Menu Costs at Work: Restaurant Prices and the Introduction of the Euro
Mathematical Appendix*

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This appendix contains the mathematical details underlying the results in the main text. The results are explained in the order that they appear in the text.

A Section III: Menu cost distribution

In the main text, we introduced the menu cost distribution function in the form it is used by DKW. That is, we used

\[
G_t(\xi) = \begin{cases} 
0 & \text{for } \xi < 0 \\
\gamma_{1t} + \gamma_{2t} \tan(\gamma_{3t}\xi + \gamma_{4t}) & \text{for } 0 \leq \xi < \xi_t \\
1 & \text{for } \xi \geq \xi_t 
\end{cases} \tag{1}
\]

In order to see how this distribution nests the menu cost distribution functions underlying the Calvo and Calvo-Taylor hybrid models, i.e. our models (i) and (ii), it turns out to be worthwhile to consider a particular reparameterization. Namely we reparametrize \(G_t\) in terms of \(\xi_t, \bar{\phi},\) and \(\phi\) as

\[
\begin{align*}
\gamma_{1t} &= -\gamma_{2t} \tan \left(\left(\bar{\phi} - 0.5\right) \pi\right) \\
\gamma_{2t} &= 1/ \left[ \tan \left(\left(\bar{\phi} - 0.5\right) \pi\right) + \tan \left(\left(\phi - 0.5\right) \pi\right) \right] \\
\gamma_{3t} &= (\bar{\phi} - \phi) \pi/\xi_t \\
\gamma_{4t} &= (\phi - 0.5) \pi
\end{align*}
\]

(2) (3) (4) (5)

where \(0 < \bar{\phi} < \phi < 1.\)

The intuition behind our parameterization is most easily explained using figure 1. The DKW distribution function is a transformation of the tangent function on \([\left(\bar{\phi} - 0.5\right) \pi, \left(\bar{\phi} - 0.5\right) \pi].\) This

*The views expressed in this paper solely reflect those of the authors and not necessarily those of the Federal Reserve Bank of New York or of the Federal Reserve System.
is the interval that is depicted in the figure by the arrow denoted by (i). The mapping of this interval on the support of the distribution function, i.e., \([0, \bar{x}_t]\), determines the value of the parameters \(\gamma_{3t}\) and \(\gamma_{4t}\). Arrow (ii) depicts how \(\gamma_{1t}\) is determined. That is, \(\gamma_{1t}\) is such that the value of the distribution function at the minimum of its support equals zero. To assure that the menu cost assumes a value within its support with probability one, \(\gamma_{2t}\) is chosen such that arrow (iii) is of length one.

The menu cost distribution function that underlies both the Calvo model and the Calvo-Taylor hybrid is

\[
G^*_t(\xi) = \begin{cases} 
0 & \text{for } \xi < 0 \\
\alpha & \text{for } 0 \leq \xi < \xi_t \\
1 & \text{for } \xi_t \leq \xi
\end{cases}
\]  

where for the Calvo model \(\xi_t\) is infinite at all \(t\), except \(T\) when the Euro is introduced, and for the Calvo-Taylor hybrid it is finite.

\(G^*_t(\cdot)\) is a limiting case of \(G_t(\cdot)\) in the sense that \(G^*_t(\cdot) \to G_t(\cdot)\) when \(\phi \to 0\) and \(\bar{\phi}\) is chosen according to

\[
\bar{\phi} = 0.5 + \frac{1}{\pi} \arctan \left( -\left( \frac{1-\alpha}{\alpha} \right) \tan \left( (\phi - 0.5) \pi \right) \right) 
\]  

**B Section IV: Equilibrium inflation dynamics**

For the derivation of the equilibrium inflation dynamics and equilibrium price adjustment behavior it turns out to be convenient to write the problem in terms of variables that are constant along the balanced growth path. For this purpose, we define

\[
\pi^S_{j,t} = \frac{\Pi^S_{j,t}}{[1+\pi](1+g)}^{t}, \quad v^S_{j,t} = \frac{V^S_{j,t}}{[1+\pi](1+g)^{t+1}},
\]

\[
p^*_{S,t} = \frac{P^*_{S,t}}{(1+\pi)^{t}}, \quad \text{and} \quad p_{i,t} = \frac{P_{i,t}}{(1+\pi)^{t}}
\]

for \(s \in \{D, E\}\) and \(j = 0, \ldots, \infty\).

