International Trade in an Uncertain World

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Motivation

- Growing concerns about *uncertainty* & *resilience* to shocks
  - Transportation disruptions in the aftermath of COVID-19
  - Uncertainty over national, regional and multilateral trade policies
  - National security concerns about access to critical goods
  - Rising geopolitical tensions (China-U.S. and Russia-Ukraine)

- Key challenge: introducing general equilibrium uncertainty into nonlinear quantitative trade models
  - Most prior research assumes that agents have perfect knowledge of the realizations of all general equilibrium variables when making import and export decisions
  - However, most public debates about resilience involve decision-making under general eqm uncertainty

- We make three main contributions
  1. Develop a tractable approach to introducing *general equilibrium uncertainty* into the class of quantitative trade models with a constant trade elasticity
  2. Allow countries to make upfront investments in bilateral import and export capacities that determine *resilience* in the face of this general equilibrium uncertainty
  3. Show that a country's *risk profile* becomes a key determinant of bilateral patterns of international trade, the terms of trade and welfare

Related Literature

- Quantitative trade models
  - Eaton and Kortum (2002), Arkolakis et al. (2012), Costinot & Rodriguez-Clare (2014), Caliendo & Parro (2015)
- Trade and uncertainty
  - Newberry & Stiglitz (1984), Koren & Tenreyro (2007, 2013), Caselli et al. (2020), Allen & Atkin (2022), Fitzgerald (2024), Caliendo et al. (2025), Castro-Vincenzi et al. (2025), Fan and Luo (2025)
- Reallocation in response to unanticipated trade policy
  - Alfaro and Chor (2023), Grossman et al. (2023), Fajgelbaum et al. (2024), Adão et al. (2025)
- International macroeconomics literature on international risk diversification
  - Cole and Obstfeld (1991), Backus et al. (1992), Backus & Smith (1993), Obstfeld and Rogoff (2000), Heathcote & Perri (2013), Coeurdacier & Gourinchas (2016), Aguiar et al. (2025), Fitzgerald (2025)
- Networks in macro and international economics
  - Acemoglu et al. (2012), Baqaee & Farhi (2019, 2024), Boehm et al. (2019), Liu (2019), Carvalho et al. (2021), Tascherau-Dumouchel (2022), Bachmann et al. (2024), Liu & Tsyvinski (2024), Nikolakoudis (2024), Elliott el. (2022), Elliot & Golub (2023), Grossman et al. (2024)
- Geoeconomics (international political economy of trade policy)
  - Hirschman (1945), Tinbergen (1962), Grossman & Helpman (1995), Broner et al. (2023), Clayton et al. (2023, 2024), Liu & Yang (2023), Thoenig (2023), Becko & Connor (2024), Kleinman et al. (2024)

Outline

- Deterministic Constant Elasticity Trade Model
- Uncertainty and Resilience
- Data
- Quantitative Results
- Conclusions

Review: Armington Trade Model

- Each country has a representative agent with labor endowment $\bar{\ell}_n$ and CES preferences
  - Goods differentiated by country of origin, subject to iceberg trade costs ($\tau_{ni} \geq 1$)

$$\mathcal{C}_n = \left(\sum_{i=1}^{N} c_{ni}^{\frac{\omega-1}{\omega}} \right)^{\frac{\omega}{\omega-1}}, \qquad \omega > 1$$

- Consumption c_{ni} by importer n of the good produced by exporter i requires:

$$c_{ni} = z_i \ell_{ni} / \tau_{ni}$$

- Labor market clearing:

$$\bar{\ell}_n = \sum_{k=1}^{N} \ell_{kn}$$

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- In a competitive equilibrium, factor prices $\{w_n\}$ satisfy

$$w_i \bar{\ell}_i = \sum_{n=1}^{N} w_n \bar{\ell}_n \frac{\left(w_i \tau_{ni} / z_i\right)^{1-\omega}}{\sum_{j=1}^{N} \left(w_j \tau_{nj} / z_j\right)^{1-\omega}}$$

- Analogous results hold throughout the class of models with a constant trade elasticity

This Paper: Economic Environment

- Each country has a representative agent with labor endowment $\bar{\ell}_n$ and CRRA preferences
  - Goods differentiated by country of origin, subject to iceberg trade costs ($\tau_{ni} \geq 1$)

$$\mathcal{U}(\mathcal{C}_n) = \frac{\mathcal{C}_n^{1-\gamma}}{1-\gamma}, \qquad \mathcal{C}_n = \left(\sum_{i=1}^N c_{ni}^{\frac{\omega-1}{\omega}}\right)^{\frac{\omega}{\omega-1}}, \qquad \omega > 1$$

- Consumption c_{ni} by importer n of the good produced by exporter i requires:
  - *Ex ante* investments by importer (ι_{ni}) and by exporter (h_{ni})
  - *Ex post* production labor by exporter (ℓ_{ni})

$$c_{ni} = \underbrace{\iota_{ni}^{\alpha}}_{\text{importer's ex-ante investment}} \underbrace{x_{ni}^{1-\alpha}}_{\text{imports}}, \qquad x_{ni} = \underbrace{h_{ni}^{1-\hat{\delta}}}_{\text{exporter's ex-ante investment}} \underbrace{\ell_{ni}^{\hat{\delta}}}_{\text{exporter's ex-post production labor}} (z_i/\tau_{ni})^{\hat{\delta}}$$

  - Equivalently, production function of consumption variety is $c_{ni} = \iota_{ni}^{\alpha} h_{ni}^{\beta} (z_i \ell_{ni}/\tau_{ni})^{\delta}$, $\alpha + \beta + \delta = 1$

- *Ex ante* investments chosen before observing random productivities $\{z_i\}$ and trade costs $\{\tau_{ni}\}$; cannot reallocate ex-post

$$\text{(labor market clearing)} \qquad \bar{\ell}_n = \sum_{j=1}^N \iota_{nj} + \sum_{k=1}^N (\ell_{kn} + h_{kn})$$

- Without uncertainty, competitive equilibrium is isomorphic to Armington

Characterizing the *Ex Ante* Equilibrium

- *Ex ante* investments depend on expectations over distribution of prices around the world
- *Ex ante* investments determine resilience in response to productivity realizations
  - Focus investments on cheaper suppliers and bigger buyers in expectation
  - Or diversify: option to trade with countries with poor expectations but good realizations

Proposition. The *ex ante* labor allocation satisfies

$$\sum_{i=1}^{N} \iota_{ni} = \alpha \bar{\ell}_n, \quad \sum_{k=1}^{N} h_{ki} = \beta \bar{\ell}_i,$$

$$\iota_{ni} \propto_n \mathbb{E}\left[\frac{\mathcal{U}'(\mathcal{C}_n)}{\mathcal{P}_n} p^x_{ni} x_{ni}\right] = \mathbb{E}\left[\mathcal{C}_n^{1-\gamma}\left(\iota_{ni}^{-\alpha/\delta} \ell_{ni}^{-\beta/\delta} \tau_{ni} w_i / z_i\right)^{1-\tilde{\omega}} \mathcal{Q}_n^{-(1-\tilde{\omega})}\right],$$

$$h_{ni} \propto_i \mathbb{E}\left[\frac{\mathcal{U}'(\mathcal{C}_i)}{\mathcal{P}_i} p^x_{ni} x_{ni}\right] = \mathbb{E}\left[\frac{\mathcal{C}_i^{1-\gamma}}{w_i}\left(\iota_{ni}^{-\alpha/\delta} h_{ni}^{-\beta/\delta} \tau_{ni} w_i / z_i\right)^{1-\tilde{\omega}} \mathcal{Q}_n^{-(1-\tilde{\omega})} w_n \bar{\ell}_n\right].$$

$$\mathcal{Q}_n \equiv \left(\sum_{j=1}^{N}\left(\iota_{nj}^{-\alpha/\delta} h_{nj}^{-\beta/\delta} \tau_{nj} w_j / z_j\right)^{1-\tilde{\omega}}\right)^{\frac{1}{1-\tilde{\omega}}},$$

$$\mathcal{W}_n = w_n \bar{\ell}_n, \qquad \mathcal{C}_n = \mathcal{Q}_n^{-\delta} \mathcal{W}_n^{\delta} \delta^{\delta}, \qquad \mathcal{P}_n = \frac{\mathcal{W}_n}{\mathcal{C}_n} = \mathcal{Q}_n^{\delta} \mathcal{W}_n^{1-\delta} \delta^{-\delta}.$$

