f_i(a_j)$. The system can be now expressed as

$$0 = -\frac{f_{i,j}s_{i,j,F}\mathbf{1}_{s_{i,j,F}^n>0} - f_{i,j-1}s_{i,j-1,F}\mathbf{1}_{s_{i,j-1,F}^n>0}}{\Delta a} - \frac{f_{i,j+1}s_{i,j+1,B}\mathbf{1}_{s_{i,j+1,B}^n<0} - f_{i,j}s_{i,,jB}\mathbf{1}_{s_{i,,jB}^n<0}}{\Delta a} - \lambda_i f_{i,j} + \lambda_{-i} f_{-i,j},$$

or equivalently

$$f_{i,j-1}\xi_{i,j-1} + f_{i,j+1}\alpha_{i,j+1} + f_{i,j}\beta_{i,j} + \lambda_{-i}f_{-i,j} = 0, \tag{70}$$

then (70) is also a system of $2 \times J$ linear equations which can be written in matrix notation as:

$$\mathbf{A^T f} = \mathbf{0}, \tag{71}$$

where $\mathbf{A^T}$ is the transpose of $\mathbf{A} = \lim_{n\to\infty} \mathbf{A}^n$. Notice that $\mathbf{A}^n$ is the approximation to the operator $\mathcal{A}$ and $\mathbf{A^T}$ is the approximation of the adjoint operator $\mathcal{A}^*$. In order to impose the normalization constraint (69) we replace one of the entries of the zero vector in equation (71) by a positive constant.[39] We solve the system (71) and obtain a solution $\hat{\mathbf{f}}$. Then we renormalize as

$$f_{i,j} = \frac{\hat{f}_{i,j}}{\sum_{j=1}^{J}\left(\hat{f}_{1,j} + \hat{f}_{2,j}\right)\Delta a}.$$

Complete algorithm The algorithm proceeds as follows.

Step 1: Individual economy problem. Given π, compute the bond price Q using (57) and solve the HJB equation to obtain an estimate of the value function $\mathbf{v}$ and of the matrix $\mathbf{A}$.

Step 2: Aggregate distribution. Given $\mathbf{A}$ find the aggregate distribution $\mathbf{f}$.

B.2 Optimal monetary policy - MPE

In this case we need to find the value of inflation that satisfies condition (23). The algorith proceeds as follows. We consider an initial guess of inflation, $\pi^{(1)} = 0$. Set $m := 1$. Then:

Step 1: Individual economy problem problem. Given $\pi^{(m)}$, compute the bond price $Q^{(m)}$ using (57) and solve the HJB equation to obtain an estimate of the value function $\mathbf{v}^{(m)}$ and of the matrix $\mathbf{A}^{(m)}$.

Step 2: Aggregate distribution. Given $\mathbf{A}^{(m)}$ find the aggregate distribution $\mathbf{f}^{(m)}$.

Step 3: Optimal inflation. Given $\mathbf{f}^{(m)}$ and $\mathbf{v}^{(m)}$, iterate steps 1-2 until $\pi^{(m)}$ satisfies[40]

$$\sum_{i=1}^{2}\sum_{j=2}^{J-1} a_j f_{i,j}^{(m)} \frac{\left(v_{i,j+1}^{(m)} - v_{i,j-1}^{(m)}\right)}{2} + \psi\pi^{(m)} = 0.$$

[39] In particular, we have replaced the entry 2 of the zero vector in (71) by 0.1.
[40] This can be done using Matlab's `fzero` function.

B.3 Optimal monetary policy - Ramsey

Here we need to find the value of the inflation and of the costate that satisfy conditions (27) and (26) in steady-state. The algorith proceeds as follows. We consider an initial guess of inflation, $\pi^{(1)} = 0$. Set $m := 1$. Then:

Step 1: Individual economy problem problem. Given $\pi^{(m)}$, compute the bond price $Q^{(m)}$ using (57) and solve the HJB equation to obtain an estimate of the value function $\mathbf{v}^{(m)}$ and of the matrix $\mathbf{A}^{(m)}$.

Step 2: Aggregate distribution. Given $\mathbf{A}^{(m)}$ find the aggregate distribution $\mathbf{f}^{(m)}$.

