INTEREST RATE EXPECTATIONS AND
THE SHAPE OF THE YIELD CURVE

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Introduction

A great deal of empirical research by market practitioners and academics on the term structure of interest rates has been concerned with determining the informational content the yield curve provides regarding future interest rates. Specifically, do current forward rates derived from the term structure represent an unbiased forecast of expected future interest rates? Understanding the mechanics of forward rates is important to investors concerned with enhancing returns, and policy makers who need to distinguish between a rising term structure and expectations for rising interest rates. According to the rational expectations hypothesis of the term structure (REHTS) long term rates should reflect market expectations for the average level of future short-term rates. As such, the rate spreads that form the basis of forward rates should predict future changes in the spot rate. In the purest interpretation of REHTS there are no term premiums in forward rates and changes in the slope of the yield curve are equivalent to what the market expects interest rates to be at a particular point in the future. However, the more generalized version of REHTS relaxes this position somewhat by assuming that forward rates are equivalent to expected rate changes, plus a constant and non-varying term premium required by investors to invest in longer maturities. This term premium is the difference between the forward rate and the corresponding expected spot rate. Unfortunately, neither expected interest rate changes nor term premiums are directly observable in the market. At the same time, however, studies have found that
forward rates unadjusted for the size and behavior of term premiums have generally proven to be poor forecasts for future spot rates.

The purpose of this paper is to examine whether REHTS assumptions conform to the term structure of outstanding U.S. Treasury securities from 1973 to 1995, and to examine the behavior of term premiums and to what extent they influence the shape of the forward curve. In section I we provide a short review of spot rates, forward rates, and the putative role of term premiums in a REHTS framework. In section II, an overview of previous studies conducted on REHTS provides the framework in which we discuss the validity of REHTS assumptions. REHTS assumptions are re-examined using familiar regression tests to determine the forecast power of forward rates for subsequent spot rates, and we use excess holding period returns, the extra return earned on a security sold prior to maturity, as the *ex post* measurement of the term premium. We argue that: (1) forward rates explain only some of the variance in future spot rates; (2) the forecast power of forward rates varies with maturity; and (3) the term premia is time-varying.

In section III, we decompose the forward rate into specific factors that influence the shape of the forward curve to further examine how the prevalence of term premiums contribute to the failure of REHTS. We demonstrate that the forward rate can be viewed as a combination of the current spot rate, a term premium, and an
expected interest rate change, where the term premium is the sum of a risk premium and a convexity premium. Using this forward rate model, we find that on average term premiums have contributed more to the shape of the forward curve than have expected rate changes. These term premiums are highest at the short-end of the yield curve but diminish at longer maturities owing to the impact of what is called a convexity bias in the term premium. In section IV, we extend our analysis on the behavior of term premiums to the nature of the yield curve itself. We examine how expected and past interest rate volatility, as well as the slope of the yield curve, may provide information on the size of expected term premiums. We find a positive correlation between volatility and the slope of the yield curve, and attribute this to the yield volatility affect on the term premium. Using an interest rate term structure model (Chen, 1990) high yield volatility tends to increase term premiums at the short-end of the yield curve by raising the required risk premium, but diminishes term premiums at longer maturities owing to the impact of the convexity premium. Taken together, the generally poor forecast power of forward rates and the variable quality of term premiums are inconsistent with the theory of rational expectations. Yet, interest rate volatility and the convexity premium may help explain why it fails.

I. Background Information on Yield Curve

Most securities can be packaged as a portfolio of zero-coupon instruments. Zero-
coupon yields, otherwise known as spot rates, are used to construct forward rates that form the basis of no-arbitrage pricing, the first step in the valuation of fixed-income instruments or other interest-rate-contingent claims such as caps, swaptions, and bond-options. Spot rates can be obtained from marketable Treasury STRIPS, but these rates may be biased owing to liquidity constraints and tax regulations which drive down the yields on long-term stripped instruments. These potential distortions in the stripped market are not a factor in the market for Treasury coupon securities. It is customary, therefore, to use outstanding coupon Treasury instruments as the basis for modeling a zero-coupon curve. Our data set consists of spot rates constructed from re-packaged outstanding coupon Treasury securities derived from zero-coupon instruments using a cubic smoothing spline function. The zero-coupon curve is free of liquidity and coupon effects that are common in outstanding Treasury securities.

The forward rate is an arbitrage-free rate that guarantees the return required for an investor to be indifferent from investing in bonds of different maturities. For example, a profit maximizing investor with a six month horizon that has the option to invest in a six-month Treasury security or two three-month Treasury securities will be indifferent if the yield after six months is the same on both investment alternatives. The investor knows the spot rate on the three-month bill and the spot rate on the six-month bill, but does not know the rate on the three-month bill in three months. However, the three-month rate in three months is implicit in the six-month rate today, and the
forward rate uses that information to construct an arbitrage-free rate at discrete intervals. The expectation is that forward rates reflect expected future spot rates. Otherwise, investors would not be indifferent to investing at the six-month spot or three-month forward rate, avoiding the former (latter) if expected future three month spot rates were above (below) the three month forward rate.

