An Analysis of Brokers' Trading, With Applications to Order Flow Internalization and Off-exchange Block Sales

Sugato Chakravarty
Purdue University
West Lafayette, Indiana, 47906
765-494-6427
Fax: 765-494-0869
EMAIL: SUGATO@PURDUE.EDU

Asani Sarkar
Federal Reserve Bank of New York
33 Liberty Street
New York, New York, 10045
212-720-8943
Fax: 212-720-1582
EMAIL: ASANI.SARKAR@NY.FRB.ORG

Current version: April 1998
Previous version: May 1997

The views expressed in the paper are those of the authors' and do not represent those of the Federal Reserve Bank of New York or the Federal Reserve System. The usual caveat applies.

A General Model of Brokers' Trading, With Applications to Order Flow Internalization and Off-exchange Block Sales

JEL Classification number: G12

Abstract

We study a wide variety of issues related to brokers' trading. In our model, multiple informed traders and noise traders trade through multiple brokers. Brokers may trade with their customers in the same transaction (simultaneous dual trading) or trade after their customers in a separate transaction (consecutive dual trading). We find that, with simultaneous dual trading, equilibrium exists only if the number of informed traders exceeds the number of brokers; and, further, relative to consecutive dual trading, brokers' profits and uninformed losses are higher and informed profits and market depth are lower. We endogenize the number of brokers and informed traders and show that informed and noise traders choose one (all) broker(s) in the simultaneous (consecutive) dual trading market. If the market entry cost is low, more informed traders enter the simultaneous dual trading market and market depth may be higher relative to the consecutive dual trading market. If the market entry cost is high, consecutive dual trading is clearly preferred. We study order flow internalization by broker-dealers, and show that, in the free entry equilibrium, internalization hurts retail customers and market quality; and that it is likely to be more prevalent in thin markets with few informed traders. Finally, we examine off-exchange block sales and find that, compared to exchange transactions, they are more liquid but less informative.

Personal trading by brokers is pervasive throughout U.S. securities and futures markets, and in financial markets around the world (Grossman (1989)). Yet, academics and policy makers are unable to conclude if brokers' trading is beneficial for customers and markets. For example, the U.S. Congress passed the Futures Trading Practice Act in 1992, which directed the Commodity Futures Trading Corporation (CFTC) to pass regulations prohibiting dual trading[1/] in high volume contracts. However, although dual trading is currently banned on the Chicago Mercantile Exchange (CME) futures markets, it is practiced freely on other futures exchanges.[2/]

In contrast, regulators have not sought to restrict personal trading by brokers in the U.S. equity markets, where the ratio of trading profits to brokers' revenue increased from 0.18 to 0.26 in 1992.[3/] Yet, even in equity markets, serious concerns remain regarding certain aspects of brokers' trading. A proposed New York Stock Exchange (NYSE) rule on front running aimed to increase restrictions on proprietary trading by member firms.[4/] Large institutional customers have long been concerned that brokers may use knowledge of their orders to trade for their own accounts.[5/] Finally, The Securities and Exchange Commission (SEC) concludes that the internalization of order flow, which occurs when a broker-dealer executes customer orders as a

---

[1/]Dual trading in futures markets is the practice whereby futures floor traders execute trades for their own and customers' accounts on the same day.

[2/]The regulations allow affected exchanges to petition for relief based on 1) an acceptable audit trail, or ability to track a floor traders' activities, or 2) a threat to the hedging utility and price discovery function of futures markets, should the practice of dual trading be prohibited. All affected exchanges have petitioned for relief. So far, only the Comex division (Comex) of the New York Mercantile Exchange and the CME's S&P 500 futures contract have received unconditional exemptions from the dual trading ban. Seven other CME contracts and 13 contracts at the Chicago Board of Trade (CBOT) have received conditional exemptions, allowing dual trading in low volume months only. See CFTC press releases, May 7 1997 and November 7 1997.

[3/]The *Securities Industry Factbook* 1995. Figures relate to broker-dealers that are members of the New York Stock Exchange (NYSE) and doing a public business, who account for the bulk of industry revenues.

[4/]The *Investment Dealer's Digest*, July 29, 1996. Specifically, NYSE has proposed widening its existing Rule 92 to allow it to consider all member firms' trades when looking for evidence of frontrunning---irrespective of whether they occurred on the NYSE floor or not.

[5/]See "Money Machine," *Business Week*, June 10, 1991, pages 81-84.

market maker, "raise significant agency-principal concerns," a feeling shared by other market participants.[6/]

The lack of a consensus on brokers' trading at the policy level is also reflected in the academic literature. Empirical studies do not agree on the effect of dual trading on market liquidity. Depending on the market studied, dual trading may increase, decrease or have no effect on liquidity (Chang and Locke (1996), Chang, Locke and Mann (1994), Fishman and Longstaff (1992), Smith and Whaley (1994)). There is no consensus in the theoretical literature, either. Whereas Grossman (1989) argues in favor of a positive effect of dual trading on liquidity, Roell (1990) and Sarkar (1995) conclude that dual trading may reduce liquidity. In Fishman and Longstaff (1992), the effect of dual trading on liquidity is ambiguous. The literature also disagrees on the effect of dual trading on customers' welfare.[7/]

To provide clearer guidance to policy makers, we attempt to provide, in this paper, a fuller examination of brokers' trading than is available in the literature, by modeling differences in the way brokers trade in different markets. Our approach draws on Grossman's (1989) distinction between ***simultaneous*** dual trading (where a broker trades for himself and a customer in the same transaction) and ***consecutive*** dual trading (where a broker trades for customers as an agent and for himself at other times, but not in the same transaction). The constraint on timing differs by markets. Simultaneous dual trading occurs in securities markets, currency and interest rate

---

[6/]The SEC comment is regarding preferencing, which is one way for broker-dealers to internalize customer order flow. More generally, internalizing occurs when a broker-dealer directs order flow to an affiliated specialist. See "Report on the practice of preferencing," the SEC, April, 1997. For reaction of market participants, see the comments of Robert Murphy, president of RPM Specialist Corp., as reported in *Securities Week*, March 11, 1996. Battalio, Greene and Jennings (1997) estimate that 7 per cent of all orders, in a set of actively-traded NYSE securities, were internalized on the Cincinnati Stock Exchange during September 1994.

[7/]In Roell (1990), uninformed traders whose trades are observed (not observed) by the dual trader have higher (lower) profits with dual trading. In Fishman and Longstaff (1992) and Sarkar (1995), the results depend upon whether brokers pass on their entire trading profits to customers in the form of lower commissions (which is assumed to be independent of the order size). Dual trading is generally bad for customers without the reduction in commissions, and good for customers with the reduction.

swap markets, and the fixed income market, but not in futures markets. Consecutive dual trading occurs in futures markets, and in all markets with simultaneous dual trading (Grossman (1989)).

While our model initially deals with dual trading, we later apply it to examine order flow internalization by brokers. We further extend the model by reinterpreting brokers as "follow-on traders"--i.e., those who trade based on the order flow of preceding traders. We use the extended model to study off-exchange sales of large blocks of stocks, which are motivated by the desire of large block sellers to hide their trades from "follow-on traders" (see below and section VI for a fuller description).

In our model, based on Kyle (1985), multiple informed traders and noise traders trade by dividing their orders equally among multiple dual traders. Later, we allow informed and noise traders to choose the number of brokers, and endogenize the number of informed traders and brokers entering the market. Informed traders observe private signals about the payoff of a single risky asset and choose a quantity to buy or sell through brokers. The brokers, on receiving orders from informed traders and noise traders, may: dual trade consecutively, by executing customers' orders in the first period as agents and trading on their personal accounts in the second period as principals; or dual trade simultaneously, by acting as agents for customers and trading for themselves as principals in the same transaction.

If brokers dual trade simultaneously, each broker submits the net order (his personal trades plus his customers' trades) to the market maker for execution, and the game ends in a single period. If brokers trade consecutively, trading occurs over two periods. In period one, each broker submits the sum of informed and period one noise trades to the market maker for execution. In period two, each broker submits his personal trade and period two noise trades to the market maker. The market maker batches brokers' trades and period two noise trades and executes them at a single price. In all cases, the market maker prices the asset to earn zero expected profits, conditional on the history of order flows realized so far.

Compared to the no-dual-trading benchmark, informed traders' period one trading is unaffected with ***consecutive*** dual trading, while noise traders' losses per period are lower. Also, price informativeness and marker depth are higher with consecutive dual trading. Thus, consecutive dual trading is unambiguously beneficial for customers and the market, compared to the no-dual-trading benchmark.

Simultaneous dual trading is generally bad for customers and the market relative to the no-dual-trading benchmark. The reason is that brokers mimic informed trades by trading with insiders in the same direction, causing insiders to trade less and at a higher price (in absolute value), and lowering informed profits. In fact, we show that if the number of brokers exceeds the number of informed traders, the simultaneous dual trading equilibrium does not exist.[8/] Uninformed losses are unaffected but, since brokers offset a portion of noise trades, market depth is reduced.

The only beneficiaries of simultaneous dual trading markets are brokers, whose expected trading profits are higher than with consecutive dual trading. However, brokers are chosen by customers, and so our results imply that markets with simultaneous dual trading may not be viable---a prediction clearly at odds with institutional reality. To resolve this puzzle, we allow informed and noise traders to choose the number of brokers and, further, we endogenize the number of brokers and informed traders in the market.

We find that, with consecutive dual trading, informed and noise traders choose all the brokers available in the market whereas, with simultaneous dual trading, they choose a single

---

[8/] Our result is consistent with the institutional fact that, in futures markets, where typically many dual traders are present in each contract, simultaneous dual trading is not permitted.

broker. These results reflect the facts that, with consecutive (simultaneous) dual trading, uninformed losses (informed profits) decrease with the number of brokers.

In the free entry equilibrium, traders pay a fixed fee to enter the market. Consistent with intuition, the number of entering brokers and informed traders is positively related to the asset volatility and the size of noise trades, and negatively related to the entry fee. In the simultaneous dual trading market, only one broker enters the market, anticipating correctly that customers will choose just one broker. In the consecutive dual trading market, multiple brokers may enter the market, but less in number than the number of informed traders.

If the entry fee is high, not enough informed traders may enter the simultaneous dual trading market to make it viable. However, if the fee is low, more informed traders enter the simultaneous dual trading market than the consecutive dual trading market. Surprisingly, if enough informed traders enter the simultaneous dual trading market, market depth may be lower compared to consecutive dual trading. Thus, with sufficiently low entry costs, and optimal choice of the number of brokers by informed and noise traders, the simultaneous dual trading market becomes viable.

In the simultaneous dual trading model, brokers act as market makers to noise traders by offsetting a portion of their order flow in the same transaction. Thus, the model is appropriate for studying order flow internalization, which primarily affects uninformed retail order flow (Easley, Kiefer and O'Hara (1996), Battalio, Greene and Jennings (1997), and the SEC, 1997). We assume that a subset of brokers are involved exclusively with internalizing the order flow of noise traders, while the remaining brokers are involved exclusively with mimicking informed traders. With internalization, broker's commissions are lower, as predicted by the SEC and the NASD research staff,[9] and, consistent with Battalio (1997), the price, net of commissions, may be lower with internalization.

In the free entry equilibrium, however, internalization reduces market depth and price informativeness, and increases uninformed losses. Further, the number of internalizing brokers is negatively related to the market depth and the number of entering informed traders. Thus, our model predicts that internalization is likely to be more prevalent in thin markets with few informed traders. If thin markets have high spreads, this result supports advocates of purchased order flow who argue that it primarily affects NYSE stocks with large spreads (Easley, Kiefer and O'Hara, 1996). However, contradicting these advocates, our results show that market quality is affected adversely.