Step 3: Costate. Given $\mathbf{f}^{(m)}$, $\mathbf{v}^{(m)}$,compute the costate $\mu^{(m)}$ using condition (26) as

$$\mu^{(m)} = \frac{1}{Q^{(m)}} \left[\sum_{i=1}^{2} \sum_{j=2}^{J-1} a_j f_{i,j}^{(m)} \frac{\left(v_{i,j+1}^{(m)} - v_{i,j-1}^{(m)} \right)}{2} + \psi \pi^{(m)} \right].$$

Step 4: Optimal inflation. Given $\mathbf{f}^{(m)}$, $\mathbf{v}^{(m)}$ and $\mu^{(m)}$, iterate steps 1-3 until $\pi^{(m)}$ satisfies

$$\left(\rho - \bar{r} - \pi^{(m)} - \delta \right) \mu^{(m)} + \frac{1}{\left(Q^{(m)}\right)^2} \left[\sum_{i=1}^{2} \sum_{j=2}^{J-1} \left(\delta a_j + y_i - c_{i,j}^{(m)} \right) f_{i,j}^{(m)} \frac{\left(v_{i,j+1}^{(m)} - v_{i,j-1}^{(m)} \right)}{2} \right].$$

C. Computational method: the dynamic case

C.1 Exogenous monetary policy

We describe now the numerical algorithm to analyze the transitional dynamics, similar to the one described in Achdou et al. (2015). With an exogenous monetary policy it just amounts to solve the dynamic HJB equation (8) and then the dynamic KFE equation (14). Define T as the time interval considered, which should be large enough to ensure a converge to the stationary distribution and discretize it in N intervals of lenght

$$\Delta t = \frac{T}{N}.$$

The initial distribution $f(0, a, y) = f_0(a, y)$ and the path of inflation $\{\pi_t\}_{t=0}^{T}$ are given. We proceed in three steps.

Step 0: The asymptotic steady-state The asymptotic steady-state distribution of the model can be computed following the steps described in Section A. Given π_N, the result is a stationary destribution $\mathbf{f}_N$, a matrix $\mathbf{A}_N$ and a bond price Q_N defined at the asymptotic time $T = N\Delta t$.

Step 1: Solution to the Bond Pricing Equation The dynamic bond princing equation (12) can be approximated backwards as

$$(\bar{r} + \pi_n + \delta) Q_n = \delta + \frac{Q_{n+1} - Q_n}{\Delta t}, \Longleftrightarrow Q_n = \frac{Q_{n+1} + \delta \Delta t}{1 + \Delta t (\bar{r} + \pi_n + \delta)}, \quad n = N - 1, .., 0, \tag{72}$$

where Q_N is the asymptotic bond price from Step 0.

Step 2: Solution to the Hamilton-Jacobi-Bellman equation The dynamic HJB equation (8) can approximated using an upwind approximation as

$$\rho \mathbf{v}^n = \mathbf{u}^n + \mathbf{A}^n \mathbf{v}^n + \frac{(\mathbf{v}^{n+1} - \mathbf{v}^n)}{\Delta t},$$

where $\mathbf{A}^n$ is constructing backwards in time using a procedure similar to the one described in Step 1 of Section B. By defining $\mathbf{B}^n = \left(\frac{1}{\Delta} + \rho\right) \mathbf{I} - \mathbf{A}^n$ and $\mathbf{d}^n = \mathbf{u}^n + \frac{\mathbf{V}^{n+1}}{\Delta t}$, we have

$$\mathbf{v}^n = (\mathbf{B}^n)^{-1} \mathbf{d}^n. \tag{73}$$

Step 3: Solution to the Kolmogorov Forward equation Let $\mathbf{A}$ defined in (66) be the approximation to the infinitesimal generator $\mathcal{A}$. Using a finite difference scheme similar to the one employed in the Step 2 of Section A, we obtain:

$$\frac{\mathbf{f}_{n+1} - \mathbf{f}_n}{\Delta t} = \mathbf{A}^{\mathbf{T}} \mathbf{f}_{n+1}, \Longleftrightarrow \mathbf{f}_{n+1} = \left(\mathbf{I} - \Delta t \mathbf{A}^{\mathbf{T}}\right)^{-1} \mathbf{f}_n, \quad n = 1, .., N \tag{74}$$

where $\mathbf{f}_0$ is the discretized approximation to the initial distribution $f_0(b)$.

