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Abstract

We modify the Laubach-Williams and Holston-Laubach-Williams models of the natural rate of interest to account for time-varying volatility and a persistent COVID supply shock during the pandemic. Resulting estimates of the natural rate of interest in the United States, Canada, and the Euro Area at the end of 2022 are close to their respective levels estimated directly before the pandemic; that is, we do not find evidence that the era of historically low estimated natural rates of interest has ended. In contrast, estimates of the natural rate of output have declined relative to those projected before the pandemic.

Key words: natural rate of output, time-varying volatility, Kalman filter, trend growth, COVID-19 pandemic

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1 Introduction

The downward trend in estimates of the natural rate of interest to historically low levels observed in many countries has garnered considerable attention and debate about their sources and consequences (Laubach and Williams, 2016, Gourinchas and Rey, 2019, Rachel and Summers, 2019).[1] The events of the past few years, including the COVID-19 pandemic and subsequent policy actions, have renewed the debate over whether historically low natural rates of interest will persist in the post-pandemic era (International Monetary Fund, 2023, Obstfeld, 2023). Answering this question using empirical models of the natural rate of interest has been challenging owing to the unprecedented macroeconomic volatility across the globe during the pandemic. This paper develops and implements a data-driven approach that addresses the extraordinary effects of the pandemic using the Holston, Laubach, and Williams (HLW, 2017) and Laubach and Williams (LW, 2003) models of the natural rate of interest. Our approach preserves to the greatest extent the basic structure and flexibility of the original models, while providing consistent model estimates of natural rates before, during, and after the pandemic period. It also has broader application to models that estimate latent variables using frequentist (as in HLW and LW) and Bayesian methods.

The HLW and LW models apply the Kalman filter to translate movements in real GDP, inflation, and short-term interest rates into estimates of trend growth, the natural rate of output, and the natural rate of interest. The model's structure is flexible and incorporates transitory and permanent shocks to supply and demand and dynamic endogenous behavior of inflation and output. However, like other models that use the Kalman or other statistical filters, identifying assumptions regarding the nature of the shock processes are imposed. In particular, the shocks are assumed to be serially uncorrelated and described by time-invariant Gaussian distributions. The COVID-19 pandemic generated extraordinary swings in macroeconomic data that are dramatically at odds with both of these assumptions.

First, the assumption of a time-invariant Gaussian distribution for the shock processes is clearly contradicted by the data. Relative to the historical experience of the prior half century, the COVID-19 pandemic is an extreme tail event in terms of its

[1] There is a related literature on the history of real interest rates over a very long time span. See, for example, Rogoff, Rossi, and Schmelzing (2022) and references therein.

effects on economies around the world. Figure 1 plots the normalized quarterly log changes in GDP in the United States, Canada and the Euro Area over 2020-2022. Each series is normalized relative to their respective means and standard deviations from 1970-2019 (1972-2019 for the Euro Area). The fluctuations in GDP growth during the first three quarters of the pandemic (2020:Q2-2020:Q4) are enormous relative to pre-pandemic behavior. In addition, an unusually high number of observations in 2021 are outliers relative to historical behavior.

Second, the assumption of serially uncorrelated shocks is inconsistent with the highly negatively-correlated sequence of swings in output associated with the shutdowns and re-openings caused by COVID-19. For example, in all three economies, GDP declined sharply in the second quarter of 2020, then rose sharply in the following quarter. If uncorrected, these two stark violations of the assumptions significantly distort the estimation of model parameters and latent variables.

We make two modifications to the HLW and LW models that address the two violations of the original models' assumptions. First, we allow for time-varying volatility of the shocks to output and inflation during the pandemic period consistent with the appearance of extreme outliers in the data. We build on the insight from Lenza and Primiceri (2022) that if the timing of increased volatility is known, one can introduce time-varying volatility in the model directly by applying a scale factor to the innovation variances. Due to the presence of shocks to both observed and latent variables, we follow Harvey and Koopman (1992) and Harvey et al (1999) in measuring outliers in terms of auxiliary residuals. These have the advantage of providing a direct interpretation and test for outliers.

Second, we incorporate a proxy for a persistent, but not permanent, supply shock that is designed to capture the effects of COVID-19 and related policy responses. Specifically, we create a COVID-adjusted measure of the natural rate of output, where the magnitude of the adjustment is proportional to the country-specific COVID-19 Stringency Index from the Oxford COVID-19 Government Response Tracker (Hale et al., 2021). The magnitude of the effect of COVID policies on the natural rate of output is estimated for each economy. This COVID shock is in addition to the full set of innovations already present in the models.