Proposition. As a second-order approximation,

$$\iota_{ni} \propto_n \mathbb{E}\left[\left(\iota_{ni}^{-\alpha/\delta}\ell_{ni}^{-\beta/\delta}\tau_{ni}w_i/z_i\right)^{1-\tilde{\omega}}\right]$$
$$\times \exp\left(Cov\left(\ln \mathcal{C}_n^{-\gamma}, \ln\left(\tau_{ni}w_i/z_i\right)^{1-\tilde{\omega}}\right)\right)$$
$$\times \exp\left(Cov\left(\ln w_n/\mathcal{P}_n, \ln\left(\tau_{ni}w_i/z_i\right)^{1-\tilde{\omega}}\right)\right)$$
$$\times \exp\left(-Cov\left(\ln \mathcal{Q}_n^{1-\tilde{\omega}}, \ln\left(\tau_{ni}w_i/z_i\right)^{1-\tilde{\omega}}\right)\right)$$

Importer n invests more into import capacity if i's goods are cheap

- In expectation
- When n's marginal utility is high
- When n has high purchasing power
- When other goods in n's consumption bundle are expensive

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$$h_{ni} \propto_i \mathbb{E}\left[\left(\iota_{ni}^{-\alpha/\delta}\ell_{ni}^{-\beta/\delta}\tau_{ni}\right)^{1-\tilde{\omega}}\mathcal{Q}_n^{-(1-\tilde{\omega})}w_n\bar{\ell}_n\right]$$
$$\times \exp\left(Cov\left(\ln \mathcal{C}_i^{-\gamma}, \ln\left(\tau_{ni}/\mathcal{Q}_n\right)^{1-\tilde{\omega}} w_n\right)\right)$$
$$\times \exp\left(-Cov\left(\ln \mathcal{P}_i, \ln\left(\tau_{ni}/\mathcal{Q}_n\right)^{1-\tilde{\omega}} w_n\right)\right)$$
$$\times \exp\left(Cov\left(\ln\left(w_i/z_i\right)^{1-\tilde{\omega}}, \ln\left(\tau_{ni}/\mathcal{Q}_n\right)^{1-\tilde{\omega}} w_n\right)\right)$$

Exporter i invests more into export capacity if importer n has higher expenditure on i

- In expectation
- When i's marginal utility is high
- When i's CPI is low (i.e., when revenue is valuable in real terms)
- When i's cost of production is low

- Holds regardless of the distribution for exogenous productivity
- Key challenge: characterization of the joint distribution of prices in Armington general equilibrium

Proposition. Suppose $\mathbb{E}[\ln \boldsymbol{z}] = \boldsymbol{\mu}_z$ and $\mathbb{V}[\ln \boldsymbol{z}] = \boldsymbol{\Sigma}_z$. Define $\boldsymbol{F}$ as the equilibrium function of the Armington wage system: $\ln \boldsymbol{w} = \boldsymbol{F}(\ln \boldsymbol{z}; \theta)$. ▸ armF ▸ Wmore ▸ Hmore ▸ Tradecosts

1. To second order,

$$\mathbb{E}[\ln w_i] \approx F_i(\boldsymbol{\mu}_z) + \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} \boldsymbol{H}^i_{jk} [\boldsymbol{\Sigma}_z]_{jk}, \qquad Var[\ln \boldsymbol{w}] \approx \boldsymbol{W} \boldsymbol{\Sigma}_z \boldsymbol{W}'$$

   where $\boldsymbol{W}$ and $\boldsymbol{H}$ are the Jacobian and Hessian of $\boldsymbol{F}$ evaluated at $\ln \boldsymbol{z} = \boldsymbol{\mu}_{z_i}$.

2. The bilateral expenditure share matrix $\boldsymbol{S}$ is a sufficient statistic for the Jacobian and Hessian:

$$S_{ni} \equiv \left. \frac{(\tau_{ni} w_i / z_i)^{-\theta}}{\sum_{k=1}^{N} (\tau_{nk} w_k / z_k)^{-\theta}} \right|_{\ln \boldsymbol{z} = \boldsymbol{\mu}_z}, \qquad T_{in} \equiv \left. \frac{w_n \bar{\ell}_n S_{ni}}{w_i \bar{\ell}_i} \right|_{\ln \boldsymbol{z} = \boldsymbol{\mu}_z}.$$

$$\boldsymbol{W} \equiv \left[\frac{d \ln w_i}{d \ln z_j} |_{\ln \boldsymbol{z} = \boldsymbol{\mu}_z} \right] = -\theta \boldsymbol{X}^{-1} (\boldsymbol{T}\boldsymbol{S} - \boldsymbol{I}), \quad \boldsymbol{X} \equiv \boldsymbol{I} - \boldsymbol{T} - \theta (\boldsymbol{T}\boldsymbol{S} - \boldsymbol{I}) + \boldsymbol{Q},$$

$$\boldsymbol{H}^i_{jk} \equiv \partial_k \boldsymbol{W}_{ij}, \qquad \partial_k \boldsymbol{W} = (\boldsymbol{I} - \boldsymbol{Q}) \left\{ \boldsymbol{X}^{-1} (\partial_k \boldsymbol{T}) ((\boldsymbol{I} + \theta \boldsymbol{S}) \boldsymbol{W} - \theta \boldsymbol{S}) + \theta \boldsymbol{X}^{-1} \boldsymbol{T} (\partial_k \boldsymbol{S}) (\boldsymbol{W} - \boldsymbol{I}) \right\},$$

$$(\partial_k \boldsymbol{S})_{ni} = \theta \boldsymbol{S}_{ni} \left(\mathbf{1}_{i=k} - \boldsymbol{W}_{ik} - \boldsymbol{S}_{nk} + (\boldsymbol{S}\boldsymbol{W})_{nk} \right),$$

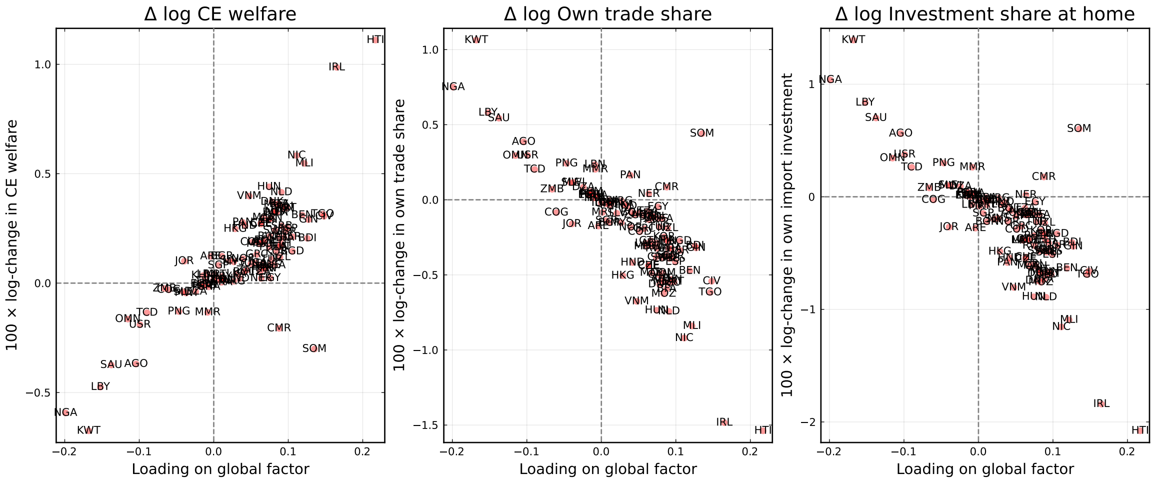
$$(\partial_k \boldsymbol{T})_{in} = \boldsymbol{T}_{in} \left[\boldsymbol{W}_{nk} - \boldsymbol{W}_{ik} + \theta \left(\mathbf{1}_{i=k} - \boldsymbol{W}_{ik} - \boldsymbol{S}_{nk} + (\boldsymbol{S}\boldsymbol{W})_{nk} \right) \right].$$

The matrix $\boldsymbol{Q}$ captures price normalization (numeraire) and $\boldsymbol{T}$ is the income share matrix.