Complete algorithm The algorith proceeds as follows:

Step 0: Asymptotic steady-state. Given π_N, compute the stationary destribution $\mathbf{f}_N$, matrix $\mathbf{A}_N$, bond price Q_N.

Step 1: Bond pricing. Given $\{\pi_n\}_{n=0}^{N-1}$, compute the bond price path $\{Q_n\}_{n=0}^{N-1}$ using (72).

Step 2: Individual economy problem. Given $\{\pi_n\}_{n=0}^{N-1}$ and $\{Q_n\}_{n=0}^{N-1}$ solve the HJB equation (73) backwards to obtain an estimate of the value function $\{\mathbf{v}_n\}_{n=0}^{N-1}$, and of the matrix $\{\mathbf{A}_n\}_{n=0}^{N-1}$.

Step 3: Aggregate distribution. Given $\{\mathbf{A}_n\}_{n=0}^{N-1}$ find the aggregate distribution forward $\mathbf{f}^{(k)}$ using (74).

C.2 Optimal monetary policy - MPE

In this case we need to find the value of inflation that satisfies condition (23)

Step 0: Asymptotic steady-state. Compute the stationary destribution $\mathbf{f}_N$, matrix $\mathbf{A}_N$, bond price Q_N and inflation rate π_N. Set $\pi^{(0)} \equiv \{\pi_n^{(0)}\}_{n=0}^{N-1} = \pi_N$ and $k := 1$.

Step 1: Bond pricing. Given $\pi^{(k-1)}$, compute the bond price path $Q^{(k)} \equiv \{Q_n^{(k)}\}_{n=0}^{N-1}$ using (72).

Step 2: Individual economy problem. Given $\pi^{(k-1)}$ and $Q^{(k)}$ solve the HJB equation (73) backwards to obtain an estimate of the value function $\mathbf{v}^{(k)} \equiv \{\mathbf{v}_n^{(k)}\}_{n=0}^{N-1}$ and of the matrix $\mathbf{A}^{(k)} \equiv \{\mathbf{A}_n^{(k)}\}_{n=0}^{N-1}$.

Step 3: Aggregate distribution. Given $\mathbf{A}^{(k)}$ find the aggregate distribution forward $\mathbf{f}^{(k)}$ using (74).

Step 4: Optimal inflation. Given $\mathbf{f}^{(k)}$ and $\mathbf{v}^{(k)}$, iterate steps 1-3 until $\pi^{(k)}$ satisfies

$$\Theta_n^{(k)} \equiv \sum_{i=1}^{2} \sum_{j=2}^{J-1} a_j f_{n,i,j}^{(k)} \frac{\left(v_{n,i,j+1}^{(k)} - v_{n,i,j-1}^{(k)}\right)}{2} + \psi \pi_n^{(k)} = 0.$$

This is done by iterating

$$\pi_n^{(k)} = \pi_n^{(k-1)} - \xi \Theta_n^{(k)},$$

with constant $\xi = 0.05$.

C.3 Optimal monetary policy - Ramsey

In this case we need to find the value of the inflation and of the costate that satisfy conditions (27) and (26)

Step 0: Asymptotic steady-state. Compute the stationary destribution $\mathbf{f}_N$, matrix $\mathbf{A}_N$, bond price Q_N and inflation rate π_N. Set $\pi^{(0)} \equiv \{\pi_n^{(0)}\}_{n=0}^{N-1} = \pi_N$ and $k := 1$.

Step 1: Bond pricing. Given $\pi^{(k-1)}$, compute the bond price path $Q^{(k)} \equiv \{Q_n^{(k)}\}_{n=0}^{N-1}$ using (72).

Step 2: Individual economy problem. Given $\pi^{(k-1)}$ and $Q^{(k)}$ solve the HJB equation (73) backwards to obtain an estimate of the value function $\mathbf{v}^{(k)} \equiv \{\mathbf{v}_n^{(k)}\}_{n=0}^{N-1}$ and of the matrix $\mathbf{A}^{(k)} \equiv \{\mathbf{A}_n^{(k)}\}_{n=0}^{N-1}$.