Given these definitions, we can write the detrended profits as

\[
\pi^S_{j,t} = \left( \frac{p^*_{S,t-j}}{(1+\pi)^{t}} - \psi \right) \left( \frac{1}{(1+\pi)^{t}} \right) \left( \frac{p^*_{S,t-j}}{p_{it}} \right)^{-\eta} \left( \frac{p_{it}}{p} \right)^{-\eta} y
\]  

where again \(s \in \{D, E\}\) and \(j = 0, \ldots, \infty\).

Furthermore, we can write the functional equations for the detrended value function as

\[
v^D_{0,t} = \max_{v^D_{0,t}} \{ \pi^D_{0,t} + \lambda E_t \max \{ v^D_{0,t+1} - w \xi, v^D_{0,t+1} + w \xi - w, v^D_{1,t+1} \} \}
\]

\[
v^E_{0,t} = \max_{v^E_{0,t}} \{ \pi^E_{0,t} + \lambda E_t \max \{ v^E_{0,t+1} - w \xi, v^E_{0,t+1} - w \xi - w, v^E_{1,t+1} \} \}
\]

\[
v^D_{j,t} = \{ \pi^D_{j,t} + \lambda E_t \max \{ v^D_{0,t+1} - w \xi, v^D_{0,t+1} + w \xi - w, v^D_{1,t+1} \} \}
\]

\[
v^E_{j,t} = \{ \pi^E_{j,t} + \lambda E_t \max \{ v^E_{0,t+1} - w \xi, v^E_{0,t+1} - w \xi - w, v^E_{1,t+1} \} \}
\]
where $\lambda = (1 + g)(1 + \pi)/(1 + r)$.

Since these value functions are defined in terms of variables that are constant along the economy’s balanced growth path, we will use this representation of the value functions to solve for the transitional path in the price level if sector $i$ that results from the announcement of the conversion to the Euro.

In order to solve the equilibrium inflation dynamics in this model, we need to define the proper state space. The structure of the state space in this model is very similar to that in DKW. The main difference is that it is not only defined as the discrete distribution of firms over the length over which they have not adjusted their prices but also over the denomination in which they charge their prices.

Let $\theta_{S,j,t}$ for $S \in \{D,E\}$ denote the fraction of firms at the start of period $t$ that changed their price $j$ periods ago and that charge their price in denomination $S$. Furthermore, let $\alpha_{S,j,t}$ for $S' \in \{D,E\}$ denote the fraction of firms that are charging a price that they set $j$ periods ago in denomination $S'$ that change their price at time $t$ and that keep on charging their price in the same denomination. Let $\alpha_{C,j,t}$ denote the fraction of firms that are charging a price that they set $j$ periods ago in their old domestic currency that change their price at time $t$ as well as switch to the Euro. Finally, let $\omega_{S,j,t}$ for $S \in \{D,E\}$ denote the fraction of firms at the end of period $t$ that changed their price $j$ periods ago and that charge their price in denomination $S$. Here, the end of period refers to the part of the period after which firms have made their pricing decisions. This is the part of the period in which revenue is generated and prices are measured.

The dynamic transition equations for the state are given by the following identities

$$\omega_{E,0,t} = \sum_{j=1}^{\infty} (\alpha_{E,j,t} \theta_{E,j,t}^E + \alpha_{C,j,t} \theta_{E,j,t}^D) \tag{15}$$

$$\omega_{D,0,t} = \sum_{j=1}^{\infty} \alpha_{D,j,t} \theta_{D,j,t}^D \tag{16}$$

$$\omega_{j,t}^E = (1 - \alpha_{E,j,t}) \theta_{j,t}^E \tag{17}$$

$$\omega_{j,t}^D = (1 - \alpha_{j,t}^D - \alpha_{C,j,t}^C) \theta_{j,t}^D \tag{18}$$

$$\theta_{j+1,t+1}^S = \omega_{j,t}^S \text{ for } S \in \{D, E\} \tag{19}$$

where, since the state represents a distribution of firms, $\omega_{j,t}^S \geq 0$ and $\sum_{s=0}^{\infty} \omega_{j,t}^S = 1$. Furthermore, since they represent transition probabilities, $0 \leq \alpha_{j,t}^S \leq 1$ for $S \in \{C, D, E\}$.

