Corrected Appendix to Nowcasting
by

Marta Bańbura, Domenico Giannone and Lucrezia Reichlin

published in Oxford Handbook on Economic Forecasting, ed. by M. P. Clements, and D. F. Hendry, pp. 63-90. Oxford University Press, 2011

B State space representation of the model

Presented here are the details for the state-space representation of equation (11) as specified by the equations (5)-(10), for $p = 1$, $r = 3$, and a single quarterly variable (y_t^Q is of dimension 1 as in the benchmark model):

$$\underbrace{\begin{pmatrix} x_t \\ y_t^Q \end{pmatrix}}_{\bar{x}_t} = \underbrace{\begin{pmatrix} \mu \\ \tilde{\mu}_Q \end{pmatrix}}_{\bar{\mu}} + \underbrace{\begin{pmatrix} \Lambda & 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ \Lambda_Q & 2\Lambda_Q & 3\Lambda_Q & 2\Lambda_Q & \Lambda_Q & 0 & 1 & 2 & 3 & 2 & 1 \end{pmatrix}}_{Z(\theta)} \underbrace{\begin{pmatrix} f_t \\ f_{t-1} \\ f_{t-2} \\ f_{t-3} \\ f_{t-4} \\ \varepsilon_t \\ \varepsilon_t^Q \\ \varepsilon_{t-1}^Q \\ \varepsilon_{t-2}^Q \\ \varepsilon_{t-3}^Q \\ \varepsilon_{t-4}^Q \end{pmatrix}}_{\bar{\xi}_t}$$

$$\begin{pmatrix} f_t \\ f_{t-1} \\ f_{t-2} \\ f_{t-3} \\ f_{t-4} \\ \varepsilon_t \\ \varepsilon_t^Q \\ \varepsilon_{t-1}^Q \\ \varepsilon_{t-2}^Q \\ \varepsilon_{t-3}^Q \\ \varepsilon_{t-4}^Q \end{pmatrix} = \underbrace{\begin{pmatrix} A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{diag}(\alpha_1, \ldots, \alpha_n) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_Q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}}_{T(\theta)} \begin{pmatrix} f_{t-1} \\ f_{t-2} \\ f_{t-3} \\ f_{t-4} \\ f_{t-5} \\ \varepsilon_{t-1} \\ \varepsilon_{t-1}^Q \\ \varepsilon_{t-2}^Q \\ \varepsilon_{t-3}^Q \\ \varepsilon_{t-4}^Q \\ \varepsilon_{t-5}^Q \end{pmatrix} + \underbrace{\begin{pmatrix} u_t \\ 0 \\ 0 \\ 0 \\ 0 \\ e_t \\ e_t^Q \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\eta_t} \tag{12}$$

where $\varepsilon_t = (\varepsilon_{1,t}, \varepsilon_{2,t}, \ldots, \varepsilon_{n,t})'$ and $e_t = (e_{1,t}, e_{2,t}, \ldots, e_{n,t})'$.

The block-specific factor structure further implies that

$$\Lambda = \begin{pmatrix} \Lambda_{N,G} & \Lambda_{N,N} & 0 \\ \Lambda_{R,G} & 0 & \Lambda_{R,R} \end{pmatrix}, \quad \Lambda_Q = \begin{pmatrix} \Lambda_{Q,G} & 0 & \Lambda_{Q,R} \end{pmatrix},$$

$$f_t = \begin{pmatrix} f_t^G \\ f_t^N \\ f_t^R \end{pmatrix}, \quad A_1 = \begin{pmatrix} A_{1,G} & 0 & 0 \\ 0 & A_{1,N} & 0 \\ 0 & 0 & A_{1,R} \end{pmatrix}, \quad Q = \begin{pmatrix} Q_G & 0 & 0 \\ 0 & Q_N & 0 \\ 0 & 0 & Q_R \end{pmatrix}.$$

Hence, the parameters of the model are:

$$\theta = (\mu, \tilde{\mu}_Q, \mathrm{vec}(\Lambda_{N,G})', \mathrm{vec}(\Lambda_{N,N})', \mathrm{vec}(\Lambda_{R,G})', \mathrm{vec}(\Lambda_{R,R})', \Lambda_{Q,G}, \Lambda_{Q,R}, A_{1,G}, A_{1,N}, A_{1,R},$$
$$Q_G, Q_N, Q_R, \alpha_1, \ldots, \alpha_n, \alpha_Q, \sigma_1, \ldots, \sigma_n, \sigma_Q)'.$$

The state space representation can be easily modified to include an arbitrary number of quarterly variables n_Q (e.g., the model with disaggregated data contains six quarterly variables). In that case, y_t^Q, $\tilde{\mu}_Q$, ε_t^Q and e_t^Q will be vectors of length n_Q. Λ_Q will be a matrix of size $n_Q \times r$ and α_Q will be a $n_Q \times n_Q$ diagonal matrix. Finally, the scalars in the lines of $Z(\theta)$ and $T(\theta)$ corresponding to y_t^Q and ε_t^Q need to be replaced by $n_Q \times n_Q$ identity or zero matrices.

