RESEARCH PAPER

TAYLOR, BLACK AND SCHOLES: SERIES APPROXIMATIONS AND RISK MANAGEMENT PITFALLS

by Arturo Estrella

Federal Reserve Bank of New York
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Taylor, Black and Scholes: Series Approximations and Risk Management Pitfalls

Abstract

Risk managers make frequent use of finite Taylor approximations to option pricing formulas, particularly of first and second order (delta and gamma). This paper shows that for a plausible range of parameter values, the Taylor series for the Black-Scholes formula diverges. Using a numerical technique developed in the paper, it is also shown that even when the series converges, finite approximations of very large order are generally necessary to achieve acceptable levels of accuracy. Implications for risk management and stress testing are discussed.

Taylor, Black and Scholes: Series Approximations and Risk Management Pitfalls

It is nearly impossible to approach the subject of options without some mention of "delta" and "gamma". In the infinitesimal world of the classic Black-Scholes (1973) derivation of the price of a European call option, these two partial derivatives are key determinants of the dynamics of the option price. Together with the time derivative ("theta"), they are sufficient to define the differential equation for the price.

In the real world of large discrete price jumps, however, delta and gamma are far from sufficient. An approximation based on these two measures may work well within a limited range of values, but fails to contain adequate information about the nonlinear behavior associated with large movements in the underlying asset. Inaccuracies may result even when the pricing formula itself is perfectly reliable.

Still, delta and gamma are widely used in practice for risk measurement and management. For example, the delta approximation is fundamental in the options treatment of the European Community's Capital Adequacy Directive, which becomes effective at the start of 1996. Also, a Group of Thirty (1994) survey on derivatives-related practices states that 98% of the 125 dealers that responded calculate deltas regularly for risk management purposes and that 91% calculate gammas.

This practice is particularly hazardous in the area of "stress testing", that is, the analysis of the behavior of the value of a portfolio under fairly extreme scenarios. According to the Group of Thirty survey, 93% of respondent dealers perform stress tests, and recent regulatory guidelines suggest that such tests will become standard practice in the near future (see Federal Reserve Board (1993), Office of the Comptroller of the Currency (1993), and Basle Committee on Bank Supervision (1994)). Although the Group of Thirty survey did not explicitly link the use of delta and gamma approximations with stress testing, some users may find it tempting to employ this combination. The reason is sheer convenience.

Armed with an option's current price, its delta and its gamma, a risk manager can construct a quadratic approximation to the price of the option corresponding to any other value of the underlying instrument. Most firms would have the capability to revalue options more precisely, but the approximation is easily computed even for exotic instruments whose pricing is calculation-intensive. More importantly, it is linear in the underlying price change and its squared value and is thus readily aggregated across instruments, portfolios and business units of the firm.

Convenience is achieved at a significant cost. This paper shows analytically that the power series for the Black-Scholes formula does not always converge. Moreover, even when it does converge, numerical results show that partial series of very high order are required to obtain a reasonable level of accuracy. The results clearly imply that the applicability of low order series approximations -- of the ubiquitous delta-gamma approximations in particular -- is very limited in the area of risk management. In the case of stress tests, these low order approximations are entirely inappropriate.

I. The Black-Scholes Formula and Its Taylor Series

Black and Scholes (1973) derived a closed form expression for the price of a call option on a non-dividend paying stock. Because most option pricing formulas developed since then -- including those for alternative price processes and for exotic instruments -- have essentially the same general analytical form as Black-Scholes (see, for example, Hull (1993)), we focus here on a slightly streamlined version of the Black-Scholes formula with little loss of generality.

Thus, consider a call option on an underlying asset whose future price is lognormally distributed. Scale the time and value units so that the time to maturity and the exercise price are both 1. Assume furthermore that the interest rate, dividends, convenience yields, etc. are all zero. The latter can be accomplished, for instance, by focusing on the forward price of the underlying rather than its spot price. Then the price of the option is:

$$v(x) = x \cdot \Phi\left(\frac{\log x + \frac{1}{2}\sigma^2}{\sigma}\right) - \Phi\left(\frac{\log x - \frac{1}{2}\sigma^2}{\sigma}\right), \qquad (1)$$

where

$$\Phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz .$$

Clearly, v is also a function of σ, and different values of this parameter will be examined in the numerical part of the paper. For the moment, however, σ will be taken as fixed. Also note that σ corresponds to $\sigma\sqrt{\tau}$ in the standard scaling with time to maturity $\tau \neq 1$. For example, using a monthly time horizon, an instrument with annual volatility of 0.35 would have $\sigma = 0.10$ under our convention. This point is important in interpreting the numerical examples of section III below.

Taylor (1715) expressed the value of an infinitely differentiable function as an infinite weighted sum of its derivatives. The complete Taylor series for $v(x)$ is:

$$T(x, x_0, \infty) = \sum_{k=0}^{\infty} \frac{v^{(k)}(x_0)}{k!} (x-x_0)^k ,$$

where $v^{(k)}(x)$ represents the k^{th} derivative of $v(x)$. In general, for an arbitrary function v, there is no assurance that T will converge to v (or to anything else) for given values of x_0 and x. In the case of (1), however, the following applies.

Proposition 1: Define the function

$$T(x, x_0, n) = \sum_{k=0}^{n} \frac{v^{(k)}(x_0)}{k!} (x-x_0)^k ,$$

the n^{th} order Taylor approximation to the function (1). Then T converges to $v(x)$ as $n \to \infty$ if $0 < x < 2x_0$ and diverges if $x > 2x_0 > 0$.