This definition of the state allows us to define the price level at the end of the period as a function of the state and the prices set by the firms. That is, we can write the measured price level at each point in time as

$$P_{it} = \left[ \sum_{S \in \{D,E\}} \sum_{j=0}^{\infty} \omega_{j,t}^S \left( \frac{1}{P_{i,t-j}^S} \right)^{\varepsilon - 1} \right]^{\frac{1}{\varepsilon - 1}} \tag{20}$$
In terms of the detrended prices, this yields
\[
p_{it} = \left[ \sum_{S \in \{ D, E \}} \sum_{j=0}^{\infty} \omega_{j,t}^S \left( \frac{(1 + \pi)^j}{P_{S,t-j}^S} \right)^{\epsilon-1} \right]^{1/(1-\epsilon)}
\] (21)
which is constant on the balanced growth path.

Solving for the firms’ optimal price setting decision involves solving for three decisions: (i) whether or not to adjust their price, (ii) whether or not to switch to the Euro (in case they are charging prices in the domestic currency), and (iii) what price to charge if the price is adjusted. We will tackle parts (i) and (ii) first and then solve (iii).

A firm that charges its price in Euros in period \( t \) and set that price \( j \) periods ago will adjust its price whenever the menu cost it draws is smaller than the gain in value that the firm obtains when it adjusts its price. Mathematically, this boils down to
\[
\xi \leq \frac{v_{0,t}^E - v_{j,t}^E}{w}
\] (22)
The probability that this happens is depends on the distribution function of menu costs. In particular
\[
\alpha_{E,j,t} = G \left( \frac{(v_{0,t}^E - v_{j,t}^E)}{w} \right)
\] (23)
We will denote the expected menu cost for such a firm, conditional on adjusting its price as
\[
\Xi_{E,j,t} = \int_0^{(v_{0,t}^E - v_{j,t}^E)/w} \xi dG(\xi)
\] This price adjustment rule is essentially the same as that in DKW.

This is not the case for the a firm that charges its price in the domestic currency, though. Rather than deciding on whether or not to change its price, such a firm decides on whether to change its price and continue to charge it in the domestic currency, change its price and start charging it in Euros, or not change its price at all.

If the firm decides to change its price, it will start charging it in Euros whenever the value of charging it in Euros net of the Euro conversion adjustment cost is higher than the value of continuing to charge it in the domestic currency. That is, if the firm adjusts its price, it will convert to the Euro whenever
\[
v_{0,t}^E - cw \geq v_{0,t}^D
\] (24)
This result implies that, if this inequality holds strictly one way or the other, either all firms that charge their prices in domestic currency and adjust their prices will change to Euros or they will all keep on charging their prices in the domestic currency. Hence, in that case \( \alpha_{D,j,t}^D \alpha_{C,j,t}^C = 0 \).

A firm that set its domestic currency denominated price \( j \) periods ago will adjust its price whenever the menu cost it draws satisfies
\[
\xi \leq \max \left\{ \frac{v_{0,t}^E - cw - v_{j,t}^D}{w}, \frac{v_{0,t}^D - v_{j,t}^D}{w} \right\}
\] (25)
This allows us to solve for the adjustment probabilities

\[ \alpha_D^{j,t} = \begin{cases} 0 & \text{whenever } v_{0,t}^D > v_{0,t}^E - cw \\ G \left( \frac{(v_{0,t}^D - v_{j,t}^D)}{w} \right) & \text{otherwise} \end{cases} \]  

(26) and

\[ \alpha_C^{j,t} = \begin{cases} 0 & \text{otherwise} \\ G \left( \frac{(v_{0,t}^E - cw - v_{j,t}^D)}{w} \right) & \text{whenever } v_{0,t}^E - cw > v_{0,t}^D \end{cases} \]  

(27)