The HLW model with these two modifications is estimated on data from the United States, Canada, and the Euro Area through 2022.[2] The estimation results

[2]In addition, we make some technical adjustments to the HLW model and the estimation pro-

demonstrate that these two modifications effectively address the two econometric issues associated with the pandemic.[3] The estimation procedure yields parameter estimates consistent with the model structure, and the pre-pandemic estimates of the natural rates of output and interest and the trend growth rate are very similar to those estimated on data ending in 2019.

The pattern of historically low estimates of trend GDP growth and the natural rate of interest experienced before the pandemic persist after the COVID-19 pandemic. In all three economies, the estimates of trend growth and the natural rate of interest in 2022 are within a few tenths of a percentage point of the corresponding estimates for 2019. In particular, these estimates provide no evidence of a reversal of the trend decline in estimates of the natural rate of interest based on data through 2022.

In all three economies, estimates of the COVID-adjusted natural rate of output in 2022 are significantly lower than what the model predicts based on pre-pandemic data. These declines reflect both the estimated effects of COVID-related restrictions and permanent negative shocks to the natural rate of output. According to the model, these declines in the natural rate of output are the most economically significant lasting effects of the COVID era.

This paper is organized as follows. Section 2 describes the HLW model and the evidence of significant departures from the model's assumptions brought on by the COVID-19 pandemic. Section 3 describes the modifications to the model to address the pandemic-related effects. Section 4 reports estimation results. Section 5 reports results from robustness exercises. Section 6 concludes.

2 Results from the Pre-Pandemic Model

In this section, we provide a short description of the original HLW model. We show that the data during the pandemic generate large outliers that are inconsistent with the assumptions of the model.

cedures for both models. Specifically, consistent with the specification of the LW model, we now estimate the relationship between trend growth and the natural rate of interest in the HLW model, instead of restricting it to unity. We also make minor technical adjustments that align the assumptions used in the various stages of the model estimation procedure, described in Appendix A1.

[3]In HLW (2017), the model was also estimated using data from the United Kingdom. Extending the sample to include the most recent years has weakened the estimated relationship between the output gap and real interest rates in the UK data, making estimates of the natural rate of interest highly unreliable. For that reason, we no longer estimate the model for the United Kingdom.

2.1 The Original HLW Model

In the HLW model, the natural rate of interest, r_t^*, is the real interest rate consistent with output equaling its natural rate, y_t^*, and stable inflation. As is standard in this literature (e.g., see Woodford, 2003), we model the output gap and inflation dynamics as a function of the real interest rate gap, $r_t - r_t^*$, using an intertemporal IS equation and Phillips curve relationship, in line with the New Keynesian framework:

$$\widetilde{y}_t = a_{y,1}\widetilde{y}_{t-1} + a_{y,2}\widetilde{y}_{t-2} + \frac{a_r}{2}\sum_{j=1}^{2}(r_{t-j} - r_{t-j}^*) + \epsilon_{\widetilde{y},t} \tag{1}$$

$$\pi_t = b_\pi \pi_{t-1} + (1 - b_\pi)\pi_{t-2,4} + b_y \widetilde{y}_{t-1} + \epsilon_{\pi,t} \tag{2}$$

The output gap is defined as $\widetilde{y}_t = 100 \cdot (y_t - y_t^*)$, where y_t and y_t^* are logarithms of real GDP and the unobserved natural rate of output, respectively, r_t is the real short-term interest rate, π_t denotes consumer price inflation, and $\pi_{t-2,4}$ is the average of the second to fourth lags of the inflation rate.[4] The stochastic disturbances $\epsilon_{\widetilde{y},t}$ and $\epsilon_{\pi,t}$ are transitory shocks to the output gap and inflation equations, respectively.

We use the Kalman filter to estimate the latent variables, which are the natural rate of output, its trend growth rate, and a process capturing other low-frequency determinants of the natural rate of interest. In keeping with the standard Kalman filter approach, the stochastic innovations to the measurement equations – the IS and Phillips curve equations – are assumed to follow a Gaussian distribution with standard deviations $\sigma_{\tilde{y}}$ and σ_π, respectively, and to be mutually and serially uncorrelated.