In its purest form (pure expectations), the rational expectations theory makes no allowance for term premiums, and changes in the slope of the yield curve are equivalent to what the market expects interest rates to be at a particular point in time. In this case one would not expect any variation between the forward rate and the future spot rate, and the one period return over any investment horizon is certain and independent of the maturity of the Treasury security, or when it is sold. In principle, therefore, all Treasury bonds should have the same one-period expected return, one equal to the spot rate for that period, and excess returns (the holding period return minus the current spot rate) should, therefore, be zero. This strict theoretical interpretation of rational expectations presumes that no compensation for longer-term investments which are generally considered more risky than short-term investments are demanded by investors. A more generalized version of the rational expectations theory (modified expectations) assumes that longer-term rates do reflect a term premium. This term premium, defined as the difference between the forward rate and the corresponding expected spot, is assumed to be constant and non-varying over time.
Forward rates should, therefore, be equivalent to the sum of expected rate changes and this constant term premium required by investors to invest in longer maturities. If term premiums are positive, forward rates should systematically exceed the expected spot rate.

II. A. Testing Rational Expectations

The preferred procedure to test the validity of the rational expectations theory of the term structure is to examine the forecast power of forward rates for future spot rates using regression analysis. Since rational expectations presumes that the forward rate is approximately equal to the expected change in the base spot rate, researchers can regress changes in the spot rate against the forward-spot premium, the difference between the forward rate and the current spot rate \((f_{t}^{\tau} - y_{t})\), to determine how well forward rates predict future spot rates. The term premium is generally defined as the difference between the forward rate and the expected spot rate with a maturity that corresponds to the forward rate, and is equivalent to the forward rate premium less the expected change in the spot rate. To set the notation, the term premium is defined as the difference between the forward rate, \(f_{t}^{\tau}\), at time \(t\), \(\tau\) periods ahead, and the expected future spot rate \(E(y_{t+\tau})\) so that

\[
f_{t}^{\tau} - E(y_{t+\tau}) = \theta_{t}^{\tau}
\]  

\(1\)
where $\theta$ is the term premium. If we subtract the spot rate, $y_t$, from both sides and rearrange terms we get

$$f_t^r - y_t = E(y_{t+\tau}) - y_t + \theta_t^r \tag{2}$$

In this equation, the forward-spot premium, $f_t^r - y_t$, equals the expected change in the spot rate plus the term premium. If we assume that forward rates are formed rationally then we can write

$$y_{t+\tau} = E(y_{t+\tau}) + \epsilon_{t+\tau} \tag{3}$$

where $\epsilon_{t+\tau}$ is a forecast error uncorrelated with other information at time $t$. Using equations (2) and (3) yields the following regression equation

$$y_{t+\tau} - y_t = \alpha_0 + \alpha_1 (f_t^r - y_t) + \epsilon_{t+\tau} \tag{4}$$

The error term $\epsilon_{t+\tau}$ has an expected value of zero under a pure expectations assumption, and a constant value under a modified expectation of a constant premium. In both cases, this term is assumed to be uncorrelated with the forward spot premium at time $t$ in order to estimate the $\alpha_1$ coefficient consistently. The presumption in both cases is that $\alpha_1$ should not be significantly different from 1.00. In this case, all variation in future spot rates is reflected in current forward rates. A significantly different value would contradict the assumption of term premiums that are constant or equal to zero.
A value of $\alpha_1 = 0.00$ would suggest that the forward rate premium has no power to forecast the change in the spot rate $\tau$ periods ahead. Alternatively, a value of $\alpha_1$ significantly greater than zero and less than 1.00 indicates that the yield curve has some predictive power for subsequent spot rates.

Testing the forecast power of forward rates using the changes in rates rather than the level of rates removes potential bias in regression results. Interest rate levels from one-month through 12 month rates tend to follow a unit root process where month-to-month levels of the short rate are highly autocorrelated suggesting nonstationarity (Fama, 1984). Because previous rate levels influence the expectation for future rate levels, using interest rate levels to examine the forecast power of the yield curve would have an upward bias on $\alpha_1$ coefficients. Changes in the spot rate, however, show little autocorrelation, and formal stationarity tests using the Augmented Dickey-Fuller (ADF) test indicate changes to be stationary (Remolona, Dziwura, and Pedraza, 1995). Term structure literature generally focus tests on the forecast power of forward rates with respect to changes in spot rates. Forward rate forecasts that yield an $\alpha_1$ coefficient close to 1.0 should be considered exceptionally impressive given that interest rates changes on maturities under 12 months appear to follow a random walk.

More recent research has examined the expectations hypothesis by testing cointegrating vectors along the whole yield curve (Campbell and Shiller, 1987 and Shea,
1992). The picture that emerges from this research is that generally short rates are cointegrated with the rest of the yield curve. However, it is difficult to impose the short rate expectations hypothesis's yield spread restrictions of cointegrating vectors on other parts of the yield curve. That is, the short rate co-moves with the rest of the yield curve but the short-rate yield spread may not.