Proof: By Proposition A.4 (see Appendix A for results labelled "A.k"), the convergence properties of the power series corresponding to $v(x)$ are the same as those of any of its derivatives. Consider the second derivative, which has the most convenient form. Thus,

$$v^{(1)}(x) = \Phi\left(\frac{\log x + \frac{1}{2}\sigma^2}{\sigma}\right)$$

and

$$v^{(2)}(x) = \frac{1}{\sigma x} \Phi'\left(\frac{\log x + \frac{1}{2}\sigma^2}{\sigma}\right) = \frac{e^{-\frac{1}{2\sigma^2}(\log x + \frac{1}{2}\sigma^2)^2}}{\sqrt{2\pi}\sigma x} . \qquad (2)$$

Apart from additive and multiplicative constants, which do not affect the convergence properties of a function, expression (2) is built from the following components: the logarithmic function, the exponential function, and the first two powers of x. It is obtained by the successive composition of e^x with x^2 and $log\ x$ and by division of the result by x. Proposition A.7 states that the exponential series is everywhere convergent. Similarly, x^2 is trivially convergent everywhere. Thus, Proposition A.6 implies that the doubly composite function in the numerator converges at least over the range of convergence of $log\ x$, namely $0<x<2x_0$. Also, Propositions A.2 and A.3 imply that division by x, which is trivially convergent everywhere, does not further limit the convergence region of the ratio.

The divergence portion of Proposition 1 does not follow directly since, in some special cases, convergence extends beyond the region supplied by the above results. An example is provided by the simple exponential and logarithmic functions, whose composition results in the identity function. The latter is everywhere convergent, even though the logarithmic function is not. Thus, consider

$$v^{(3)}(x) = -\frac{h+\sigma}{\sigma x}\cdot v^{(2)}(x) \;,$$

$$where \quad h \equiv \frac{1}{\sigma}\log x + \frac{1}{2}\sigma \;.$$

Solving for h, one obtains

$$h = -\sigma\left(1 + \frac{x\cdot v^{(3)}(x)}{v^{(2)}(x)}\right). \tag{3}$$

Now suppose the series corresponding to $v^{(2)}(x)$ converges for $x>2x_0>0$. If that is the case, then by Propositions A.4, the series representation of $v^{(3)}(x)$ converges for those values. Furthermore, Propositions A.2 and A.3 for products and reciprocals imply that the right hand side of equation (3) has a convergent series representation as well. But the left hand side of (3) is essentially the logarithmic function, which is known (Proposition A.7) not to converge for those values. Hence, $v^{(2)}(x)$ must diverge for $x > 2x_0 > 0$. ***Q.E.D.***

Proposition 1 has important practical implications for risk management systems that rely on Taylor series approximations to $v(x)$ for large changes in x. It is not unlikely that the upper bound required for convergence may be breached in realistic applications. Specifically, convergence is guaranteed only if

$$\log(x/x_0) < \log 2 = 0.693 .$$

Note that this inequality does not depend on σ and is more likely to fail when looking at large σ or multiples of σ. The condition does not hold, for instance, when $\sigma=0.35$ and $\log(x/x_0)=2\sigma$ or when $\sigma=0.25$ and $\log(x/x_0)=3\sigma$. In such cases, the Taylor approximation is not very precise, and its accuracy deteriorates indefinitely as the order of the series increases.

II. High-Order Taylor Approximations to $v(x)$

Proposition 1 suggests that much caution should be used in approximating the Black-Scholes formula (1) with Taylor series, and raises questions regarding the magnitude of the problem. In order to obtain numerical estimates of the approximation errors, series with many terms must be computed. The expressions for high-order derivatives of the Black-Scholes formula $v(x)$ are complicated and cumbersome, but Proposition 2 below provides a recursive formula to compute $v^{(n)}(x)$ for large values of n in a relatively straightforward way. The method is somewhat akin to the use of Hermite polynomials to calculate the derivatives of the normal distribution function (see, e.g., Abramowitz and Stegun (1964) or Lebedev (1972)).

Proposition 2: The derivatives of $v(x)$ of second and higher order may be expressed as

$$v^{(n+2)}(x) = \frac{(-1)^n P_n(h)\, \Phi'(h)}{(\sigma x)^{n+1}}\,, \tag{4}$$

where $n = 0,1,\ldots,$

$$h = \frac{\log x + \frac{1}{2}\sigma^2}{\sigma}$$

and $P_n(h)$ is a sequence of polynomials satisfying $P_0(h) \equiv 1$ and computable by either of the recursive relationships:

$$P_n(h) = (h+n\sigma)\cdot P_{n-1}(h) - P'_{n-1}(h)$$

or

$$P_n = (h+n\sigma)\cdot P_{n-1} - [(n-1)P_{n-2} + \frac{(n-1)(n-2)}{2}\sigma P_{n-3} + \ldots + \frac{(n-1)!}{n-1}\sigma^{n-2}P_0]\,.$$

Proof: The first recursive relationship may be easily derived by induction using equation (4). The second relationship, which does not involve differentiation and thus provides a purely numerical method of calculation, may be obtained by the use of a generating function. Let

$$G(h,t) = \sum_{n=0}^{\infty} \frac{P_n(h)}{n!} t^n \tag{5}$$

be the exponential generating function for the polynomials $P_n(h)$. By making the substitutions

$$h = \frac{\log x_0 + \frac{1}{2}\sigma^2}{\sigma}$$

and

$$t = \frac{x_0 - x}{\sigma x_0}$$

in (5) and using (4) as the definition of $P_n(h)$, (5) becomes the Taylor series for $v^{(2)}(x)/v^{(2)}(x_0)$ around x_0. Algebraic manipulation shows that if the series converges, then

$$G(h, t) = \frac{1}{1-\sigma t} e^{-\frac{h \log(1-\sigma t)}{\sigma} - \frac{1}{2\sigma^2}\log^2(1-\sigma t)}.$$

