Specialization and Efficiency with Labor-Market Matching

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Abstract

This paper constructs a labor-market matching model with heterogeneous workers. Due to matching frictions, there may be a mismatch of talents within a production team, forcing a worker to specialize in a task at which she is not talented. We consider a partnership model where production takes place in teams consisting of two workers. We characterize the steady-state of the matching equilibrium. The constrained efficiency of the matching equilibrium depends on the distribution of talents. The constrained-efficient allocation can always be implemented by a type-specific tax. We also examine an alternative model with Diamond-Mortensen-Pissarides type matching between firms and workers.

Keywords: matching, heterogeneity, specialization

JEL Classifications: E24, J41, J64, J65

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1 Introduction

Economists have long emphasized the social benefit of people specializing in what they are good at. In Plato’s Republic, Socrates argues:

it occurred to me that, in the first place, no two of us are born exactly alike. We have different natural aptitudes, which fit us for different jobs. ... Quantity and quality are therefore more easily produced when a man specializes appropriately on a single job for which he is naturally fitted, and neglects all others. (370, pp. 56–57)

In a modern society, a large part of production is conducted in teams. In team production, specialization requires finding a good partner because when someone specializes in one task, she must find someone else to specialize in the other tasks. Very often, finding this partner is not easy, and if one cannot find a good partner then she would have to perform tasks at which she is not good, creating inefficiency in the economy. The more frictions there are in searching for a partner, the more inefficiency will result.

We construct a model of labor-market matching to analyze patterns of specialization. Under matching friction, how often do people specialize in tasks at which they are not good? Do people accept inefficient matches too frequently or too infrequently compared to the socially-desirable level (the constrained-efficient solution)? How is the economy’s aggregate output affected by this friction? How do government policies change the patterns of specialization? What is the resulting income distribution? How can we implement the constrained-efficient solution? Our paper focuses on these questions.

This paper constructs a natural framework for analyzing the “mismatch” of talents (skills) under heterogeneous agents and matching frictions. We consider an economy which consists of workers who search for partners with whom to produce (a partnership model). In our model, the mismatch of talents produces an inefficient outcome through one of two mechanisms. First, a mismatch can lead to unemployment if two workers decide not to form a match because their talents are not compatible with each other. Second, a mismatch can lead to inefficient specialization if two workers decide to form a match even though their talents are not compatible with each other. We find that the
relative population of agents with different talents is an important determinant of the efficiency of the equilibrium.

In labor economics, there is a long tradition of analyzing specialization patterns using the Roy model (Roy, 1951).\(^1\) In this literature, it is commonly assumed that workers are price-takers and choose their tasks (or sectors) based on their talent and the market wage. Sattinger (2003) extends the standard single-agent search model and allows the workers to choose which sector within which to search prior to engaging in a search. In our model, search activity is entirely random, and the workers can decide whether to accept or reject the match after a meeting occurs, depending on the matched partner’s characteristics. Another important difference of our work from Sattinger’s model is that our model is an equilibrium model (while Sattinger analyzes a single worker’s decision problem) and therefore one worker’s decision influences the matching environment of the others.

For the analysis of the constrained-efficient solution, we build on the work of Shimer and Smith (2001a). Our model falls into the ‘submodular’ case of their general framework. In addition to the analysis of the constrained-efficient solution, we provide a comparison between the matching equilibrium and the constrained-efficient solution.

Shimer and Smith (2001b) study the efficiency of the search and matching equilibrium in a model with heterogeneous workers. They consider an environment in which workers are ranked according to their productivity levels. In this environment, when the output of the match between agent \(i\) and agent \(k\), \(f_{ik}\), is larger than \(f_{jk}\) for some agent \(j\), then it is always the case that \(f_{i\ell} > f_{j\ell}\) for any agent \(\ell\). In our framework, it is possible that \(f_{ik} > f_{jk}\) but \(f_{i\ell} < f_{j\ell}\) for \(k \neq \ell\). In other words, we allow for \(k\) to be a better partner for \(i\) than for \(j\), but \(\ell\) to be a better partner for \(j\) than for \(i\). In Shimer and Smith (2001b), inefficiency is caused by differences in workers’ productivity levels, while in our environment inefficiency is related to asymmetry of the talent distribution. We also find, similarly to Shimer and Smith (2001b), that a type-specific tax can implement the constrained-efficient allocation.

We also consider an alternative model where production takes place in firm-worker matches. In this model, the creation of vacancies by the firms influences the frequency of the matching and an aggregate matching function à la Diamond-Mortensen-Pissarides is introduced. We show that many

\[\text{See, for example, the survey by Sattinger (1993).} \]
of the results from the partnership model are extended to this setting.

Marimon and Zilibotti (1999) consider a Diamond-Mortensen-Pissarides type matching model with ex-ante heterogeneous workers. Their model is similar to our alternative model in Section 6, and since they assume that the distribution of the talents is symmetric, their efficiency result is consistent with ours. They consider a lump-sum tax on all workers for financing the unemployment benefit, while we tax only the employed workers—this generates multiple equilibria in our model. Our partnership model emphasizes that the asymmetry in talent distribution is an important determinant of the inefficiency. This aspect is not present in Marimon and Zilibotti (1999), since they assume that the distribution is symmetric.

A recent paper by Berliant, Reed, and Wang (2006) also examines matching of heterogeneous agents in an environment with frictions. Similar to our main model, their model is a partnership model. They assume an increasing-returns-to-scale matching function, while our model features constant-returns-to-scale in matching. Their focus is on the decision to participate in the market, and how the matching externality and the congestion externality affect the total population. Since they assume symmetry in the talent distribution, their analysis abstracts from the additional source of inefficiency that arises from asymmetry in population.

There is a recent literature in monetary economics that analyzes specialization under market frictions. These papers consider search frictions in trading, rather than in forming a long-run relationship as we do.

This paper is organized as follows. The next section sets up the model. In Section 3, we characterize the matching equilibrium when worker types are symmetrically distributed across the population. In Section 4, we show that the matching equilibrium in Section 3 is constrained efficient. In Section 5, we show that when worker types are asymmetric, the matching equilibrium is not constrained-efficient. We also show that the constrained-efficient outcome can be implemented by a type-specific tax. Section 6 endogenizes the meeting probability by considering a Diamond-Mortensen-Pissarides type model. Section 7 concludes.

\[^2\text{See, for example, Kiyotaki and Wright (1993) and Camera, Reed, and Waller (2003).}\]
2 Model

Time is continuous. We consider a production process that requires two tasks, $x$ and $y$. Each task has to be performed by one worker who specializes in that task; therefore, production requires two workers working together (a partnership). There are two types of infinitely-lived workers, $X$-type and $Y$-type. The two types of workers have different talents (skills): an $X$-type worker is talented in performing task $x$ and a $Y$-type worker is talented in performing task $y$. As a consequence, when an $X$-type and a $Y$-type match, each worker can specialize in the task at which she is good, which means that a match between $X$ and $Y$ is very productive. In contrast, when two workers with the same types match, one worker has to specialize in the task at which she is not good, resulting in a lower output. We denote the flow output from the match between $i$-type and $j$-type as $f_{ij}$. Let $f_{XY} = f_{YX} = 2p$ and $f_{XX} = f_{YY} = 2q$, where $p > q > 0$.

There is a continuum of workers with a measure $n_X$ of $X$-type workers and a measure $n_Y$ of $Y$-type workers. The workers’ types are exogenously given and do not change over time. The total population is normalized to one, therefore $n_X + n_Y = 1$. We call the workers who are not matched with anyone “unemployed workers” and the workers who are matched “employed workers.” The share of $X$-type workers in the unemployment pool is denoted as $\pi_X$. The share of $Y$-type workers in the unemployment pool is $\pi_Y = 1 - \pi_X$.