Next, we consider off-exchange block sales---i.e., the sale of very large blocks of stocks by big investors or institutional holders to broker-dealers away from U.S. exchange floors. In a typical off-exchange sale, a dealer offers to buy the block of stock at a discount to the closing price on the exchange, acting as a principal. The recent popularity of such sales is due to the fact that, if the sales were made on an exchange in piecemeal fashion (as predicted by Kyle (1985)), the first sale may alert other traders, such as short sellers, who may then drive the price down.[10] Proponents say off-exchange sales provide liquidity for block trading, while opponents claim that it reduces transparency (since the trades occur away from the floor of the exchange).

---

[9]The SEC suggests this possibility in its "Market 2000" report. The NASD research staff argues that many brokers who use preferencing are discount brokers (see "The introduction of Nacquess into the Nasdaq stock market: Intent and expectation," June 1996).

[10]For example, Carl Icahn, who sold off-exchange 19.9 million shares in RJR Nabisco Holding Corp. to Goldman, Sachs & Co, is quoted as saying that, with a piecemeal sale on the exchange floor, "word would have spread, giving short-sellers a chance to drive the price down at his expense." See "More big stocks are handled 'off-Board'," the *Wall Street Journal*, Wednesday, April 9, 1997, page C1.

Since the off-exchange block sale does not involve brokers' trading, we use the model of Holden and Subrahmanyam (1992)---where insiders sell shares piecemeal over time--- to analyze the off-exchange transaction. We use our consecutive dual trading model to describe the exchange transaction, reinterpreting brokers as follow-on traders (such as short-sellers). When insiders sell a second time on the exchange floor, we assume they have to sell at the period two price determined by follow-on traders. Our results show that, compared to the exchange market, the off-exchange transaction may be more liquid but the off-exchange price is less informative. Thus, while off-exchange sales are good for insiders, they are bad for the exchange (since the exchange price would be more informative if insiders had traded on the floor).

The existing literature on dual trading does not consider multiple informed traders and multiple brokers in the same model, nor do they endogenize the number of traders. Fishman and Longstaff (1992) study consecutive dual trading in a model with a single broker and a fixed order size. Roell (1990) and Chakravarty (1994) have multiple dual traders but a single informed trader. Sarkar (1995) studies simultaneous dual trading with multiple informed traders and a single broker. Our results imply that the coexistence of consecutive and simultaneous dual trading markets cannot be explained by models where traders do not optimally choose the number of brokers, and where market entry is fixed.

The idea of "follow-on" trading goes all the way back to Keynes (1937), who viewed traders' orders as being determined primarily by the observed or expected trades of other market participants. Scharfstein and Stein (1990) and Shiller (1990, 1995) also suggest that traders track each other. Welch (1992) shows that cascades are possible in a model where traders observe whether prior traders bought or sold and then decide to buy or sell only one unit. In Chakrabarti and Roll (1997), traders observe the trades and expenditures of prior large (but not small) traders and, consequently, make a noisy inference of the large traders' information. Their simulation exercise shows that return volatility is lower and price efficiency is higher when traders observe other traders.

Regarding internalization, Battalio and Holden (1996) show that, if brokers can distinguish between informed and uninformed orders (as in our model), they can profit from internalizing uninformed orders. However, unlike our model, they do not focus on the effect of internalization on market quality. Dutta and Madhavan (1997) find that a collusive equilibrium is easier to sustain with preferencing arrangements. In contrast to the theoretical results, the empirical studies of Battalio (1997), Battalio, Greene and Jennings (1997) and Lightfoot, Martin, Peterson, and Sirri (1997) and the experimental study of Bloomfield and O'Hara (1996) find no adverse effect on market quality. As Battalio, Greene and Jennings (1997) state, their result may imply that broker-dealers are not systematically "skimming" uninformed order flows or, alternatively, that internalizing brokers have a cost advantage in executing orders. Macey and O'Hara (1997) survey the literature on preferencing and internalization.

The remainder of the paper is organized as follows. Section I describes a trading model with multiple customers and many brokers, when the number of informed traders and brokers is fixed. Sections II and III solve the consecutive and simultaneous dual trading models. Section IV endogenizes the number of informed traders and brokers. Section V analyzes order flow internalization, while section VI studies off-exchange stock sales. Section VII concludes. All proofs are in the appendix.

I. A Model of Trading With Multiple Customers and Multiple Brokers.

We consider an asset market structured along the lines of Kyle (1985). There is a single risky asset with random value v, drawn from a normal distribution with mean 0 and variance Σ_v. There are n informed traders, each of whom receives a signal about the true asset value and submits market orders. For an informed trader i, i=1,..,n, the signal is $s^i = v + e^i$, where e^i is drawn from a normal distribution with mean 0 and variance Σ_e. A continuum of noise traders also submit aggregate market orders u, where u is normally distributed with mean zero and variance Σ_u. All random variables are independent of one another.

All customers, informed and uninformed, must trade through brokers. There are m brokers in the market, who submit customer orders to the market maker. m and n are common knowledge. We assume that orders are split equally among the m brokers. By observing informed orders, brokers can infer the informed traders' signals. By observing orders of noise traders, they are aware of the size of uninformed trades. Consequently, brokers have an incentive to trade based on their customers' orders. However, brokers are not allowed to trade ahead of (i.e., front run) their customers.

Brokers may trade in two possible ways. They may execute their customers' orders first, and trade for their own accounts second in a separate transaction---i.e., engage in ***consecutive*** dual trading. Alternatively, they may trade with their customers in the same transaction--i.e., engage in ***simultaneous*** dual trading. The sequence of events is as follows: in stage one, informed trader i, i=1,..,n observes s^i and chooses a trading quantity x^i. In stage two, broker j, j=1,..,m trade consecutively or simultaneously, and places orders of an amount z^j. In subsequent sections, we describe in more detail how simultaneous and consecutive dual trading differ. Finally, all trades (including brokers' personal trades) are batched and submitted to a market maker, who sets a price that earns him zero expected profits conditional on the history of net order flows realized.

Initially, the number of informed traders and brokers is fixed. Later, we allow informed traders to choose the number of brokers to allocate their orders to, and study the free entry equilibrium where informed traders and brokers decide whether to enter the market, depending on a market entry cost and their expected profits upon entering.

II. Consecutive Dual Trading.

In this section, we solve for the equilibrium in a market with consecutive dual trading. We assume that brokers do not trade with their customers in the same transaction--i.e., simultaneous dual trading is not allowed, as in futures markets.

A. The Consecutive Dual Trading Model

Trading occurs in two periods. In period one, brokers receive market orders from n informed traders and the noise traders, which they then submit to the market maker. In period two, brokers trade for themselves, along with period two noise traders. Each period, a market marker observes the history of net order flow realized so far and sets a price such to earn zero expected profits, conditional on the order flow history.

The sequence of events is as follows: in period one, informed trader i, i=1,..,n observes s^i and chooses $x^{i,d}$, knowing that his order will be executed in the first period. Accordingly, informed trader i, i=1,..,n, chooses $x^{i,d}$ to maximize conditional expected profits $E[(v-p_1)x^{i,d}|s^i]$, where the period one price is $p_1 = \lambda_1 y_1$, the period one net order flow is $y_1 = x_d+u_1$, the aggregate informed trade is $x_d = \Sigma_i x^{i,d}$ and u_1 is the period one noise trade.

In period two, brokers choose their personal trading quantity after observing the n-vector of informed trades $\{x^{1,d},..,x^{n,d}\}$, u_1 and p_1. Thus, broker j, j=1,..,m, chooses z^j to maximize conditional expected profits $E[(v-p_2)z^j | \{x^{1,d}/m ,..,x^{n,d}/m\}, u_1/m, p_1]$, where the conditioning is based on each broker observing his portion of the informed and uninformed orders received, plus the period one price.

The m brokers submit their personal trades to the market maker, who sets $p_2 = \lambda_2 y_2 + \mu_2 y_1$ where $y_2 = \Sigma_j z^j + u_2$, $\Sigma_j z^j$ is the aggregate trade of all brokers and u_2 is the noise trade in period two. Finally, the liquidation value v is publicly observed and both informed traders as well as brokers realize their respective profits (if any).

Define $t=\Sigma_v/\Sigma_s$, where t is the unconditional precision of s^i, i=1,..,n. Note that $0 \leq t \leq 1$. Further, define $Q = 1+t(n-1)$, where $(Q-1)s^i$ represents informed trader i's conjecture (conditional on s^i) of the remaining (n-1) informed traders' signals. Since informed traders have different information realizations, they also have different conjectures about the information of other informed traders. t also measures the correlation between insider signals. For example, if t=1 (perfect information), informed signals are perfectly correlated, Q=n, and informed trader i conjectures that other informed traders know $(n-1)s^i$--i.e., the informed trader believes other informed traders have the same information as he.

Proposition 1 below solves for the unique linear equilibrium in this market.

Proposition 1: In the consecutive dual trading case, there is a unique linear equilibrium for t>0. In period one, informed trader i, i=1,..,n, trades $x^{i,d}=A_d s^i$, and the price is $p_1=\lambda_1 y_1$. In period two, each broker j, j=1,..,m trades $z = B_1 x_d + B_2 u_1$, where $x_d = \sum_i x^{i,d}$ and the price is $p_2 = \lambda_2 y_2 + \mu_2 y_1$, where:

$$A_d = \frac{\sqrt{t\Sigma_u}}{\sqrt{n\Sigma_v}} \tag{1}$$

$$\lambda_1 = \frac{\sqrt{nt\Sigma_v}}{\sqrt{\Sigma_u}(1+Q)} \tag{2}$$

$$B_1 = \frac{1}{\sqrt{mQ(1+Q)}} \tag{3}$$

$$B_2 = -\frac{\sqrt{Q}}{\sqrt{m(1+Q)}} \tag{4}$$

$$\lambda_2 = \frac{\sqrt{nt\Sigma_v}}{\sqrt{Q(1+Q)\Sigma_u}}\frac{\sqrt{m}}{1+m} \tag{5}$$

$$\mu_2 = \frac{\sqrt{nt\Sigma_v}}{(1+Q)\sqrt{\Sigma_u}} \tag{6}$$

W_d is the expected trading revenue per broker, as given by:

$$W_d = \frac{\sqrt{nt\Sigma_u\Sigma_v}}{\sqrt{Q(1+Q)}}\frac{1}{\sqrt{m}(1+m)} \tag{7}$$

The period one solution is identical to Kyle's (1985) single period equilibrium, extended to include multiple informed traders and noisy signals.[11/] Thus, informed trades are not affected by dual trading in period two since informed traders trade only in period one.[12/] In period two, dual traders piggyback on period one informed trades (B_1>0) and offset noise trades ($B_2<0$).

Corollary 1: (1) $\delta B_1/\delta n < 0$, and $\delta B_1/\delta t < 0$ for n>1. $\delta|B_2|/\delta n > 0$ and $\delta|B_2|/\delta t > 0$ for n>1.

(2) $\delta B_1/\delta m < 0$, $\delta|B_2|/\delta m < 0$. $\delta(mB_1)/\delta m > 0$, $\delta(m|B_2|)/\delta m > 0$.

[11/] See, for example, lemma one in Admati and Pfleiderer (1988).

[12/] The result is similar to the consecutive dual trading equilibrium in Fishman and Longstaff (1992), where the first trade is not affected by dual trading.