Here, we assume that, in case of indifference, firms will switch to the Euro. Its expected menu cost, conditional on adjusting its price and still charging it in the domestic currency

\[ \Xi_D^{j,t} = \frac{R(v_{0,t}^D - v_{j,t}^D)}{w} \xi_d G(\xi) \]  

whenever \( v_{0,t}^D > v_{0,t}^E - cw \)

(28) and its expected menu cost, conditional on adjusting its price and switching to the Euro equals

\[ \Xi_C^{j,t} = \frac{R(v_{0,t}^E - cw - v_{j,t}^D)}{w} \xi_d G(\xi) \]  

whenever \( v_{0,t}^E - cw > v_{0,t}^D \)

(29)

The solution of these adjustment probabilities and expected adjustment costs now allows us to solve for the optimal price \( P_{S,t}^* \) for \( S \in \{D, E\} \). However, we will solve for the optimal price detrended price, \( p_{S,t}^* \), rather than for \( P_{S,t}^* \). In order to do so, it is convenient to first rewrite the functional equations that define the value function by substituting in the optimal price adjustment decisions. This yields

\[ v_{0,t}^D = \max_{P_{D,t}} \left\{ \pi_D^{0,t} + \lambda \alpha_D^{1,t+1} v_{0,t+1}^D + \lambda \alpha_C^{1,t+1} (v_{0,t+1}^E - wc) \\
+ \lambda (1 - \alpha_D^{1,t+1} - \alpha_C^{1,t+1}) v_{1,t+1}^D - \lambda w \Xi_D^{1,t+1} - \lambda w \Xi_C^{1,t+1} \right\} \]  

(30)

\[ v_{0,t}^E = \max_{P_{E,t}} \left\{ \pi_E^{0,t} + \lambda \alpha_E^{1,t+1} v_{0,t+1}^E + \lambda (1 - \alpha_E^{1,t+1}) v_{1,t+1}^E - \lambda w \Xi_E^{1,t+1} \right\} \]  

(31)

\[ v_{j,t}^D = \left\{ \pi_D^{j,t} + \lambda \alpha_D^{j+1,t+1} v_{0,t+1}^D + \lambda \alpha_C^{j+1,t+1} (v_{0,t+1}^E - wc) \\
+ \lambda (1 - \alpha_D^{j+1,t+1} - \alpha_C^{j+1,t+1}) v_{j+1,t+1}^D - \lambda w \Xi_D^{j+1,t+1} - \lambda w \Xi_C^{j+1,t+1} \right\} \]  

(32)

\[ v_{j,t}^E = \left\{ \pi_E^{j,t} + \lambda \alpha_E^{j+1,t+1} v_{0,t+1}^E + \lambda (1 - \alpha_E^{j+1,t+1}) v_{j+1,t+1}^E - \lambda w \Xi_E^{j+1,t+1} \right\} \]  

(33)

which allows us to derive the first order necessary conditions for the optimal prices \( p_{D,t}^* \) and \( p_{E,t}^* \).
The first order necessary condition for the choice of $p_{D,t}^*$ reads
\[
0 = \frac{\partial}{\partial p_{D,t}^*} \left\{ \pi_{0,t}^D + \lambda \alpha_{1,t+1}^D v_{0,t+1} + \lambda \alpha_{1,t+1}^D (v_{0,t+1}^E - wc) \right. \\
+ \lambda (1 - \alpha_{1,t+1}^D - \alpha_{1,t+1}^C) v_{1,t+1}^D - \lambda w \xi_{1,t+1}^D - \lambda w \xi_{1,t+1}^D \right\} \\
= \frac{\partial \pi_{0,t}^D}{\partial p_{D,t}^*} + \lambda \frac{\partial \alpha_{1,t+1}^D}{\partial p_{D,t}^*} (v_{0,t+1}^D - v_{1,t+1}^D) - \lambda w \frac{\partial \xi_{1,t+1}^D}{\partial p_{D,t}^*} \\
+ \lambda (1 - \alpha_{1,t+1}^D - \alpha_{1,t+1}^C) \frac{\partial v_{1,t+1}^D}{\partial p_{D,t}^*} \\
\text{(34)}
\]