Since one cannot produce alone, an unemployed worker must find a partner in order to start producing. We assume that there are frictions in the matching process. The probability of meeting somebody during a short time period $dt$ is $\alpha dt$. Meeting is entirely random—the conditional probability of meeting with an $X$-type worker is $\pi_X$ and the conditional probability of meeting with a $Y$-type worker is $\pi_Y$.

After meeting, the two unemployed workers decide whether to form a match. If they agree to match, they start producing immediately. If they do not, they stay unemployed. While being matched, these workers do not meet with any other workers. The matched (employed) workers split the present

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3Mukoyama and Sahin (2006) analyze a related model in which workers can choose to acquire either general skills or specialized skills.

4That is, there is no “on-the-job search.”
value of the surplus from production by Nash bargaining. Thus, the matching decision is mutually
agreeable and a match is formed if and only if the total surplus is positive. A matched pair is separated
with the Poisson rate $\delta$ and separated workers go back into the unemployment pool.

An unemployed worker receives $h + b$ amount of goods while unemployed. Here, $h$ is the value of
leisure and home production, and $b$ is unemployment compensation. The government collects tax $\tau$
from the employed workers to finance the unemployment compensation.\(^5\) The following government
budget constraint has to be satisfied:

$$ b(u_X n_X + u_Y n_Y) = \tau((1 - u_X)n_X + (1 - u_Y)n_Y), \quad (1) $$

where $u_i$ is the unemployment rate of type $i$. In our analysis, we consider $b$ to be the policy instrument
and $\tau$ to be set by (1) in equilibrium.\(^6\)

Below, the equilibrium of this model is characterized. We focus on the steady-state equilibrium of
the model.

3 Matching Equilibrium in the Symmetric Case

In this and the next section we consider the case in which $n_X = n_Y = 1/2$. In other words, the two
types are completely symmetric. This symmetry allows us to completely characterize the matching
equilibrium and the constrained-efficient solution in a simple manner. In this section, we focus on the
symmetric pure-strategy equilibrium.

Note that there are two pure strategies from which an agent chooses. The first strategy is to
accept a match with anyone she meets with: we call this “non-selective strategy.” The second is to
accept a match only with the opposite type: we call this “selective strategy.” Therefore, there are two

\(^5\)Note that we do not tax home production ($h$) or unemployment insurance ($b$). It is standard to assume that home
production output is not taxed—see, for example, McGrattan, Rogerson, and Wright (1997). Whether to tax $b$ is more
controversial—in the U.S., unemployment compensation is taxable. However, we believe that our assumption is more
realistic than applying the same lump-sum tax to both employed and unemployed workers, as in Marimon and Zilibotti
(1999). Even with a flat labor tax (more with a progressive labor tax), since the wage is usually larger than $b$, the
amount of tax for an employed worker would be larger than the tax paid by an unemployed worker. (Since we don’t
have an intensive margin in our model, the effect of a flat tax is the same as the lump-sum tax differentiated by the
employment status and productivity.) A more straightforward interpretation is that this is a (partly) progressive labor
income tax with zero tax for income weakly smaller than $b$ and a positive (and constant amount of) tax for income
strictly larger than $b$.

\(^6\)An alternative is to consider $\tau$ to be the policy instrument and $b$ to be set by (1) in equilibrium.
kinds of symmetric pure-strategy equilibria: one in which everyone plays the non-selective strategy, the “non-selective equilibrium,” and one in which everyone plays the selective strategy, the “selective equilibrium.”

### 3.1 Non-selective equilibrium

To examine the conditions for the equilibrium to be non-selective, we first suppose that all of the workers are playing the non-selective strategy. Then we derive the conditions under which the non-selective strategy would be the optimal response.

Let the value of an employed $i$-type worker matched with a $j$-type worker be $W^i_j$ and the value of an unemployed $i$-type worker be $U_i$. From the Nash bargaining,

$$W^i_j - U_i = W^j_i - U_j \quad \text{where } i, j = X, Y$$

has to be satisfied. Here, from symmetry, the Nash bargaining results in splitting the output in half. Thus, a worker who is matched with a different type of worker receives $p - \tau$ and a worker who is matched with a same type worker receives $q - \tau$.

The Bellman equation for an $X$-type worker matched with an $X$-type worker is

$$r W^X_X = q - \tau - \delta (W^X_X - U_X),$$

where $r$ is the discount rate. Similarly, the Bellman equation for an $X$-type worker matched with an $Y$-type worker is

$$r W^Y_X = p - \tau - \delta (W^Y_X - U_X),$$

and the Bellman equation for an unemployed $X$-type worker is

$$r U_X = h + b + \alpha (\pi_X (W^X_X - U_X) + \pi_Y (W^Y_X - U_X)).$$

The Bellman equations for $Y$-type workers are symmetric to these equations. Note that $\pi_X = \pi_Y (= 1/2)$ holds from symmetry.

From the above Bellman equations, we can derive the following equations.

$$\left( r + \delta \right) (W^X_X - U_X) = q - \tau - h - b - \frac{\alpha}{\alpha + r + \delta} \left( \frac{p + q}{2} - \tau - h - b \right), \quad (2)$$
\[(r + \delta)(W^Y_X - U_X) = p - \tau - h - b - \frac{\alpha}{\alpha + r + \delta} \left( \frac{p + q}{2} - \tau - h - b \right).\]

From \(p > q\), we can see that \(W^X_X \geq U_X\) implies \(W^Y_X \geq U_X\).\(^7\) Thus, it is optimal to play the non-selective strategy if and only if \(W^X_X \geq U_X\). From (2), this is equivalent to

\[h + b \leq q - \tau - \frac{\alpha(p - q)}{2(r + \delta)}.\]

(3)

From the symmetry between \(X\) and \(Y\), the same condition applies to \(Y\)-type. Therefore, the non-selective equilibrium exists if and only if (3) holds. Here, \(\tau\) is determined by the government budget constraint (1). In the non-selective equilibrium, the Poisson rate of finding a partner is \(\alpha\) and the Poisson rate of separation is \(\delta\). In the steady-state,

\[\alpha u_i = \delta(1 - u_i), \quad i = X, Y\]

holds. Therefore,

\[u_i = \frac{\delta}{\alpha + \delta}.\]

(4)

From (1) and \(n_X = n_Y = 1/2\), the tax rate in the non-selective equilibrium is

\[\tau_n \equiv \frac{\delta}{\alpha}.\]

With this tax rate, condition (3) becomes

\[h + \left(1 + \frac{\delta}{\alpha}\right)b \leq q - \frac{\alpha(p - q)}{2(r + \delta)}.\]

3.2 Selective equilibrium

Now we consider the selective equilibrium. As in the previous section, we analyze the best response of a worker when everyone else is playing the selective strategy.

The Bellman equation for an \(X\)-type worker matched with a \(Y\)-type worker is

\[rW^Y_X = p - \tau - \delta(W^Y_X - U_X).\]

The Bellman equation for an unemployed \(X\)-type worker is

\[rU_X = h + b + \alpha \pi_Y(W^Y_X - U_X).\]

\(^7\)Note that we need to check only these two inequalities since the options for an unemployed worker who met another worker is either to accept a match with this worker or to stay unemployed.
Note that now only the match with $Y$ is formed. From these equations,

$$(r + \delta)(W_X^Y - U_X) = \frac{2(r + \delta)}{2(r + \delta) + \alpha}(p - \tau - h - b)$$

holds. $W_X^Y > U_X$ holds if and only if $p - \tau - h - b > 0$. We assume that $p$ is large enough so that this inequality is satisfied with the equilibrium tax rate that is derived later. (Otherwise, the selective equilibrium does not exist.)