The extent of piggybacking B_1 decreases in the number of informed traders n and the information precision t. Competition between informed traders leads insiders to use their information less, making piggybacking less valuable. An increase in t increases the correlation between insiders' signals, reducing (for n>1) the value of observing multiple informed orders. Since the marginal value of observing noise trades is higher when the marginal value of observing informed trades is lower, $|B_2|$, the extent to which noise trades are offset, increases in n and t. Competition between brokers reduces the extent to which each broker exploits customer trades. However, brokers in the aggregate exploit customer trades more.

B. The Impact of Consecutive Dual Trading on Traders' Profits and the Market

Since the period one equilibrium is identical to one without dual trading, we can compare it with the period two equilibrium to obtain the impact of consecutive dual trading.

Corollary 2. (1) $\mu_2 = \lambda_1$ and $\lambda_2 < \lambda_1$.

(2) Price informativeness is higher with consecutive dual trading.

(3) Uninformed losses per period are lower while informed profits are unaffected with consecutive dual trading.

$\mu_2 = \lambda_1$ because covariance(y_1,y_2)=0: to the market maker, the period one order flow is not informative about the period two order flow. The reason is that dual traders, by piggybacking on informed trades, induce a positive serial correlation between the order flows. But, by offsetting noise trades, dual traders also induce a negative serial correlation between the order flows. These two effects are exactly offsetting. Also, $\lambda_2 < \lambda_1$ implies that the market maker's adverse selection costs due to brokers' trading in period two is less than the adverse selection cost from insider trading in period one.

Since the covariance between the order flows of the two periods is zero, information revealed by the period two price is simply the information revealed by the period one price plus the additional information revealed by brokers' trading.

Informed trades and, therefore, informed profits, are not affected by dual trading. Since $\mu_2=\lambda_1$, it follows that the period two price $p_2 = p_1 + \lambda_2 y_2$: the dual trader trades at worse prices than the informed trader. Thus, dual trading profits are lower than informed profits in the first period and, consequently, uninformed losses to dual trading are lower than losses to informed traders.

We now solve for the simultaneous dual trading equilibrium.

III. Simultaneous Dual Trading.

A. The Simultaneous Dual Trading Model

Simultaneous dual trading is modeled in a single period Kyle (1985) framework. The notations are the same as in section II. All variables and parameters related to simultaneous dual trading are denoted either with superscript *s* or subscript *s*.

A group of *n* informed traders receive signals s^i about the unknown value *v*, and choose quantities $x^{i,s}$ knowing that his order will be executed along with the orders of brokers and noise traders in the same transaction. Accordingly, informed trader *i*, *i*=1,..,*n*, chooses $x^{i,s}$ to maximize conditional expected profits $E[(v-p_s)x^{i,s}|s^i]$, where the price is $p_s = \lambda_s y_s$, the net order flow is $y_s = x_s+mz+u$, the aggregate informed trade is $x_s=\Sigma_i x^{i,s}$, *z* is the amount each broker trades and *u* is the noise trade.

Upon receiving the orders of their informed and uninformed customers, brokers choose their personal trading quantity after observing the *n*-vector of informed trades $\{x^{1,s},..,x^{n,s}\}$ and *u*.

Thus, broker j, j=1,..,m, chooses $z^{j,s}$ to maximize expected profits E[$(v-p_s)z^{j,s}$|$\{x^{1,s}/m,..,x^{n,s}/m\}$, u/m], where the conditioning is based on each broker observing his portion of the informed and uninformed orders.

The m brokers submit their customer trades and personal trades to the market maker, who sets the price that earns him zero expected profits conditional on the net order flow realized. Finally, the liquidation value v is publicly observed and both informed traders as well as brokers realize their respective profits (if any).

Proposition 2 below solves for the unique linear equilibrium in this market.

Proposition 2: In the simultaneous dual trading case, there is a unique linear equilibrium for t>0 and Q>m. Informed trader i, i=1,..,n, trades $x^{i,s}=A_s s^i$, broker j, j=1,..,m trades $z = B_{1,s}x_s+B_{2,s}u$, where $x_s=\Sigma_i x^{i,s}$, and the price is $p_s = \lambda_s y_s$, where:

$$A_s = \frac{1}{1+m}\frac{(Q-m)}{Q}\frac{\sqrt{t\Sigma_u}}{\sqrt{n\Sigma_v}} \tag{8}$$

$$\lambda_s = (1+m)\frac{\sqrt{nt\Sigma_v}}{\sqrt{\Sigma_u}(1+Q)} \tag{9}$$

$$B_{1,s} = \frac{1}{(Q-m)} \tag{10}$$

$$B_{2,s} = -\frac{1}{(1+m)} \tag{11}$$

W_s, the expected trading revenue per broker, satisfies:

$$W_s = \frac{\sqrt{nt\Sigma_u\Sigma_v}}{Q(1+m)} \tag{12}$$

Equilibrium exists if $Q>m$. Since $n \geq Q$, existence implies $n>m$: the number of informed traders must exceed the number of brokers. The intuition behind this result is as follows. Suppose an insider buys. Dual traders also buy in the same transaction, piggybacking on the insider trade (i.e. $B_1>0$), and increasing the price paid by the insider for his purchase. From corollary three below, the order of an individual insider is exploited less as the number of insiders increases, and is exploited more as the number of brokers increases. If the number of brokers is too large relative to the number of insiders, the adverse price effect makes it too costly for insiders to trade.

Corollary 3: **(1)** $\delta B_{1,s}/\delta n < 0$**, and** $\delta B_{1,s}/\delta t < 0$ **for** $n>1$**.** $\delta |B_{2,s}|/\delta n = 0$ **and** $\delta |B_{2,s}|/\delta t = 0$**.**

(2) $\delta B_{1,s}/\delta m > 0$**,** $\delta |B_{2,s}|/\delta m < 0$**.** $\delta (mB_{1,s})/\delta m > 0$**,** $\delta (m|B_{2,s}|)/\delta m > 0$**.**

Unlike the consecutive dual trading case, the extent to which noise trades are offset depends inversely on the number of brokers but is independent of the number of informed traders n. In the consecutive dual trading model, knowledge of period one noise trades is relevant to brokers because it influences the way the market maker prices the asset in period two. The market maker's pricing problem, in turn, depends on the extent of private information. This interaction between information and noise trades is absent in the case of simultaneous dual trading.

B. The Impact of Simultaneous Dual Trading on Traders' Profits and the Market

In this section, we compare the simultaneous dual trading equilibrium with one where dual trading is banned (i.e., the period one solution for the consecutive dual trading equilibrium).

Corollary 4. Compared to the no-dual-trading benchmark,

(1) Market depth is lower with dual trading.

(2) Price informativeness is the same with or without dual trading.

(3) Uninformed losses are the same and informed profits are lower with simultaneous dual trading.

Market depth is lower with dual trading because brokers reduce the net order flow of noise traders, increasing the market maker's adverse selection costs. Price informativeness depends positively on the variance of the net order flow and inversely on market depth. Dual trading reduces depth and decreases the variance of the net order flow in the same proportion. Hence, price informativeness is invariant to dual trading.

Informed profits are lower with simultaneous dual trading because the dual traders' piggybacking forces insiders to reduce the size of their orders (for the same level of information). However, dual trading profits make up the reduction in informed profits exactly, and, consequently, uninformed losses to dual traders are the same.

C. Comparison Between Simultaneous And Consecutive Dual Trading

By combining results from propositions one and two, and corollaries two and four, we can compare market parameters for the two kinds of dual trading, holding the number of informed traders and brokers fixed.

Corollary 5. Suppose m and n are fixed. Then:

(1) If $Q \leq m$, only the consecutive dual trading equilibrium is viable.

(2) If $Q > m$, then both types of dual trading equilibrium are viable. Relative to simultaneous dual trading, uninformed losses and brokers' profits are lower, while informed profits, market depth and price informativeness are higher with consecutive dual trading.

Corollary five implies that only brokers prefer the simultaneous dual trading market. However, since brokers are chosen by customers, it is puzzling that markets with simultaneous dual trading should continue to exist. To resolve this puzzle, in the next section, we allow informed traders to choose brokers, which may be one way for informed traders to protect themselves from excessive piggybacking by brokers. We also allow endogenous market entry by traders.

IV. Free Entry by Informed Traders and Brokers

In this section, we study the two forms of dual trading, given that informed and uninformed traders optimally choose the number of brokers to give their orders to, and given that there is free entry into the asset market. The decision-making sequence of agents is as follows: 1) Informed traders and brokers simultaneously decide whether to enter the market; (2) Informed and noise traders choose the number of brokers to give the order to; (3) Informed and noise traders divide their orders equally among the chosen brokers. From then on, the game continues as before.

Informed traders choose the number of brokers to maximize expected profits. Uninformed noise traders choose the number of brokers to minimize their expected losses to informed traders and brokers. The following proposition describes traders' choice of the number of brokers to trade with.

Proposition 3. (1) When brokers trade consecutively, informed and noise traders choose all the brokers available.

(2) Suppose there are at least two informed traders, so that the simultaneous dual trading equilibrium exists. Then, with simultaneous dual trading, informed and noise traders give orders to only one broker.

With consecutive dual trading, informed traders' profits are independent of m but noise trader losses are decreasing in m. Thus, noise traders choose all available brokers while informed traders are indifferent to the choice of m. When brokers trade simultaneously, the situation is reversed: informed profits are decreasing in m whereas uninformed losses are independent of m. Thus, informed traders choose one broker and noise traders are indifferent to the choice of m. For concreteness, we assume that informed and noise traders choose the same number of brokers in each case.

Next, consider free entry by informed traders and brokers. Let k_i be the cost of entering market i, $i=d$ (consecutive dual trading),s (simultaneous dual trading). To obtain analytic solutions, we assume $t=1$. The free entry equilibrium satisfies two conditions. Traders enter a market until their expected profits, net of the entry cost, are zero. And the cost is low enough so that entry is profitable for the minimum number of traders necessary to sustain equilibrium.

In the consecutive dual trading equilibrium, the equilibrium number of informed traders, n_d, entering the market is independent of m, since informed profits are independent of m. But broker profits are dependent on n_d, and so the equilibrium number of brokers entering the market, m_d, depends on n_d. In the free entry equilibrium, we show below that $m_d < n_d$. Thus, costs must be low enough to sustain at least one broker and two informed traders.

In the simultaneous dual trading equilibrium, proposition three implies that only one broker enters the market and, proposition two implies that at least two informed traders must enter the market, or the market will not be viable. Again, costs must be low enough to sustain at least one broker and two informed traders.

Proposition four solves for the free entry equilibrium in the two markets.

Proposition 4. (1) In the market with consecutive dual trading, n_d informed traders and m_d brokers enter the market, where $m_d < n_d$, and m_d and n_d are given by:

$$\sqrt{n_d}(1 + n_d) = \frac{\sqrt{\Sigma_u \Sigma_v}}{k_d} \tag{13}$$

$$\sqrt{m_d}(1 + m_d) = \frac{\sqrt{\Sigma_u \Sigma_v}}{k_d} \frac{1}{\sqrt{1 + n_d}} \tag{14}$$

At least *two* informed traders and at least *one* broker enter the market if:

$$k_d \le \frac{\sqrt{\Sigma_u \Sigma_v}}{3\sqrt{2}} \tag{15}$$

(2) In the market with simultaneous dual trading, n_s informed traders and *one* broker enter the market, where n_s is given by:

$$\frac{\sqrt{n_s}(1 + n_s)}{(n_s - 1)} = \frac{\sqrt{\Sigma_u \Sigma_v}}{2k_s} \tag{16}$$

At least *two* informed traders and *one* broker enter the market if:

$$k_s < \frac{\sqrt{\Sigma_u \Sigma_v}}{12\sqrt{2}} \tag{17}$$

Proposition four is intuitive: the number of informed traders and brokers entering a market is inversely related to the cost of entering the market, and positively related to the volatility of the asset value and the size of noise trades. In the consecutive dual trading market, a broker expects to make less trading profits than an informed trader since he trades second. Thus, the equilibrium number of brokers is less than the equilibrium number of informed traders in this market.