A host of studies have tested the theory of rational expectations using various maturities in the above regression formula. Although forward rates have demonstrated some ability to forecast subsequent spot rates in some studies, the overwhelming conclusion in the literature is that long rates do not equal the average level of current and future short-term rates. In addition, the predictive value of forward rates appears to vary at different sections of the yield curve. Tests using three and six month forward rates (Hamburger and Platt, 1975, Mankiw and Miron, 1986, and Shiller, Campbell, and Schoenholtz, 1983) provide the clearest rejection of the ability of the yield curve to predict subsequent three and six month spot rates, reporting values for \( \alpha_t \) that are not significantly different from zero. Shiller, Campbell, and Schoenholtz (1983) conclude that "the simple theory that the slope of the term structure can be used to forecast the direction of future changes in interest rates seems worthless." However, the predictive power of forward rates improves at other maturities. The cumulative predictive power of one-month rates using maturities up to six months has been found to be significantly different from zero for one-month rates up to three months (Fama, 1984), and up to two
months into the future (Mishkin, 1988). The forecasting power of forward rates also appears to approve when testing in intervals of years. Fama and Bliss (1987) find that the yield curve from 1 to 5 years has had significant forecast power, with the one-year forward rate four years in the future explaining almost 48 percent of the variance of the four-year change in the one year rate. Campbell and Shiller (1989) derive similar results using a weighted average change in the one year rate.

We run the regression tests using equation (4) to update results testing rational expectation assumptions using our data set for changes in one-month and 12 month spot rates first conducted by Fama (1984) and Fama and Bliss (1987). Our data consists of spot rates and forward rates continuously compounded and based on end-of-month observations derived from zero-coupon yields observed between October 1982 to January 1995. Our results reported in Table 1 also finds $\alpha_1$ - coefficients well under 1.00 and significantly different from zero. In the one-month regression set, forward rates do explain a great deal of the variance in subsequent spot rates as indicated by $\alpha_1$ - coefficients that range from 0.34 to 0.53. This forecast power deteriorates at longer horizons, during months 10-12, as evidenced by progressively lower $\alpha_1$ - coefficients, and $R^2$ values that are negatively correlated with maturity. Our results for the one year regressions are similar to Fama and Bliss (1987). Long-maturity forward rates explain more of the variance in the spot rate than those reported in the one month regressions, and are particularly strong for 3 to 4 years out. In contrast to findings by Fama (1984)
and Mishkin (1988), however, we find the forecast power of one-month forward rates for subsequent one-month rates to be significantly different from zero out to 11 months, indicating that the ability of the yield curve to forecast changes in one-month spot rates beyond the three month horizon may be stronger than originally believed. Nevertheless, those forecast errors remain significant. In Figure 1 we plot a series of one-month forward rates observed at the end of December 1994 against actual spot rates observed through 1995. This discrepancy between forward rate forecasts and subsequent spot rates is emblematic of the wider conclusion derived from the regressions.

We can draw two basic conclusions from the regression results in Table 1. First, although forward rates may have some predictive power for future spot rates, even the strongest results cannot support rational expectation theory assumptions. Second, the forecast power of forward rates varies depending on the intervals examined (month, quarter, annual rates) and with maturity. The principal reason forward rates cannot forecast future changes in the spot rate according to theory is due to the variability of term premiums which we infer from our regression estimates of \( \alpha \) from equation (4) that are far lower than the theoretical value of 1.00. If we assume that market participants efficiently use all available information in forming expectations, then the tendency to systematically over-or-under forecast future interest rates suggests that rational expectation theory assumptions cannot adequately explain the changing slope
II. B. Term Premiums

Since the predictive value of forward rates is influenced by the variation in expected term premiums and expected interest rate changes, reliable information on future term premiums would enhance our ability to extrapolate rate expectations from the yield curve. Unfortunately, these term premiums cannot be reliably quantified ex ante without knowing precisely the level of interest rate expectations. At the same time, however, interest rate expectations cannot be isolated without knowing the size of the term premium reflected in the yield curve. It is possible, however, to obtain reliable information on past term premiums by calculating the excess holding period returns from holding n-month securities for m-months compared to the return from holding m-month securities. This excess return reflects a premium to investors for holding longer n-month securities and it enables researchers to observe the distinction between the term premium and past expected interest rate changes, and how the term premium behaves on average over time. To set the notation, define $P$ as the price at time $t$ of a pure discount Treasury bond that matures at time $t + \tau$ and pays $1$ at maturity. Define $y_t$ as the spot rate of interest at time $t$ so that using continuous compounding

$$
P_t = \exp(-ty_t)$$

(5)
The one-period holding period return, $h$, from $t$ to $t+1$ on a security with $\tau$ periods to maturity is

$$h_{t+1}^{\tau} = \ln\left(\frac{P_{t+1}^{\tau-1}}{P_t^{\tau}}\right) = \ln(P_{t+1}^{\tau-1}) - \ln(P_t^{\tau}) \quad (6)$$

where $P_{t+1}^{\tau-1}$ is the price of a discount Treasury security at $t+1$. Subtracting the guaranteed spt rate, $y$, from both sides of equation (6) yields excess holding period return, $h_{t+1}^{\tau} - y$, or the term premium.