However, the envelope theorem implies that
\[
0 = \lambda \frac{\partial \alpha_{1,t+1}^D}{\partial p_{D,t}^*} (v_{0,t+1}^D - v_{1,t+1}^D) - \lambda w \frac{\partial \xi_{1,t+1}^D}{\partial p_{D,t}^*} \\
\text{(37)}
\]
\[
0 = \lambda \frac{\partial \alpha_{1,t+1}^C}{\partial p_{D,t}^*} (v_{0,t+1}^E - v_{1,t+1}^D) - \lambda w \frac{\partial \xi_{1,t+1}^C}{\partial p_{D,t}^*} \\
\text{(38)}
\]

Hence, the first order condition simplifies to
\[
0 = \frac{\partial \pi_{0,t}^D}{\partial p_{D,t}^*} + \lambda (1 - \alpha_{1,t+1}^D - \alpha_{1,t+1}^C) \frac{\partial v_{1,t+1}^D}{\partial p_{D,t}^*} \\
\text{(39)}
\]

The partial $\frac{\partial v_{j,t+1}}{\partial p_{D,t}^*}$ can be derived in a similar way as the above condition. It equals
\[
\frac{\partial v_{j,t+1}}{\partial p_{D,t}^*} = \frac{\partial \pi_{0,t}^D}{\partial p_{D,t}^*} + \lambda (1 - \alpha_{j+1,t+1}^D - \alpha_{j+1,t+1}^C) \frac{\partial v_{j+1,t+1}^D}{\partial p_{D,t}^*} \\
\text{(40)}
\]

Solving the optimality condition through forward recursion yields that $p_{D,t}^*$ is chosen such that
\[
0 = \frac{\partial \pi_{0,t}^D}{\partial p_{D,t}^*} + \sum_{j=1}^{\infty} \lambda^j \prod_{s=1}^{j} (1 - \alpha_{j+s,t+s}^D - \alpha_{j+s,t+s}^C) \frac{\partial \pi_{j,t+j}}{\partial p_{D,t}^*} \\
\text{(41)}
\]

Since
\[
\frac{\partial \pi_{j,t+j}}{\partial p_{D,t}^*} = \left[ \frac{1 - \varepsilon}{(1 + \pi)^j} + \varepsilon \frac{\psi}{p_{D,t}^*} \right] \left( \frac{1}{(1 + \pi)^j} \frac{p_{D,t}^*}{p_{i,t+j}} \right)^{-\varepsilon} \left( \frac{p_{i,t+j}}{p} \right)^{-\eta} \\
\text{(42)}
\]

This first order condition implies that
\[
p_{D,t}^* = \frac{\varepsilon}{\varepsilon - 1} \psi \sum_{j=0}^{\infty} \lambda_{j,t}^D (1 + \pi)^j \\
\text{(43)}
\]

where
\[
\lambda_{j,t}^D = \left\{ \begin{array}{ll}
\frac{1}{\lambda^j} \prod_{s=1}^{j} (1 - \alpha_{s,t+s}^D - \alpha_{s,t+s}^C) (1 + \pi)^{\varepsilon j} p_{i,t+j}^{-\eta} & \text{for } j = 0 \\
\lambda^j \prod_{s=1}^{j} (1 - \alpha_{s,t+s}^D - \alpha_{s,t+s}^C) (1 + \pi)^{\varepsilon j} p_{i,t+j}^{-\eta} & \text{for } j > 0
\end{array} \right.
\text{(44)}
\]
Similarly, we can solve for the optimal price charged in Euros as being

\[
P^*_{E,t} = \frac{\varepsilon}{\varepsilon - 1} \sum_{j=0}^{\infty} \frac{\lambda^E_j (1 + \pi)^j}{\sum_{j=0}^{\infty} \lambda^E_j} \tag{45}
\]

where

\[
\chi^D_{j,t} = \begin{cases} 
1 & \text{for } j = 0 \\
\lambda^j \prod_{s=1}^{\infty} (1 - \alpha_{s,t+s}) (1 + \pi)^j p_{s+t+j} \chi^D_{s,t} & \text{for } j > 0 
\end{cases} \tag{46}
\]

In terms of the non-transformed prices, we thus obtain for \( S \in \{D, E\} \) that

\[
P^*_{S,t} = \frac{\varepsilon}{\varepsilon - 1} \sum_{j=0}^{\infty} \Omega^S_{j,t} \Psi_{t+j} \text{ where } \Omega^S_{j,t} = \frac{\chi^E_{j,t}}{\sum_{q=0}^{\infty} \chi^E_{q,t}} \tag{47}
\]

which is the result used in the main text.