An important observation is that the participation constraint (17) is more restrictive than the inequality (15), implying that the simultaneous dual trading equilibrium is viable only at lower market entry cost, relative to the consecutive dual trading equilibrium. In the following proposition, we compare the two types of dual trading markets given free entry by brokers and informed traders and the optimal choice of the number of brokers by informed traders.

Proposition 5. In the free entry equilibrium of proposition four,

1) Suppose $k_c = k_s = k$. If k is high (low), then n_c is greater (less) than n_s.

2) Market depth is higher (lower) with simultaneous dual trading if n_s is high (low) relative to n_c and m_c.

The proposition says that, if entry costs are low, the simultaneous dual trading market may have *more* informed traders and lower market depth in a free entry equilibrium. The reason is that informed traders protect themselves from excessive piggybacking by choosing only one broker, thus maximizing their expected profits in the by simultaneous dual trading market (see corollary three). Consequently, if entry costs are low enough, many informed traders enter the simultaneous dual trading market and market depth is higher. Thus, the proposition provides some intuition as to why we see markets with simultaneous dual trading exist in the real world.

V. Internalization of order flow by broker-dealers

Internalization is the direction of order flow by a broker-dealer to an affiliated specialist or order flow executed by that broker-dealer as market maker. Broker-dealers can internalize order flow in several ways. For example, large broker-dealer firms, particularly NYSE member firms, purchase specialist units on regional exchanges and direct small retail customer orders to them.[13/] In off-exchange internalizations, NYSE firms execute orders of their retail customers against their own account, with the transaction taking place in the so-called third market or over-the-counter market. Such transactions, also called 19c-3 trading,[14/] has become a major source of profits for broker-dealers.

We use our simultaneous dual trading model to analyze order flow internalization. Since internalizing brokers typically handle order flows of small retail investors, we assume that there are m_1 internalizing brokers who handle all of the noise trades, and m_2 piggybacking brokers who handle all the informed traders, with $m_1+m_2 = m$. As before, the market maker sees the pooled order, and, further, cannot distinguish between internalizing brokers and informed-order brokers.

Let z_1 be the trade of an internalizing broker and let z_2 be the trade of informed-order brokers. From (10) and (11):

$$z_1 = -\frac{u}{(1+m_1)} \tag{18}$$

$$z_2 = \frac{x_s}{(Q - m_2)} \tag{19}$$

We compare uninformed losses, brokers' commissions and the market quality between the internalization model and a model with no brokers' trading. For simplicity, we assume t=1. The following proposition shows the effect of order flow internalization on noise trader losses and the market, for both the fixed entry and free entry equilibrium solutions.

[13/] See "Report on the practice of preferencing," the SEC, April, 1997.

[14/] The name refers to Rule 19c-3 allowing NYSE stocks listed after April 26, 1979 to be traded off-exchange. Broker-dealers were rumored to have earned over $500 million in 1994 from 19c-3 trading. See "In-House trades can be costly for small investors," the *Wall Street Journal*, December 20, 1994, page C1.

Proposition 6. (1) Suppose market entry is fixed. With internalization of order flow, market depth is lower while uninformed losses and price informativeness is unchanged. The competitive broker's commission is lower with order flow internalization.

(2) Suppose market entry is free. Then, with internalization of order flow, (i) one piggybacking broker, n_o informed traders and m_o internalizing brokers enter the market, where m_o varies inversely with n_o, and $m_o < n_o$. (ii) Relative to a market with no order flow internalization, market depth, price informativeness and the number of informed traders are lower, while uninformed losses are higher.

With fixed entry, the results on market depth, price informativeness and uninformed losses follow from corollary four. Brokers make positive profits, reducing the competitive commission. The expected profits of internalizing brokers are inversely related to the market depth. As more informed traders enter the market, market depth increases, and so internalizing brokers' expected profits are lower. Thus, in the free entry equilibrium, there is less internalization when there are more informed traders. Further, the number of entering informed traders is lower with order flow internalization, since internalizing brokers reduce market depth by offsetting the uninformed order flow. Thus, compared to the market without internalization, price informativeness is lower and uninformed losses are higher.

Based on proposition six and the discussion above, we state the following corollary.

Corollary 6. The extent of order flow internalization is inversely related to market depth and the number of informed traders.

VI. Off-exchange stock sales by broker-dealers.

In this section, we re-interpret the brokers in our model as follow-on traders--i.e., traders whose trading is derived from observing the order flow of preceding traders**.** In the context of this broader definition, we study off-exchange sales of large blocks of stocks.

In a typical off-exchange sale, the market maker offers to buy a block of stock at a discount to the closing price on the exchange, acting as a principal. Thus, an off-exchange transaction involves no dual trading or follow-on trading. The appropriate model for the off-exchange transaction is Holden and Subrahmanyam (1992), since insiders optimally dribble a large trade over time. We simplify by assuming two periods of trading and letting the time between two auctions in the Holden and Subrahmanyam (1992) model (Δt_n in their notation) equal one. Thus, n insiders commit to sell their block of stock off-exchange over two periods. In deciding how much stock to sell, insiders maximize their total expected profits in the current and future period. The market is structured along the lines of Kyle's (1985) sequential auctions model.

We use our consecutive dual trading model to describe the insiders' alternative of selling on the exchange floor. In period one, insiders sell stock on an exchange. m follow-on traders (equivalent to the consecutive dual traders) observe the stock sale in period one, and engage in "follow-on" trading in period two. When insiders sell a second time, we assume they have to sell at the price p_2 determined by follow-on traders in period two.

To determine whether off-exchange trading benefits insiders, we compare market depth and price informativeness in period two for the two cases. We assume t=1 (perfect information) in the consecutive dual trading model and**.** Since analytic solutions are not possible, we use numerical illustrations for our results. The following proposition (illustrated in figure one) compares the market quality in period two for off-exchange and exchange trading by insiders.

Proposition 7. Price informativeness is lower but market depth higher with off-exchange trading.

Off-exchange trading provides more liquidity to insiders since "follow-on" trading drives up period two prices, making it costly for insiders when they trade a second time on the exchange. However, greater liquidity is obtained at the expense of the price efficiency of the exchange market.

VII. Conclusion.

In this article, we study a wide variety of issues related to brokers' trading. Multiple informed traders and noise traders trade through multiple brokers, who either trade in the same transaction as their customers (simultaneous dual trading) or in a separate transaction (consecutive dual trading).

While the consecutive dual trading equilibrium always exists, the simultaneous dual trading equilibrium fails when the number of brokers is greater than the number of informed traders. The reason is that brokers trade with informed traders in the same direction, thus worsening informed traders' terms of trade. This effect is magnified with many brokers, leading informed traders to stop trading. When both equilibria exist, informed profits and market depth are lower, while uninformed losses and brokers' profits are higher with simultaneous dual trading, relative to consecutive dual trading. Thus, only brokers prefer simultaneous dual trading.

We allow informed and noise traders to choose the number of brokers, and endogenize the number of brokers and informed traders in the market. In the simultaneous dual trading market, informed and noise traders choose only one broker whereas, with consecutive dual trading, informed and noise traders choose all available brokers. If the market entry cost is low, more informed traders enter the simultaneous dual trading market, and market depth may be lower, relative to the consecutive dual trading market. Thus, the adverse effects of simultaneous dual trading on customers and the market are mitigated in the free entry equilibrium. If the market entry costs are high, consecutive dual trading is better.

In the simultaneous dual trading model, we allow some brokers to internalize the uninformed order flow, by selling to noise traders as dealers, out of inventory. In the free entry equilibrium, we find that although internalization results in lower brokers' fees, market depth and price informativeness are lower, and uninformed losses are higher. In addition, the number of internalizing brokers is inversely related to market depth and the number of informed traders.

We reinterpret brokers as "follow-on traders", who trade based on the order flows of preceding traders, and apply our consecutive dual trading model to explain large block sales conducted off the exchange floor to avoid "follow-on" trading. We find that such off-exchange transactions may provide additional liquidity for large block sales, but at the expense of reducing the informativeness of the exchange market.

References

Admati, A. and P. Pfleiderer (1988), A theory of intraday patterns: Volume and price variability, *Review of Financial Studies*, 1, 3-40.

Battalio, Robert and Craig W. Holden (1996), Why doesn't decimal trading eliminate payment for order flow and internalization? Working Paper, Indiana University.

Robert Battalio (1997), Third market broker-dealers: Cost competitors or cream skinners? *Journal of Finance*, 52, 341-352.

Battalio, Robert, Greene, Jason and Robert Jennings (1997), Do competing specialists and preferencing dealers affect market quality? *Review of Financial Studies*, 10, 969-993.

Bloomfield, R. and M. O'Hara (1996), Does order preferencing matter? Working Paper, Johnson Graduate School of Management, Cornell University.

Chakrabarti, Rajesh and Richard Roll (1997), Learning from others, reacting and market quality, Working Paper, University of California, Los Angeles.

Chakravarty, S. (1994), Should actively traded futures contracts come under the dual trading ban?, *Journal of Futures Markets*, 14, 661-684.

Chang, E. C. and P. R. Locke (1996), The performance and market impact of dual trading: CME Rule 552, *Journal of Financial Intermediation*, 5, 23-48.

Chang, E. C., Locke, P. R. and S. C. Mann (1994), The effect of CME Rule 552 on dual traders, *Journal of Futures Markets*, 14(4), 493-510.

Dutta, P. and A. Madhavan (1997), Competition and collusion in dealer markets. *Journal of Finance*, 52, 245-276.

Easley, David, Kiefer, Nicholas M. and Maureen O'Hara (1996), Cream-skimming or profit-sharing? The curious role of purchased order flow. *Journal of Finance*, 51, 3, 811-833.

Fishman, M. and F.A. Longstaff (1992), Dual trading in futures markets, *Journal of Finance*, 47, 643-672.

Grossman, S.J. (1989), An economic analysis of dual trading, working paper, University of Pennsylvania.

Holden, C.W., and A. Subrahmanyam (1992), Long-lived private information and imperfect competition, *Journal of Finance*, 47, 247-270.

John Maynard Keynes (1937), The general theory of employment, interest and money, (New York: Harcourt, Brace).

Kyle, A.S. (1985), Continuous auctions and insider trading, *Econometrica*, 53, 1315-1335.

Lightfoot, Lois E., Martin, Peter G., Peterson, Mark A., and Erik Sirri (1997), Preferencing and market quality on U.S. equity exchanges, Working Paper, the Securities and Exchange Commission.

Macey, Jonathan R. and Maureen O'Hara (1997), The law and economics of best execution, forthcoming, *Journal of Financial Intermediation*.

Roell, A. (1990), Dual-capacity trading and the quality of the market, *Journal of Financial Intermediation*, 1, 105-124.

Sarkar, A. (1995), Dual trading: winners, losers and market impact, *Journal of Financial Intermediation*, 4, 77-93.

Scharfstein, David S. and Jeremy C. Stein (1990), Herd behavior and investment, *American Economic Review*, June 1990, 80, 3, 465-479.

Shiller, Robert J. (1990), Market volatility and investor behavior, *AEA Papers and Proceedings*, May 1990, 80, 2, 58-62.