Using this methodology and ignoring transaction costs we model the term premium in Figure 2 for one-month ($\tau = 1$) and one-year ($\tau = 12$) zero-coupon Treasuries from 1973 to 1995. Contrary to rational expectations, the level of the term premium can be very volatile and are not constant through time or between one-month and one-year securities. The variability of the term premium compromises the forecast power of the yield curve because the fraction of expected interest rate changes implied by the forward-spot premium is not constant, and will be large or small depending on the size of the term premium factored into the yield curve. Although the average money-market term premium is positive, they can assume negative-value at times when subsequent interest rates rise significantly higher than expected rate changes anticipated by the forward-spot premium. In this case, excess holding period returns become negative when upward shifts in the yield curve between $t$ and $t+1$ are sufficient.
to render the second side of equation (6) negative. Negative term premiums indicate that an investor would have been better off buying consecutive m-month securities than a single n-month security which had less value at t + 1.

A complementary regression method to test REHTS is to replace the change in rates in equation (4) with the excess holding period return, \( h_{t+1} - y_t \), as the dependent variable to test whether variation in expected term premiums causes variation in the shape of the yield curve. This can be written as

\[
h_{t+1} - y_t = \beta_0 + \beta_1 (f_t^x - y_t) + \epsilon_{t,t}
\]  

(7)

Rational expectations would expect the value of the \( \beta_1 \) coefficients to be 0.00 because all variation in the forward spot premium is presumed to be due to expected interest rate changes and not variation in the term premium. Conversely, a positive value for \( \beta_1 \) is an indication that the forward rate premium has forecasting power for excess returns that vary through time. Our regression results in Table 2 indicate that \( \beta_1 \) coefficients are significantly different from 0.00. The implication is that variation in the slope of the yield curve is affected by term premiums that change over time in a systematic manner that shows up in the forward-spot premium. The higher the fraction of the variance in the forward spot premium due to the variance in expected term premiums, the smaller the fraction due to variance of expected rate changes, and the greater will be the departure from rational expectations.
II. C. Interest Rate Expectations

The manner in which interest rate expectations are formed may help explain the variation in the power of forward rate forecasts at different parts of the yield curve owing to its affect on the balance of expectations and term premiums that constitute the forward rate premium. Any forecast of future interest rates must consider monetary policy because interest rate targeting by the Federal Reserve for the overnight rate for federal funds has an enormous influence on the level of other money-market rates. Since the market must forecast future Fed behavior, the process by which interest rate expectations are formed may produce a systematic bias, one that could affect the forecast power of the yield curve. If the variance in expected rate changes is smaller than it otherwise would be in the absence of Fed policy, the ratio of the variance of the term premium to the variance of the expected change in interest rates will rise resulting in larger forecast errors for subsequent spot rates.

Mankiw and Miron (1986) argue that this is what has happened since the founding of the Federal Reserve in 1914. Using equation (4), the authors find \( \alpha_1 \) - coefficients on the spread between the 3 and 6 month rates close to a value of 1.00 as predicted by theory in the period between 1890 and 1914 to support this claim. They argue that the reason the market has had a harder time forecasting short-term rates since 1914 is because at any given time the market rationally expects the fed funds rate
to be at the level the Federal Reserve expects to maintain given all available information, and short rates thus assume a random walk where changes in the short rate are interpreted to be close to zero. Predictable variation in short rates that would otherwise be incorporated into yield spreads is diminished, leaving little change in the slope of the yield curve. Rudebusch (1995) developed a daily targeting model of the Fed funds rate and supports the Mankiw and Miron (1986) view that the failure of rational expectations at the 3 to 12 month range can be attributed to this "target persistence." However, Rudebusch (1995) demonstrates that if the market perceives the target rate as persistent but not permanent (i.e., expected changes in the target rate is not zero), the spread between the 6 and 3 month rates do have forecast power, but only when probabilities of negative and positive target rate changes are factored into the 6 and 3 month yields.

The view that market participants have a harder time factoring for expected rate changes due to Federal Reserve interest rate-targeting can help explain some of the poor forecasting power of the yield curve in the 3 to 12 month range, but is inconsistent with superior forecasts generated by the yield curve up to 3 months and from 1 to 5 years. The reason for this discrepancy is that the reaction of interest rates to the process that generates interest-rate targeting differs in these sections of the yield curve. The slope of the yield curve between 3 and 12 months varies less than it does for rates between 1 week to 3 months in response to information influencing policy expectations.
(Cook and Hahn, 1989). The greater variation in expected interest rate changes as a fraction of the forward rate premium explains why the yield curve from 1 to 3 months generates better interest rate forecasts than those derived from the yield curve in the 3 to 12 month range. One would expect interest rate movements to be most predictable at the shortest-end of the curve because market participants can better gauge the likely influence of economic data on Fed policy. However, it is counter-intuitive that the forecast of 12 month rates in 4 years is superior to that of one-month rates in 4 months (see Table 1) or 3 month rates in 6 months. The long horizon forecasting power may be attributed to market expectations for slow mean reversion of short rates at long horizons. The expectation is that the Federal Reserve policy eventually forces interest rates to revert to a mean level over time, and if the market factors in the expectation for "Fed smoothing" some of the uncertainty surrounding the level of future short-term spot rates is reduced.