C  Section V: Calibration of price adjustment frequencies

Because our calibration pertains to the steady state in which firms do not switch the denomination of their prices, we ignore the currency denomination dimension in this derivation. Our goal is to derive an expression for the probability of adjusting the price \( x \) times in 12 months.

Let \( q_{x,y,j} \) denote the probability that a firm adjusts its price \( x \) times over the next \( y \) periods conditional on not having adjusted its price for \( j \) periods. Let \( \alpha_j \) denote the probability of adjusting the price when a firm has not adjusted its price for \( j \) periods.

These probabilities can be derived using a version of the Chapman-Kolmogorov equation for discrete time and discrete state Markov processes. The particular application here yields that

\[
q_{x,y,j} = (1 - \alpha_j) q_{x,y-1,j+1} + \alpha_j q_{x-1,y-1,0} \tag{48}
\]

The intuition for this result is that there are two ways to adjust \( x \) times from now during the next \( y \) periods. The first, which happens with probability, \( 1 - \alpha_j \), is that a firm does not adjust its price in the current period and will thus have to adjust its price \( x \) times in the remaining \( y - 1 \) periods. The second, which happens with probability \( \alpha_j \), is that the firm adjusts its price in the current period and has \( y - 1 \) periods left to adjust its price another \( x - 1 \) times.

Since the model implies that a firm can at maximum adjust its price once per period, \( q_{x,y,j} = 0 \) for \( x > y \) and all \( j \). Note also that \( q_{0,0,j} = 1 \) for all \( j \). These latter two results can be used to initialize the recursion implied by the Chapman-Kolmogorov equation above.

If a period is a month, then the probability that a firm that hasn’t adjusted its price for \( j \) months adjusts its price \( x \) times in the subsequent year is given by \( q_{x,12,j} \). Let \( \omega_j \) be the steady state fraction of firms that have not adjusted their prices for \( j \) months, then the fraction of firms that will adjust its price \( x \) times in a year in steady state, which is what we use for our calibration and which is what we will denote by \( Q_{x,12} \) equals

\[
Q_{x,12} = \sum_{j=0}^{\infty} \omega_j q_{x,12,j} \tag{49}
\]
In particular, the data for the Dutch restaurant sector in which we based our calibration reports the empirical equivalents of $Q_{0,12}$, $Q_{1,12}$, and $Q_{2,12} + Q_{3,12} + Q_{4,12}$.

D Section V: Numerical solution method

For the numerical solution of our model we use the ‘extended path’ method. This method has been applied in other studies of transitional dynamics, like Greenwood and Yorukoglu (1997). We will assume that our economy starts off in period 0 in the steady state in which everyone charges their prices in the domestic currency and charging prices in Euros is not an option. In period 0 the conversion to the Euro at time $T$ is announced. We will solve for the transitional path of the economy under the assumption that at time $T > T > 0$ the sector has converged to its new steady state. This new steady state is the one in which all firms charge their prices in Euros.

The numerical solution method basically works as follows

1. We start with a guess for the equilibrium price path $\{p_{i,t}\}_{t=0}^{T}$.

2. We solve the optimal price setting response for the firms. This is done using the value function iterations, (30) through (33), the optimal price setting rules, (43) and (45), and the transition equations for the state space, (15) through (19) and (23), (26) and (27).

3. The new path of the prices and the state space is then used to solve the price level identity, (21) and obtain a new equilibrium price path $\{p'_{i,t}\}_{t=0}^{T}$.

4. Steps 2 and 3 above are repeated until $\{p_{i,t}\}_{t=0}^{T} \rightarrow \{p'_{i,t}\}_{t=0}^{T}$.

GAUSS code that implements this numerical procedure is available upon request.
Figure 1: DKW menu cost distribution