Data

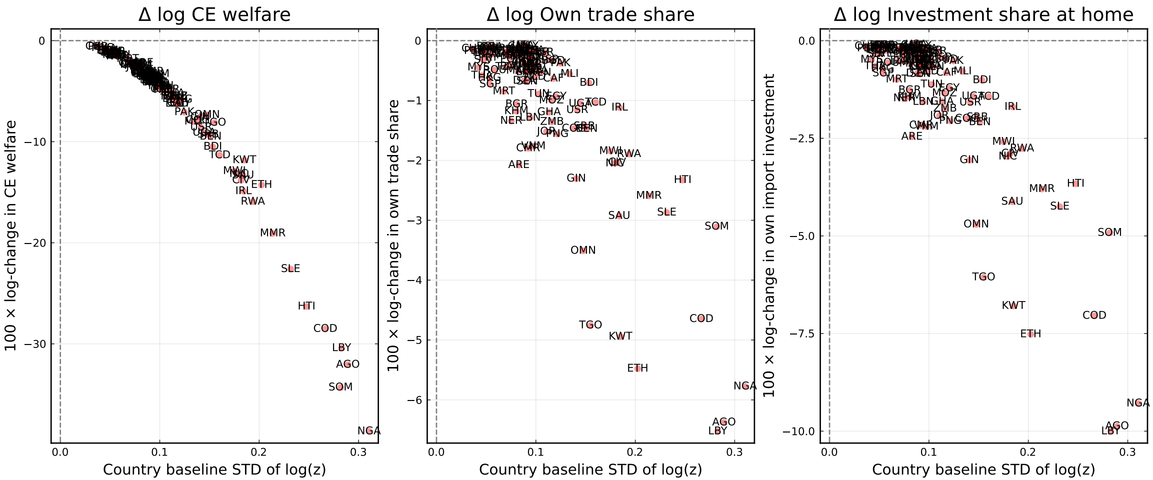
- Balanced panel of 106 countries over 1993-2017
- IMF Direction of Trade Statistics
  - Bilateral value of trade
- Global Macro Database (GMD)
  - Country GDP, price indexes and population
- World Input-Output Database (WIOD)
  - Country gross output for a sample of countries and years
  - Predict gross-output to GDP ratio out-of-sample using WIOD sample

Counterfactual 1: Eliminating Correlations in Productivity (Holding Variance Constant)



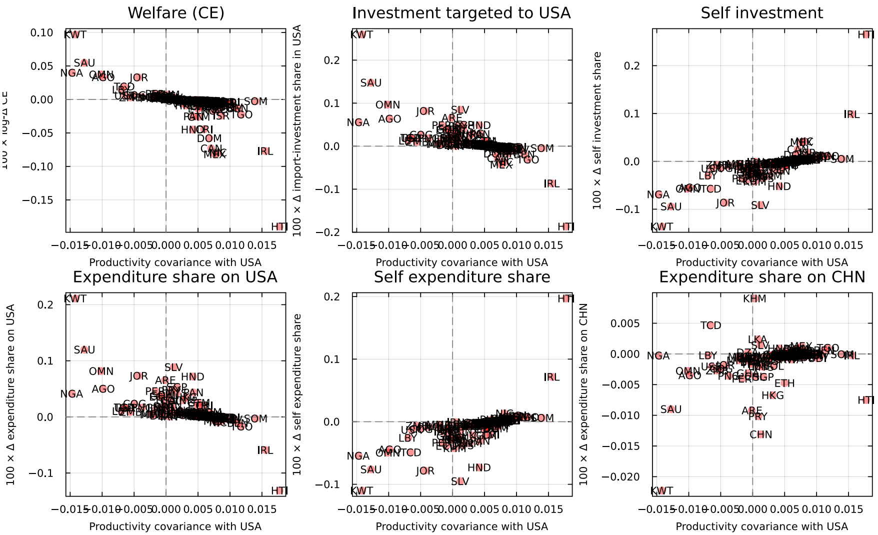
- Welfare ↑ in countries with positive loadings and ↓ in countries with negative loadings
- Removing correlation raises hedging opportunities through trade for countries with positive loadings
- Removing correlation reduces hedging opportunities through trade for countries with negative loadings

Counterfactual 2: Doubling Global Uncertainty (Holding Constant Expected Productivity)



- Welfare goes down in all countries because of risk aversion
- Countries trade off domestic goods (with zero trade costs) against foreign goods (with positive trade costs but imperfectly correlated productivity)
- As global uncertainty ↑ ⇒ domestic investments ↓ and domestic trade shares ↓

Counterfactual 3: Mean Preserving Spread for USA Productivity



- Holding constant mean U.S. productivity and cross-country correlations of productivity
- Countries with +ve covariance with U.S.: welfare ↓, US investment ↓, domestic investment ↑
- Countries with -ve covariance with U.S.: welfare ↑, US investment ↑, domestic investment ↓

Assess the Quality of the Approximation for the Distribution of Wages

- Invert conventional Armington model to recover productivity (z_i) and trade costs (τ_{ni})
  - Use data for the 50 largest countries by GDP in 2005
  - Invert model to recover bilateral trade costs in 2005
  - Invert model to recover productivity annually from 2001-2010

- We find: 2nd-order Taylor expansions provide close approximations to 1st and 2nd moments
  - Estimate the mean and covariance ($\hat{\boldsymbol{\mu}}_z$ and $\hat{\boldsymbol{\Sigma}}_z$) of log productivities (detrend for covariance)
  - Simulate $\ln \boldsymbol{z} \sim \mathcal{N}\left(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}}_z\right)$, solve for wages, and estimate their empirical moments

$$\frac{\left|\left|\ln \hat{w}_i - \hat{\mathbb{E}}\left[\ln w_i\right]\right|\right|}{\left|\left|\hat{\mathbb{E}}\left[\ln w_i\right]\right|\right|} = 0.00002\%, \qquad \frac{\left|\left|W \Sigma_z W' - \hat{\Sigma}_w\right|\right|}{\left|\left|\hat{\Sigma}_w\right|\right|} = 0.94\%$$

  - Cannot detect non-normality for more than 5% of sample by Kolmogorov-Smirnov test
    - in a seeded test of simulating 100 times with $nreps = 1,000$ each time, the distribution for the number of countries rejected for normality is

# countries rejected for normality	0	1	2	3	4	8	9
# of simulations (out of 100)	44	39	10	3	2	1	1

Conclusions

- Growing concerns about uncertainty & resilience to shocks
  - Transportation disruptions in the aftermath of COVID-19
  - Uncertainty over national, regional and multilateral trade policies
  - National security concerns about access to critical goods
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- Key challenge: introducing general equilibrium uncertainty into nonlinear quantitative trade models
  - Most prior research assumes that agents have perfect knowledge of the realizations of all general equilibrium variables when making import and export decisions
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- We make three main contributions
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  2. Allow countries to make upfront investments in bilateral import and export capacities that determine *resilience* in the face of this general equilibrium uncertainty
  3. Show that a country's *risk profile* becomes a key determinant of bilateral patterns of international trade, the terms of trade and welfare