In contrast to the transitory shocks to the output gap and inflation equations, movements in r_t^* reflect highly persistent, or permanent, shifts in the relationship between the real short-term interest rate and the output gap (Williams, 2003). The law of motion for the natural rate of interest is given by

$$r_t^* = c \cdot g_t + z_t \tag{3}$$

where g_t is the trend growth rate of the natural rate of output, and z_t captures other determinants of r_t^*.[5] We specify the three latent variables in our state-space model

[4]See HLW (2017) Section 2 and Appendix A for details of the model specification. We take as a starting point the open-economy New Keynesian model specification as in Galí (2008) and relax two standard restrictions to work with reduced-form IS and Phillips curve equations.

[5]Note that, consistent with LW (2003), we relax the assumption in HLW (2017) of a one-for-one

as follows. The logarithm of the natural rate of output follows a random walk with a stochastic drift, g_t, that itself follows a random walk,

$$y_t^* = y_{t-1}^* + g_{t-1} + \epsilon_{y^*,t} \tag{4}$$

$$g_t = g_{t-1} + \epsilon_{g,t} \tag{5}$$

and the component z_t capturing other determinants of r_t^*, which is assumed to follow a random walk as well,

$$z_t = z_{t-1} + \epsilon_{z,t} \tag{6}$$

We assume that the disturbances $\epsilon_{y^*,t}$, $\epsilon_{g,t}$, and $\epsilon_{z,t}$ are normally distributed with standard deviations σ_{y^*}, σ_g, and σ_z, respectively, and are serially and contemporaneously uncorrelated with all other disturbances.

Equations 1 and 2 make up the measurement equations in our state-space model and can be expressed as

$$\mathbf{y}_t = \mathbf{A}' \cdot \mathbf{x}_t + \mathbf{H}' \cdot \xi_t + \epsilon_t \tag{7}$$

with stochastic innovations ϵ_t. Equations 4, 5, and 6 make up the state equations in our state-space model, written as

$$\xi_t = \mathbf{F} \cdot \xi_{t-1} + \eta_t \tag{8}$$

where ξ_t is the state vector of latent variables and η_t is the vector of stochastic innovations. See Appendix A1 for the full state-space representation of the model.

2.2 Outliers in Estimation with 2019:Q4 Model Parameters

We now analyze how the extreme movements in GDP and inflation during the COVID-19 pandemic yield large outliers in the standard HLW model. We then show that estimates of the latent variables are heavily affected by these sizable outliers, even when we constrain the model parameters at their pre-pandemic values, and demonstrate that modifications to the HLW and LW models are necessary.

relationship between trend growth and the natural rate of interest and estimate this relationship. See Appendix A1 for details on changes to the HLW (2017) model.

Before making any adjustments to the models, we begin by estimating the standard HLW model in Section 2.1 with data through 2019:Q4, prior to the onset of the COVID-19 pandemic.[6] For reference, the upper three panels of Figure 2 show the full samples of data. We fix the model parameters at their estimated values and re-estimate the latent variables through 2022:Q4 using the Kalman filter, taking all parameter values as given from the 2019:Q4 estimated model. We also fix the initial vector of unobserved states and its covariance matrix at the 2019:Q4 values.[7] This exercise is equivalent to dropping observations beginning in 2020:Q1 through the end of the sample during the maximum likelihood estimation of model parameters, while allowing the Kalman updating procedure to continue without modification through the end of the sample. In other words, we make no modifications to the state-space model, except that the model coefficient matrices and covariance matrices in the Kalman filtering procedure are fixed at their 2019:Q4 values. This would be a suitable approach if we take the view that the pandemic period is not informative for the model parameters, such as the slopes of the IS and Phillips curve equations, but *is informative* for the latent variables.

The final step of the Kalman filtering procedure to estimate the vector of unobserved state variables at time t (denoted as $\hat{\xi}_{t|t}$, and conditional on the information set at time t) is given by the Kalman updating equation,

$$\hat{\xi}_{t|t} = \hat{\xi}_{t|t-1} + \mathbf{K}_t \cdot \underbrace{(y_t - \mathbf{A}' \cdot x_t - \mathbf{H}' \cdot \hat{\xi}_{t|t-1})}_{\text{one-step-ahead prediction error}} \tag{9}$$

where $\hat{\xi}_{t|t-1}$ is the initial estimate of the state vector during the period, conditional on the information set at time $t-1$, and $\mathbf{K}_t$ is the Kalman gain matrix. The final term contains the one-step-ahead prediction errors (or forecast errors) corresponding to the measurement equations in the model. These one-step-ahead prediction errors are the residuals to the IS and Phillips curve equations, using the forecast of $\mathbf{y}_t$ (the vector of contemporaneous endogenous variables, that is, the output gap and inflation) based

[6]Throughout the paper, we use the current data vintage at the time of publication, regardless of the sample period.