Now, to check that a deviation is not beneficial, suppose that a pair of workers deviates and accepts the match between $X$-type and $X$-type. In this case, the Bellman equation is

$$rW_X^X = q - \tau - \delta(W_X^X - U_X).$$

This deviation is undesirable if and only if $W_X^X < U_X$. This is equivalent to

$$h + b > q - \tau - \alpha\frac{(p - q)}{2(r + \delta)}.$$  \hspace{1cm} (5)

This is the necessary and sufficient condition for the selective equilibrium to exist. The tax rate can be calculated using the government budget constraint (1). Note that from symmetry, $\pi_X = \pi_Y = 1/2$. The Poisson rate of finding a partner for an $i$-type worker is $\alpha\pi_j = \alpha/2$ and the Poisson rate of separation is $\delta$. In the steady-state,

$$\frac{\alpha}{2}u_i = \delta(1 - u_i), \quad i = X, Y$$

holds. Therefore,

$$u_i = \frac{2\delta}{\alpha + 2\delta}$$  \hspace{1cm} (6)

holds. From (1) and $n_X = n_Y = 1/2$, the tax rate in the selective equilibrium is

$$\tau_s \equiv b \frac{2\delta}{\alpha}.$$ 

Note that the tax rate is twice as high as in the non-selective equilibrium. With this tax rate, the condition (5) becomes

$$h + \left(1 + \frac{2\delta}{\alpha}\right)b > q - \alpha\frac{(p - q)}{2(r + \delta)}.$$ 

Comparing (3) and (5), we can immediately see that when $b = 0$, which implies $\tau = 0$, there always exists a unique equilibrium. This equilibrium is non-selective if (3) holds and selective if the
opposite inequality, (5), holds. The uniqueness is somewhat surprising, provided that the matching process creates externalities (see Example 3 in Burdett and Coles, 1997). As is explained in Section 4.2, the uniqueness here is because two externalities exactly offset each other in the symmetric case considered here.

When $b > 0$, $\tau_s > \tau_n$ holds. The inequalities (3) and (5) imply that when the parameters satisfy

$$q - \tau_n - \frac{\alpha(p - q)}{2(r + \delta)} \geq h + b > q - \tau_s - \frac{\alpha(p - q)}{2(r + \delta)},$$

there are multiple equilibria. This multiple-equilibria situation is created by the active government policy. Through the government’s budget constraint, a worker’s matching decision imposes an externality on the other agents. Intuitively, when more workers choose the selective strategy, the unemployment rate increases, and the tax burden for each employed worker increases. This makes working unattractive, so more people choose the selective strategy.

### 3.3 Comparison of two equilibria

As is demonstrated above, the unemployment rate is lower in the non-selective equilibrium. At the same time, however, the average productivity of a worker in the non-selective equilibrium is lower than in the selective equilibrium, since some workers specialize in the wrong task in the non-selective equilibrium. The selective equilibrium is characterized by a higher unemployment rate and and a higher tax rate than the non-selective equilibrium. Since all workers specialize in the correct task, however, average labor productivity is higher. If the latter effect is strong enough, it is possible that the total output in the selective equilibrium could be higher than the total output in the non-selective equilibrium. Since a higher $b$ makes the selective equilibrium more likely to be realized, it is possible that a higher total output can be achieved through a more generous unemployment compensation.\(^8\)

The steady-state total output (including $h$) in the non-selective equilibrium is

$$O_n = \frac{1}{2} \frac{p}{\alpha + \delta} + \frac{1}{2} \frac{q}{\alpha + \delta} + h \frac{\delta}{\alpha + \delta},$$

\(^8\)Acemoglu and Shimer (1999) suggest a different mechanism that makes the total output higher when the unemployment compensation is more generous. Lagos (2006) constructs a model where policies can affect the productivity through search frictions. In his paper, it is also the case that a more generous unemployment insurance increases average productivity.
since half of the employed worker specialize in the wrong task. In the selective equilibrium, the steady-state total output is

\[ O_s = p \frac{\alpha}{\alpha + 2\delta} + h \frac{2\delta}{\alpha + 2\delta}. \]

Some algebra reveals that \( O_n \geq O_s \) if and only if

\[ h \leq q - \frac{\alpha(p - q)}{2\delta}. \]  

(7)

\( O_s > O_n \) if this inequality does not hold. In this case, a more generous unemployment insurance can increase the total output.\(^9\)

The two equilibria also have different implications for the earnings distribution. In the selective equilibrium, everyone engages in the task at which she is good, and the earnings level is the same across workers. In contrast, in the non-selective equilibrium, some workers engage in ‘wrong’ tasks for them, and this mismatch creates lower earnings. In particular, there are two earning levels in the non-selective equilibrium: half of the employed workers receive \( p \) and half of the employed workers receive \( q \). Note that this earnings difference is observed among the workers with the same characteristics.

Therefore, in the data, this difference in wages would be observed as the within-group wage inequality. In this and the following section, we only consider the symmetric case; therefore, there is no difference in earnings distributions across different types. In Section 5, we will see that the scarcity of skills also generates differences in earnings.

When is the selective equilibrium most likely to be realized? To answer this, we perform a simple comparative statics exercise on the inequalities (3) and (5). It is clear that a high \( b \) (and, therefore, a high \( \tau \)) makes the selective equilibrium more likely. This is because a high \( b \) makes the unemployment state more attractive, and a high \( \tau \) makes the employment state less attractive. An increase in \( h \) has a similar effect as an increase in \( b \). An increase in \( q \) (keeping \( p - q \) constant) makes the non-selective equilibrium more likely to be realized, since a worker compares \((h + b)\) and \((q - \tau)\) when considering

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\(^9\)Since \( r > 0 \), the threshold on \( h \) in (7) is lower than the thresholds in (3) and (5) with \( b = 0 \). This does not imply that the equilibrium solution with \( b = 0 \) is inefficient—in fact, we will show later that if we consider the present-value of output instead of steady-state output, the equilibrium solution with \( b = 0 \) maximizes the present-value of output in the steady-state. This difference corresponds to the difference between the golden rule and the modified golden rule in the neoclassical growth theory. Similarly to the neoclassical growth theory, the two solutions coincide when there is no discounting (i.e. when \( r \to 0 \)).
Table 1: Comparison between the non-selective and selective equilibria for the symmetric case

<table>
<thead>
<tr>
<th></th>
<th>Non-selective equilibrium</th>
<th>Selective equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unemployment rate</td>
<td>$\frac{\alpha + \delta}{\alpha + p + q}$</td>
<td>$\frac{\alpha + 2\delta}{\alpha + \frac{q}{p}}$</td>
</tr>
<tr>
<td>Average labor productivity</td>
<td>$\frac{\alpha (p + q)}{2 (\alpha + q)} + \frac{\delta h}{\alpha + q}$</td>
<td>$\frac{\alpha p}{\alpha + 2\delta} + \frac{2\delta h}{\alpha + 2\delta}$</td>
</tr>
<tr>
<td>Output (including home production)</td>
<td>$\frac{\alpha (p + q)}{2 (\alpha + q)} + \frac{\delta h}{\alpha + q}$</td>
<td>$\frac{\alpha p}{\alpha + 2\delta} + \frac{2\delta h}{\alpha + 2\delta}$</td>
</tr>
<tr>
<td>Wage dispersion (standard deviation)</td>
<td>$\frac{p - q}{2}$</td>
<td>0</td>
</tr>
</tbody>
</table>

Whether to match with a worker of the same type. Similarly, a larger value of $(p - q)$ makes the selective equilibrium more likely, since the reward for finding right match is higher. A higher $\alpha$ also makes the selective equilibrium more likely: since opportunities for a better match come more frequently, the workers are more willing to wait. A higher $r$ makes workers value the present more, which makes the non-selective equilibrium more likely since workers would rather grasp an immediate opportunity rather than wait for a better match. Finally, when $\delta$ is high, matches last for a shorter period on average, so the reward for a better match is smaller; therefore, workers are more willing to accept any match.