Thank You

This Paper

- Analytically characterize the first and second moments of endogenous variables induced in general equilibrium by productivity and trade cost shocks using second-order Taylor-series expansions
  - Endogenous variables are high-dimensional and non-linear functions of the exogenous variables
- Allow countries to make *ex ante* investments in bilateral import and export capacities prior to the resolution of uncertainty, which determine *resilience* in the face of this uncertainty
  - Solve for investment decisions and trade shares in a competitive equilibrium
  - Solution coincides with a domestic planner solving a "portfolio choice" problem over trade partners in GE
- *Ex ante* investments are affected by variances and covariances of income, prices, and expenditures
  - *Ex ante* investments affect *ex post* trade flows
  - A country's risk profile becomes a key determinant of bilateral trade, the terms of trade, and welfare
  - Opening the closed economy to trade affects welfare through both first and second moments
- Quantitative counterfactuals for changes in global and country uncertainty on bilateral patterns of trade, the terms of trade and welfare
- Second-order approximations are close to the empirical distributions of the endogenous variables from numerical solutions of the non-linear model for simulated productivity distributions

Competitive Equilibrium with Uncertainty: *Ex Ante* Stage

- The representative consumer is indifferent between allocating labor *ex ante* and *ex post*

$$w_n^- = \mathbb{E}\left[\frac{U'(\mathcal{C}_n)}{\mathcal{P}_n} w_n\right]$$

  - No cross-country risk sharing (financial autarky)
  - Mainly focus in presentation on productivity uncertainty (paper incorporates trade cost uncertainty)

- Firms choose *ex ante* investments to maximize profits using consumer's marginal utility as the SDF
  - Profits or losses are rebated back to the domestic consumer

importer firm in country n: $\max_{\iota_{ni}} \left\{ \mathbb{E}\left[\frac{\mathcal{U}'(\mathcal{C}_n)}{\mathcal{P}_n} \max_{x_{ni}} \left(p^c_{ni} c_{ni} - p^x_{ni} x_{ni}\right)\right] - w_n^- \iota_{ni} \right\}$

exporter firm in country i: $\max_{h_{ni}} \left\{ \mathbb{E}\left[\frac{\mathcal{U}'(\mathcal{C}_i)}{\mathcal{P}_i} \max_{h_{ni}} \left(p^x_{ni} x_{ni} - w_i \ell_{ni}\right)\right] - w_i^- h_{ni} \right\}$

- Incomplete markets: countries may differ in relative marginal utility across states

- In the *ex post* stage, agents take *ex ante* investments $\{\iota_{ni}, h_{ni}\}$ as given
- In general equilibrium, trade balance $\implies$ wages must satisfy

$$w_i \left(\bar{\ell}_i - \sum_{k=1}^{N} \iota_{ik} - \sum_{n=1}^{N} h_{ni} \right) = \sum_{n=1}^{N} \frac{\left(\iota_{ni}^{-\alpha/\delta} h_{ni}^{-\beta/\delta} \tau_{ni} w_i / z_i \right)^{1-\tilde{\omega}}}{\sum_{j=1}^{N} \left(\iota_{nj}^{-\alpha/\delta} h_{nj}^{-\beta/\delta} \tau_{nj} w_j / z_j \right)^{1-\tilde{\omega}}} w_n \left(\bar{\ell}_n - \sum_{k=1}^{N} \iota_{nk} - \sum_{m=1}^{N} h_{mn} \right)$$

  - Isomorphic to Armington model with trade elasticity $\tilde{\omega} \equiv \frac{\omega}{\omega - \delta(\omega - 1)}$
  - *Ex ante* investments are absorbed into bilateral iceberg trade costs: $\tilde{\tau}_{ni} \equiv \iota_{ni}^{-\alpha/\delta} h_{ni}^{-\beta/\delta} \tau_{ni}$

Solving for an Equilibrium Given $\boldsymbol{\mu}_z, \boldsymbol{\Sigma}_z$ and Bilateral Trade Costs $\boldsymbol{\tau}$

Initialize a guess of *ex ante* bilateral investments $\{\iota_{ni}, h_{ni}\}$

1. Solve for wages and expenditure shares in the *ex post* equilibrium where $\ln \boldsymbol{z} = \boldsymbol{\mu}_z$
2. Derive first and second moments of endogenous variables using Jacobian and Hessian
3. Update *ex ante* investments and go to step 1

Iterate until convergence

Certainty-equivalent Welfare and Gains from Trade

- Certainty-equivalent welfare:

$$\mathcal{W}_n \equiv \mathcal{U}^{-1}\left(\mathbb{E}\left[\mathcal{U}\left(\mathcal{C}_n\right)\right]\right) = \mathbb{E}\left[\mathcal{C}_n^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

- To second-order,

$$\ln \mathcal{W}_n \approx \mathbb{E}\left[\ln \mathcal{C}_n\right] + \frac{1-\gamma}{2}\mathbb{V}\left[\ln \mathcal{C}_n\right]$$

- Gains from trade:

$$\begin{aligned} \ln \mathcal{W}_n/\mathcal{W}_n^{aut} &\approx \mathbb{E}\left[\ln \mathcal{C} - \ln \mathcal{C}^{aut}\right] + \frac{1-\gamma}{2}\left(\mathbb{V}\left[\ln \mathcal{C}\right] - \mathbb{V}\left[\ln \mathcal{C}^{aut}\right]\right) \\ &= \mathbb{E}\left[\alpha \ln\left(\frac{\iota_{nn}}{\alpha \bar{\ell}_n}\right) + \beta \ln\left(\frac{h_{nn}}{\beta \bar{\ell}_n}\right) + \delta \ln S_{nn}^{\frac{1}{1-\tilde{\omega}}}\right] \\ &\quad + \frac{1-\gamma}{2}\left(\mathbb{V}\left[\ln \mathcal{C}\right] - \mathbb{V}\left[\ln \mathcal{C}^{aut}\right]\right) \end{aligned}$$

- **Alternative interpretation**: equilibrium coincides with solution of domestic planners choosing within-country allocations, taking the distribution of *ex post* tradable prices as given
- *Ex ante* investments: the domestic planner's "portfolio choice" over trade partners

Lemma. (Campbell and Viceira 2001, Allen and Atkin 2022) Let $\boldsymbol{x}$ denote a $N \times 1$ lognormal random vector, $\ln \boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x)$, and $\boldsymbol{a}$ is an $N \times 1$ deterministic vector. Then $\boldsymbol{a}'\boldsymbol{x}$ is approximately lognormally distributed (up to second-order) with the following moments:

$$\mathbb{E}\left[\ln \boldsymbol{a}'\boldsymbol{x}\right] \approx \ln \sum_{i=1}^{N} a_i \exp\left(\mu_{x_i}\right) + \frac{1}{2}\sum_{i=1}^{N} m_i \sigma_i^2 - \frac{1}{2}\boldsymbol{m}'\boldsymbol{\Sigma}_{\mathbf{x}}\boldsymbol{m}, \qquad \mathbb{V}\left[\ln \boldsymbol{a}'\boldsymbol{x}\right] \approx \boldsymbol{m}'\boldsymbol{\Sigma}_{\mathbf{x}}\boldsymbol{m}, \qquad m_i \equiv \frac{a_i \exp\left(\mu_{x_i}\right)}{\sum_{j=1}^{N} a_j \exp\left(\mu_{x_j}\right)}.$$

- Earlier propositions hold independently of this Lemma
- Lemma $\rightarrow$ sum of lognormal random variables is approximately lognormal
- Lemma $\rightarrow$ derive mean and variance of this sum using a second-order Taylor-series expansion
- Assume productivity is lognormally distributed
- Iteratively apply to Armington equilibrium conditions