III. Components of the Forward Rate

In this section we decompose the forward rate into component parts to examine in more detail how the interaction of factors that lend shape to the forward curve prevent forward rates from being unbiased forecasts of expected spot rates. We equate the forward rate to the current spot rate, market expectations for changes in the spot rate, and the term premium which is equivalent to the sum of a risk premium and a
convexity premium. This procedure is in the spirit of the fixed-income pricing term
structure models of Black and Karinski (1991) and Heath, Jarrow, and Morton (1992)
and Hull and White (1990). Decomposing the forward rate in this way enables us to
observe how the structural components of term premiums change through time and
contribute to the time-varying quality of term premia documented earlier. Again to set
the notation, define \( P \) as the price at time \( t \) of a pure discount Treasury bond that
matures at time \( t + \tau \) and pays \( \$1 \) at maturity, and define \( y_t \) as the spot rate of interest at
time \( t \) so that using continuous compounding we can rearrange equation (6) so that the
one period holding period return, \( h_t \), is

\[
h_{t-1}^\tau = \left[ \ln(P_{t-1}^{\tau-1}) - \ln(P_t^{\tau-1}) \right] + \left[ \ln(P_t^{\tau-1}) - \ln(P_t^{\tau}) \right]
\]  

or

\[
h_{t-1}^\tau = \Delta \ln(P_{t-1}^{\tau-1}) + \ln\left(\frac{P_t^{\tau-1}}{P_t^{\tau}}\right)
\]

The first term on the right can be written as a Taylor series expansion and
approximated by

\[
\Delta \ln(P_t^{\tau-1}) = - \frac{d}{d y_{t-1}} \ln P_t^{\tau-1} \Delta y_{t-1} + .5 \frac{d^2}{d y_{t-1}^2} \ln P_t^{\tau-1} (\Delta y_{t-1})^2
\]
where \( d \ln P/d y \) is equal to a bond's duration and \( d^2 \ln P/d y^2 \) is equal to a bond's convexity. Duration is an indication of a bond's interest rate sensitivity, measuring the linear relationship between the change in the price of a bond for a change in interest rates. Convexity measures the rate of change of duration given a change in interest rates. We assume parallel shifts in the yield curve in calculating the duration and convexity of bond’s across various maturities. Although yield changes across the Treasury curve are not perfectly correlated, using duration and convexity in this framework allows us to analyze and interpret the shape of the yield in a consistent manner.

Convexity and duration are necessary to consider because changing expected returns across maturities will alter the balance between the term premium and the rate expectation component of the forward rate. The impact of rate expectations captured by duration signifies the expected capital gain or loss caused by the expected change in rates, and the convexity premium captures the expected capital gain caused by rate uncertainty. All option-free fixed-income securities have, to varying degrees, a positive convexity so that the price of a bond will rise more as yields decline then it will decline in value given an equivalent rise in yields. The greater the uncertainty of future interest rate movements (volatility), the greater is the effect that convexity will have on the value of the bond, and the higher will be expected returns. As the impact of convexity increases as a function of maturity and interest rate volatility, investors holding longer
maturities will require a lower yield owing to the prospect of enhancing returns as a result of convexity. The influence that this nonlinearity between a bond’s price and yield exerts on the forward rate is the convexity premium. We account for the downward sloping effect that convexity imparts on the forward rate by adding the negative convexity premium to the forward rate.

The second term in equation (9) is the forward rate, $f_t$, at time $t$ for a $\tau$ period investment and can be expressed as

$$\ln\left(\frac{P_{t-1}^\tau}{P_t^\tau}\right) = f_t^\tau$$  \hspace{1cm} (11)

Plugging equations (10) and (11) into (9) gives

$$h_{t+1}^\tau = -\frac{d}{d y_{t-1}} \ln P_{t-1}^\tau \Delta y_{t-1} + .5 \frac{d^2}{d y_{t-1}^2} \ln P_{t-1}^\tau (\Delta y_{t-1})^2 + f_t^\tau$$  \hspace{1cm} (12)

Subtracting the spot rate, $y_t$, from both sides we can write

$$h_{t+1}^\tau - y_t = -\frac{d}{d y_{t-1}} \ln P_{t-1}^\tau \Delta y_{t-1} + .5 \frac{d^2}{d y_{t-1}^2} \ln P_{t-1}^\tau (\Delta y_{t-1})^2 + f_t^\tau - y_t$$  \hspace{1cm} (13)

Again rearranging terms and taking expectations conditional on all publicly available information
\[ f^T_i = y_i + E(h^t_{i+1} - y_i) + \frac{d}{dy_{i-1}} \ln P_{i-1} \cdot E(\Delta y_{i-1}) + 0.5 \cdot \frac{d^2}{dy_{i-1}^2} \ln P_{i-1} \cdot E(\Delta y_{i-1})^2 \]  

(14)

where \( E(h^t_{i+1} - y_i) \) is the expected risk premium and \( E(\Delta y_{i-1}) \) is the expected change in the spot rate. Since the term premium is the sum of a risk premium and a convexity premium, we can estimate the term premium by adding the convexity premium represented by the last term in equation (14) to the expected risk premium \( E(h^t_{i+1} - y_i) \).