The two different equilibria have different implications for labor market outcomes. Table 1 summarizes the differences between the non-selective equilibrium and the selective equilibrium in terms of the unemployment rate, the tax rate, average labor productivity, output, and wage dispersion. The selective equilibrium features a higher unemployment rate, a higher tax rate, a higher average labor productivity, and a lower wage dispersion than the non-selective equilibrium. However, output in the selective equilibrium could be higher or lower than the that of the non-selective equilibrium depending on the parameter values.

In order to examine the empirical relevance of these findings we present some summary statistics for France, Italy, and the U.S. for the 1990–1995 period in Table 2. Comparison of France with the U.S. and then France with Italy provides some useful insights.

As Table 2 shows, the U.S. has a less generous unemployment insurance system than France. According to our analysis, workers in an economy with a less generous unemployment insurance system tend to be less selective. In the data, the U.S. labor market is characterized by a lower unemployment rate, a lower average labor productivity, and a higher wage dispersion than France.

<table>
<thead>
<tr>
<th></th>
<th>France</th>
<th>Italy</th>
<th>U.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benefit replacement rate (1990-1995)</td>
<td>58%</td>
<td>32%</td>
<td>26%</td>
</tr>
<tr>
<td>Average unemployment rate (1990-1995)</td>
<td>10.8%</td>
<td>8.1%</td>
<td>6.4%</td>
</tr>
<tr>
<td>GDP per hour worked (1990)</td>
<td>108.3</td>
<td>99.4</td>
<td>100.0</td>
</tr>
<tr>
<td>GDP per hour worked (1995)</td>
<td>109.9</td>
<td>105.3</td>
<td>100.0</td>
</tr>
<tr>
<td>Wage dispersion (D9/D5)</td>
<td>1.99</td>
<td>1.57</td>
<td>2.07</td>
</tr>
<tr>
<td>Wage dispersion (D5/D1)</td>
<td>1.64</td>
<td>1.67</td>
<td>2.08</td>
</tr>
<tr>
<td>GDP per capita( 1990-1995)</td>
<td>79.3</td>
<td>78.6</td>
<td>100.0</td>
</tr>
</tbody>
</table>
These observations are consistent with the workers in the U.S. economy being less selective. Output, measured as GDP per capita, is higher in the U.S. than in France.

Italy, similarly to the U.S., has lower unemployment insurance benefits than France, which makes its workers likely to be less selective than those in France. Table 2 also shows the labor market statistics in Italy. Italy has a lower unemployment rate and lower average labor productivity than France. Wage dispersion is higher at the lower end of the wage distribution in Italy, while it is lower at the upper end of the distribution. Output, however, is lower than in France.

Although the labor market statistics in Table 2 can be affected by many institutional differences in these economies, the comparisons of France with the U.S. and Italy are mostly consistent with the predictions of our model summarized in Table 1. In particular, the effect of higher unemployment benefits on labor productivity is unambiguously positive. Both Italy and the U.S. have lower average labor productivity than France. However, the effect of unemployment benefits on output is ambiguous. The U.S. has higher GDP per capita than France while Italy’s GDP per capita is lower than France’s.

4 Constrained-Efficient Solution in the Symmetric Case

In this section, we characterize the constrained-efficient outcome in the symmetric environment. By the constrained-efficient outcome we mean the solution that maximizes the present value of total output, taking the search frictions as given. More precisely, we characterize a solution for the social planner’s problem where her objective is to maximize the present value of total output, and her decision is whether to form a match when a meeting occurs.

More formally, the planner maximizes

$$\mathcal{W} = \int_0^\infty e^{-rt} \left( \frac{g_{XX}}{2} m_{XX}(t) + \frac{g_{XY}}{2} m_{XY}(t) + \frac{g_{YX}}{2} m_{YX}(t) + \frac{g_{YY}}{2} m_{YY}(t) + h \right) dt$$

subject to

$$\frac{dm_{ij}(t)}{dt} \leq \alpha \left( \frac{u_i(t)u_j(t)}{u_X(t) + u_Y(t)} - \delta m_{ij}(t) \right), \quad i, j = X, Y,$$

and

$$m_{ij}(t) \geq 0, \quad i, j = X, Y$$
in addition to the initial conditions on \( m_{ij} \). \( m_{ij} \) is the number of \( i \)-type agents matched with \( j \)-type agents. Thus, \( u_i = n_i - m_{iX} - m_{iY} \). We defined \( g_{ij} \equiv f_{ij} - 2h \). As in the previous section, \( f_{XX} = f_{YY} = 2q \) and \( f_{XY} = f_{YX} = 2p \).

In the following, we focus on a steady-state constrained-efficient solution, where \( m_{ij}(t) \) is time invariant. Shimer and Smith’s (2001a) Proposition 2 guarantees that such a solution exists in our setting.

Clearly, the social planner should accept the “mixed” match between a \( X \)-type worker and \( Y \)-type worker. The question is whether she should accept matches between workers of the same type. Thus, the optimum is either “non-selective” (accepting all matches) or “selective” (accepting only mixed matches). In the following, we derive the conditions under which the non-selective solution is optimal and the conditions under which the selective solution is optimal.

4.1 Non-selective solution

Shimer and Smith (2001a) derive the first-order necessary conditions for constrained-efficiency in a framework that encompasses our model as a special case. In our case, these conditions are also sufficient (see Shimer and Smith, 2001a, Proposition 2). The steady-state version of their conditions (in our notation) is

\[
0 = \alpha \frac{u_i u_j}{u_X + u_Y} - \delta m_{ij}, \quad \text{if } S_{ij} > 0, \\
m_{ij} \geq 0, \quad \text{if } S_{ij} = 0
\]

for \( i, j = X, Y \),

\[
\Phi_i = \alpha \{\pi_X [S_{iX} + \pi_X (S_{iX} - S_{XX}) + \pi_Y (S_{iX} - S_{YX})] \\
+ \pi_Y [S_{iY} + \pi_X (S_{iY} - S_{XY}) + \pi_Y (S_{iY} - S_{YY})]\}, \quad i = X, Y; \tag{8}
\]

and

\[
S_{ij} = \max \left\{ 0, \frac{g_{ij} - \Phi_i - \Phi_j}{2(\tau + \delta)} \right\}, \quad i, j = X, Y. \tag{9}
\]

Intuitively, \( S_{ij} \) is (half of) the present-value of creating a match between an \( i \)-type and a \( j \)-type worker. \( \Phi_i \) is the flow value of keeping an \( i \)-type worker unemployed. The terms \( \pi_X (S_{iX} - S_{XX}) + \pi_Y (S_{iX} - S_{YX}) \) and \( \pi_X (S_{iY} - S_{XY}) + \pi_Y (S_{iY} - S_{YY}) \) in (8) represent the externality from creating a match.
For example, when \( i = Y \), the first term is \( \pi_X(S_{YX} - S_{XX}) \). This term captures the effect that, when this \( Y \)-type worker matches with an \( X \)-type worker, the \( X \)-type worker loses the opportunity of (with probability \( \pi_X \)) matching with another \( X \)-type worker (creating the surplus \( S_{XX} \)) by accepting the current match (which creates \( S_{YX} \)).