  $\Longrightarrow$ Obtain approximate lognormal distribution of wages as a fixed point

$$w_i \bar{\ell}_i = \sum_{n=1}^{N} w_n \bar{\ell}_n \frac{\left(w_i \tau_{ni}/z_i\right)^{-\theta}}{\sum_{j=1}^{N}\left(w_j \tau_{nj}/z_j\right)^{-\theta}}$$

Additional Theoretical Implications and Extensions

- Certainty-equivalent welfare and gains from trade
- Multi-sector
- Complete markets and risk-sharing

Multi-sector

$$\mathcal{U}_n = \mathbb{E}\left[\frac{\mathcal{C}_n^{1-\gamma}}{1-\gamma}\right], \qquad \mathcal{C}_n = \prod_{k=1}^{K} \left(C_n^k\right)^{\eta_n^k}, \qquad C_n^k = \left(\sum_{i=1}^{N} \left(c_{ni}^k\right)^{\frac{\omega-1}{\omega}}\right)^{\frac{\omega}{\omega-1}}$$

- Each country needs to make *ex ante* investments against each trade partner in each sector
  - *Ex post* equilibrium isomorphic to Costinot, Donaldson, and Komunjer (2012)

Proposition. As a second-order approximation,

$$\iota_{ni}^k \propto_{n,k} \mathbb{E}\left[\left(\left(\iota_{ni}^k\right)^{-\alpha/\delta}\left(h_{ni}^k\right)^{-\beta/\delta}\tau_{ni}^k w_i/z_i^k\right)^{1-\tilde{\omega}}\right]$$
$$\times \exp\left(Cov\left(\ln \mathcal{C}_n^{-\gamma}, \ln\left(\tau_{ni}^k w_i/z_i^k\right)^{1-\tilde{\omega}}\right)\right)$$
$$\times \exp\left(Cov\left(\ln w_n/\mathcal{P}_n, \ln\left(\tau_{ni}^k w_i/z_i^k\right)^{1-\tilde{\omega}}\right)\right)$$
$$\times \exp\left(-Cov\left(\ln\left(\mathcal{Q}_n^k\right)^{(1-\tilde{\omega})}, \ln\left(\tau_{ni}^k w_i/z_i^k\right)^{1-\tilde{\omega}}\right)\right)$$

$$h_{in}^k \propto_n \mathbb{E}\left[\left(\left(\iota_{in}^k\right)^{-\alpha/\delta}\left(h_{in}^k\right)^{-\beta/\delta}\tau_{in}^k/\mathcal{Q}_i^k\right)^{1-\tilde{\omega}} w_i\bar{\ell}_i\eta_i^k\right]$$
$$\times \exp\left(Cov\left(\ln \mathcal{C}_n^{-\gamma}, \ln\left(\tau_{in}^k/\mathcal{Q}_i^k\right)^{1-\tilde{\omega}} w_i\right)\right)$$
$$\times \exp\left(-Cov\left(\ln \mathcal{P}_n, \ln\left(\tau_{in}^k/\mathcal{Q}_i^k\right)^{1-\tilde{\omega}} w_i\right)\right)$$
$$\times \exp\left(Cov\left(\ln\left(w_n/z_n^k\right)^{1-\tilde{\omega}}, \ln\left(\tau_{in}^k/\mathcal{Q}_i^k\right)^{1-\tilde{\omega}} w_i\right)\right)$$

Complete Markets and Risk-sharing

- Competitive equilibrium with financial autarky: generically inefficient due to incomplete markets
  - Countries value states differently (i.e., $\frac{\mathcal{U}'(\mathcal{C}_n(s))/\mathcal{P}_n(s)}{\mathcal{U}'(\mathcal{C}_m(s))/\mathcal{P}_m(s)}$ not constant across states of the world)

- Under complete markets, there exists state price density $\varphi(s)$ such that

$$\mathcal{U}'(\mathcal{C}_n(s))/\mathcal{P}_n(s) = \zeta_n \varphi(s)$$

- *Ex post* wages in state s satisfy

$$\delta \sum_{n=1}^{N} \mathcal{P}_n(s) \mathcal{C}_n(s) \frac{\left[\iota_{ni}^{\alpha} h_{ni}^{\beta} \left(z_i(s) \ell_{ni}(s)/\tau_{ni}\right)^{\delta}\right]^{\frac{\omega-1}{\omega}}}{\sum_{j=1}^{N} \left[\iota_{nj}^{\alpha} h_{nj}^{\beta} \left(z_j(s) \ell_{nj}(s)/\tau_{nj}\right)^{\delta}\right]^{\frac{\omega-1}{\omega}}} = w_i(s) \sum_{n=1}^{N} \ell_{ni}(s)$$

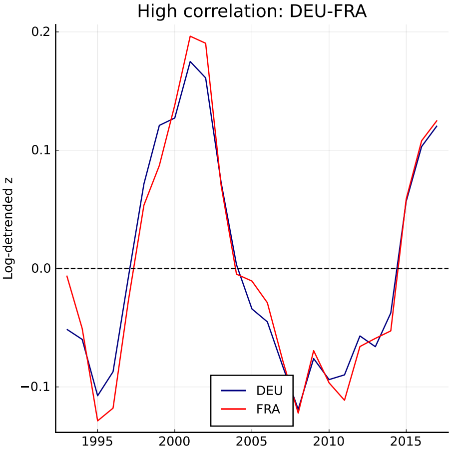
- *Ex ante* budget balance implies

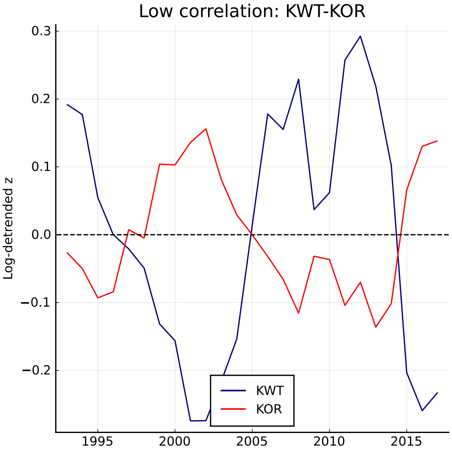
$$\mathbb{E}\left[\varphi(s) w_n(s)\right] \bar{\ell}_n = \mathbb{E}\left[\varphi(s) \mathcal{P}_n(s) \mathcal{C}_n(s)\right]$$

Quantitative exercises

- Set baseline expected log-productivities (μ_z) and trade frictions based on inverted values in 2017
  - Covariance matrix Σ_z based on the full-sample
  - Inversion is based on the deterministic Armington model ▸ more
- Parameters
  - $\beta = 0$ (no exporter ex-ante investments)
  - $\alpha = 0.5$, and $\omega = 9$: match short-run and long-run trade elasticities of $\theta^{SR} = \frac{\delta(\omega-1)}{(1-\delta)\omega+\delta} = 0.8$ and $\theta^{LR} = \alpha + \beta + \delta\omega - 1 = 4.0$
  - Risk aversion: $\gamma = 10$ (Collin-Dufresne et al. 2016)
- Undertake principle components decomposition of the variance-variance matrix ▸ princcomp