Step 3: Aggregate distribution. Given $\mathbf{A}^{(k)}$ find the aggregate distribution forward $\mathbf{f}^{(k)}$ using (74).

Step 4: Costate. Given $\mathbf{f}^{(k)}$ and $\mathbf{v}^{(k)}$, compute the costate $\mu^{(k)} \equiv \{\mu_n^{(k)}\}_{n=0}^{N-1}$ using (27):

$$\begin{aligned}\mu_{n+1}^{(k)} &= \mu_n^{(k)}\left[1+\Delta t\left(\rho-\bar{r}-\pi^{(k)}-\delta\right)\right] \\ &\quad +\frac{\Delta t}{\left(Q_n^{(k)}\right)^2}\left[\sum_{i=1}^{2}\sum_{j=2}^{J-1}\left(\delta a_j+y_i-c_{n,i,j}^{(k)}\right)f_{n,i,j}^{(k+1)}\frac{\left(v_{n,i,j+1}^{(k)}-v_{n,i,j-1}^{(k)}\right)}{2}\right],\end{aligned}$$

with $\mu_0^{(k)} = 0$.

Step 5: Optimal inflation. Given $\mathbf{f}^{(k)}$, $\mathbf{v}^{(k)}$ and $\mu^{(k)}$ iterate steps 1-4 until $\pi^{(k)}$ satisfies

$$\Theta_n^{(k)} \equiv \sum_{i=1}^{2}\sum_{j=2}^{J-1} a_j f_{n,i,j}^{(k)}\frac{\left(v_{n,i,j+1}^{(k)}-v_{n,i,j-1}^{(k)}\right)}{2}+\psi\pi_n^{(k)}-Q_n^{(k)}\mu_n^{(k)}=0.$$

This is done by iterating

$$\pi_n^{(k)} = \pi_n^{(k-1)} - \xi\Theta_n^{(k)}.$$

D. An economy with costly price adjustment

In this appendix, we lay out a model economy with the following characteristics: (i) firms are explicitly modelled, (ii) a subset of them are price-setters but incur a convex cost for changing their nominal price, and (iii) the social welfare function and the equilibrium conditions are the same as in the model economy in the main text.

Final good producer

In the model laid out in the main text, we assumed that output of the single consumption good in each country k, y_{kt}, is exogenous. From now on we ignore k subscripts for ease of exposition. Consider now an alternative setup in which the single consumption good is produced by a representative, perfectly competitive final good producer with the following Dixit-Stiglitz technology,

$$y_t = \left(\int_0^1 y_{it}^{(\varepsilon-1)/\varepsilon}di\right)^{\varepsilon/(\varepsilon-1)}, \tag{75}$$

where $\{y_{it}\}$ is a continuum of intermediate goods and $\varepsilon > 1$. Let P_{it} denote the nominal price of intermediate good $i \in [0,1]$. The firm chooses $\{y_{it}\}$ to maximize profits, $P_t y_t - \int_0^1 P_{it}y_{it}di$, subject to (75). The first order conditions are

$$y_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\varepsilon} y_t, \tag{76}$$

for each $i \in [0,1]$. Assuming free entry, the zero profit condition and equations (76) imply $P_t = (\int_0^1 P_{it}^{1-\varepsilon} di)^{1/(1-\varepsilon)}$.

Intermediate goods producers

Each intermediate good i is produced by a monopolistically competitive intermediate-good producer, which we will refer to as 'firm i' henceforth for brevity. Firm i operates a linear production technology,

$$y_{it} = z_t n_{it}, \tag{77}$$

where n_{it} is labor input and z_t is productivity. The latter is assumed to follow a 2-state Poisson process: $z_t \in \{z_1, z_2\}$, with $z_1 < z_2$, where the process jumps from z_1 to z_2 with intensity λ_1 and vice versa with intensity λ_2.