In a non-selective solution, \( u_X = u_Y = \delta/(\alpha + \delta) \) and \( m_{XX} = m_{XY} = m_{YY} = \alpha/[2(\alpha + \delta)] \) hold. Therefore, \( \pi_X = \pi_Y = 1/2 \). From symmetry, \( \Phi_X = \Phi_Y \). In this case, it follows from (9) that \( S_{XX} = S_{YY} \) and \( S_{XY} = S_{YX} \). Note that (9) implies \( S_{XY} \geq S_{XX} \).

From (8),

\[
\Phi_X = \Phi_Y = \frac{1}{2} \alpha(S_{XX} + S_{XY}).
\]

Now we need to check that in (9),

\[
\frac{g_{ij} - \Phi_i - \Phi_j}{2(r + \delta)} \geq 0.
\]

Since \( S_{XY} \geq S_{XX} \), we need to check this inequality for the same-type match: we will check whether \( g_{XX} \geq 2\Phi_X \). From (8) and (9), this is equivalent to

\[
h \leq q - \frac{\alpha(p - q)}{2(r + \delta)}. \tag{10}
\]

Therefore, the constrained-optimal solution is non-selective when this inequality is satisfied.

### 4.2 Selective solution

Next we consider the selective solution. Note that under this solution, \( u_X \) and \( u_Y \) are higher, but \( \pi_X = \pi_Y = 1/2 \) still holds. This time, we have to verify that

\[
\frac{g_{XX} - \Phi_X - \Phi_X}{2(r + \delta)} < 0.
\]

Suppose that this holds. Then, \( S_{XX} = S_{YY} = 0 \), therefore (8) becomes

\[
\Phi_X = \Phi_Y = \frac{1}{2} \alpha S_{XY}.
\]

From (9),

\[
S_{XY} = \frac{2(p - h) - \Phi_X - \Phi_Y}{2(r + \delta)}.
\]
From these two equations,

$$S_{XY} = \frac{p - h}{r + \delta + \alpha/2}$$

holds. Therefore, $g_{XX} - 2\Phi_X < 0$ is equivalent to

$$h > q - \frac{\alpha(p - q)}{2(r + \delta)}. \quad (11)$$

Clearly, conditions (10) and (11) are identical to conditions (3) and (5) with $b = 0$ (therefore $\tau = 0$).

Thus, the equilibrium characterized in Section 3 is constrained efficient when $b = 0$, when we focus on the steady-state. This is a surprising result since the matching process creates externalities. The intuition for this result is that two types of externalities exactly cancel out. When an $X$-type worker goes out of the unemployment pool, she imposes a positive externality on other $X$-type workers, since it reduces their chance of matching with another $X$-type worker. At the same time, she imposes a negative externality on $Y$-type workers by decreasing their chance of matching with an $X$-type worker. When the population of types is symmetric, these two effects cancel out exactly. When this is not the case, however, the equilibrium is no longer constrained-efficient. We will see this in the next section.

5 Inefficiency in the Asymmetric Case

Now we consider the case where $n_i \neq 1/2$. Although this case is substantially more complex, it is still possible to show that the equilibrium with $b = 0$ is not efficient by focusing on the non-selective equilibrium.\footnote{Characterizing other types of equilibria is substantially more difficult and beyond the scope of this paper.}

5.1 Equilibrium

First we derive the necessary and sufficient condition for the existence of a non-selective equilibrium. We set $b = 0$. Note that in the non-selective equilibrium, $\pi_X = n_X$ and $\pi_Y = n_Y$. We only consider the case in which $n_X < n_Y$.\footnote{The solution is the same, but with reversed notation for $n_Y < n_X$.}

The Bellman equations for an $X$-type worker are

$$rW_X^X = q - \delta(W_X^X - U_X),$$

\footnotesize

\[10]\text{Characterizing other types of equilibria is substantially more difficult and beyond the scope of this paper.}\]

\[11]\text{The solution is the same, but with reversed notation for } n_Y < n_X.\]
\[ rW_X^X = z - \delta(W_X^X - U_X), \]
\[ rU_X = h + \alpha (\pi_X(W_X^X - U_X) + \pi_Y(W_Y^X - U_X)). \]

The second equation is different from the previous section. Here, \( z \) is the flow wage for an \( X \)-type worker matched with a \( Y \)-type worker. Now \( \pi_X \) and \( \pi_Y \) will be different, and \( z \) may be different from \( p \) (but will still be determined by Nash bargaining). Solving these three equations yields

\[ (r + \delta)(W_X^X - U_X) = q - h - \frac{\alpha}{\alpha + r + \delta} (\pi_X q + (1 - \pi_X)z - h) \] (12)
and
\[ (r + \delta)(W_Y^X - U_X) = z - h - \frac{\alpha}{\alpha + r + \delta} (\pi_X q + (1 - \pi_X)z - h). \] (13)

Similarly, for a \( Y \)-type worker,

\[ (r + \delta)(W_Y^Y - U_Y) = q - h - \frac{\alpha}{\alpha + r + \delta} ((1 - \pi_X)q + \pi_X z - h) \] (14)
and
\[ (r + \delta)(W_Y^X - U_Y) = 2p - z - h - \frac{\alpha}{\alpha + r + \delta} ((1 - \pi_X)q + \pi_X z - h). \] (15)

It turns out that \( z > q \) and \( 2p - z > q \) (we will check this later). Thus, for the non-selective equilibrium to exist, it is sufficient to check that \( W_X^X \geq U_X \) and \( W_Y^Y \geq U_Y \) hold.

The value of \( z \) is determined by Nash bargaining. That is,

\[ W_X^Y - U_X = W_Y^X - U_Y \]
is satisfied. From (13) and (15),

\[ z = q + \frac{2(r + \delta + (1 - \pi_X)\alpha)(p - q)}{2r + 2\delta + \alpha}, \] (16)
and
\[ 2p - z = q + \frac{2(r + \delta + \pi_X \alpha)(p - q)}{2r + 2\delta + \alpha}. \]

The right-hand-side of both equations is positive; therefore we have confirmed that \( z > q \) and \( 2p - z > q \). The value of \( z \) is decreasing in \( \pi_X \). That is, when skill \( x \) is more scarce, \( X \)-type workers receive a greater reward. Since we are looking at the case where \( \pi_X = n_X < 1/2, z > 2p - z \) holds.
Now we derive the conditions for $W_X^X \geq U_X$ and $W_Y^Y \geq U_Y$ from (12) and (14). The conditions are

$$h \leq h_X \equiv q - \frac{\alpha}{r + \delta}(1 - \pi_X)(z - q)$$

and

$$h \leq h_Y \equiv q - \frac{\alpha}{r + \delta}\pi_X(2p - z - q).$$

Since $z > 2p - z$, $h_X < h_Y$ holds. Therefore, the necessary and sufficient condition for the existence of the non-selective equilibrium is, using (16),

$$h \leq h_{eq} \equiv q - \frac{\alpha(r + \delta + (1 - \pi_X)\alpha)}{(r + \delta)(2r + 2\delta + \alpha)}(p - q),$$

where $h_{eq}$ is the maximum value of $h$ for which a non-selective equilibrium exists.