Examples - High and Low Covariance

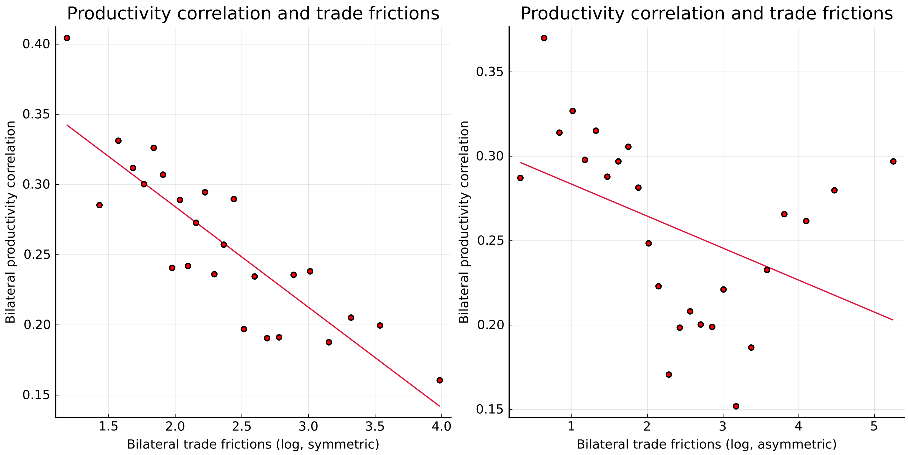




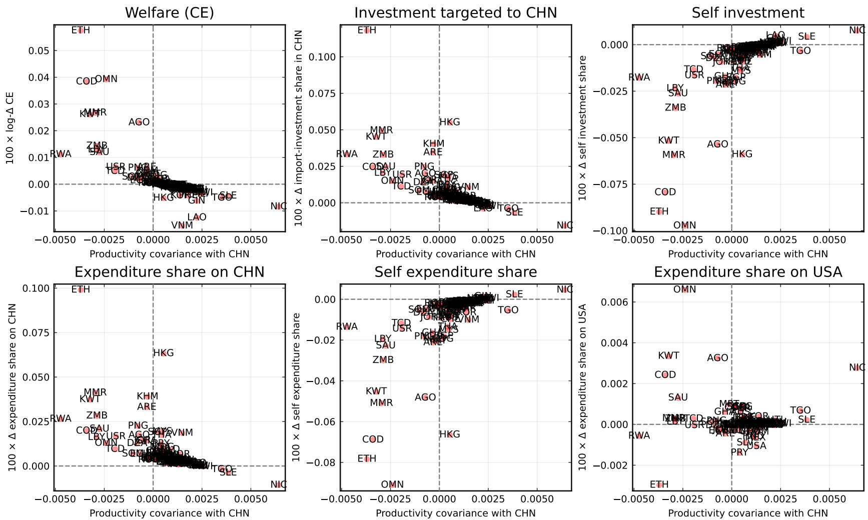
Global Factor

- Global factor (1st principal component) accounts for 43% of productivity variance
- First 5 factors account for 80%

Bilateral Covariance and Bilateral Frictions



Counterfactual 4: Mean Preserving Spread for CHN Productivity



- Holding constant mean Chinese productivity and cross-country correlations of productivity
- Countries with +ve covariance with China: welfare ↓, US investment ↓, domestic investment ↑
- Countries with -ve covariance with China: welfare ↑, US investment ↑, domestic investment ↓

Armington Wage System

- The equality between a country's income and expenditure on its good in the deterministic Armington model implies the following equilibrium function

$$\ln \boldsymbol{w} = F\left(\ln \boldsymbol{z}; \theta\right)$$

$$\ln w_i = \ln \sum_n \xi_{ni} w_n, \qquad \xi_{ni} \equiv \frac{\ell_n \left(\tau_{ni} w_i / \exp\left(\ln z_i\right)\right)^{-\theta}}{\sum_{k=1}^{N} \ell_i \left(\tau_{ni} w_k / \exp\left(\ln z_k\right)\right)^{-\theta}}$$

back

Jacobian

- Totally differentiating the income accounting condition around an initial equilibrium, holding constant labor endowments (ℓ_n) and trade costs (τ_{ni}), and allowing country productivities (z_n) to change, we obtain:

$$\mathrm{d}\ln w_i = \sum_{n=1}^{N} t_{in}\,\mathrm{d}\ln w_n + \theta\left(\sum_{h=1}^{N}\sum_{n=1}^{N} t_{in}s_{nh}\left[\mathrm{d}\ln w_h - \mathrm{d}\ln z_h\right] - \left[\mathrm{d}\ln w_i - \mathrm{d}\ln z_i\right]\right),$$

- where t_{in} is the share of exporter i's income from market n:

$$t_{in} \equiv \frac{s_{ni}w_n\ell_n}{w_i\ell_i}, \qquad s_{nit} = \frac{\left(\tau_{ni}w_i/z_i\right)^{-\theta}}{\sum_{j=1}^{N}\left(\tau_{nj}w_j/z_j\right)^{-\theta}}.$$

- This market clearing condition has the following matrix representation:

$$\mathrm{d}\ln \mathbf{w} = \boldsymbol{T}\,\mathrm{d}\ln \mathbf{w} + \theta\left(\boldsymbol{TS} - \boldsymbol{I}\right)\left(\mathrm{d}\ln \mathbf{w} - \mathrm{d}\ln \mathbf{z}\right),$$

- which can be re-written as:

$$\left(\boldsymbol{I} - \boldsymbol{T} - \theta\left(\boldsymbol{TS} - \boldsymbol{I}\right)\right)\mathrm{d}\ln \mathbf{w} = -\theta\left(\boldsymbol{TS} - \boldsymbol{I}\right)\mathrm{d}\ln \mathbf{z}.$$

back

Jacobian

- To solve for wages we use our choice of world GDP as numeraire, which implies $\boldsymbol{Q}\,\mathrm{d}\ln\mathbf{w} = 0$. Using this choice of numeraire in the market clearing condition above, we have:

$$(\boldsymbol{I} - \boldsymbol{T} - \theta\left(\boldsymbol{TS} - \boldsymbol{I}\right) + \boldsymbol{Q})\,\mathrm{d}\ln\mathbf{w} = -\theta\left(\boldsymbol{TS} - \boldsymbol{I}\right)\mathrm{d}\ln\mathbf{z}.$$

which can be re-written as:

$$\mathrm{d}\ln\mathbf{w} = \mathbf{W}\,\mathrm{d}\ln\mathbf{z},$$

$$\mathbf{W} \equiv -\left(\boldsymbol{I} - \boldsymbol{T} - \theta\left(\boldsymbol{TS} - \boldsymbol{I}\right) + \boldsymbol{Q}\right)^{-1}\theta\left(\boldsymbol{TS} - \boldsymbol{I}\right).$$

back

Hessian

- To derive the Hessian, i.e., $H^i_{jk} \equiv \frac{d^2 \ln w_i}{d \ln z_j \, d \ln z_k}$, we need to take derivative of Jacobian matrix $(\boldsymbol{W})$ with respect to $d \ln z_k$. Define $\boldsymbol{Z} \equiv \boldsymbol{I} - \boldsymbol{T} - \theta^{DA} (\boldsymbol{TS} - \boldsymbol{I}) + \boldsymbol{Q}$. Note that:

$$d\boldsymbol{Z} = -d\boldsymbol{T} - \theta^{DA} (d\boldsymbol{TS} + \boldsymbol{T}d\boldsymbol{S}) \, .$$

  Using this result, we have:

$$\begin{aligned} dW &= (\boldsymbol{I} - \boldsymbol{Q}) \left[\boldsymbol{Z}^{-1} (d\boldsymbol{Z}) \boldsymbol{Z}^{-1} \theta^{DA} (\boldsymbol{TS} - \boldsymbol{I}) - \theta^{DA} \boldsymbol{Z}^{-1} d (\boldsymbol{TS} - \boldsymbol{I})\right], \\ &= (\boldsymbol{I} - \boldsymbol{Q}) \left[-\boldsymbol{Z}^{-1} (d\boldsymbol{Z}) \boldsymbol{W} - \theta^{DA} \boldsymbol{Z}^{-1} (d\boldsymbol{TS} + \boldsymbol{T}d\boldsymbol{S})\right], \\ &= (\boldsymbol{I} - \boldsymbol{Q}) \left[\boldsymbol{Z}^{-1} \left(d\boldsymbol{T} + \theta^{DA} (d\boldsymbol{TS} + \boldsymbol{T}d\boldsymbol{S})\right) \boldsymbol{W} - \theta^{DA} \boldsymbol{Z}^{-1} (d\boldsymbol{TS} + \boldsymbol{T}d\boldsymbol{S})\right] \\ & (\boldsymbol{I} - \boldsymbol{Q}) \left[\boldsymbol{Z}^{-1} d\boldsymbol{T} \left[\left(\boldsymbol{I} + \theta^{DA} \boldsymbol{S}\right) \boldsymbol{W} - \theta \boldsymbol{S}\right] + \theta^{DA} \boldsymbol{Z}^{-1} \boldsymbol{T} d\boldsymbol{S} (\boldsymbol{W} - \boldsymbol{I})\right]. \end{aligned}$$