[7]We store the estimated parameter vector θ from the final (stage 3) model as well as the signal-to-noise ratios λ_g and λ_z from the median unbiased estimation procedures following stages 1 and 2, respectively. See HLW (2017) for a description of the estimation procedure and footnote 6 for the initialization process of the vector of unobserved states, its conditional expectation $\xi_{1|0}$ in the first period, and the covariance matrix $P_{1|0}$.

on the data at time t and information at time $t-1$, corresponding to the state vector $\hat{\xi}_{t|t-1}$:

$$\mathbf{y}_t - \mathbb{E}[\mathbf{y}_t|\mathbf{x}_t, \zeta_{t-1}] = \mathbf{y}_t - (\mathbf{A}' \cdot \mathbf{x}_t + \mathbf{H}' \cdot \hat{\xi}_{t|t-1}) \tag{10}$$

In Equation 9, the Kalman gain matrix dictates the weight placed on the one-step-ahead prediction errors during the latent variable estimation. A larger Kalman gain $\mathbf{K}_t$ indicates that the final estimates of the latent variables are more heavily influenced by the gap between the realized data and the model's prediction, relative to $\hat{\xi}_{t|t-1}$, the prior estimate of ξ_t conditional on information in the preceding period.

In this initial exercise, the coefficient matrices $\mathbf{H}'$ (on the state vector $\hat{\xi}_{t|t-1}$) and $\mathbf{A}'$ (on the data $\mathbf{x}_t$) are fixed at their 2019:Q4 values. The resulting one-step-ahead prediction errors are very large during much of the pandemic period. These large forecast errors translate directly to the estimated vector of unobserved latent variables (y_t^*, g_t, z_t), so that the data during this period has large effects on estimates of these latent variables.

Because the HLW model is an unobserved-components model, there exists another set of model residuals in addition to the one-step-ahead prediction errors that are commonly used for diagnostic testing. These auxiliary residuals are smoothed estimators of the disturbances to the measurement equations, ϵ_t, and to the state equations, η_t, meaning that they incorporate all available information over the full sample period and provide a different interpretation of the stochastic innovations (Harvey and Koopman, 1992; Harvey et al. 1999). They have the advantage in applying a test for outliers: under the assumption that the stochastic innovations are from a Gaussian distribution, standardized auxiliary residuals greater than 2 (in absolute value) indicate either the presence of outliers or structural change.

We use the algorithm from Koopman and Durbin (2000) to obtain the standardized auxiliary residuals to the measurement equations, $\widetilde{\epsilon}_t/\hat{\sigma}_{\epsilon,t}$, and to the state equations, $\widetilde{\eta}_t/\hat{\sigma}_{\eta,t}$. The gold lines in Figure 3 shows that the standardized auxiliary residuals to the measurement equations, given by

$$\frac{\widetilde{\epsilon}_t}{\hat{\sigma}_{\epsilon,t}} = \frac{\mathbb{E}[\epsilon_t|\mathbf{y}_T, \mathbf{x}_T, \zeta_T]}{SD[\epsilon_t|\mathbf{y}_T, \mathbf{x}_T, \zeta_T]}, \tag{11}$$

indicate extreme outliers to the output gap equation in all economies in our sample, under the standard HLW model using 2019:Q4 parameter values. In the United States and Canada, standardized auxiliary residuals to the IS equation are 9 to 10 times the outlier threshold for a Gaussian model in the second quarter of 2020, and 17 times the threshold in the Euro Area. While not as extreme, the model residuals also detect outliers to the inflation equation in each of the three economies, with standardized auxiliary residuals to the Phillips curve equation reaching double the outlier threshold in the United States.