Allowing \( E(\Delta y_{i-1})^2 \) to be spot rate volatility, \( \sigma^2 \), we can then define the forward rate as equal to the current spot rate, a risk premium, \( RPM \), the impact of rate expectations (duration * the expected spot rate), and a convexity premium. Equation (14) becomes:

\[ f^T_i = y_i + RPM + \frac{d}{dy_{i-1}} \ln P_{i-1} \cdot E(\Delta y_{i-1}) + 0.5 \cdot \frac{d^2}{dy_{i-1}^2} \ln P_{i-1} \cdot \sigma^2 \]  

(15)

This forward rate model is instructive for depicting the interaction of components shaping forward rates and how term premiums behave, on average, over time. Using estimates of historical averages for each of the relevant variables in equation (15) can tell us the percentage yield impact these components have had on the average forward curve during the same period. Using data for zero-coupon treasuries from 1973 to 1995, we quantify average risk premiums and convexity premiums and then back out expected interest rate changes according to the forward rate equality.
shown by equation (15). Our results depicted in Figure 3 indicate that negative and positive expected interest rate changes effectively wash out over time and on average account for only a small portion of the forward rate. The average term premium, however, is positive and is the principal reason that the average forward curve is upward sloping. The term premium is dominated by the risk premium at shorter maturities. The convexity premium is negligible at the short-end because the expected capital gain caused by rate uncertainty is only significant at longer maturities. As the convexity effect increases, it reduces the overall term premium required by investors to be risk-neutral across different maturities. As a result, term premiums rise less significantly, and begin to flatten around 5 years before decreasing at longer maturities.

Knowing how term premiums behave on average, however, cannot be used as a reliable prescription for expected rate changes incorporated in current yields. Using historical average premiums to back out expected rate changes ex ante can be misleading owing to the historical sample period selected to determine average term premiums. Because term premiums swing positive and negative through time as illustrated in Figure 2, different sample periods will generate a different average term premium, and expected rate changes that we back-out from forward rates will not be consistent. This is illustrated in Figure 4. Changing average premiums from 1973-1994, to one observed between 1990-1994, changes the putative rate changes expected in the one-year forward rates observed on January 1994. The term premium (TP in Figure) is
longer at all maturities during the 1990-1995 period, and despite the upward sloping forward curve, rate expectations assume a downward trajectory in contrast to an expectation for a modest increase in rates using average term premiums from 1973-1995. To reliably extrapolate expected interest rate changes \textit{ex ante} requires a better gauge of the expected level and behavior of future term premiums.

IV. Yield Volatility and the Term Premium

In this section, we examine how future and past interest rate volatility, as well as the slope of the yield curve, may provide information on the size of expected term premiums. We assume that the time-variant nature of term premiums is an indication that a bond's market price of risk also varies through time. Intuitively, the volatility of interest rates should, therefore, play an important role on the size of expected excess returns and by extension on the shape of the yield curve. The principle of volatility-driven term premiums is conceptually appealing. It is consistent with modern asset pricing theories in which the rate of return on an asset is positively correlated with the riskiness of that asset. When volatility is high the risk premium factor embedded in forward rates should assume a greater importance relative to the expectations component, and vice-versa when volatility is low. At the same time, however, interest rate volatility affects the magnitude of the convexity premium (see Equation (15)). For a given increase (decrease) in volatility, we would not expect term premiums to rise (fall)
equally along all points of the maturity spectrum.

In general, excess returns tend to be high during recessions and low during a business expansion. Bonds tend to have higher excess returns from declining interest rates around cyclical troughs, and declining excess returns from rising interest rates around cyclical peaks (Fama and French, 1989). On the one hand, this may be attributed to monetary policy targeting of the spot rate in response to changing economic conditions. During recessions the Federal Reserve typically lowers rates in the Federal Funds market, thereby increasing holding period returns for all non-parallel shifts in the yield curve. On the other hand, the required premia or degree of risk aversion also tends to increase (decrease) during periods of slow (strong) economic growth. Assuming that markets are efficient, the slope of the yield curve should incorporate all available financial and macroeconomic information in a shape that reflects a combination of prevailing expectations for interest rates and premiums. Excess returns or term premiums should, therefore, be related to the steepness of the yield curve.