Shiller, Robert J. (1995), Conversation, information and herd behavior, *AEA Papers and Proceedings*, May 1995, 85, 2, 181-185.

Smith, T. and R. E. Whaley (1994), Assessing the cost of regulation: The case of dual trading, *Journal of Law and Economics*, 37(1), 329-36.

Spiegel, M. and A. Subrahmanyam (1992), Informed speculation and hedging in a noncompetitive securities market, *Review of Financial Studies*, 5(2), 307-329.

Subrahmanyam, A. (1991), Risk aversion, market liquidity and price efficiency, *Review of Financial Studies*, 4, 417-441.

Table 1

Trading Volume and Price Impact Around Boesky's Trading in Carnation Stock

We estimate the following regression:

$$STKPR/SIZE_t = a_0 + a_1 STDPR_t + a_2 BKVOL_t + a_3 NONBKVOL_t + a_4 PREI_t + a_5 CURI_t + a_6 POSTI_t$$

where STKPR is the hourly average of Carnation stock trading price, SIZE is the total hourly volume divided by the number of (hourly) trades, STDPR is the hourly standard deviation of Carnation's stock price, BKVOL is the hourly Boesky volume, NONBKVOL is the difference between the hourly Boesky volume and the hourly total volume, PREI equals 1 in the hour immediately before the hour Boesky trades on a day and zero otherwise, CURI equals 1 in the hour when Boesky trades and zero otherwise and POSTI equals 1 in the hour immediately following the Boesky trading hour and 0 otherwise.
The p-values of the coefficient estimates are in parentheses below the coefficients. The number of observations are 336. The sample period is June 1, 1984 through August 31, 1984.

Independent variables	Dependent variable: STKPR/ SIZE
Intercept	12.0050
	(0.0001)
STDPR	-5.2618
	(0.1103)
BKVOL	0.000082
	(0.9895)
NONBKVOL	-0.010043
	(0.0001)
PREI	-1.3124
	(0.4233)
CURI	-6.8576
	(0.0074)
POSTI	-3.1477
	(0.0436)
R-Squared	0.22

An important aspect of market competition, analysed in our next result, is the number of traders--in our case, the relative number of insiders and brokers on the market.

Corollary 4: (1) μ_2 is independent of m. μ_2 is decreasing (increasing) in n if n is greater (less) than (2-t)/t.

(2) λ_2 is decreasing in m. λ_2 is decreasing (increasing) in n if n is greater (less) than $[(2-t)(1-t)]^{1/2}/t$.

(3) The informativeness of period two prices is increasing in n and m.

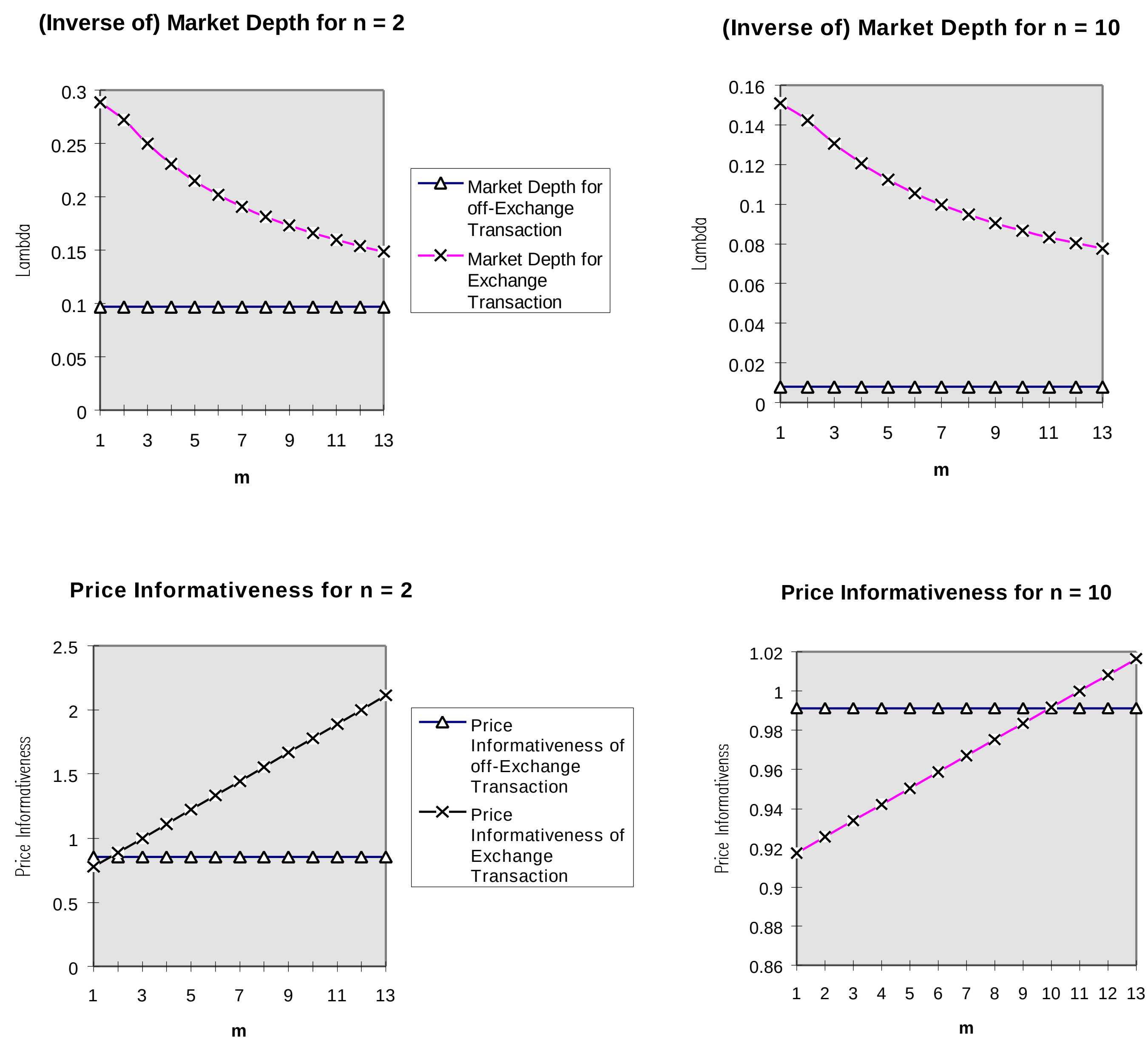
(4) Suppose the fixed cost of brokerage per trade is small. Then net informed profits and net uninformed losses are decreasing in m and n.

λ_2 is decreasing in m because of increased competition between brokers. The effect of increasing the number of informed traders on λ_2 and μ_2 is non-monotonic, echoing earlier results by Subrahmanyam (1991) in the context of risk-averse informed traders. Here, the "switch" value of n depends on information precision. If information is perfect (t=1), market depth increases with n, as in Holden and Subrahmanyam (1992). For very imprecise information (t close to 0), market depth decreases with n. The reason is as follows. As information becomes more precise, each insider's conjecture about the information of other insiders becomes more highly correlated, increasing the variability of aggregate insider trades and the market depth.[15/]

An increase in the number of insiders and brokers increases the informativeness of period two prices because increased competition makes insiders and brokers use more private information in the aggreagate. Increased competition also drives down informed and brokers' profits, reducing commissions and uninformed losses.

[15/] If t=1, the variance of aggregated insider trades is proportional to the square of n. If t=0, the variance is proportional to n.

Figure 1: (Inverse of) Market Depth and Price Informativeness with off-Exchange Block Sales



Appendix

Proof of Proposition 1:

Consecutive dual trading equilibrium.

In period one, the insider's problem is identical to the single period model in Kyle (1985), but with multiple informed traders. The solution to this equilibrium is in Lemma 1 of Admati and Pfleiderer (1988). This gives A_d and l_1.

In period two, dual trader *j* observes p_1, x_d / m and u_1 / m and trades z^j. The net order flow is $y_2 = \sum_{j=1}^{m} z^j + u_2 = mz + u_2$. We will show that $z^j = z$ for each *j* since each z^j is a function of x_d *and* u_1.

Using the law of iterated projections, it is easy to show that
$$\mathrm{E}\left[v \middle| \left\{\frac{x^{1,d}}{m}, \frac{x^{2,d}}{m}, \ldots, \frac{x^{n,d}}{m}\right\}, \frac{u_1}{m}, p_1\right] = \mathrm{E}\left[v \middle| \left\{\frac{x^{1,d}}{m}, \frac{x^{2,d}}{m}, \ldots, \frac{x^{n,d}}{m}\right\}, \frac{u_1}{m}\right].$$

The *j*[th] dual trader's problem is given by:

$$\underset{z^j}{Max} \quad E\left[(v - p_2)z^j \middle| \frac{x^{1,d}}{m}, \frac{x^{2,d}}{m}, \ldots., \frac{x^{n,d}}{m}, \frac{u_1}{m}, p_1\right]$$

or, $$\underset{z^j}{Max} \quad E\left[\left(v - l_2 z^j - l_2 z^{-j} - l_2 u_2 - p_1\right) z^j \middle| \frac{x^{1,d}}{m}, \frac{x^{2,d}}{m}, \ldots., \frac{x^{n,d}}{m}, \frac{u_1}{m}, p_1\right] \qquad (1.1)$$

where $z^{-j} = \sum_{j \neq k} z^k$. From above, the first and second order conditions are:

$$\text{Foc: } E\left[v \middle| \frac{x^{1,d}}{m}, \frac{x^{2,d}}{m}, \ldots.., \frac{x^{n,d}}{m}\right] - 2\, l_2 z^j - l_2 z^{-j} - m_2 A_d s - m_2 u_1 = 0 \qquad (1.2)$$

where s = $\sum_{i=1}^{n} s^i$

$$\text{Second Order Condition: } -2\, l_2 < 0, \quad \Rightarrow l_2 > 0 \qquad (1.3)$$

$$E\left[v\left|\frac{x^{1,d}}{m},\frac{x^{2,d}}{m},\ldots,\frac{x^{n,d}}{m}\right.\right]=\left(\frac{x_d}{m}\right)\frac{\frac{A_d}{m}n\Sigma_v}{\left(\frac{A_d}{m}\right)^2 Var(s)}=\left(\frac{x_d}{m}\right)\left(\frac{m}{A_d}\right)\frac{n\Sigma_v}{nQ\Sigma_s}=\frac{t}{Q}s \tag{1.4}$$

Substituting (1.4) in (1.1) and solving for $z=z^j$, for every *j*, we get

$$z=\frac{ts}{l_1Q(1+Q)}\cdot\frac{T_m}{l_2(1+m)}-\frac{m_2}{l_2(1+m)}u_1 \tag{1.5}$$

where, $T_m = \; l_1(1+Q)-m_2Q$ (1.6)