Differentiating both sides with respect to t results in the partial differential equation

$$(1-\sigma t)\frac{\partial G}{\partial t} = (h+\sigma)G + \frac{\log(1-\sigma t)}{\sigma}G. \tag{6}$$

By: (i) substituting G and $\partial G/\partial t$ from expression (5) into this differential equation, (ii) expanding the logarithmic function in the last term of (6) into a power series as in Proposition A.8 in Appendix A and multiplying out, and (iii) focusing on the coefficients associated with t^{n-1} on either side of the equation, the second recursive expression for P_n is obtained. *Q.E.D.*

Examples of the polynomials $P_n(h)$ for $n = 0,...,4$ are provided in Appendix B. The second recursive formula of Proposition 2 is used below to obtain numerical results for high-order Taylor approximations.

III. Finite Taylor Approximations to the Black-Scholes Formula

The divergence result of Proposition 1 (section I) may be illustrated by computing a sufficiently large number of terms in the Taylor approximation for appropriate values of the arguments. Figure 1 presents results for $\sigma=0.25$, using the Taylor series around $x_0=1$ to approximate the value of $v(x)$ at $x=e^{3\sigma}$. Terms of up to order 40 are used. In this case, $log(x/x_0) = 0.75 > log\ 2$, and it is clear from the figure that the Taylor series diverges quickly and decisively. Note that for the parameter values in the figure $v(x)=1.117$, which is of a different order of magnitude from most of the approximate values displayed.

Not all parameter values will lead to series approximations as ill-behaved as those illustrated in Figure 1. In particular, for cases in the which the condition $x<2x_0$ is met, convergence will be achieved. The question remains, however, as to how many terms will be required to attain some specified level of accuracy. The numerical experiments described below show that, in general, fairly high order approximations are required, even when the difference between x and x_0 is relatively small and the infinite series converges.

A. Varying n for fixed x and x_0

Figures 2 and 3, which are analogous to Figure 1, illustrate the Taylor approximation results for lower volatility levels of 0.05 and 0.15, respectively. Although in both cases the series converge, it is clear that low order approximations produce large proportional errors in either direction. Furthermore, successive approximations tend to get worse before they get better, and one or two dozen terms are required for the approximation to finally close in on the actual value. This phenomenon is analyzed more systematically in Tables I-III.

Table I presents the results of the following experiment. Consider a move in the underlying price from 1 (the strike price) to $e^{\pm k\sigma}$ for $k=1,2,$ *or* 3. How accurate are the Taylor approximations of orders 2,4, and 6? Is there an improvement in accuracy as the order increases and, if so, what is the magnitude of the improvement? Accuracy is measured by the absolute approximation error scaled by the at-the-money value of the option.

Scaling is useful in comparing the errors associated with different levels of volatility. For these purposes, the at-the-money value is used in all cases because the obvious alternative -- using $v(x)$ -- would lead to misleadingly large errors for out-of-the-money values. The results are reported only for even values of n because they tend to correspond to the largest marginal improvements in accuracy. This phenomenon is related to the fact that, for the cumulative normal distribution, every other term in the Taylor series around the mean is equal to zero.

Table I shows that for changes of up to one standard deviation, the magnitude of the errors falls in generally acceptable ranges. The gamma (second-order) approximations are a bit imprecise for large σ, but the use of the sixth order approximation leads to results comfortably below 1%. For changes of 2 or 3 standard deviations, the results are much less encouraging. Particularly when an increase in the underlying price is considered, the errors are quite large, ranging from 15% to 12.5 times the at-the-money amount (1,250%).

The results for decreases in the underlying price tend to be less dramatic because, for a given absolute proportional change, the associated arithmetic decline is smaller than the corresponding arithmetic increase. Nevertheless, for a 3σ drop, the error is of the order of 30% or higher even with a sixth-order approximation.

In Table I, all changes take the strike price of the option as the starting value. Table II, which is analogous to Table I, shows that the results may be even worse when the starting point is away from the money and the endpoint is the strike price or beyond. The table illustrates a phenomenon denoted loosely as "out-of-the-money gamma." Even though the option may appear almost linear at the initial point, a large change in the underlying value makes approximations very inaccurate. Note, however, that taking account of gamma with $n=2$, or even using $n=6$ is not sufficient to reduce the errors to acceptable levels.

Table III shows the results for the dual experiment to the one reported in Tables I and II. Specifically, it shows the order n that would be necessary to achieve a level of accuracy of 5% or 1%, respectively, for the changes in the underlying value considered in the foregoing tables. If an error no greater than 5% is desired for the volatilities considered, a gamma approximation may suffice within one standard deviation. Otherwise, the orders required are substantial. For changes of $\pm 3\sigma$, a minimum order of 15 to 20 is generally required. Furthermore, in the higher volatility case, these levels of accuracy are unattainable at any order for reasons stated in section I above.

B. Varying x for fixed n and x_0

Some further insight into the nature of the approximation errors may be obtained by looking at a given order n over a range of underlying values. Figure 4 plots the Taylor approximations of orders one through six for $\frac{1}{2} \leq x \leq 1\frac{1}{2}$ and $\sigma=0.15$. The function $T(x,x_0,n)$ is denoted in the figure as "Tn".

In a neighborhood of the base point $x=1$, the approximations are very close to the value of the function. Nevertheless, success within this neighborhood does not guarantee extrapolative accuracy. Away from the base point, the higher order Taylor terms have effects that can be large and unhelpful.