Standardized auxiliary residuals to the unobserved state equations are given by

$$\frac{\widetilde{\eta}_t}{\hat{\sigma}_{\eta,t}} = \frac{\mathbb{E}[\eta_t|\mathbf{y}_T, \mathbf{x}_T, \zeta_T]}{SD[\eta_t|\mathbf{y}_T, \mathbf{x}_T, \zeta_T]} \tag{12}$$

As shown by the gold lines in Figure 4, these residuals demonstrate the presence of extreme outliers to the natural rate of output in the standard HLW model across all of the economies in our sample.

This observation is not unique to the HLW model. Figure 1 shows that GDP realizations during the pandemic are outliers with respect to historical data. Macroeconomic fluctuations of this magnitude would result in outliers in any standard macroeconomic model. As expected, even when pandemic-era GDP and inflation data are excluded during parameter estimation, using the Kalman filter with these extreme outliers present in the sample significantly distorts estimates of the latent variables during the pandemic period. The gold lines in Figure 5 displays large swings in the estimates of the latent variables using the model with 2019:Q4 parameter values. The extreme volatility in these estimates is in conflict with the specification of these latent variables as reflecting lower-frequency movements.

3 COVID-adjusted Model

The objective of this paper is to estimate the natural rate of interest following the COVID-19 pandemic in a way that is consistent with the HLW model outlined in Section 2.1. The large movements in economic activity and the persistent supply shocks during the pandemic period violate two standard, but important, model assumptions in HLW. As we show in Section 2.2, dropping the observations from this period during the maximum likelihood estimation of model parameters is insufficient to overcome these violated model assumptions, with model residuals indicating the presence of large outliers. The resulting estimates of the natural rates of output and interest are extremely volatile and inconsistent with our specification of r_t^* as a medium-run concept that is driven by low-frequency movements.

This section details two adjustments to the HLW model that, taken together, address the violations to the original model. The first is the introduction of time-varying volatility in the model, which we implement by allowing the variances of the stochastic innovations to the output gap and inflation equations to be higher in the COVID era relative to the non-pandemic period. The second is the introduction of a persistent, but ultimately temporary, COVID supply shock, in addition to the transitory and permanent demand and supply shocks that are already present in the model. Each of these modifications alone is insufficient to overcome the estimation challenges posed by the pandemic, but in conjunction with each other they allow for continued estimation of the natural rate of interest.

Importantly, because we are estimating latent variables that are specified as random walks, simply dropping the observations during the pandemic period would not only understate the true uncertainty associated with this period, but would also necessitate interpolating these latent variables from their pre-pandemic values. Instead, rather than imposing that these variables do not change as a result of the COVID-19 pandemic, we are able to let the data inform the estimated natural rates of interest and output after the pandemic has abated. While the introduction of time-varying volatility has a similar effect on the latent variables as dropping the observations from 2020, our approach provides more flexibility in the later years of the pandemic and does not require the strong assumption that several years of data has no effect on the latent variables. Additionally, the HLW model with time-varying volatility but without serially correlated shocks to supply would constrain how the natural rate of output could evolve in response to the pandemic data. Modeling the persistent COVID supply shock is necessary in order to capture the effects of the pandemic on the natural rate of output.

3.1 Time-Varying Volatility during COVID-19

COVID-19 represents an extreme tail event relative to the assumption of Gaussian disturbances. As we show in Section 2.2, the resulting outliers contaminate estimates of the latent variables even when we exclude them from the estimation of model parameters. This is a statistical problem that is not unique to our model or to estimation of the natural rate of interest. We present a straightforward approach to account for the substantial increase in volatility during this period by introducing time-varying volatility in the model during the window of time associated with the COVID-19 pandemic. We build on an insight from Lenza and Primiceri (2022): if the timing of increased volatility is known – as is the case for the COVID pandemic – we can introduce time-varying volatility in the model directly by applying a scale factor to the innovation variances during the period of increased volatility. We apply this insight to our unobserved-components model in order to estimate the natural rate of interest, but our approach can generalize to any state-space model with latent variables.

In particular, we introduce three new model parameters, κ_{2020}, κ_{2021}, and κ_{2022}. These are the variance scale parameters for 2020, 2021, and 2022, respectively, which multiply the variances of the innovations to the output gap and inflation equations. We define the vector κ_t of variance scale parameters at time t, that takes the values

$$\kappa_t = \begin{cases} \kappa_{2020} & \text{2020:Q2} \leq t \leq \text{2020:Q4} \\ \kappa_{2021} & \text{2021:Q1} \leq t \leq \text{2021:Q4} \\ \kappa_{2022} & \text{2022:Q1} \leq t \leq \text{2022:Q4} \\ 1 & \text{otherwise} \end{cases} \tag{13}$$

We estimate the three variance scale parameters by maximum likelihood together with the other model parameters, with the constraints $\kappa_{2020} \geq 1$, $\kappa_{2021} \geq 1$, and $\kappa_{2022} \geq 1$[8]. κ_t takes the value of 1 before the pandemic period and in 2023 and beyond. Section 5.1 considers alternative specifications of time-varying volatility.