C EM algorithm

The parameters θ of the state-space form of equation (12) are estimated by the expectation maximization (EM) algorithm. The algorithm is a popular solution to problems for which latent or missing data yield a direct maximization of the likelihood function intractable or computationally difficult.[13] The basic principle behind the EM is to write the likelihood in terms of observable as well as latent variables (in our case, in terms of $\bar{x}_t$ and $\bar{\xi}_t$, $t = 1, \ldots, T_v = \max_i T_{i,v}$) and, given the available data Ω_v,[14] obtain the maximum likelihood estimates in a sequence of two alternating steps. Precisely, iteration $j+1$ would consist of the following steps:

- E-step - the expectation of the log-likelihood conditional on the data is calculated using the estimates from the previous iteration, $\theta(j)$,
- M-step - the new parameters, $\theta(j+1)$, are estimated through the maximization of the expected log-likelihood (from the previous iteration) with respect to θ.

Below we provide the details of the implementation of the EM algorithm for the state-space representation of equation (12) (based on the results in Bańbura and Modugno, 2010).

We first estimate μ and $\tilde{\mu}_Q$ by sample means and use the de-meaned data throughout the EM steps.

To deal with missing observations in $\bar{x}_t$ we follow Bańbura and Modugno (2010) and introduce selection matrices W_t and W_t^Q. They are diagonal matrices of size n and 1, respectively, with 1s corresponding to the nonmissing values in x_t and y_t^Q, respectively.

For the sake of simplicity, we first consider the case without restrictions on Λ, Λ_Q, A_1 and Q implied by block-specific factors.

The maximization of the expected likelihood (M-step) with respect to θ in the $(r+1)$-iteration would yield the following expressions:

- The matrix of loadings for the monthly variables:

$$\begin{aligned}\operatorname{vec}\big(\Lambda(j+1)\big) \quad &= \quad \left(\sum_{t=1}^{T} \mathbb{E}_{\theta(j)}\left[f_t f_t' | \Omega_v\right] \otimes W_t\right)^{-1} \\ &\quad \operatorname{vec}\left(\sum_{t=1}^{T} W_t x_t \mathbb{E}_{\theta(j)}\left[f_t' | \Omega_v\right] - W_t \mathbb{E}_{\theta(j)}\left[\varepsilon_t f_t' | \Omega_v\right]\right).\end{aligned} \tag{13}$$

[13]See Dempster, Laird, and Rubin (1977) for a general EM algorithm and Shumway and Stoffer (1982) or Watson and Engle (1983) for application to state-space representations.

[14]$\Omega_v \subseteq \{y_1, \ldots, y_{T_v}\}$ because some observations in y_t are missing.

- The matrix of loadings for the quarterly variables:

  Let $f_t^{(p)} = [f_t', \dots, f_{t-p+1}']'$ and $D = \sum_{t=1}^{T} \mathbb{E}_{\theta(j)}\big[f_t^{(5)} f_t^{(5)'} | \Omega_v\big] W_t^Q$ and $\tilde{\epsilon}_t^Q = \epsilon_t^Q + 2\epsilon_{t-1}^Q + 3\epsilon_{t-2}^Q + 2\epsilon_{t-3}^Q + \epsilon_{t-4}^Q$. The unrestricted row vector of factor loadings for y_t^Q is given by

$$\big(\bar{\Lambda}_Q^{ur}(j+1)\big)' = D^{-1}\left(\sum_{t=1}^{T} W_t^Q y_t^Q \mathbb{E}_{\theta(j)}\big[f_t^{(5)'}|\Omega_v\big] - W_t^Q \mathbb{E}_{\theta(j)}\big[\tilde{\epsilon}_t^Q f_t^{(5)'}|\Omega_v\big]\right)'.$$