In addition, the direction of the error is not necessarily symmetrical for a given approximation. The linear (delta) approximation does underestimate systematically since gamma, which determines the sign of the remainder, is always positive in this case (see equation (2)). However, the fifth and sixth order approximations illustrate the possibility of errors of opposite sign for increases and decreases in the underlying price. Furthermore, as in the previous section, the results clearly demonstrate that higher order approximations may not lead to more accurate results.

C. Average over x for fixed n and x_0

In the foregoing analysis, the accuracy of the n^{th} order Taylor approximation has been examined for various specific values of the underlying price. A natural and convenient means of reducing all the relevant cases to a single number is provided by the probability distribution implicit in the Black-Scholes approach. More precisely, there is a probability distribution from which the price of the option may be derived as the expected value of the conditional option payments:

$$v(x) = \int_1^{\infty} (y-1)\, f(y)\, dy\,,$$

where y is the price of the underlying instrument at maturity conditional on a current price x.

For the specific option form analyzed in this paper,

$$f(y) = \frac{1}{y \cdot \sqrt{2\pi} \cdot \sigma} \cdot e^{-\frac{1}{2} \cdot \left(\frac{\log(y/x) + \frac{1}{2}\sigma^2}{\sigma} \right)^2} .$$

This distribution may be applied to the results of a Taylor approximation of any given order to calculate the "average error" produced by the approximation. If the error of order n is defined as

$$r_n(x) = v(x) - T(x, x_0, n) ,$$

three average measures are reported:

(i) the mean error

$$\int_{-\infty}^{\infty} r_n(y) \, f(y) \, dy ,$$

(ii) the mean absolute error

$$\int_{-\infty}^{\infty} |r_n(y)| \, f(y) \, dy ,$$

(iii) the root-mean-squared error

$$\sqrt{\int_{-\infty}^{\infty} r_n(y)^2 f(y) \, dy} .$$

The results are reported in Table IV for approximation orders 1 through 6 and for $\sigma = 0.15$. As in earlier sections, the figures are scaled by dividing by the at-the-money value of the option.

Regardless of the average measure used, it is clear that the usual delta or delta plus gamma approximations do not produce very attractive results. The delta approximation overestimates the option price by over 40% and the delta plus gamma approximation underestimates it by over 9%. The fifth order approximation leads to more acceptable results using the mean or absolute mean measures, but the sixth order is worse and none of the root-mean-squared results are particularly good. The errors in each tail may be very large, and the fact that the tails may offset each other is not necessarily useful in the context of risk management.

IV. Interpretation and Implications

The Black-Scholes formula has two essential mathematical components -- the logarithmic function and the normal probability distribution -- whose analytical properties differ. The logarithmic function has a series representation that converges relatively quickly, but only for a limited range of values of the argument. In contrast, the series for the normal distribution converges everywhere, but may take a large number of terms to reach desired levels of accuracy, and its precision may initially deteriorate with the addition of higher order terms. The best closed-form approximations to the normal distribution typically consist of polynomials in nonlinear transformations of the argument (see Abramowitz and Stegun (1964)).

The results obtained for the Black-Scholes formula are -- not surprisingly -- a combination of the above features. For some parameter values that may be encountered in practice, the series fails to converge as a result of the properties of the logarithmic component. In other plausible cases for which the series converges, a large number of terms is necessary to achieve a reasonable level of accuracy. The latter result is largely attributable to the convergence properties of the normal distribution.

Thus, the nonlinearity of options is not limited to "gamma risk" as is often assumed in risk management parlance. The nonlinearity is generally of very high order and not susceptible to simplified treatment.

The implications for the use of Taylor approximations in risk management applications are clear. First, <u>Taylor approximations should not be used in the context of stress testing</u>. Substantial inaccuracies are to be expected when approximating moves of 2 or 3 standard deviations, which constitute a lower bound for the movements assumed in the course of stress tests. Such tests routinely make use of 5 or 10 standard deviation movements, which are far beyond any reasonable range for the approximations.

Second, <u>in risk management applications involving a preponderance of relatively small moves, it may be feasible -- though sometimes risky -- to use Taylor approximations</u>. For moves no larger than one standard deviation, the accuracy of gamma approximations seems generally adequate. Problems may arise, however, if attention is focused on the tails of the distribution as is often the case in risk management applications (e.g., in simulations). Whereas computed means may be reasonably accurate, some order statistics may be significantly misstated. Special care should be used when approximating the values of highly nonlinear options, such as near-the-money short maturity options.

If the ease of aggregation of the Taylor series approach is desirable, there may be feasible alternatives. For example, <u>accurate results may be obtained by using a large number of terms of the Taylor approximation (say, 30 or 40), as long as the series converges</u>.

Depending on the functional forms of the option pricing formulas for the particular portfolio, the method of Proposition 2 of section II could be used to generate quickly a large number of terms. The method may be applied to each of several additive components of some option pricing formulas. Unfortunately, in some cases the method may not be directly applicable or would require further derivations. Also, the series may simply not converge for the desired values, making the use of higher order terms counterproductive.

A straightforward alternative is to use the exact formula with a predetermined underlying movement or set of movements. The costs of using this method are the loss of flexibility from having to specify the underlying movements in advance and the additional computational burden. The need for a full model is not in itself an issue because a model is also required for the computation of the derivatives in the case of series approximations.

Appendix A: Some Results on Analytic Functions

The proofs of Propositions 1 and 2 in the text require the use of several well-established results in the theory of power series of analytic functions. For convenience, those results are stated without proof in this appendix. Proofs of all the results, with the exception of Proposition A.3, may be found in Knopp (1990). Apostol (1974) provides a more modern but less comprehensive treatment. Proposition A.3 is proved by Bromwich (1991).