The covariance matrix of the stochastic innovations to the output gap and inflation equations is now time-varying and is given by

$$\mathbf{R}_t = \kappa_t^2 \cdot \mathbf{R} = \begin{bmatrix} (\kappa_t \sigma_{\tilde{y}})^2 & 0 \\ 0 & (\kappa_t \sigma_\pi)^2 \end{bmatrix} \tag{14}$$

with time-varying innovation variances to the IS curve and Phillips curve equations of $(\kappa_t \sigma_{\tilde{y}})^2$ and $(\kappa_t \sigma_\pi)^2$, respectively.

[8]The restriction that $\kappa_t \geq 1$ is necessary to ensure that the likelihood estimation cannot *down-weight* the variance of certain observations, which would in effect allow it to place more weight on favorable observations. Instead, the estimated κ_t factors can only increase the innovation variances during the pandemic period.

Outside of the pandemic period, the innovation variances are specified exactly as in HLW (2017). That is, the innovation variances to the output gap equation, $\sigma^2_{\tilde{y}}$, and inflation equation, σ^2_{π}, are constant over the sample prior to 2020 and after 2022. During the year 2021, for example, the innovation variances take the values $(\kappa_{2021} \cdot \sigma_{\tilde{y}})^2$ and $(\kappa_{2021} \cdot \sigma_{\pi})^2$, respectively. Therefore κ_t is a ratio of the standard deviations of the disturbances to the measurement equations (the output gap and inflation equations) at time t relative to the standard deviations in the non-pandemic sample. When $\kappa_t > 1$, as we find for 2020 through 2022 in all economies in our sample, the innovation variances are greater than in the non-pandemic sample.

Introducing time-varying volatility to the stochastic innovations via the variance scale factors in κ_t has the effect of down-weighting extreme outlier observations in the maximum likelihood estimation of model parameters as well as in estimation of the latent variables via the Kalman filter. When $\kappa_t > 1$, the diagonal covariance matrix $\mathbf{R}_t$ of the disturbances to the output gap and inflation equations is larger relative to the case where $\kappa_t = 1$, and the resulting Kalman gain is smaller. As shown in Equation 9, the Kalman gain dictates the weight placed on the one-step-ahead prediction errors – the difference between realized values of the output gap and inflation in a given period and the model's predicted values based on information in the prior period – in updating the filtered estimates of the latent variables. As the innovation variances in a given period become large, the Kalman gain shrinks, so that the Kalman filter places relatively little weight on these new observations and the estimates of the latent variables in the state vector (that is, y^*_t, g_t, z_t and therefore r^*_t) remain close to the estimates from the prior period.

In the limit as the innovation variances tend toward infinity, the Kalman gain approaches zero, so that no weight is placed on the time-t observations in estimating the state vector. In effect, the model does not make use of time-t information, so that the forecast of the state vector at time t given the time-t information set is unchanged from the forecast given the information set at time $t-1$. This limiting case is equivalent to dropping the COVID-19 observations when estimating the latent variables. The same holds for parameter estimation: when κ_t is large, the model forecast error in this period is down-weighted when computing the log likelihood function, such that the data in this period have relatively little impact on the set of parameters that maximize the log likelihood function.