back

Hessian

- Therefore Hessian ($H^i_{jk} \equiv d_k W \equiv \frac{d^2 \ln w_i}{d \ln z_j \, d \ln z_k}$) of wages to productivities is:

$$d_k \boldsymbol{W} = (\boldsymbol{I} - \boldsymbol{Q}) \left\{ \boldsymbol{Z}^{-1} (d_k \boldsymbol{T}) \left(\left(\boldsymbol{I} + \theta^{DA} \boldsymbol{S} \right) \boldsymbol{W} - \theta^{DA} \boldsymbol{S} \right) + \theta^{DA} \boldsymbol{Z}^{-1} \boldsymbol{T} (d_k \boldsymbol{S}) (\boldsymbol{W} - \boldsymbol{I}) \right\},$$

$$(d_k \boldsymbol{S})_{ni} = \theta^{DA} \boldsymbol{S}_{ni} \left(\boldsymbol{\delta}_{ik} - \boldsymbol{J}_{ik} - \boldsymbol{S}_{nk} + (\boldsymbol{S}\boldsymbol{J})_{nk} \right),$$

$$(d_k \boldsymbol{T})_{in} = \boldsymbol{T}_{in} \left[\boldsymbol{J}_{nk} - \boldsymbol{J}_{ik} + \theta^{DA} \left(\boldsymbol{\delta}_{ik} - \boldsymbol{J}_{ik} - \boldsymbol{S}_{nk} + (\boldsymbol{S}\boldsymbol{J})_{nk} \right) \right],$$

where δ_{ik} is the Kronecker delta:

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}.$$

back

Proposition. Define $\boldsymbol{F}$ as the equilibrium function in the deterministic Armington model that maps log productivities ($\ln \boldsymbol{z}$) and log trade costs ($\ln \boldsymbol{\tau}$) into log wages ($\ln \boldsymbol{w}$) given the trade elasticity (θ): $\ln \boldsymbol{w} = \boldsymbol{F}(\ln \boldsymbol{z}, \ln \boldsymbol{\tau}; \theta)$. back

(A) To second order,

$$\begin{aligned} \mathbb{E}[\ln w_k] \quad \approx \quad & F_k(\boldsymbol{\mu}_z, \boldsymbol{\mu}_\tau) + \frac{1}{2} \sum_{ni,mj} \frac{\mathrm{d}^2 \ln \boldsymbol{F}_k}{\mathrm{d} \ln \tau_{ni} \, \mathrm{d} \ln \tau_{mj}} [\boldsymbol{\Sigma}_\tau]_{ni,mj} \\ & + \sum_{i,j} \frac{\mathrm{d}^2 \ln \boldsymbol{F}_k}{\mathrm{d} \ln z_i \, \mathrm{d} \ln z_j} [\boldsymbol{\Sigma}_z]_{i,j} + \frac{1}{2} \sum_{i,mj} \frac{\mathrm{d}^2 \ln \boldsymbol{F}_k}{\mathrm{d} \ln z_i \, \mathrm{d} \ln \tau_{mj}} [\boldsymbol{\Sigma}_{z\tau}]_{i,mj} \end{aligned}$$

$$V[\ln \boldsymbol{w}] \approx \boldsymbol{W} \boldsymbol{\Sigma}_z \boldsymbol{W}' + \boldsymbol{M} \boldsymbol{\Sigma}_\tau \boldsymbol{M}' + \boldsymbol{W} \boldsymbol{\Sigma}_{z\tau} \boldsymbol{M}' + \boldsymbol{M} \boldsymbol{\Sigma}_{z\tau}' \boldsymbol{W}$$

where the Jacobian matrices $\boldsymbol{W} \equiv \left[\frac{\mathrm{d} \ln w_k}{\mathrm{d} \ln z_i}\right]$ and $\boldsymbol{M} \equiv \left[\frac{\mathrm{d} \ln w_k}{\mathrm{d} \ln \tau_{ni}}\right]$ and the Hessians (the second derivatives) are evaluated at $(\ln \boldsymbol{z}, \ln \boldsymbol{\tau}) = (\boldsymbol{\mu}_z, \boldsymbol{\mu}_\tau)$.

(B) The bilateral expenditure share matrix $\boldsymbol{S}$ is a sufficient statistic for the Jacobian and Hessian:

$$S_{ni} \equiv \left. \frac{(\tau_{ni} w_i / z_i)^{-\theta}}{\sum_{k=1}^{N} (\tau_{nk} w_k / z_k)^{-\theta}} \right|_{\ln \boldsymbol{z} = \boldsymbol{\mu}_z} .$$

Deterministic Armington Inversion

- We observe nominal GDP (y_{it}), population (ℓ_{it}), expenditure shares (s_{nit}) and price indexes (p_{it})
- We recover nominal wages (w_{it}) as:

$$w_{it} = \frac{y_{it}}{\ell_{it}}.$$

- We recover country productivities (z_{it}) using the above solutions for nominal wages (w_{it}), together with observed bilateral expenditure shares (s_{nit}) and price indexes (p_{it}):

$$\frac{w_{it}}{p_{it}} = z_{it}\left(\frac{1}{s_{iit}}\right)^{\frac{1}{\omega-1}}.$$

- We recover bilateral trade costs (τ_{nit}) using our solutions for wages (w_{it}) and productivities (z_{it}), together with observed expenditure shares (s_{nit}) and price indexes (p_{it}):

$$s_{nit} = \left(\frac{\tau_{nit} w_{it}}{z_{it} p_{it}}\right)^{1-\omega}.$$

- Normalization: $\tau_{iit} = 1$

back

Principal Components

- We have a dataset $\boldsymbol{X}$ of n observations and j variables, written as an $n \times j$ matrix
- The columns of $\boldsymbol{X}$ are mean-centered, such that each column has zero mean
- Step 1: Compute the covariance matrix:

$$\boldsymbol{\Sigma} = \frac{1}{n-1}\boldsymbol{X}^T\boldsymbol{X}$$

- Step 2: Solve the eigenvalue problem:

$$\boldsymbol{\Sigma}\boldsymbol{v}_k = \lambda_k \boldsymbol{v}_k$$

  - $\boldsymbol{v}_k$ is the kth eigenvector (a principal component direction)
  - λ_k is the eigenvalue (the amount of variance in that direction)
  - Sort eigenvalues from largest to smallest: $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_j$
- Step 3: Form Principal Components
  - The k-th principal component score for observation i is:

$$\boldsymbol{z}_{ik} = \boldsymbol{x}_i^T \boldsymbol{v}_k$$

  - where $\boldsymbol{x}_i$ is the i-th observation (as a column vector)
  - $\boldsymbol{v}_k$ tells you the direction (i.e., weights on original variables)
  - $\boldsymbol{z}_{ik}$ tells you the value along that direction for each observation

back

Principal Components

- Dimensionality reduction
  - We can approximate the original data using only the first few components:

$$X \approx Z_k V_k^T$$

  - $\boldsymbol{Z}_k$ is an $n \times k$ matrix of the first k principal component scores
  - $\boldsymbol{V}_k$ is a $j \times k$ matrix of the first k eigenvectors
- We keep most of the variance while reducing the number of variables from j to $k < j$

back

Campbell and Viceira (2001)

Lemma. (Campbell and Viceira 2001, Allen and Atkin 2022) Let $\boldsymbol{x}$ denote a $N \times 1$ lognormal random vector, $\ln \boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x)$, and $\boldsymbol{a}$ is an $N \times 1$ deterministic vector.