Proposition A.1 (Knopp § 18): Let

$$\sum_{j=0}^{\infty} a_j (x-x_0)^j$$

be a power series and let

$$\mu = \text{limsup} \sqrt[k]{|a_k|} \; .$$

Then the power series converges absolutely for every x such that

$$|x-x_0| < \frac{1}{\mu}$$

and diverges for every x such that

$$|x-x_0| > \frac{1}{\mu} \; .$$

Note: If $\mu=0$, the series converges for all real x. If $\mu=\infty$, the series converges only for $x=0$. When $0<\mu<\infty$, the region of convergence is an open ball centered at the point x_0.

Proposition A.2 (Knopp § 21): Let $f(x)$ and $g(x)$ be real-valued functions of a real variable x. Further, let

$$f(x) = \sum_{j=0}^{\infty} a_j (x-x_0)^j \quad for \ |x-x_0| < R_f$$

and

$$g(x) = \sum_{j=0}^{\infty} b_j (x-x_0)^j \quad for \ |x-x_0| < R_g .$$

Then for

$$|x-x_0| < \min\{R_f, R_g\} ,$$

we have

$$1. \quad f(x) \pm g(x) = \sum_{j=0}^{\infty} (a_j \pm b_j)(x-x_0)^j$$

and

$$2. \quad f(x) \cdot g(x) = \sum_{m=0}^{\infty} \sum_{j+k=m} (a_j \cdot b_k)(x-x_0)^m .$$

Proposition A.3 (Bromwich § 89): Let $f(x)$ be as in Proposition A.2. Then, for

$$|x-x_0| < R_f ,$$

provided that $f(x) \neq 0$ for any such x, there is a convergent series representation

$$\frac{1}{f(x)} = \sum_{j=0}^{\infty} c_j (x-x_0)^j .$$

Proposition A.4 (Knopp § 20): A function represented by a power series is differentiable at every interior point of its interval of convergence and its derivative within that interval may be obtained by means of term-by-term differentiation.

Proposition A.5 (Knopp § 20): If a function $f(x)$ is represented by a convergent power series centered at x_0, then the coefficients of the power series are related to the derivatives of the function by

$$a_k = \frac{f^{(k)}(x_0)}{k!}.$$

Proposition A.6 (Knopp § 21): Let $f(x)$ and $g(x)$ be as in Proposition A.2. Then $g(f(x))$ has a convergent power series representation for all

$$|x-x_0| < R_f$$

for which the series for $f(x)$ is absolutely convergent to some y such that

$$|y-f(x_0)| < R_g .$$

Proposition A.7 (Knopp § 23): The exponential series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges for any real x.

Proposition A.8 (Knopp § 26): The logarithmic series

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

converges absolutely for $|x| < 1$ and diverges for $|x| > 1$.

Appendix B: Examples of the Polynomials $P_n(h)$

Section II of the text defines a sequence of polynomials $P_n(h)$, $n=0,1,\ldots$ implicitly in the equation

$$v^{(n+2)}(x) = \frac{(-1)^n P_n(h)\,\Phi'(h)}{(\sigma x)^{n+1}},$$

where

$$h = \frac{\log x + \frac{1}{2}\sigma^2}{\sigma}.$$

The polynomials are computable by either of the recursive relationships:

$$P_n(h) = (h+n\sigma)\cdot P_{n-1}(h) - P'_{n-1}(h)$$

or

$$P_n = (h+n\sigma)\cdot P_{n-1} - [(n-1)P_{n-2} + \frac{(n-1)(n-2)}{2}\sigma P_{n-3} + \ldots + \frac{(n-1)!}{n-1}\sigma^{n-2}P_0].$$

Examples of these polynomials for $n \leq 4$ are:

$P_0(h) = 1$

$P_1(h) = h + \sigma$

$P_2(h) = h^2 + 3\sigma h + 2\sigma^2 - 1$

$P_3(h) = h^3 + 6\sigma h^2 + (11\sigma^2 - 3)h + 6\sigma^3 - 6\sigma$

$P_4(h) = h^4 + 10\sigma h^3 + (35\sigma^2 - 6)h^2 + (50\sigma^3 - 30\sigma)h + 24\sigma^4 - 35\sigma^2 + 3$

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Table I
Proportional errors
Projections from at-the-money value

	δ_1	σ: .05	.15	.25
For n = 2	$+\sigma$	.052	.083	.12
	$-\sigma$	.025	.002	.017
	$+2\sigma$	.61	.94	1.4
	-2σ	.36	.17	.041
	$+3\sigma$	2.3	3.7	5.9
	-3σ	1.3	.70	.30
For n = 4	$+\sigma$	.007	.015	.024
	$-\sigma$	.001	.004	.007
	$+2\sigma$	.32	.66	1.1
	-2σ	.098	.028	.089
	$+3\sigma$	2.7	6.2	12.3
	-3σ	.90	.069	.25
For n = 6	$+\sigma$	.001	.002	.003
	$-\sigma$	.0001	.001	.001
	$+2\sigma$	.15	.38	.49
	-2σ	.014	.041	.045
	$+3\sigma$	3.0	8.5	12.5
	-3σ	.44	.29	.32

The figures in the table are:

$$\frac{|v(x_1) - T(x_1, 1, n)|}{v(1)},$$

where $v(x)$ is the exact Black-Scholes call option formula, $T(x,x_0,n)$ is its n^{th} order Taylor approximation based at x_0, and $\delta_1 = \log x_1$.