Now,

$$\begin{aligned} y_2 &= mz+u_2 \\ &= \frac{ts}{l_1Q(1+Q)}T_m\frac{m}{1+m}\frac{1}{l_2} \quad - \quad \frac{m}{1+m}\frac{1}{l_2}m_2u_1 \quad + \quad u_2 \\ &= \frac{ts}{l_1Q(1+Q)}T_mT_l \quad - \quad T_lm_2u_1 \quad + \quad u_2 \end{aligned} \tag{1.7}$$

where, $T_l = \frac{m}{1+m}\frac{1}{l_2}$ (1.8)

$$\begin{aligned} y_1 &= A_ds+u_1 \\ &= \frac{ts}{l_1(1+Q)}+u_1 \end{aligned} \tag{1.9}$$

$$\Sigma_{01}=Cov(v,y_1)=\frac{nt\Sigma_v}{l_1(1+Q)} \tag{1.10}$$

$$\Sigma_{02}=Cov(v,y_2)=\frac{nt\Sigma_v}{l_1Q(1+Q)}T_mT_l \tag{1.11}$$

$$\begin{aligned} \Sigma_{12} &= Cov(y_1,y_2) \\ &= \frac{t^2T_mT_l}{Q\left[\,l_1^2(1+Q)^2\right]}Var(s)-m_2T_l\Sigma_u \\ &= \frac{t^2T_mT_l}{Q\left[\,l_1^2(1+Q)^2\right]}nQ\Sigma_s-m_2T_l\Sigma_u \\ &= \frac{nt\Sigma_v}{\left[\,l_1(1+Q)\right]^2}T_mT_l-m_2T_l\Sigma_u \end{aligned} \tag{1.12}$$

$$
\begin{aligned}
\Sigma_{11} &= Var(y_1) \\
&= \frac{t^2}{[l_1(1+Q)]^2} Var(s) + \Sigma_u \\
&= \frac{t^2}{[l_1(1+Q)]^2} nQ\Sigma_s + \Sigma_u \qquad (1.13)\\
&= \frac{nt\Sigma_v Q}{[l_1(1+Q)]^2} + \Sigma_u
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_{22} &= Var(y_2) \\
&= \frac{t^2 Var(s)}{[l_1Q(1+Q)]^2} (T_m T_l)^2 + (T_l m_2)^2 \Sigma_u + \Sigma_u \qquad (1.14)\\
&= \frac{t^2 nQ\Sigma_s}{[l_1Q(1+Q)]^2} (T_m T_l)^2 \quad + \quad \left[(T_l m_2)^2 + 1\right]\Sigma_u
\end{aligned}
$$

Next, we solve for m_2 and l_2. First, we show that, if $\Sigma_{12} = 0,$ then $l_1 = m_2$. From (1.12),

$$\Sigma_{12} = 0 \Rightarrow \frac{nt\Sigma_v T_m}{l_1^2(1+Q)^2} = m_2\Sigma_u$$

Substituting for l_1, $\Sigma_{12} = 0 \Rightarrow T_m = m_2$

Substituting for T_m from (1.6), $\Sigma_{12} = 0 \Rightarrow l_1 = m_2$

Further, if $l_1 = m_2$, then $\Sigma_{12} = 0$.

If $\Sigma_{12} = 0,$ then $l_2 = \Sigma_{02} / \Sigma_{22}$. Substituting for Σ_{02} and Σ_{22} from (1.11) and (1.14), we get (5).

Finally, it is easy to check that, if l_1, l_2 and m_2 are as given in (2), (5) and (6), $\Sigma_{12} = 0.$

Alternatively, instead of assuming $\Sigma_{12} = 0,$ we can solve for m_2 and l_2 directly from the projection formulas

$$
\begin{aligned}
l_2 &= \frac{\Sigma_{02}\Sigma_{11} - \Sigma_{01}\Sigma_{12}}{D} \\
m_2 &= \frac{\Sigma_{01}\Sigma_{22} - \Sigma_{02}\Sigma_{12}}{D}
\end{aligned}
$$

where $D = \Sigma_{11}\Sigma_{22} - (\Sigma_{12})^2$.

The calculations are long and involved, but the same result obtains.

$$
\begin{aligned}
z &= \frac{ts}{Q}\frac{1}{l_1(1+Q)}\frac{T_m}{l_2(1+m)} \quad - \quad \frac{m_2}{l_2}\frac{u_1}{1+m} \\
&= \frac{\sqrt{\Sigma_u}\sqrt{Q(1+Q)}}{\sqrt{m}\sqrt{nt\Sigma_v}}\left\{\frac{ts}{Q(1+Q)} - \frac{\sqrt{nt\Sigma_v}}{\sqrt{\Sigma_u}}\frac{u_1}{1+Q}\right\} \\
\therefore z &= \left[\frac{1}{\sqrt{m}}\frac{1}{\sqrt{Q(1+Q)}}\right]x_d \quad - \quad \left[\frac{1}{\sqrt{m}}\frac{\sqrt{Q}}{\sqrt{(1+Q)}}\right]u_1
\end{aligned}
\tag{1.15}
$$

Therefore, comparing coefficients:

$$
\begin{aligned}
B_1 &= \frac{1}{\sqrt{m}}\frac{1}{\sqrt{Q(1+Q)}} \\
B_2 &= \frac{-1}{\sqrt{m}}\frac{\sqrt{Q}}{\sqrt{(1+Q)}}
\end{aligned}
\tag{1.16}
$$

The j^{th} broker profits:

$$
\begin{aligned}
W_d &= E\left[(v - p_2(z))z\right] \\
&= E\left[(v - l_2 y_2 - m_2 y_1)z\right] \\
&= E\left[\{v - l_2(mz + u_2) - m_2(x_d + u_1)\}z\right]
\end{aligned}
$$

Substituting for z from above, taking expectations and simplifying we obtain:

$$
W_d = \frac{\sqrt{nt\Sigma_u\Sigma_v}}{\sqrt{Q(1+Q)}}\cdot\frac{1}{\sqrt{m}(1+m)}
\tag{1.17}
$$

Proof of Corollary 1:

(1) We use the results that $\partial Q / \partial n > 0$ and $\partial Q / \partial t > 0$ for $n>1$. The proof follows since $\partial B_1 / \partial Q < 0$ and $\dfrac{\partial |B_2|}{\partial \sqrt{Q}} = \dfrac{1}{(1+Q)^{\frac{3}{2}}} > 0.$

(2) It is clear that B_1 and B_2 are decreasing in m from definition. Further, $mB_1 = \dfrac{\sqrt{mQ}}{\sqrt{1+Q}}$ and $m|B_2| = \dfrac{\sqrt{m}\sqrt{Q}}{\sqrt{1+Q}}$ are increasing in m.

Proof of Corollary 2:

(1) $m_2 = l_1$ follows from (2) and (6).

$$l_2 = l_1 \frac{\sqrt{1+Q}}{\sqrt{Q}} \left(\frac{\sqrt{m}}{1+m} \right)$$

Note that $Q \geq 1$ and $m \geq 1$. For $Q = 1$ and $m = 1$, $l_2 = \frac{l_1}{\sqrt{2}} < l_1$.

Further, $\frac{\sqrt{1+Q}}{\sqrt{Q}}$ is decreasing in Q and $\sqrt{m}/(1+m)$ is decreasing in m. Thus, $l_2 < l_1$ for $Q > 1$ and $m > 1$.

(2) Define period one price informativeness as: $PI_1 = \Sigma_v - Var(v|p_1) = 2\, l_1 \Sigma_{01} - (l_1)^2 \Sigma_{11}$, where $\Sigma_{01} = Cov(v, y_1)$ and $\Sigma_{11} = Var(y_1)$. Thus, $PI_1 = \frac{nt\Sigma_v}{1+Q}$.

Similarly, period two price informativeness is defined as:

$$PI_2 = \Sigma_v - \mathrm{var}(v|p_1, p_2)$$

$PI_2 > PI_1$ because $\mathrm{var}(v|p_1, p_2) < \mathrm{var}(v|p_1)$.

(3) The expected informed profit for insider i is

$$E[(v - p_1)x^{i,d}] = \frac{\sqrt{t\Sigma_u \Sigma_v}}{\sqrt{n}(1+Q)}$$

which is the same as with no dual trading.

Now, the aggregate insider profits $\Pi_I = \frac{n\sqrt{t\Sigma_u \Sigma_v}}{\sqrt{n}(1+Q)} = \frac{\sqrt{nt\Sigma_u \Sigma_v}}{(1+Q)}$. (1)

Additionally, aggregate broker profits are given by

$$\Pi_B = m\,W_d = \frac{\sqrt{nt\Sigma_u \Sigma_v}}{\sqrt{1+Q}\sqrt{Q}} \frac{\sqrt{m}}{1+m} \quad (2)$$

Hence, from (1) and (2), the noise traders' per period losses are: $U_d = \frac{1}{2}(\Pi_I + \Pi_B)$.

Noise traders' losses without dual trading $\Pi_I > U_d$ if: $\Pi_B < \Pi_I$ or, $\frac{\sqrt{m}}{1+m} \cdot \frac{\sqrt{1+Q}}{\sqrt{Q}} < 1$. The last inequality holds since $l_2 < l_1$

Proof of Proposition 2:

Simultaneous dual trading equilibrium

There are m dual traders and n informed traders. The net order flow y_s is given by

$$y_s = \sum_{j=1}^{m} z^j + x_s + u \tag{2.1}$$

where $x_s = \sum_{i=1}^{n} x^{i,s}$.

The dual trader j's problem:

$$\underset{z^j}{Max} \quad E\left[(v - p_s) z^j \,\middle|\, \frac{x^{1,s}}{m}, \frac{x^{2,s}}{m}, \ldots, \frac{x^{n,s}}{m}, \frac{u}{m} \right] \tag{2.2}$$

or, $\underset{z^j}{Max} \quad E\left[(v - \lambda_s z^j - \lambda_s z^{-j} - \lambda_s x_s - \lambda_s u) z^j \,\middle|\, \frac{x^{1,s}}{m}, \frac{x^{2,s}}{m}, \ldots, \frac{x^{n,s}}{m}, \frac{u}{m} \right]$

where $z^{-j} = \sum_{k \neq j} z^k$

The first order condition gives us

$$E\left[v \,\middle|\, \frac{x^{1,s}}{m}, \frac{x^{2,s}}{m}, \ldots, \frac{x^{n,s}}{m}, \frac{u}{m} \right] - 2\lambda_s z^j - \lambda_s z^{-j} - \lambda_s x_s - \lambda_s u = 0 \tag{2.3}$$

where $E\left[v \,\middle|\, \frac{x^{1,s}}{m}, \frac{x^{2,s}}{m}, \ldots, \frac{x^{n,s}}{m}, \frac{u}{m} \right] = \frac{t}{Q}\hat{s};\ and\ \hat{s} = \frac{\sum_{i=1}^{n} x^{i,s}}{A_s} = \frac{x_s}{A_s}$.

Substituting $z = z^j$, for each j, and solving for z, we obtain

$$z = \frac{x_s}{\lambda_s (1+m)} \left(\frac{t}{QA_s} - \lambda_s \right) - \frac{u}{1+m} \tag{2.4}$$

Insider i's problem:

$$\underset{x^{i,s}}{Max} \quad E\left[(v - \lambda_s m z - \lambda_s x^{i,s} - \lambda_s x^{-i,s} - \lambda_s u) x^{i,s} \,\middle|\, s^i \right] \tag{2.5}$$

where $x^{-i,s} = \sum_{k \neq i} x^{k,s}$

or, $\underset{x^{i,s}}{Max} \quad E\left[\left(v - \frac{x_s m}{1+m} \left(\frac{t}{QA_s} - \lambda_s \right) - \lambda_s x^{i,s} - \lambda_s x^{-i,s} - \frac{\lambda_s u}{1+m} \right) x^{i,s} \,\middle|\, s^i \right]$

$$\underset{x^{i,s}}{Max}\left\{ x^{i,s}\left(ts^i - x^{i,s}\left(l_s + \frac{m}{1+m}\left(\frac{t - l_s QA_s}{QA_s} \right) \right) - E\left[x^{-i,s} \middle| s^i\right]\left(l_s + \frac{m}{1+m}\left(\frac{t - l_s QA_s}{QA_s} \right) \right) \right) \right\}$$

From above, the first order condition gives us

$$2x^{i,s}\left(\frac{l_s}{1+m} + \frac{m}{1+m} \cdot \frac{t}{QA_s} \right) = -\frac{(n-1)A_s ts^i}{1+m}\left(l_s + \frac{mt}{QA_s} \right) + ts^i$$

$$= \frac{ts^i}{Q(1+m)}\left[Q(1+m) - l_s A_s (n-1)Q - mt(n-1) \right] \qquad (2.6)$$

$$= \frac{ts^i}{Q(1+m)}\left[Q + m - l_s A_s (n-1)Q \right]$$

The second order condition is:

$$l_s + \frac{mt}{QA_s} > 0.$$

Thus, given $l_s > 0$, A_s has to be positive to satisfy the second order condition.