At each point in time, firms can change the price of their product but face quadratic price adjustment cost as in Rotemberg (1982). Letting $\dot{P}_{it} \equiv dP_{it}/dt$ denote the change in the firm's price, price adjustment costs in units of the final good are given by

$$\Psi_t\left(\frac{\dot{P}_{it}}{P_{it}}\right) \equiv \frac{\psi}{2}\left(\frac{\dot{P}_{it}}{P_{it}}\right)^2 \tilde{C}_t, \tag{78}$$

where $\tilde{C}_t$ is aggregate consumption. Let $\pi_{it} \equiv \dot{P}_{it}/P_{it}$ denote the rate of increase in the firm's price. The instantaneous profit function in units of the final good is given by

$$\begin{aligned} \Pi_{it} &= \frac{P_{it}}{P_t} y_{it} - w_t n_{it} - \Psi_t(\pi_{it}) \\ &= \left(\frac{P_{it}}{P_t} - \frac{w_t}{z_t}\right)\left(\frac{P_{it}}{P_t}\right)^{-\varepsilon} y_t - \Psi_t(\pi_{it}), \end{aligned} \tag{79}$$

where w_t is the perfectly competitive real wage and in the second equality we have used (76) and (77). Without loss of generality, firms are assumed to be risk neutral and have the same discount factor as households, ρ. Then firm i's objective function is

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} \Pi_{it} di,$$

with Π_{it} given by (79). The state variable specific to firm i, P_{it}, evolves according to $dP_{it} = \pi_{it} P_{it} dt$. We conjecture that the aggregate state relevant to the firm's decisions can be summarized by $(a_t, P_t, z_t, t) \equiv (S_t, z_t, t)$.[41] Then firm i's *value function* $V(P_{it}, S_t, z_h, t) \equiv V_h(P_{it}, S_t, t)$ must satisfy

[41] In particular, we later show that in equilibrium $y_t = z_t$, whereas w_t and $\tilde{C}_t$ are also functions of (a_t, P_t, z_t, t). The states P_t and a_t follow the same laws of motion as in the main text.

the following Hamilton-Jacobi-Bellman (HJB) equation,

$$\begin{aligned}(\rho+\lambda_h)V_h(P_i,S,t) &= \max_{\pi_i}\left\{\left(\frac{P_i}{P}-\frac{w}{z_h}\right)\left(\frac{P_i}{P}\right)^{-\varepsilon}y-\Psi(\pi_i)+\pi_i P_i\frac{\partial V_h}{\partial P_i}(P_i,S,t)\right\}\\ &\quad +\frac{\partial V_h}{\partial t}(P_i,S,t)+\lambda_h V_{h'}(P_i,S,t)+s_h(a,t)\frac{\partial V_h}{\partial a}(P_i,S,t)+\pi P\frac{\partial V_h}{\partial P}(P_i,S,t),\end{aligned}$$

for $h,h'=1,2,h'\neq h$, where $s_h(a,t)$ is the drift of the country's net assets a as defined in section 2.1 of the main text. The first order and envelope conditions of this problem are (we omit the arguments of V_h to ease the notation),

$$\psi\pi_i\tilde{C}=P_i\frac{\partial V_h}{\partial P_i}, \tag{80}$$

$$\begin{aligned}(\rho+\lambda_h)\frac{\partial V_h}{\partial P_i} &= \left[\varepsilon\frac{w}{z_h}-(\varepsilon-1)\frac{P_i}{P}\right]\left(\frac{P_i}{P}\right)^{-\varepsilon}\frac{y}{P_i}+\pi_i\left(\frac{\partial V_h}{\partial P_i}+P_i\frac{\partial^2 V_h}{\partial P_i^2}\right)\\ &\quad +\frac{\partial^2 V_h}{\partial t\partial P_i}+\lambda_h\frac{\partial V_{h'}}{\partial P_i}+s_h(a,t)\frac{\partial^2 V_h}{\partial a\partial P_i}+\pi P\frac{\partial^2 V_h}{\partial P\partial P_i}.\end{aligned} \tag{81}$$