Table II

Proportional errors

Projections from in- or out-of-the-money values

	δ_0	δ_1	σ: .05	.15	.25
For n = 2	-3σ	0	.93	.93	.91
	3σ	0	.94	.95	.95
	-1.5σ	1.5σ	1.7	1.6	1.3
	1.5σ	-1.5σ	1.8	1.9	1.8
For n = 4	-3σ	0	.42	.22	.044
	3σ	0	.55	.65	.72
	-1.5σ	1.5σ	2.1	2.9	3.1
	1.5σ	-1.5σ	1.4	.76	.26
For n = 6	-3σ	0	.38	.58	.53
	3σ	0	.13	.10	.29
	-1.5σ	1.5σ	1.6	3.7	4.8
	1.5σ	-1.5σ	.17	.44	.57

The figures in the table are:

$$\frac{|v(x_1) - T(x_1, x_0, n)|}{v(1)},$$

where $v(x)$ is the exact Black-Scholes call option formula, $T(x,x_0,n)$ is its n^{th} order Taylor approximation based at x_0, and $\delta_i = \log x_i$.

Table III

Minimum order such that proportional error < 5%, 1%

	δ_0	δ_1	σ: .05	.15	.25
<5%	0	σ	3	3	3
	0	$-\sigma$	2	2	2
	0	2σ	9	5	15
	0	-2σ	6	4	2
	0	3σ	20	30	$\nexists$*
	0	-3σ	16	9	7
	-3σ	0	5	16	4
	3σ	0	12	11	8
	-1.5σ	1.5σ	20	26	$\nexists$
	1.5σ	-1.5σ	14	10	12
<1%	0	σ	4	5	5
	0	$-\sigma$	4	2	4
	0	2σ	11	15	18
	0	-2σ	8	9	7
	0	3σ	22	40	$\nexists$
	0	-3σ	19	14	13
	-3σ	0	16	23	54
	3σ	0	16	11	13
	-1.5σ	1.5σ	24	34	$\nexists$
	1.5σ	-1.5σ	19	16	16

* Does not exist; series diverges.

The proportional error is defined as:

$$\frac{|v(x_1) - T(x_1, x_0, n)|}{v(1)},$$

where $v(x)$ is the exact Black-Scholes call option formula, $T(x,x_0,n)$ is its n[th] order Taylor approximation based at x_0, and $\delta_i = \log x_i$.

Table IV

Average Proportional errors

$\sigma = 0.15$

Order of Approximation	(A) Mean Error	(B) Mean Absolute Error	(C) Root Mean Squared Error
1	.413	.413	.657
2	-.092	.093	.324
3	-.074	.074	.198
4	.057	.062	.514
5	.011	.019	.189
6	-.052	.061	1.13

The error is defined as:

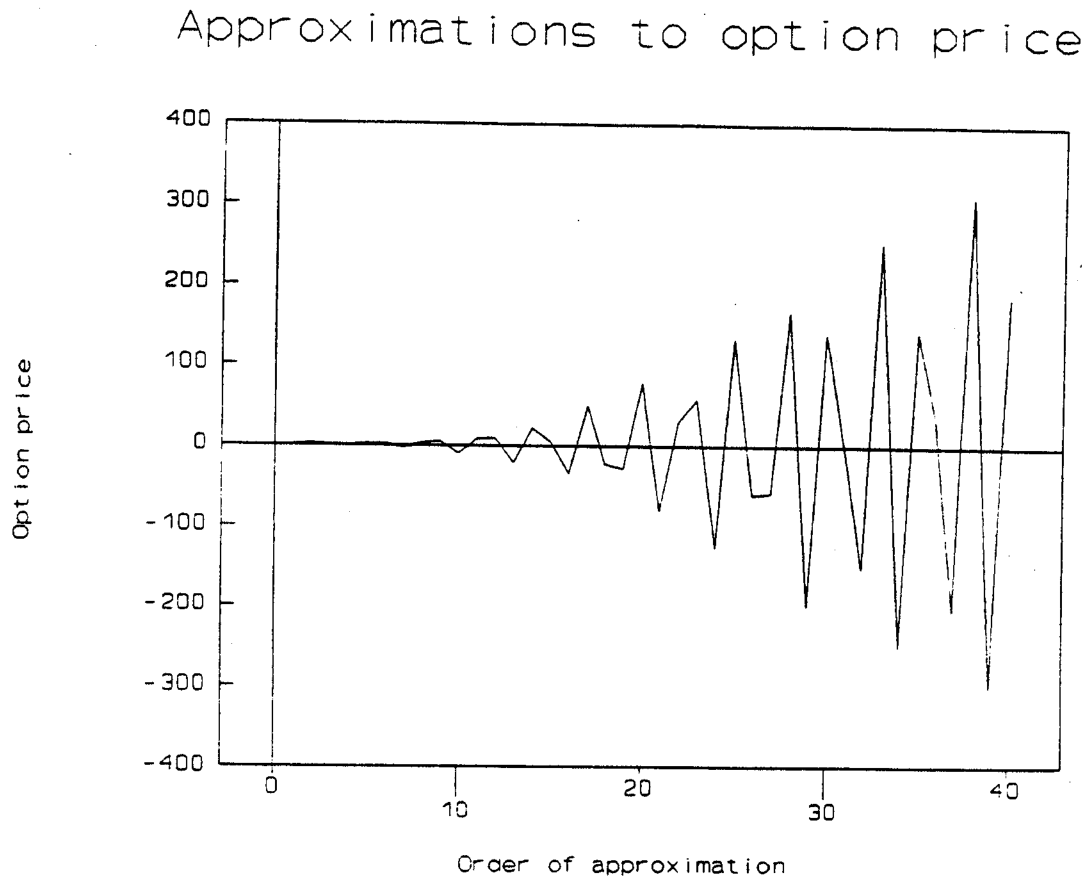
$$r_n(x) = \frac{v(x) - T(x,1,n)}{v(1)}$$

with the associated probability density function

$$f(x) = \frac{1}{x \cdot \sqrt{2\pi} \cdot \sigma} e^{-\frac{1}{2}\left(\frac{\log x + \frac{1}{2}\sigma^2}{\sigma}\right)^2} .$$