Simplifying (2.6), we get

$$x^{i,s} = \frac{ts^i}{Q(1+m)} \cdot \frac{\left[m + Q\{1 - l_s A_s (n-1)\} \right](1+m)QA_s}{2\left[tm + l_s QA_s \right]} \qquad (2.7)$$

Comparing coefficients,

$$A_s = \frac{A_s t\left[m + Q\{1 - l_s A_s (n-1)\} \right]}{2\left[tm + l_s QA_s \right]} \qquad (2.8)$$

Simplifying the above,

$$A_s = \frac{t(Q-m)}{l_s Q(1+Q)} > 0 \text{ if } Q > m \text{ and } l_s > 0 \qquad (2.9)$$

Note that for t=1, $Q=1 + t(n-1) = n$. Thus, $A_s > 0$ iff $n > m$.

$$z = \frac{t}{l_s Q(1+m)} \cdot \frac{x_s}{A_s} - \frac{x_s}{1+m} - \frac{u}{1+m}$$

$$= \frac{x_s}{1+m}\left(\frac{t}{l_s A_s Q} - 1 \right) - \frac{u}{m+1} \qquad (2.10)$$

$$= \frac{x_s}{Q-m} - \frac{u}{m+1}, \text{ after substituting for } A_s$$

Therefore, $B_{1,S} = \frac{1}{(Q-m)}$ and $B_{2,S} = \frac{-1}{(1+m)}$

Now,

$$
\begin{aligned}
y_s &= x_s + mz + u \\
&= x_s + \frac{x_s m}{Q-m} + \frac{1}{1+m}u \qquad (2.11)\\
&= x_s\left(\frac{Q}{Q-m}\right) + \frac{1}{1+m}u
\end{aligned}
$$

$$
\begin{aligned}
l_s &= \frac{\left(\frac{Q}{Q-m}\right) nA_s\Sigma_v}{\frac{Q^2}{(Q-m)^2}(A_s)^2 nQ\Sigma_s + \left(\frac{1}{1+m}\right)^2\Sigma_u} \\
&= \frac{\left(\frac{Q}{Q-m}\right)\cdot\frac{t(Q-m)}{l_s Q(1+Q)}\cdot n\Sigma_v}{\frac{Q^2}{(Q-m)^2}\cdot\frac{t^2(Q-m)^2}{l_s{}^2 Q^2(1+Q)^2} nQ\Sigma_s + \left(\frac{1}{1+m}\right)^2\Sigma_u} \qquad (2.12)\\
&= \frac{nt\Sigma_v\, l_s(1+Q)}{ntQ\Sigma_v + \left(\frac{1}{1+m}\right)^2\Sigma_u\, l_s{}^2(1+Q)^2}
\end{aligned}
$$

After further simplification, we obtain

$$
l_s = \frac{(1+m)\sqrt{nt\Sigma_v}}{\sqrt{\Sigma_u}(1+Q)} \qquad (2.13)
$$

$$
\begin{aligned}
A_s &= \frac{t(Q-m)}{l_s Q(1+Q)} \\
&= \frac{t(Q-m)}{Q(1+Q)}\cdot\frac{(1+Q)\sqrt{\Sigma_u}}{\sqrt{nt\Sigma_v}(1+m)} \qquad (2.14)\\
&= \frac{\sqrt{t}(Q-m)\sqrt{\Sigma_u}}{Q(1+m)\sqrt{n\Sigma_v}} > 0,\ \text{for } Q > m
\end{aligned}
$$

Finally, the j^{th} broker profits

$$
\begin{aligned}
W_s &= E[(v - p_s)z] \\
&= E\left[\left(v - \frac{l_s Q}{Q-m}x_s - \frac{l_s u}{1+m}\right)z\right] \\
&= \frac{nA_s\Sigma_v}{Q-m} - \frac{l_s Q}{Q-m}\cdot\frac{(A_s)^2 nQ\Sigma_s}{(Q-m)} + \frac{l_s\Sigma_u}{(1+m)^2} \\
&= \frac{nA_s\Sigma_v}{Q-m}\cdot\frac{1}{1+Q} + \frac{\sqrt{nt\Sigma_u\Sigma_v}}{(1+Q)(1+m)} \\
&= \frac{\sqrt{n\Sigma_v}}{(Q-m)}\cdot\frac{(Q-m)\sqrt{t\Sigma_u}}{Q(1+m)}\cdot\frac{1}{1+Q} \quad + \quad \frac{\sqrt{nt\Sigma_u\Sigma_v}}{(1+Q)(1+m)} \\
&= \frac{\sqrt{nt\Sigma_u\Sigma_v}}{(1+m)}\cdot\frac{1}{Q}
\end{aligned}
\qquad (2.15)
$$

Proof of Corollary 3:

Follows immediately from the definitions of $B_{1,s}$ and $|B_{2,s}|$ in (10) and (11) in the text, and from $\partial Q/\partial n > 0$ and $\partial Q/\partial t > 0$ for n>1.

Proof of Corollary 4:

(1) $l_s = \frac{\sqrt{nt\Sigma_v}}{\sqrt{\Sigma_u}}\frac{1+m}{1+Q} = l_1(1+m)$

Thus, $l_s > l_1$. The above implies that market depth is lower with simultaneous dual trading compared to the no-dual trading benchmark (i.e., the inverse of l_1).

(2)

$$
\begin{aligned}
PI_s &= l_s Cov(v, y_s) \\
&= l_s A_s n\Sigma_v \frac{Q}{Q-m} \\
&= \frac{t(Q-m)}{Q(1+Q)} n\Sigma_v \frac{Q}{Q-m} \\
&= \frac{nt\Sigma_v}{1+Q}
\end{aligned}
$$

which is the same as when there is no dual trading.

(3)

Individual insider profit = $\frac{\sqrt{t\Sigma_u\Sigma_v}}{\sqrt{n}(1+Q)}\frac{1}{1+m}\frac{Q-m}{Q} < \frac{\sqrt{t\Sigma_u\Sigma_v}}{\sqrt{n}(1+Q)}$,

where the term to the right of the inequality is an insider's profit without dual trading. The inequality follows since $\left(\frac{1}{1+m}\right) < 1$ and $\left(\frac{Q-m}{Q}\right) < 1$.

Uninformed losses with simultaneous dual trading equals the sum of aggregate insider and broker profits, or

$$\frac{m\sqrt{nt\Sigma_u\Sigma_v}}{(1+m)Q} + \frac{(Q-m)\sqrt{nt\Sigma_u\Sigma_v}}{Q(1+Q)(1+m)} = \frac{\sqrt{nt\Sigma_u\Sigma_v}}{1+Q}$$

Uninformed losses with no dual trading equals

$$\frac{\sqrt{nt\Sigma_u\Sigma_v}}{(1+Q)}$$

The final result follows directly.

<u>Proof of Proposition 3:</u>

(1) Under consecutive dual trading, informed profits (I_d) is $I_d = \frac{\sqrt{t\Sigma_u\Sigma_v}}{\sqrt{n}(1+Q)}$, which is independent of m.

Noise trader losses $U_d = nI_d + mW_d = \frac{\sqrt{nt\Sigma_u\Sigma_v}}{1+Q}\left[1+\frac{\sqrt{1+Q}}{\sqrt{Q}}\cdot\frac{\sqrt{m}}{1+m}\right]$ which is decreasing in *m*.

(2) Informed profits with simultaneous dual trading

$$I_s = \frac{\sqrt{t\Sigma_u\Sigma_v}}{\sqrt{n}Q(1+Q)}\cdot\left[\frac{Q-m}{1+m}\right]$$

Consider the term in the square bracket above

$$\frac{\partial}{\partial m}\left[\frac{Q-m}{1+m}\right] = \frac{-(1+Q)}{(1+m)^2} < 0.$$

Noise trader losses $U_s = \frac{\sqrt{nt\Sigma_u\Sigma_v}}{1+Q}$,

which is independent of *m*.

<u>Proof of Proposition 4:</u>

1) In the consecutive dual trading model, the profit of a broker is $W_d = \frac{\sqrt{nt\Sigma_v\Sigma_u}}{\sqrt{Q(1+Q)}}\cdot\frac{1}{\sqrt{m}}\cdot\frac{1}{1+m}$

When $t=1$, $W_d = \frac{\sqrt{\Sigma_v\Sigma_u}}{\sqrt{(1+n)}}\cdot\frac{1}{\sqrt{m}}\cdot\frac{1}{1+m}$

The profits of an informed trader in the consecutive dual trading model, $I_d = \frac{\sqrt{t\Sigma_v\Sigma_u}}{\sqrt{n}(1+Q)}$

When $t=1$, $I_d = \frac{\sqrt{\Sigma_v\Sigma_u}}{\sqrt{n}(1+n)}$

The equilibrium number of informed traders, n_d, satisfies $I_d = k_d$ since I_d is decreasing in *n*. So

$$\sqrt{n_d}(1+n_d) = \frac{\sqrt{\Sigma_v\Sigma_u}}{k_d} \tag{4.1}$$

Similarly, the equilibrium number of brokers, m_d, satisfies:

$$\sqrt{m_d}(1+m_d) = \frac{\sqrt{\Sigma_v\Sigma_u}}{k_d} \cdot \frac{1}{\sqrt{1+n_d}} \tag{4.2}$$

From (4.1) and (4.2), $m_d < n_d$.

Thus, for the equilibrium outcome to be non-null, m_d =1 and n_d =2.

m_d =1 requires:

$$k_d \le \frac{\sqrt{\Sigma_v\Sigma_u}}{2\sqrt{3}} \tag{4.3}$$

where the RHS of (4.3) is W_d evaluated at *m* = 1 and *n* = 2.

n_d =2 requires:

$$k_d \le \frac{\sqrt{\Sigma_v\Sigma_u}}{3\sqrt{2}} < \frac{\sqrt{\Sigma_v\Sigma_u}}{2\sqrt{3}} \tag{4.4}$$

where the RHS of (4.4) is I_d evaluated at *n* = 2. Hence, m_d =1 and n_d =2 hold simultaneously if (4.4) is satisfied.

2) In the simultaneous dual trading model, only one broker enters the market. Since $Q > m$ in equilibrium, at least two informed traders enter the market.

The profits of a broker with simultaneous dual trading, W_s, is

$W_s = \frac{\sqrt{nt\Sigma_v\Sigma_u}}{Q(1+m)} = \frac{\sqrt{\Sigma_v\Sigma_u}}{\sqrt{n}(m+1)}$, at $t=1.$.