**5.2 Constrained-efficient solution**

Now we solve for the conditions under which the constrained-efficient solution is non-selective. From (8), noting that $S_{XY} = S_{YX}$,

$$\Phi_X = \alpha[\pi_X(2 - \pi_X)S_{XX} + 2(1 - \pi_X)^2S_{XY} - (1 - \pi_X)^2S_{YY}]$$

and

$$\Phi_Y = \alpha[-\pi_X^2S_{XX} + 2\pi_X^2S_{XY} + (1 - \pi_X)(1 + \pi_X)S_{YY}].$$

From these two equations and (9),

$$(r + \delta)S_{XX} = \frac{g_{XX}}{2} - \alpha[\pi_X(2 - \pi_X)S_{XX} + 2(1 - \pi_X)^2S_{XY} - (1 - \pi_X)^2S_{YY}],$$

$$(r + \delta)S_{YY} = \frac{g_{YY}}{2} - \alpha[-\pi_X^2S_{XX} + 2\pi_X^2S_{XY} + (1 - \pi_X)(1 + \pi_X)S_{YY}],$$

and

$$(r + \delta)S_{XY} = \frac{g_{XY}}{2} - \alpha[\pi_X(1 - \pi_X)S_{XX} + (\pi_X^2 + (1 - \pi_X)^2)S_{XY} + \pi_X(1 - \pi_X)S_{YY}].$$

To simplify these three equations, we use the relationship (which can be verified from (18), (19), (20), and $g_{XX} = g_{XY}$)

$$2S_{XY} - (S_{XX} - S_{YY}) = \frac{g_{XY} - g_{XX}}{r + \delta}.$$
Then, (18), (19), and (20) can be rewritten as

\[(r + \delta + \alpha)S_{XX} = \frac{g_{XX}}{2} - \frac{\alpha(1 - \pi_X)^2}{r + \delta} (g_{XY} - g_{XX}), \tag{21}\]

\[(r + \delta + \alpha)S_{YY} = \frac{g_{XX}}{2} - \frac{\alpha\pi_X^2}{r + \delta} (g_{XY} - g_{XX}),\]

and

\[(r + \delta + \alpha)S_{XY} = \frac{g_{XY}}{2} - \frac{\alpha\pi_X(1 - \pi_X)}{r + \delta} (g_{XY} - g_{XX}).\]

Thus, when \(\pi_X < 1/2, S_{XX} < S_{YY}\) and \(S_{XX} < S_{XY}\) hold. Therefore, the non-selective solution is optimal if and only if \(S_{XX} \geq 0\). From (21), this is the case when

\[h \leq h_{ce} \equiv q - \frac{2\alpha(1 - \pi_X)^2}{r + \delta} (p - q) \tag{22}\]

where \(h_{ce}\) is the maximum value of \(h\) for which the non-selective solution is constrained efficient.

Comparing (17) and (22), it is clear that the equilibrium is not necessarily constrained efficient. It can easily be shown that \(h_{eq} = h_{ce}\) only when \(\pi_X = 1/2\). When \(\pi_X < 1/2\), it is always the case that \(h_{eq} > h_{ce}\). Therefore, when \(h\) is between \(h_{eq}\) and \(h_{ce}\) the equilibrium is non-selective, but the non-selective solution is not constrained-efficient. In this case, in equilibrium \(X\)-type workers accept matches too often. This is because an individual worker does not take the externalities into account. The negative externality of matching (imposed on the opposite type) overwhelms the positive externality (imposed on the same type), since \(X\)-type workers are the scarce type. It is better to keep the \(X\)-type worker unemployed because they face a high chance of matching with a \(Y\)-type worker, since there are many \(Y\)-type workers in the unemployment pool.

### 5.3 Decentralizing the constrained-efficient outcome

Similarly to Shimer and Smith (2001b), we can decentralize the constrained-efficient outcome when a type-specific tax is available.\(^{12}\) To see this, set \(b = 0\) (thus \(\tau = 0\)) and instead charge a tax \(t_i\) to an \(i\)-type unemployed agent.

Denote

\[\tilde{S}_{ij} \equiv \max \left\{0, W^*_j - U_i\right\}, \quad i, j = X, Y. \tag{23}\]

\(^{12}\)Note that our setting is different from Shimer and Smith (2001b) in the sense that their agents match ‘once-and-for-all’ (i.e. the agents receive output and exit from the market once they match).
Then, we can write the value functions in equilibrium as

$$rW^j_i = \tilde{z}_{ij} - \delta(W^j_i - U_i), \quad i, j = X, Y, \quad (24)$$

and

$$rU_i = h - t_i + \alpha(\pi_X \tilde{S}_{iX} + \pi_Y \tilde{S}_{iY}), \quad i = X, Y. \quad (25)$$

Here, $\tilde{z}_{ij}$ is the flow wage for an $i$-type worker matched with a $j$-type worker, and is determined by the Nash bargaining:

$$W^j_i - U_i = W^i_j - U_j, \quad i, j = X, Y. \quad (26)$$

We can calculate $(W^j_i - U_i)$ and $(W^i_j - U_j)$ by subtracting (25) from (24). For example,

$$(r + \delta)(W^j_i - U_i) = \tilde{z}_{ij} - h + t_i - \alpha(\pi_X \tilde{S}_{iX} + \pi_Y \tilde{S}_{iY}),$$

and calculate $\tilde{z}_{ij}$ by (26) and the fact that $\tilde{z}_{ij} = f_{ij} - \tilde{z}_{ji}$. Plugging this $\tilde{z}_{ij}$ into (27), we obtain

$$(r + \delta)(W^j_i - U_i) = g_{ij} - \frac{\tilde{\Phi}_i - \tilde{\Phi}_j}{2(r + \delta)},$$

where we used $g_{ij} = f_{ij} - 2h$. From (23), $\tilde{S}_{ij}$ can be expressed as a solution of the system of equations

$$\tilde{S}_{ij} = \max \left\{ 0, g_{ij} - \frac{\tilde{\Phi}_i - \tilde{\Phi}_j}{2(r + \delta)} \right\}, \quad i, j = X, Y. \quad (28)$$

Let the amount of the type-specific tax be

$$t_i = -\alpha \left\{ \pi_X [\pi_X (\tilde{S}_{iX} - \tilde{S}_{XX}) + \pi_Y (\tilde{S}_{iX} - \tilde{S}_{XY})] + \pi_Y [\pi_X (\tilde{S}_{iY} - \tilde{S}_{XY}) + \pi_Y (\tilde{S}_{iY} - \tilde{S}_{YY})] \right\},$$

for $i = X, Y$. Then, from (8), (9), (28), and (29), $\Phi_i = \tilde{\Phi}_i$ and $S_{ij} = \tilde{S}_{ij}$ for $i, j = X, Y$. Therefore, the matching equilibrium and the constrained-efficient solution coincide. This implies that we can
decentralize the constrained-efficient outcome when a type-specific tax is available. The amount of the tax that an unemployed $X$-type worker pays is

$$t_X = -\alpha \{ \pi_X [\tilde{S}_{XX} - \tilde{S}_{XY}] + \pi_Y [\tilde{S}_{XY} - \tilde{S}_{YY}] \},$$

and the amount of the tax that an unemployed $Y$-type worker pays is

$$t_Y = -\alpha \{ \pi_X [\tilde{S}_{YX} - \tilde{S}_{XX}] + \pi_Y [\tilde{S}_{YY} - \tilde{S}_{XY}] \},$$

which implies that

$$\pi_X t_X + \pi_Y t_Y = 0. \quad (30)$$

Equation (30) implies that $t_X$ and $t_Y$ have opposite signs. When one type is taxed, the other type is subsidized. Since $\pi_i = u_i / (u_X + u_Y)$ for $i = X, Y$, (30) implies that $u_X t_X + u_Y t_Y = 0$. Since $(u_X t_X + u_Y t_Y)$ is the total amount of tax, this type-specific tax is budget balancing.