- Define $\epsilon_i \equiv \ln x_i - \mu_{x_i}$. Note that $\mathbb{E}[\epsilon_{x_i}] = -\frac{1}{2}\sigma_i^2$ from the properties of the lognormal distribution. We have

$$\begin{aligned}
\ln \boldsymbol{a}'\boldsymbol{x} &\equiv \ln \sum_i a_i x_i, \\
&= \ln \sum_i a_i \exp\left(\mu_{x_i}\right) \frac{x_i}{\exp\left(\mu_{x_i}\right)}, \\
&= \ln \sum_i a_i \exp\left(\mu_{x_i}\right) + \ln \sum_i \frac{a_i \exp\left(\mu_{x_i}\right)}{\sum_j a_j \exp\left(\mu_{x_j}\right)} \frac{x_i}{\exp\left(\mu_{x_i}\right)}, \\
&= \ln \sum_i a_i \exp\left(\mu_{x_i}\right) + \ln \sum_i m_{x_i} \exp\left(\ln x_i - \mu_{x_i}\right),
\end{aligned}$$

where we have defined

$$m_{x_i} \equiv \frac{a_i \exp\left(\mu_{x_i}\right)}{\sum_j a_j \exp\left(\mu_{x_j}\right)}$$

Campbell and Viceira (2001)

- Note that $\ln \boldsymbol{a}'\boldsymbol{x}$ is a linear combination of lognormally distributed random variables and hence is approximately lognormally distributed (e.g., Mehta et al. 2007)
- Taking a second-order approximation around $\epsilon_i = 0$, we have

$$
\begin{aligned}
\ln \boldsymbol{a}'\boldsymbol{x} &\approx \ln \sum_i a_i \exp\left(\mu_{x_i}\right) + \ln \sum_i m_{x_i}\left(1 + \epsilon_i + \frac{1}{2}\epsilon_i^2\right), \\
&\approx \ln \sum_i a_i \exp\left(\mu_{x_i}\right) + \sum_i m_{x_i}\left(\epsilon_i + \frac{1}{2}\epsilon_i^2\right) - \frac{1}{2}\sum_i \sum_j m_{x_i} m_{x_j} \epsilon_i \epsilon_j.
\end{aligned}
$$

- Taking expectations, we have:

$$
\mathbb{E}\left[\ln \boldsymbol{a}'\boldsymbol{x}\right] \approx \ln \sum_i a_i \exp\left(\mu_{x_i}\right) + \frac{1}{2}\sum_i m_{x_i}\sigma_i^2 - \frac{1}{2}\boldsymbol{m}_{\mathbf{x}}'\boldsymbol{\Sigma}_{\mathbf{x}}\boldsymbol{m}_{\mathbf{x}},
$$

$$
\mathbb{V}\left[\ln \boldsymbol{a}'\boldsymbol{x}\right] \approx \boldsymbol{m}_{\mathbf{x}}'\boldsymbol{\Sigma}_{\mathbf{x}}\boldsymbol{m}_{\mathbf{x}}.
$$

- Optimization of each country's representative consumer

$$\max_{c_{ni}(\omega)} u_n = \frac{1}{1-\gamma} \int \left(\sum_i c_{ni}(\omega)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}(1-\gamma)} \mathrm{d}F(\omega)$$

$$\text{s.t.} \quad \int p_{ni}(\omega)\, c_{ni}(\omega)\, \mathrm{d}F(\omega) = \int w_n(\omega)\, \ell_n\, \mathrm{d}F(\omega)$$

General equilibrium condition

$$w_i(\omega)\, \ell_i(\omega) = \sum_n S_{ni}(\omega)\, E_n(\omega)$$

where $S_{ni}(\omega) \equiv \frac{p_{ni}(\omega) c_{ni}(\omega)}{\sum_j p_{nj}(\omega) c_{nj}(\omega)}$, $E_n(\omega) = \sum_j p_{nj}(\omega)\, c_{nj}(\omega)$

Complete Markets

- Within a state:

$$\frac{p_{ni}(\omega)\, c_{ni}(\omega)}{\sum_j p_{nj}(\omega)\, c_{nj}(\omega)} = \frac{p_{ni}(\omega)^{1-\sigma}}{\sum_j p_{nj}(\omega)^{1-\sigma}}$$

  Across states:

$$\frac{p_n(\omega)\, c_n(\omega)}{\int p_n(\omega)\, c_n(\omega)\, \mathrm{d}F(\omega)} = \frac{p_n(\omega)^{1-\frac{1}{\gamma}}}{\int p_n(\omega)^{1-\frac{1}{\gamma}}\, \mathrm{d}F(\omega)}$$

  where

$$p_n(\omega) = \left(\sum_j p_{nj}(\omega)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}, \qquad c_n(\omega) = \left(\sum_i c_{ni}(\omega)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

- Define the following variables

$$S_{ni}(\omega) = \frac{p_{ni}(\omega)^{1-\sigma}}{\sum_j p_{nj}(\omega)^{1-\sigma}}$$

$$E_n(\omega) = \int w_n(\omega)\,\ell_n \,\mathrm{d}F(\omega) \frac{\left(\sum_j p_{nj}(\omega)^{1-\sigma}\right)^{\frac{1}{1-\sigma}\frac{\gamma-1}{\gamma}}}{\int \left(\sum_j p_{nj}(\omega)^{1-\sigma}\right)^{\frac{1}{1-\sigma}\frac{\gamma-1}{\gamma}} \mathrm{d}F(\omega)}$$

$$p_{ni}(\omega) = w_i(\omega)\,\tau_{ni}(\omega)\,/\,z_i(\omega)$$

- We can write the general equilibrium condition as

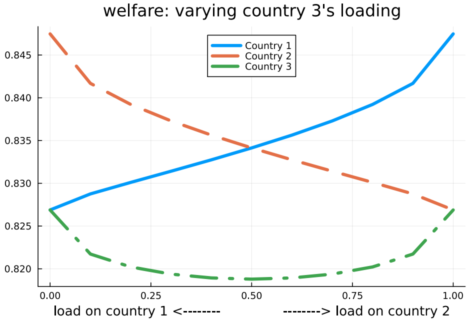
$$w_i(\omega)\,\ell_i = \sum_n \frac{\left[\frac{w_i(\omega)\tau_{ni}(\omega)}{z_i(\omega)}\right]^{1-\sigma}}{\sum_j \left[\frac{w_j(\omega)\tau_{nj}(\omega)}{z_j(\omega)}\right]^{1-\sigma}} \frac{\left(\sum_j \left[\frac{w_j(\omega)\tau_{nj}(\omega)}{z_j(\omega)}\right]^{1-\sigma}\right)^{\frac{1}{1-\sigma}\frac{\gamma-1}{\gamma}}}{\int \left(\sum_j \left[\frac{w_j(\omega')\tau_{nj}(\omega')}{z_j(\omega')}\right]^{1-\sigma}\right)^{\frac{1}{1-\sigma}\frac{\gamma-1}{\gamma}} \mathrm{d}F(\omega')} \int w_n(\omega')\,\ell_n \,\mathrm{d}F(\omega')$$

Example: Risk Profile as a Source of Comparative Advantage

- Consider a world with three countries, with identical sizes and $\mathbb{E}[\ln z]$ but different risk profiles

$$\mathbf{\Sigma}_z = \begin{bmatrix} 1 & 0 & \sqrt{1-x} \\ 0 & 1 & \sqrt{x} \\ \sqrt{1-x} & \sqrt{x} & 1 \end{bmatrix}, \qquad x \in [0,1]$$

  - Log-utility, no trade costs, complete markets



- As $x \nearrow$ from 0 to 1, country 3 becomes more dissimilar to country 1 and more similar to country 2
  - Countries ↑ *ex ante* investments towards importing from country 1, improving its terms of trade
- More broadly, risk profile is a source of comparative advantage & affects ToT and trade flows