We now examine the relation between yield curve steepness and term premiums. Here, we regress the premium using annual returns on the slope of the yield curve at the short end using:
\[ h_{t+1} = 12y_t + \beta_0 + \beta_1 (y_t - 3y) + \epsilon_{t+1} \] \hspace{1cm} (16)

where \( y \) is the \( \tau \) year rate and \( 12y \) and \( 3y \) are the 12-month and 3-month spot rates, respectively. The results are reported in Table 3 for the October 1982 through January 1995 period. Entries in the first column set the spread between the \( \tau \) year spot rate and the 3-month spot rate to explain the variation in the premium earned on a \( \tau \) year zero-coupon bond and sold in 1 year. In forecasting for 12-month premiums we only use yield curve information observed at time \( t \). We find that the slope of the yield curve is significantly related to subsequent premiums at each maturity in our sample. The \( R^2 \) coefficients indicate a progressively better fit through our sample period, and by year 5 the slope of the yield curve is explaining 10 percent of the monthly variation in excess returns. The positive relationship indicates that the premiums trend higher when the yield curve is steep, and lower when the yield curve is flat. Although 90 percent of the premium variation is unpredictable, we should expect higher premiums with steeper yield curves, and lower expected rate changes as a component of forward rates.

If the volatility of interest rate changes is positively correlated to the slope of the yield curve, then we can infer a positive relationship between volatility and premiums on the strength of preceding regressions. Our regression for the slope of the yield curve on volatility using data from October 1982 through January 1995 is specified as:
\[ y_t^r - 3y_t = \beta_0 + \beta_1 (\Delta y_t^r) + \epsilon_t \] (17)

where \( y_t^r \) is the yield on a \( r \) period instrument, \( 3y_t \) is the 3-month spot rate, \( \Delta y_t^r \) is the volatility of the yield, measured by the absolute percentage change in the yield. The regression results are reported in Table 4. As shown, the regression results are significant through year 3. The implication is that volatility has a positive and significant impact on the shape of the yield curve at the short-end, and a diminishing affect at the longer end of our sample. The positive correlation between yield volatility and the slope of the yield curve indicates that we should expect a steeper yield curve at the short-end when the expected level of interest rate volatility increases. The predominant factor driving the rise in yields at the short-end is a rising term premium, one that is dominated by a rising risk premium required by investors as compensation for greater interest rate uncertainty. The other component to the term premium, the convexity premium, has a negligible affect at the short-end. As earlier graphs illustrated, the convexity premium is significant only for maturities greater than 4 years and is not adequately captured by this sample.

A host of term-structure models allow for the decomposition of the forward rate into an expectations component, a risk premium, and a convexity premium. These models differ according to assumptions on factors influencing interest rates, the behavior of factors such as mean reversion, the risk premium, and the behavior of
volatility. One factor models (where the one factor is typically the spot rate) imply that rate changes and returns are perfectly correlated across bonds. However, we know that correlations across bond returns are not perfect owing to the time variation of term premiums and the imperfect correlation of yield changes across the Treasury yield curve. Multi-factor term structure models provide a richer theoretical framework. These models, which typically include different factor combinations relating to the spot rate, long rate, slope, and curvature among others, generate yield curves that are intended to simulate the imperfect correlation of bond returns and the changing shape of the yield curve at different maturities. As such, multi-factor models can be useful in demonstrating the nexus among yields, volatility, and premiums.

Chen (1995) created a three factor term-structure model where the factors are the current spot rate, the short-term mean of the spot rate, and the current volatility of the spot rate. Unlike most term-structure models, Chen makes spot-rate volatility a stochastic factor that is not directly tied to the level of the spot rate. This is a useful innovation because it provides a theoretical framework in which to examine how yield-levels can be a function of volatility owing to the affect it has on term premiums. Holding other parameters constant, Chen demonstrates that changing volatility affects the slope of the yield curve through the term premium as illustrated in Figure 5. At the short-end of the yield curve, an increase in volatility increases the risk premium and pushes the front end of the yield curve higher. Note, the steeper slope is a function of
the higher risk premium and not due to expectations for higher interest rates because the expected spot-rate remains unchanged.

This model is consistent with our regression results which related steepness and volatility to higher risk premiums at the short-end driving up the term premium component. At the longer end of the yield curve, the term premium is increasingly dominated by the convexity component which grows as a function of maturity and volatility. As an increase in volatility enhances the prospect of higher returns due to rate uncertainty, the convexity premium increases. The negative convexity values effectively reduce the required term premium for risk neutral investors to invest in longer maturities, thereby reducing required yields. As a result, when expected volatility increases (decreases) the yield curve will peak higher (lower) and at shorter (longer) maturities and exhibit greater (less) curvature at longer maturities. Litterman, Scheinkman, and Weiss (1991) find a similar affect due to an increase in yield volatility.

We can also analyze the behavior of time-varying premiums as a function of volatility in more detail with an autoregressive conditional heteroscedasticity (ARCH) model. This framework can be used to model the time-varying conditional variance of excess holding period yields in order to forecast rates of return which are, in practice, unforecastable. The time-varying premia are treated solely as risk premia where risk is due to unanticipated interest rate changes and is measured by the conditional variance
of one period excess holding period yields. In the ARCH framework, future volatility
depends on past volatility. If recent volatility is high the model will predict high future
volatility and high excess holding period yields. Using monthly data from 1973 to 1996,
we estimate an ARCH model using the format of Engle, Lilien, and Robins (1987). The
ARCH format is

\[ y_t = \beta_0 + \delta h_t + \varepsilon_t \]  \hspace{1cm} (18)

\[ h_t^2 = \gamma + \alpha \varepsilon_t^2, \]  

where \( y_t \) is the excess holding period yield, \( h_t \) the premium, \( h_t^2 \)
equal to a weighted sum of past squared yield changes, and \( \varepsilon_t \) an error term.