6 Diamond-Mortensen-Pissarides Matching Model

In this section, we depart from the partnership model and consider a model where production takes place in firm-worker pairs. We follow the standard Diamond-Mortensen-Pissarides model (see, for example, Pissarides, 2000) in assuming an aggregate matching function, vacancy-posting decision by firms, and wage determination by Nash bargaining. As in the partnership model, there are two types of workers: $X$-type and $Y$-type. The number of $i$-type unemployed workers is denoted as $u_i$. Instead of matching with other workers, a worker matches with a job. There are two types of jobs: $x$-type and $y$-type. The firm decides which type of job to create when it posts the vacancy. A vacancy posting requires $\xi$ amount of flow cost. Let the number of vacancies posted for $i$-type be $v_i$. When an $i$-type worker matches with a $j$-type job, the match can produce $P_{ij}$ amount of output. Assume that $P_{X}^x = P_{Y}^y = p$ and $P_{X}^y = P_{Y}^x = q$, where $p > q$.

The meeting between a vacancy and an unemployed worker is dictated by a constant-returns-to-scale matching function. Let the vacancy-unemployment ratio $\theta$ be defined by

$$\theta \equiv \frac{v_x + v_y}{u_X + u_Y}.$$
and the probability of a vacancy finding an unemployed worker be $\alpha(\theta)$. From the constant-returns-to-scale assumption, $\theta \alpha(\theta)$ is the probability of an unemployed worker finding a vacancy. A commonly-used example of the $\alpha(\cdot)$ function is (from the Cobb-Douglas matching function):

$$
\alpha(\theta) = \alpha_0 \theta^{-\eta},
$$

(31)

where $\alpha_0 > 0$ and $\eta \in (0, 1)$ are constants. The meeting process is random, and the probability of meeting with a specific type depends on the relative population of each type. The probability of a vacancy meeting with an $i$-type worker is $\pi_i \alpha(\theta)$, where $\pi_i \equiv u_i/(u_i + u_Y)$. Similarly, the probability of a worker meeting with an $i$-type vacancy is $\lambda_i \theta \alpha(\theta)$, where $\lambda_i \equiv v_i/(v_x + v_y)$. The separation is exogenous at the rate $\delta$, and the discount rate is $r$.

### 6.1 Non-selective equilibrium

Let $W_i^j$ be the value of an employed $i$-type worker matched with a $j$-type job, $U_i$ be the value of an unemployed an $i$-type worker, $J_i^j$ be the value of a $j$-type job matched with $i$-type worker, and $V_j$ be the value of a $j$-type vacancy. The Bellman equations in a non-selective equilibrium are (for $i = X, Y$ and $j = x, y$):

$$
\begin{align*}
    rW_i^j &= w_i^j - \tau - \delta(W_i^j - U_i), \\
    rU_i &= h + b + \lambda_x \theta \alpha(\theta)(W_i^x - U_i) + \lambda_y \theta \alpha(\theta)(W_i^y - U_i), \\
    rJ_i^j &= P_i^j - w_i^j - \delta(J_i^j - V_j), \\
    rV_j &= -\xi + \pi_X \alpha(\theta)(J_i^X - V_j) + \pi_Y \alpha(\theta)(J_i^Y - V_j),
\end{align*}
$$

and

$$
W_i^j - U_i = \gamma(J_i^j + W_i^j - V_j - U_i),
$$

where $w_i^j$ is the wage for an $i$-type worker matched with a $j$-type job. The wage is determined by the Nash bargaining as in standard Diamond-Mortensen-Pissarides models. It satisfies

$$
W_i^j - U_i = \gamma(J_i^j + W_i^j - V_j - U_i),
$$

where $\gamma \in (0, 1)$ is a constant parameter. As in the standard model, we assume a free entry to the vacancy creation, so $V_j = 0$ for $j = x, y$ in equilibrium.
Below, we consider a case where the population of $X$-type workers and $Y$-type workers are the same. In the steady-state of the symmetric equilibrium, $W_X^X = W_Y^Y$, $W_X^Y = W_Y^X$, $J_X^X = J_Y^Y$, $J_X^Y = J_Y^X$, $U_X = U_Y$, $V_x = V_y$, $w_X^X = w_Y^Y$, $w_X^Y = w_Y^X$, $\lambda_x = \lambda_y = 1/2$, $u_X = u_Y$, $v_x = v_y$, and $\pi_X = \pi_Y = 1/2$ hold. We can exploit the symmetry by defining the following variables:

\[
\bar{W} \equiv \frac{(W_{ji} + W_{ij})}{2}, \quad \bar{P} \equiv \frac{(P_{ji} + P_{ij})}{2}, \quad \bar{J} \equiv \frac{(J_{ji} + J_{ij})}{2}, \quad \bar{w} \equiv \frac{(w_{ji} + w_{ij})}{2},
\]

where $(i, j) = (X, y)$ or $(Y, x)$. Then the above equations can be rewritten as (the subscript on $U$ and $V$ are dropped because of the symmetry):

\[
\begin{align*}
    r\bar{W} &= \bar{w} - \tau - \delta(\bar{W} - U), \\
    rU &= h + b + \theta\alpha(\theta)(\bar{W} - U), \\
    r\bar{J} &= \bar{P} - \bar{w} - \delta(\bar{J} - V), \\
    rV &= -\xi + \alpha(\theta)(\bar{J} - V),
\end{align*}
\]

and

\[
\bar{W} - U = \gamma(\bar{J} + \bar{W} - V - U).
\]

The structure of this transformed problem (five equations and five unknowns: $\bar{W}$, $\bar{J}$, $U$, $\bar{w}$, and $\theta$; $V$ is always zero) is exactly the same as the standard model with homogeneous types of workers and jobs. Therefore, the solution of the standard problem can be used for the determination of $\theta$ (see equation (1.24) of Pissarides, 2000):

\[
(1 - \gamma)(\bar{P} - \tau + h - b) - \frac{r + \delta + \gamma\theta\alpha(\theta)}{\alpha(\theta)}\xi = 0. \tag{32}
\]

Similarly, it can be shown that the wage is given by (see equation (1.23) of Pissarides, 2000)

\[
\bar{w} = (1 - \gamma)(h + b + \tau) + \gamma\bar{P} + \gamma\theta\xi. \tag{33}
\]

The individual values can be solved by going back to the original system of equations. For example, it can be shown that

\[
w_i^i - w_i^j = \gamma(p - q), \tag{34}
\]

for $(i, j) = (X, y)$ or $(Y, x)$.

Finally, note that the government budget constraint implies that

\[
\tau = b \frac{\delta}{\alpha(\theta)}. \tag{35}
\]
6.2 Selective equilibrium

In a selective equilibrium, X-type workers only match with x-type jobs, and Y-type workers only match with y-type jobs. In the symmetric steady-state equilibrium, only half of the meetings materialize into an actual match. Thus, the Bellman equations are (subscripts are dropped because of the symmetry):

\[ rW = w - \tau - \delta(W - U), \]  
\[ rU = h + b + \frac{1}{2}\theta\alpha(\theta)(W - U), \]  
\[ rJ = p - w - \delta(J - V), \]

and

\[ rV = -\xi + \frac{1}{2}\alpha(\theta)(J - V). \]

The wage satisfies

\[ W - U = \gamma(J + W - V - U). \]

Again, this is similar to the standard model, and it can easily be shown that \( \theta \) solves

\[(1 - \gamma)(p - \tau - h - b) - \frac{2(r + \delta) + \gamma\theta\alpha(\theta)}{\alpha(\theta)}\xi = 0 \] (38)

and wages are

\[ w^x_X = w^y_Y = (1 - \gamma)(h + b + \tau) + \gamma p + \gamma\theta\xi. \] (39)

From the government budget constraint, the tax rate is

\[ \tau = b\frac{2\delta}{\alpha(\theta)}. \] (40)

6.3 Efficiency

To examine the constrained-efficient solution, we limit our attention to the stationary symmetric solution. The choice of the firm and the worker who meet is restricted to either “match with all types” or “accept only the good match, that is, within the same types (x and X, y and Y)”. Since we only consider the steady-state with the symmetric choice, the probability of a meeting being a good match is always 1/2. Therefore, the outcome of this model is the same as the model where all jobs
and workers are ex-ante homogeneous and the quality of the match is drawn after the worker and
the firm meet. This type of model is analyzed in Chapter 6 of Pissarides (2000) as “stochastic job
matchings.”