The profits of an informed trader with simultaneous dual trading

$I_s = \frac{\sqrt{t\Sigma_v\Sigma_u}}{\sqrt{n}(1+Q)} \cdot \frac{Q-m}{Q(1+m)} = \frac{\sqrt{\Sigma_v\Sigma_u}}{\sqrt{n}(1+n)} \cdot \frac{n-m}{n(1+m)}$ at $t=1.$

Given that two informed traders are in the market, at least one broker enters the market if:

$$k_s \le \frac{\sqrt{\Sigma_v \Sigma_u}}{2\sqrt{2}} \tag{4.5}$$

Given that one broker enters the market, at least two informed traders enter the market if:

$$k_s \le \frac{\sqrt{\Sigma_v \Sigma_u}}{12\sqrt{2}} \tag{4.6}$$

Thus, if (4.6) is satisfied, the participation constraints of both brokers and the informed traders are simultaneously satisfied.

The equilibrium number of informed traders, n_s (given $m_s = 1$) is:

$$\frac{2n_s\sqrt{n_s}\left(1+n_s\right)}{\left(n_s-1\right)} = \frac{\sqrt{\Sigma_v \Sigma_u}}{k_s} \tag{4.7}$$

Proof of Proposition 5:

1) Let $k_s \ge k_d$. Suppose k_s satisfies (4.6). Then both equilibrium outcomes, consecutive and simultaneous, are non-null. From (4.1) and (4.7) in the proof of proposition 4:

$$\sqrt{\frac{n_s}{n_d}} \cdot \frac{\left(1+n_s\right)}{\left(1+n_d\right)} = \frac{\left(n_s-1\right)}{2\,n_s}\frac{k_d}{k_s} < 1 \tag{5.1}$$

From (2) and (11) in the text,

$$\begin{aligned}\frac{l_s}{l_2} &= 2\,\sqrt{1+m_d} \cdot \sqrt{\frac{1+m_d}{1+n_s}} \cdot \sqrt{\frac{n_s}{m_d}} \cdot \sqrt{\frac{1+n_d}{1+n_s}} \\ &> 1 \text{ since } n_s < n_d\end{aligned} \tag{5.2}$$

(2) From (5.1), if $k_s < \frac{k_d}{2}$ and $n_s \ge 2$, then $n_s > n_d$. Consider (5.2) again. Suppose m_d is large.

Then approximately, $\frac{\left(1+m_d\right)}{m_d} \cong 1$, and

$$\begin{aligned}\frac{l_s}{l_2} &\cong \frac{2\sqrt{\left(1+m_d\right)\left(1+n_d\right)}}{\left(1+n_s\right)} \\ &< 1 \text{ if } n_s \text{ is large relative to } m_d \text{ and } n_d.\end{aligned}$$

For example, if $n_s > 2n_d$, then $\frac{l_s}{l_2} < 1$.

Proof of Proposition 6:

1) Let the subscript "o" refer to outcomes in the order flow internalization model. Analogous to proposition 2, we can show that:

$$A_o = \frac{(Q - m_2)}{Q(1 + m_1)} \cdot \frac{\sqrt{t\Sigma_u}}{\sqrt{n\Sigma_v}} \qquad (6.1)$$

$$l_o = \frac{(1 + m_1) \cdot \sqrt{nt\Sigma_v}}{\sqrt{\Sigma_u}(1 + Q)} \qquad (6.2)$$

In the market without internalization, $m_2 = m$ and $m_1 = 0$. Since the market depth without internalization is:

$$l_1 = \frac{\sqrt{nt\Sigma_v}}{\sqrt{\Sigma_u}(1 + Q)}, \quad l_o \ > \ l_1.$$

$$PI_o = \ l_o \operatorname{cov}(v, y_o), \text{ where}$$

$$y_o = x_o \cdot \frac{Q}{Q - m_2} + \frac{u_1}{1 + m_1},\ x_o = A_o s,\ s = \sum_{i=1}^{n} s^i.$$

From (6.1) and (6.2), and $t = 1$,

$$PI_o = \frac{n\Sigma_v}{1+n} = PI_1 \text{ (price informativeness without internalization).}$$

Uninformed losses with order flow internalization is:

$$U_o = n\mathrm{E}[(v - p_o)x_o] + m_2\mathrm{E}[(v - p_o)z_2] + m_1\mathrm{E}[(v - p_o)z_1]$$

where $p_o = \ l_o y_o$. From (6.1) and (6.2),

$$U_0 \ = \ \frac{\sqrt{nt\Sigma_u\Sigma_v}}{(1+Q)(1+m_1)} \cdot \frac{Q - m_2}{Q} \ + \ \frac{\sqrt{nt\Sigma_u\Sigma_v}}{(1+Q)(1+m_1)} \cdot \frac{m_2}{Q} \ + \ \frac{\sqrt{nt\Sigma_u\Sigma_v}}{(1+Q)(1+m_1)} \cdot m_1$$

$$= \ \frac{\sqrt{nt\Sigma_u\Sigma_v}}{1+Q}$$

= uninformed losses without dual trading.

Consider the fixed entry equilibrium. Suppose each broker charges \$$c$ per trade, independent of the order size. Also the fee is paid irrespective of whether the customer makes the trade. The aggregate of noise trades is considered one trade, so there is one trade per internalizing broker. Brokers have a cost of *f* per trade, where *f* > 0. Following Fishman and Longstaff (1992), each broker chooses *c* so that the broker's expected revenues from trading plus the expected commissions equal zero.

Assume:

$$c < \frac{\sqrt{t\Sigma_u\Sigma_v}}{(1+Q)(1+m_1)} \cdot \frac{Q-m_2}{\sqrt{n}\cdot Q} \tag{6.3}$$

Then, informed trader's expected profits, net of commissions are positive. Let c_o be the competitive commission with order flow internalization. m_1 brokers internalize the order flow of noise traders, and each obtains trading profits given by.

$$W_o = \frac{\sqrt{nt\Sigma_u\Sigma_v}}{(1+Q)(1+m_1)} \tag{6.4}$$

where W_o was derived in the proof of part one. By assumption, noise trades are considered one trade, and so

$$c_o = f - W_o \tag{6.5}$$

Without order flow internalization, the competitive commission is c_n where

$$c_n = f \tag{6.6}$$

Since $W_o > 0$, $c_o < c_n$. Further, $p_s + c_0 < p_s + c_n$.

2) (i) From the proof of part one, informed profits are decreasing in m_2, so informed traders choose $m_2 = 1$. However, informed traders do not choose the internalizing brokers, and uninformed losses are independent of m_1, so m_1 is determined in equilibrium. Equilibrium requires $n > m_2$, so at least two informed traders must enter.

Assume $t = 1$. For at least two informed traders to enter,

$$k_o \leq \frac{\sqrt{\Sigma_v\Sigma_u}}{12\sqrt{2}} \tag{6.7}$$

where k_o is the entry cost in the market with order flow internalization.
For at least one piggybacking broker to enter (i.e., $m_2 = 1$):

$$k_o \leq \frac{\sqrt{\Sigma_v \Sigma_u}}{6\sqrt{2}} \tag{6.8}$$

For at least one internalizing broker to enter (i.e., $m_1 = 1$):

$$k_o \leq \frac{\sqrt{\Sigma_v \Sigma_u}}{3\sqrt{2}} \tag{6.9}$$

Hence, the constraint (6.7) is binding.

Given entry by m_o internalizing brokers and one piggybacking broker, the number of informed traders, n_o, who enter is:

$$k_o = \frac{\sqrt{\Sigma_u \Sigma_v}}{n_o \sqrt{n_o}} \cdot \frac{n_o - 1}{n_o + 1} \cdot \frac{1}{1 + m_o} \tag{6.10}$$

Given entry by n_o informed traders, the number of internalizing brokers entering is given by:

$$k_o = \sqrt{\Sigma_u \Sigma_v} \cdot \frac{\sqrt{n_o}}{1 + n_o} \cdot \frac{m_o}{1 + m_o} \tag{6.11}$$

From (6.10) and (6.11),

$$m_o = \frac{n_o - 1}{(n_o)^2} < n_o$$

And so the number of entering brokers varies negatively with the number of entering informed traders for $n_o > 1$.

(ii) In the market with no internalization, $m_2 = 1$ and $m_1 = 0$, and so the profits of an informed trader is

$$\frac{1}{1+n} \cdot \frac{\sqrt{\Sigma_u \Sigma_v}}{\sqrt{n}} \cdot \frac{n-1}{n}.$$

Assume the market entry cost is k_w, and n_w informed traders enter. Then

$$k_w = \frac{\sqrt{\Sigma_u \Sigma_v}}{n_w \sqrt{n_w}} \quad \frac{n_w - 1}{n_w + 1} \tag{6.12}$$

Assume $k_o = k_w = k$. Comparing (6.10) and (6.12), $n_o < n_w$.

Evaluate l_w (market depth with no internalization) at $n = n_w$ and l_o at $n = n_o$. Since l_w is simply l_o evaluated at $m_1 = 0$, we have:

$$l_w = \frac{\Sigma_v}{k}\left[\frac{1}{1+n_w}\right]^2 \frac{n_w - 1}{n_w} \tag{6.13}$$

$$l_o = \frac{\Sigma_v}{k}\frac{(n_o - 1)}{n_o(n_o + 1)} \tag{6.14}$$

Therefore,

$$\frac{l_o}{l_w} = \frac{(n_w+1)}{(n_o+1)} \cdot (n_o - 1) \cdot \frac{n_w}{n_o} \cdot \frac{(n_w+1)}{(n_w-1)}$$
$$> 1 \text{ since } n_w > n_o \geq 2.$$

From the proof of part (i) of this proposition, informativeness of price

$$PI_o = \frac{n_o}{1+n_o}\Sigma_v \tag{6.15}$$

$$PI_w = \frac{n_w}{1+n_w}\cdot\Sigma_v \tag{6.16}$$

Hence, $PI_o - PI = \frac{(n_o - n_w)}{(1+n_o)(1+n_W)}\cdot\Sigma_V \quad < \quad 0$.

From the proof of part (i) of this proposition, uninformed losses in the two markets are:

$$U_o = \frac{\sqrt{n_o}}{1+n_o}\sqrt{\Sigma_u\Sigma_v} \tag{6.17}$$

$$U_w = \frac{\sqrt{n_w}}{1+n_w}\sqrt{\Sigma_u\Sigma_v} \tag{6.18}$$

and $$U_o - U_w = \frac{\left(\sqrt{n_w} - \sqrt{n_o}\right)\left(\sqrt{n_w n_o} - 1\right)}{(1+n_o)(1+n_w)}\cdot\sqrt{\Sigma_u\Sigma_v}$$
$$> 0$$

Proof of Proposition 7:

Consider the Holden-Subrahmanyam model (1992), specialized to two periods and $\Delta t_n \equiv 1$. From propositions one and two of their paper, the informed trading intensity in period two, b_2, the period two market depth, l_2, and price informativeness $(\Sigma_v - \Sigma_2)$ are:

$$\Sigma_1 = \frac{\Sigma_v}{n(1-2q_1)+1} \tag{7.1}$$

$$\Sigma_2 = \frac{\Sigma_1}{1+n} \tag{7.2}$$

$$l_2 = \frac{n\Sigma_2}{\Sigma_u(1+n)} \tag{7.3}$$

$$b_2 = \frac{\Sigma_u}{\Sigma_2(1+n)} \tag{7.4}$$

and $q_1 \in \left(0, \frac{1}{2}\right)$ is the unique root satisfying the cubic equation:

$$0 = 2n(q_1)^3 - (1+n)(q_1)^2 - \frac{2q_1}{(1+n)^2} + \frac{1}{(1+n)^2} \tag{7.5}$$

We numerically compare (7.3) with (4.5), (7.4) with (4.1), and $(\Sigma_v - \Sigma_2)$ with $PI_2 = \frac{n\Sigma_v}{(1+n)} \cdot \left[1 + \frac{m}{n(1+n)}\right]$.

♥