The estimates generated by the ARCH model are outlined in Table 5 for various
time horizons extending to two years. The non-zero value for the coefficient on the
time-varying premium, \( \delta \), indicates the existence of time-varying premium that
increase with an increase in volatility of the excess holding yield. Term premiums
generally increase over time becoming significant at the five-month through 12-month
horizons, after which the risk premium starts to decline. These results are consistent
with the presumption that term premiums are time-varying, as the risk premium
predominates at the short-end of the yield curve but becomes less significant at longer
horizons owing to the convexity premium.
Conclusion

Financial market prices are frequently used by investors and policy makers to define market expectations for future interest rates. The rational expectations theory of the term structure suggests that forward rates should provide reliable information on the course of future spot rates. However, changing term premiums that reflect a changing ordering of risks and expectations across maturities indicate that the traditional intuition based on rational expectations provides only limited insight on the curvature of the term structure observed on any given day. Neither the presumption that future spot rates must somehow achieve a level anticipated by the forward rate curve, nor that term premiums are constant is supported by the data presented in this paper and others in the field. The level of term premiums vary across maturities and through time, is the principal reason that the average forward rate curve slopes upward, and is a significant component of the forward rate premium. The operating assumption among academics and market practitioners is that interest rate expectations are formed rationally using all available information. Interest-rate targeting by the Federal Reserve influences the manifestation of rate expectations to varying degrees, but the systematic forecast errors captured by regression tests is not an indication that expectations are formed irrationally. Rather, it largely reflects the effect that variable term premiums have on the changing slope of the yield curve and the relative composition of the forward rate premium used to forecast subsequent spot rates.
Decomposing the forward rate illustrates how expected changes in interest rates is only one of several factors lending shape to the forward rate curve, and how one's interpretation of forward rates at any given time is affected by assumptions regarding the unobservable term premium. We know that term premiums are consigned to change over time, in part owing to the offsetting interaction of the risk premium and convexity premium, and in part to the correlation between yield volatility and term premiums. Higher expected volatility should increase term premiums at the short-end due to a higher risk premium, and decrease term premiums at the long-end owing to the greater impact of convexity. A steep yield curve may be less an indication of higher expected short term rates then it is an indication of high yield volatility and higher term premiums. However, since yield volatility and the slope of the yield curve cannot account for most of the variance in excess returns, more research is required to more accurately model future term premiums. Extrapolating precise information from the term structure is difficult, and in large measure, it is easier to determine what information forward rates do not reveal as opposed to deriving a reliable prescription for expected interest rate behavior. Given the behavior of term premiums discussed in this paper, however, market practitioners and policy makers cannot make reliable judgments on the putative path of spot rates by interpreting forward rates according to rational expectations theory assumptions.
References


Atlanta, Federal Reserve Bank of Minneapolis, and University of Iowa.


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\( R_f = \left( \frac{1}{K - 2} \right) + \alpha = \frac{1}{K - 1} + \alpha \)

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Monthly returns on the forward premium, \( \kappa \rightarrow \eta \text{ and } \kappa \rightarrow \eta - \eta \tfrac{\text{f}}{\text{f}} \text{ are the forward premium returns on the forward premium.} \)

Regressions of the excess holding period returns, \( \kappa \rightarrow \eta \text{ with } \kappa \rightarrow \eta - \eta \tfrac{\text{f}}{\text{f}} \text{ and } \kappa \rightarrow \eta \tfrac{\text{f}}{\text{f}} \text{ are the forward premium returns on the forward premium.} \)

Table 2
Significance at the 5 percent level. Parentheses indicate standard errors are in parentheses.

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October 1982 - January 1995

Regressions of the slope of the yield curve, %Y = β0 + β1%Y

The Slope of the Yield Curve and Volatility

Table 4

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October 1982 - January 1995

Regressions of the risk premium on the slope of the yield curve, %Y = β0 + β1%Y

Risk Premiums and the Slope of the Yield Curve

Table 3
Note: \(\hat{y}\) is the excess holding yield. Maximum likelihood estimates. Newey-West corrected standard errors in parentheses.

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March 1973 - January 1996

Monthly

\[
\begin{align*}
\alpha + \gamma & = \gamma' \\
\gamma + \phi' \gamma & = \gamma'
\end{align*}
\]

ARCH Term Premium Model

Table 5
The following papers were written by economists at the Federal Reserve Bank of New York either alone or in collaboration with outside economists. Single copies of up to six papers are available upon request from the Public Information Department, Federal Reserve Bank of New York, 33 Liberty Street, New York, NY 10045-0001 (212) 720-6134.


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