In what follows, we will consider a symmetric equilibrium in which all firms choose the same price: $P_i=P,\pi_i=\pi$ for all i. After some algebra, it can be shown that the above conditions imply the following pricing Euler equation,[42]

$$\left[\rho-\frac{d\tilde{C}_h/dt}{\tilde{C}_h(a,t)}+\lambda_h\left(1-\frac{\tilde{C}_{h'}(a,t)}{\tilde{C}_h(a,t)}\right)\right]\pi(t)=\frac{\varepsilon-1}{\psi}\left(\frac{\varepsilon}{\varepsilon-1}\frac{w}{z_h}-1\right)\frac{z_h}{\tilde{C}_h(a,t)}+\pi'(t), \tag{82}$$

where $\tilde{C}_h\equiv\tilde{C}_h(a,t)$ is consumption in state $h=1,2$, and similarly for $h'\neq h$. Equation (82) determines the market clearing wage w as a function of the country's state: $w=w_h(a,t),h=1,2$.

Households

The representative household's preferences are given by

$$\mathbb{E}_0\int_0^\infty e^{-\rho t}\log\left(\tilde{C}_t\right)dt,$$

[42] The proof is available upon request.

where $\tilde{C}_t$ is household consumption of the final good. Define *total real spending* as the sum of household consumption and price adjustment costs,

$$\begin{aligned} c_t &\equiv \tilde{C}_t + \int_0^1 \Psi_t(\pi_{it})\,di \\ &= \tilde{C}_t + \frac{\psi}{2}\pi_t^2\tilde{C}_t, \end{aligned} \tag{83}$$

where in the second equality we have used the definition of Ψ_t (eq. 78) and the symmetry across firms in equilibrium. Instantaneous utility can then be expressed as

$$\begin{aligned} \log(\tilde{C}_t) &= \log(c_t) - \log\left(1 + \frac{\psi}{2}\pi_t^2\right) \\ &= \log(c_t) - \frac{\psi}{2}\pi_t^2 + O\left(\left\|\frac{\psi}{2}\pi_t^2\right\|^2\right), \end{aligned} \tag{84}$$

where $O(\|x\|^2)$ denotes terms of order second and higher in x. Expression (84) is the same as the utility function in the main text (eq. 18), up to a first order approximation of $\log(1+x)$ around $x = 0$, where $x \equiv \frac{\psi}{2}\pi^2$ represents the percentage of aggregate spending that is lost to price adjustment. For our baseline calibration ($\psi = 5.5$), the latter object is relatively small even for relatively high inflation rates, and therefore so is the error in computing the utility losses from price adjustment. Therefore, the utility function used in the main text provides a fairly accurate approximation of the welfare losses caused by inflation in the economy with costly price adjustment described here.

We assume that the household supplies one unit of labor input inelastically: $n_t = 1$. It also receives firms' profits in a lump-sum manner. Thus the household's real income equals $w_t + \int_0^1 \Pi_{it} di$. In the symmetric equilibrium, each firm's real profits equal $\Pi_{it} = y_t - w_t - \frac{\psi}{2}\pi_t^2\tilde{C}_t$. Therefore, real primary surplus equals

$$\begin{aligned} w_t + \int_0^1 \Pi_{it} di - \tilde{C}_t &= y_t - \frac{\psi}{2}\pi_t^2\tilde{C}_t - \tilde{C}_t \\ &= y_t - c_t, \end{aligned}$$

as in the main text, where in the second equality we have used (83). It is then trivial to show that the household's maximization problem is exactly the same as in the main text. As a result, the policy function for consumption is also the same: $c_t = c_h(a_t, t)$, $h = 1, 2$.

Market clearing and equilibrium

In the symmetric equilibrium, each firm's labor demand is $n_{it} = y_{it}/z_t = y_t/z_t$. Since labor supply equals one, labor market clearing requires

$$\int_0^1 n_{it} di = y_t/z_t = 1 \Leftrightarrow y_t = z_t.$$

Therefore, in equilibrium output is simply equal to exogenous productivity z_t.

Notice finally that $\tilde{C}_t = c_h(a,t)/[1+\frac{\psi}{2}\pi(t)^2] \equiv \tilde{C}_h(a,t)$, $h = 1,2$. Likewise, the pricing Euler equation derived above (equation 82) determines the market clearing wage given the country's state: $w_t = w_h(a,t)$, $h = 1,2$. We thus verify our previous conjecture that (a, z_h, P, t) are the relevant aggregate states for firms.