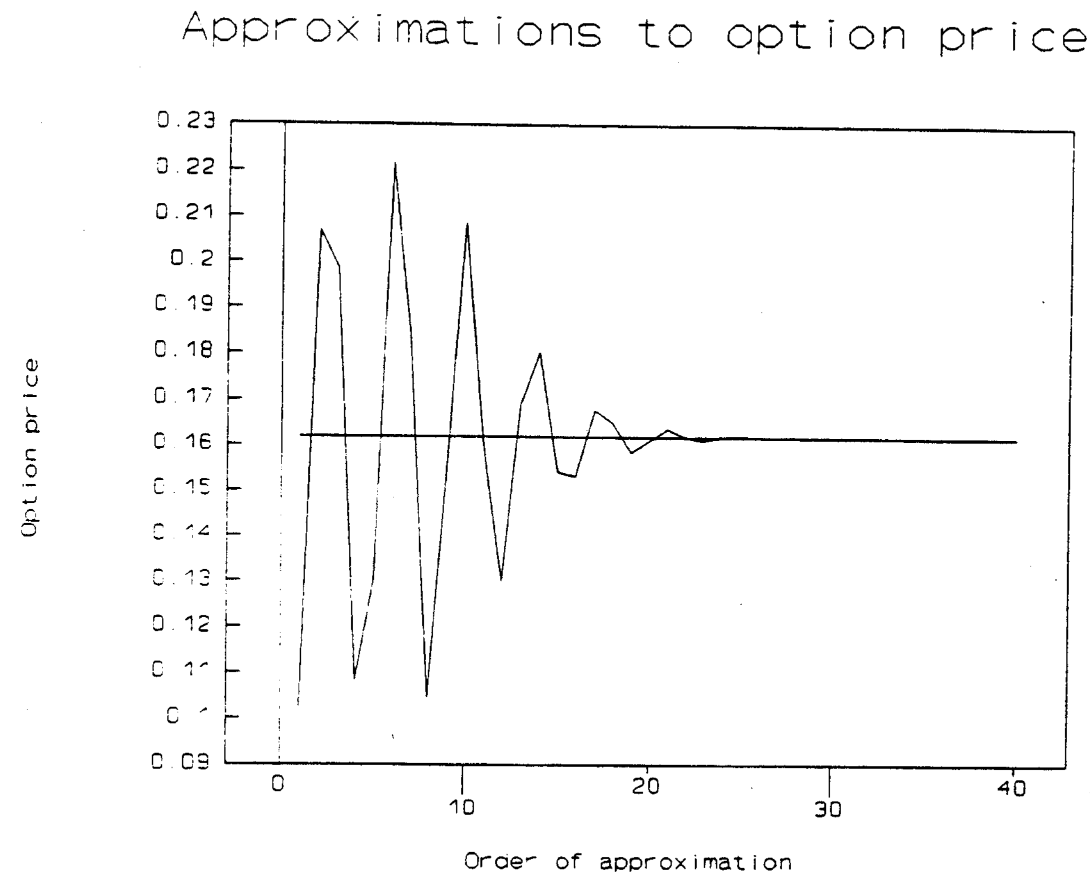
(A), (B), and (C) are defined in the usual way (see text).

Figure 1



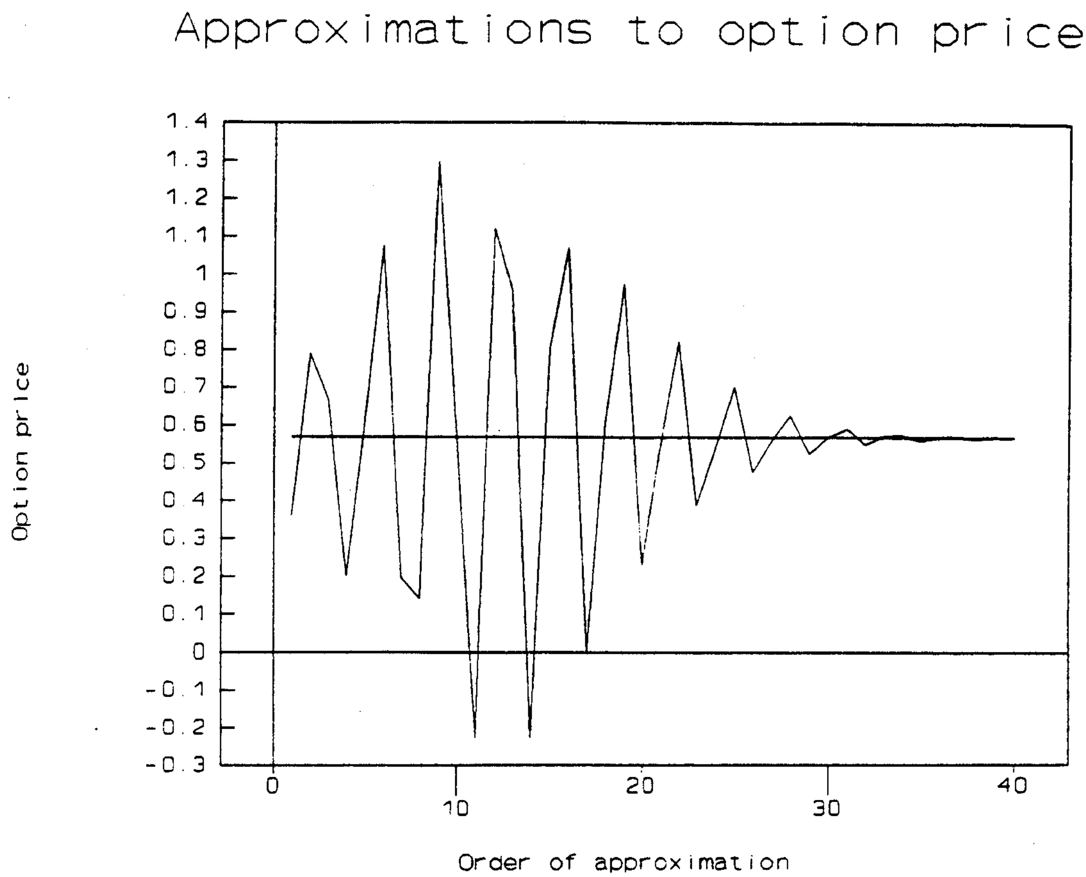
Taylor series approximations of orders 1 to 40 for the Black-Scholes option pricing formula $v(x)$ with $\sigma = 0.25$. The series is centered around $x_0 = 1$ and evaluated at the fixed value $x = e^{3\sigma}$. The exact option price is $v(x) = 1.117$.