Pissarides (2000, Section 8.4) shows that the constrained efficiency of the stochastic job matching
model is achieved by the Hosios (1990) condition:

\[ \gamma = \tilde{\eta}(\theta), \tag{41} \]

where \( \tilde{\eta}(\theta) \) is the absolute value of the elasticity of the \( \alpha(\theta) \) function. When \( \alpha(\theta) \) takes the form of
(31), \( \tilde{\eta}(\theta) \) is the constant parameter \( \eta \). It follows that in our model (with symmetry), the constrained
efficiency is achieved by the market equilibrium with no tax as long as (41) is satisfied.

6.4 Possibility of multiple equilibria with tax

Similarly to the partnership model, when \( b \) is positive (therefore \( \tau \) is positive), there is a possibility of
multiple equilibria. To show this, we first describe the conditions for the existence of each equilibria
(non-selective and selective) in the current setting.

To show that a non-selective equilibrium exists, it is sufficient to show that when everyone else is
non-selective, being selective is not beneficial, i.e. \( J_y^X > 0 \). This condition implies that

\[ J_y^X = \frac{q - w_y^X}{r + \delta} > 0, \]

where the individual wage can be solved by (33) and (34).

To show that selective equilibrium exists, it is sufficient to show that when everyone else is selective,
being non-selective is not beneficial. Formally, the deviation is not beneficial if \( W_y^X + J_y^X - U_X < 0 \).
The Bellman equations for a deviating match with an \( X \)-type worker and a \( y \)-type job is:

\[ rW_y^X = w_y^X - \tau - \delta(W_y^X - U_X), \tag{42} \]
\[ rJ_y^X = q - w_y^X - \delta J_y^X. \tag{43} \]

\[ ^{13} \text{Pissarides (2000) shows that the equilibrium solution is unique. Thus, it follows that also in our model, the} \]
\[ \text{symmetric steady-state equilibrium is unique when there is no tax.} \]
Here, $U_X$ is a part of the solution in Section 6.2. From (37), (42), and (43),

$$(r + \delta)(W_X^y + J_y^X - U_X) = q - \tau - h - b - \frac{1}{2}\theta\alpha(\theta)(W_X^y - U_X).$$

Using (36) and (37), this can be simplified as

$$(r + \delta)(W_X^y + J_y^X - U_X) = q - \tau - h - b - \frac{1}{2}\theta\alpha(\theta)\frac{w_X^y - \tau - h - b}{r + \delta + \theta\alpha(\theta)/2}. \quad (44)$$

All of the variables on the right-hand side can be solved using (38), (39), and (40). Thus we can evaluate the sign of $W_X^y + J_y^X - U_X$ by (44).

We illustrate the possibility of multiple equilibria with a numerical example. Multiple equilibria exist when $J_y^X > 0$ and $W_X^y + J_y^X - U_X < 0$ are both satisfied. We set $r = 0.005$, $\delta = 0.05$, $\eta = 0.72$, $a_0 = 0.675$, $b = 0$, $\gamma = 0.72$, $\xi = 1.5$, $p = 7.0$, and $q = 5.9$.\footnote{Note that here, the Hosios (1990) condition is satisfied. Thus, the equilibrium with $b = 0$ is constrained-efficient. With this value of $\xi$, $\theta = 1$ with the (selective) equilibrium when $b = 2.0$.} It turns out that for $b < 0.535$, $J_y^X > 0$ when everyone is behaving non-selectively, and a non-selective equilibrium exists. When $b > 0.492$, $W_X^y + J_y^X - U_X < 0$ when everyone is behaving selectively, and a selective equilibrium exists. Therefore, there are multiple equilibria for $b \in (0.492, 0.535)$. Figure 1 depicts $J_y^X/(1 - \gamma)$ and $W_X^y + J_y^X - U_X$ to show this possibility of multiple equilibria.

### 6.5 Endogenous probability of meeting

In the partnership model, the probability of meeting is exogenous. In particular, the probability of meeting somebody during a short time period $dt$ is $\alpha dt$. In the Diamond-Mortensen-Pissarides matching model, the probability of meeting is endogenous, i.e. depends on the reaction of the firms through the vacancy-unemployment ratio. Specifically, the probability that an unemployed worker finds a vacancy is $\theta\alpha(\theta)dt$. The Diamond-Mortensen-Pissarides model has this additional channel through which changes in the economic environment affect the model’s outcomes. For example, equations (32) and (38) show that both in the non-selective and selective equilibria, the value of $\theta$ declines with $b$. Higher $b$ raises the wage rate, which reduces the profitability of posting a vacancy. Therefore, a worker’s job-finding probability, $\theta\alpha(\theta)$, declines with $b$. This tends to make the worker less selective. This effect was absent in the partnership model, since the meeting probability $\alpha$ was
Figure 1: The possibility of multiple equilibria
assumed to be constant. However there is also the direct effect of the increase in \( b \) and \( \tau \) which makes the worker more selective as \( b \) increases. In our numerical example, the direct effect of the increase in \( b \) and \( \tau \) is stronger, and the worker tends to become more selective as \( b \) increases. Even though the meeting probability is endogenous, we still obtain a similar result to the partnership model in our numerical example.

### 6.6 Investment by workers

Suppose that, instead of assuming the workers’ type distribution to be fixed, a worker can invest in one of two skills at time zero: \( X \) and \( Y \). Then, even when there is no tax, serious coordination problems may arise. When the investment cost is zero (everyone can choose one of the skills for free at time zero), both “all workers invest in \( X \) skill and all vacancies are of \( x \)-type” and “all workers invest in \( Y \) skill and all vacancies are of \( y \)-type” are equilibria. Moreover, “half of the workers invest in \( X \) skill and half of the workers invest in \( Y \) skill” is also an equilibrium. This is because, given this type distribution of workers, half of the vacancies are created as \( x \)-type and half of the vacancies are created as \( y \)-type, and workers are indifferent between both skills. This equilibrium is not robust to a small perturbation, however. Suppose, for example, there are more \( X \)-type workers than \( Y \)-type workers. Then, the firm has more incentive to post \( z \)-type vacancies, since the probability of meeting with \( X \)-type worker is higher. This, in turn, increases the incentive for a worker to invest in \( X \)-type skill. Acemoglu (1996) illustrates a similar feedback process as a microfoundation of increasing returns to human capital investment.

### 7 Conclusion

In this paper, we consider the patterns of specialization in a labor market with matching frictions. We construct a two-type partnership model and characterize both the matching equilibrium and the constrained-efficient solution. The matching process creates externalities in two directions. When the population of workers is symmetric across types, the two externalities offset each other exactly, and the matching equilibrium with no unemployment compensation is constrained efficient. When the population is not symmetric, the two externalities do not offset each other. In the non-selective
equilibrium, workers with the scarce skill accept matches too often compared to the constrained-efficient outcome, but the constrained-efficient allocation can be implemented with a type-specific tax on the unemployed workers.

We also extend our model to incorporate the Diamond-Mortensen-Pissarides type matching mechanism. This extension allows us to endogenize the probability of meeting. We show that the implications of this model are in line with those of the partnership model.

We analyze how matching friction, distribution of skills, and unemployment compensation policies affect unemployment, output, and income distribution. Quantitatively evaluating these effects is an important future research topic.
References