  For the restricted $\bar{\Lambda}_Q = (\Lambda_Q \;\; 2\Lambda_Q \;\; 3\Lambda_Q \;\; 2\Lambda_Q \;\; \Lambda_Q)$ it holds that $C\bar{\Lambda}_Q' = 0$ with

$$C = \begin{bmatrix} I_r & -\frac{1}{2}I_r & 0 & 0 & 0 \\ I_r & 0 & -\frac{1}{3}I_r & 0 & 0 \\ I_r & 0 & 0 & -\frac{1}{2}I_r & 0 \\ I_r & 0 & 0 & 0 & -I_r \end{bmatrix}.$$

  Consequently the restricted $\bar{\Lambda}_Q$ is given by:

$$\big(\bar{\Lambda}_Q(j+1)\big)' = \big(\bar{\Lambda}_Q^{ur}(j+1)\big)' - D^{-1}C'(CD^{-1}C')^{-1}C\big(\bar{\Lambda}_Q^{ur}(j+1)\big)'.$$

- The autoregressive coefficients in the factor VAR:

$$A_1(j+1) = \left(\sum_{t=1}^{T} \mathbb{E}_{\theta(j)}\big[f_t f_{t-1}'|\Omega_v\big]\right)\left(\sum_{t=1}^{T} \mathbb{E}_{\theta(j)}\big[f_{t-1} f_{t-1}'|\Omega_v\big]\right)^{-1}. \tag{14}$$

- The covariance matrix in the factor VAR:

$$Q(j+1) = \frac{1}{T}\left(\sum_{t=1}^{T} \mathbb{E}_{\theta(j)}\big[f_t f_t'|\Omega_v\big] - A_1(j+1)\sum_{t=1}^{T} \mathbb{E}_{\theta(j)}\big[f_{t-1} f_t'|\Omega_v\big]\right). \tag{15}$$

- The autoregressive coefficients in the AR representation for the idiosyncratic component of the monthly variables:

$$\alpha_i(j+1) = \left(\sum_{t=1}^{T} \mathbb{E}_{\theta(j)}\big[\varepsilon_{i,t}\varepsilon_{i,t-1}|\Omega_v\big]\right)\left(\sum_{t=1}^{T} \mathbb{E}_{\theta(j)}\big[(\varepsilon_{i,t-1})^2|\Omega_v\big]\right)^{-1} \qquad i = 1, \dots, n, Q.$$

- The variance in the AR representation for the idiosyncratic component of the monthly variables:

$$\sigma_i^2(j+1) = \frac{1}{T}\left(\sum_{t=1}^{T} \mathbb{E}_{\theta(j)}\big[(\varepsilon_{i,t})^2|\Omega_v\big] - \alpha_i(j+1)\sum_{t=1}^{T} \mathbb{E}_{\theta(j)}\big[\varepsilon_{i,t-1}\varepsilon_{i,t}|\Omega_v\big]\right) \qquad i = 1, \dots, n, Q\,.$$

The conditional expectations (the E-step) in the expressions above are computed using the Kalman smoother on the state-space representation of equation (12) with the previous iteration parameters $\theta(j)$. The initial parameters $\theta(0)$ are obtained on the basis of principal components analysis (in the spirit of the two-step method of Doz, Giannone, and Reichlin, 2006b).

To account for the restrictions imposed by group-specific factors, we would split the parameters in Λ into $\Lambda_N = (\Lambda_{N,G} \quad \Lambda_{N,N})$ and $\Lambda_R = (\Lambda_{R,G} \quad \Lambda_{R,R})$ and obtain the $j+1$-iteration of Λ_N by modifying the formula in equation (13) as

$$\begin{aligned}\text{vec}\big(\Lambda_N(j+1)\big) &= \left(\sum_{t=1}^{T} \mathbb{E}_{\theta(j)}\big[f_t^{G,N} f_t^{G,N'} |\Omega_v\big] \otimes W_t^N\right)^{-1} \\ &\quad \text{vec}\left(\sum_{t=1}^{T} W_t^N x_t^N \mathbb{E}_{\theta(j)}\big[f_t^{G,N'} |\Omega_v\big] - W_t^N \mathbb{E}_{\theta(j)}\big[\varepsilon_t^N f_t^{G,N'} |\Omega_v\big]\right),\end{aligned}$$

where $f_t^{G,N} = (f_t^{G'}\ f_t^{N'})'$, x_t^N and ε_t^N are the subvectors of x_t and ε_t containing only nominal variables and idiosyncratic components, respectively. W_t^N can be obtained from W_t by discarding all the rows and columns corresponding to the real data. The updating formulas for Λ_R can be obtained in an analogous fashion. To obtain restricted versions of A_1 and Q we can use equations (14) and (15) for each of the factors f_t^G, f_t^N, f_t^R, separately.