Figure 2



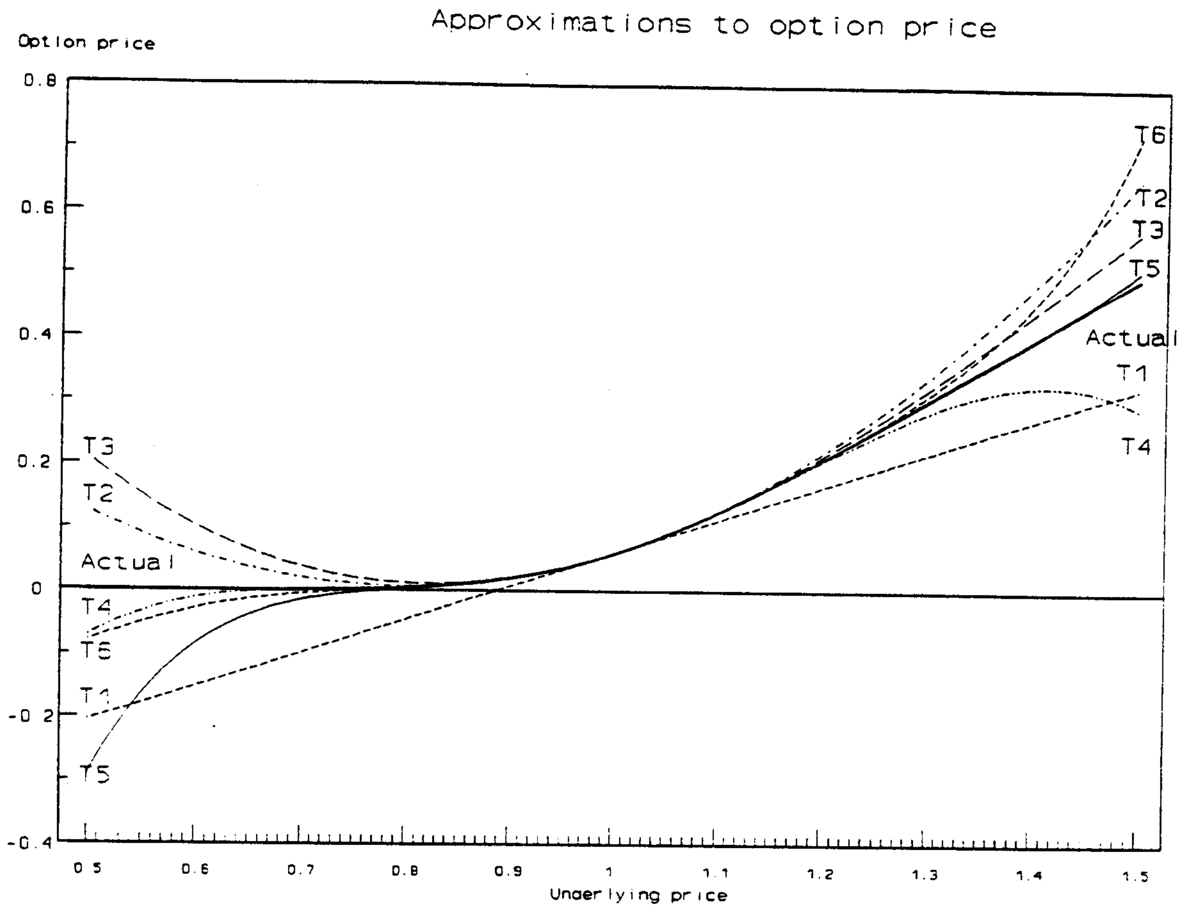
Taylor series approximations of orders 1 to 40 for the Black-Scholes option pricing formula $v(x)$ with $\sigma = 0.05$. The series is centered around $x_0 = 1$ and evaluated at the fixed value $x = e^{3\sigma}$. The exact option price is $v(x) = 0.162$.

Figure 3



Taylor series approximations of orders 1 to 40 for the Black-Scholes option pricing formula $v(x)$ with $\sigma = 0.15$. The series is centered around $x_0=1$ and evaluated at the fixed value $x = e^{3\sigma}$. The exact option price is $v(x) = 0.568$.

Figure 4



Taylor series approximations of orders $n=1,2,\ldots,6$ for the Black-Scholes option pricing formula $v(x)$ with $\sigma= 0.15$. For each n, the series is centered around $x_0=1$ and presented over a range $\frac{1}{2} \leq x \leq 1\frac{1}{2}$. The n^{th} order approximation is labelled "Tn".