We see our approach as preferable to outright discarding the COVID-19 outliers by treating them as missing data for several reasons. It is not our primary goal to provide estimates of r_t^* during the COVID-19 pandemic, and we treat these estimates with extreme caution. Rather, our objective is to deliver a framework for estimation of the natural rate of interest that is consistent with our approach in HLW, so that we are able to parse permanent changes to r_t^* from transitory shocks once the pandemic has abated. While the timing of the onset of the pandemic is clear, selecting an end date for the set of observations to discard would not be straightforward, and estimation of r_t^* may be sensitive to this choice. Additionally, a binary decision to drop or keep pandemic-related observations necessitates treating the entire period universally. Our approach is flexible in that we *allow* for increased volatility during the three years following the onset of the pandemic, but we do not *impose* higher innovation variances. We also do not impose any relationship between κ_{2020}, κ_{2021}, and κ_{2022}. By estimating these parameters together with the remaining model parameters, including the innovation variances during the non-pandemic period, our approach instead allows the data to inform the choice of variance scale factors, so that more extreme outliers are more heavily down-weighted. Indeed, when we allow the model to treat the later quarters of the pandemic differently from the earlier quarters, we find that it chooses to do so. Finally, it is well-known that estimates of the natural rate of interest are highly uncertain. Excluding data associated with the COVID-19 pandemic would understate the true uncertainty about future r_t^* estimates (Lenza and Primiceri, 2022). Our approach preserves estimation of the model standard errors over the full sample.

Lenza and Primiceri (2022) first implement this approach of applying scale factors to the covariance matrix of model forecast errors in order to introduce time-varying volatility within a fixed period to estimation of a vector autoregression. In the VAR setting, multiplying the innovations by a scale factor is equivalent to transforming the (observed) data directly by dividing by the scale factor. One can estimate the VAR with transformed data with no other modifications. In contrast, estimating a model with latent variables requires an additional modeling choice: treatment of the stochastic innovations to the state equations. To see this, note that the measurement equations in a model with latent variables, written in state-space form as Equation 7, include the vector of unobserved state variables ξ_t. Because these latent variables appear in the measurement equations, we cannot simply transform the observed variables without also transforming the latent variables. Transforming the latent variables would have the effect of applying the same variance scale factor to the stochastic innovations to the latent variables. This embeds the assumption that the variances of the permanent shocks to the natural rate of output, its trend growth rate, and the other determinants of the natural rate of interest have increased in the same proportion as the transitory shocks to the output gap and inflation. Instead, we model the increase in shock volatility within the covariance matrix of the transitory shocks only, so that we do not introduce this assumption.

The choice to allow for increased volatility in the output gap and inflation innovations during the pandemic, but *not* increased volatility in the innovations to the latent variables, is consistent with the original HLW approach. We maintain the ability of the model to distinguish transitory shocks that do not affect the latent variables from low-frequency movements that do. The outsized movements in GDP and inflation during the early pandemic period, while extraordinarily large, were ultimately short-lived. Rather than explicitly modeling an increase in the volatility of the unobserved latent variables, we let the data speak by modeling an increase in the volatility of the transitory shocks only and leaving the specification of the latent variables unchanged. In the event that pandemic-induced movements in GDP and inflation are long-lasting, the Kalman filter will infer this accordingly and will ascribe permanent changes to r_t^* and y_t^*.

3.2 COVID-19 Supply Shock

The second feature of the COVID-19 pandemic that is at odds with the HLW model specification is the persistence of the associated supply shock. The HLW model incorporates transitory shocks to supply as innovations in the Phillips curve equation. Stochastic innovations to the IS and Phillips curve equations are assumed to be mutually and serially uncorrelated. Because the effects on supply of the sequence of shutdowns and re-openings associated with COVID-19 are highly serially correlated during the pandemic, they are not adequately captured by a sequence of serially uncorrelated transitory shocks. This is the case even once we have accounted for the increased variances of the stochastic innovations during the pandemic. We modify the model to incorporate a persistent, but ultimately temporary, COVID supply shock, in addition to the transitory and permanent demand and supply shocks already present in HLW.

The direct effects of COVID-19 on the economy are incorporated in the model as an adjustment to the natural rate of output in the output gap specification. We introduce one new variable, denoted d_t, as a proxy for the direct effects of the government restrictions and shutdowns implemented in response to the pandemic. We set this COVID indicator variable equal to the quarterly average of the COVID-19 Stringency Index from the Oxford COVID-19 Government Response Tracker (OxCGRT) for each country or region, as shown in the lower panel of Figure 2.[9] The stringency index, which ranges between 0 and 100 with larger numbers indicating stricter restrictions, aggregates measures of government containment and closure policies such as school and workplace closures, travel restrictions, bans or limits on public gatherings, and shutdowns of public transportation.

We choose this indicator because it is comprehensive and publicly available for all of the economies in our sample. We recognize that such an index of government responses cannot capture the full set of behavioral responses or compliance; nonetheless, it should provide a reasonable first-order approximation to the time-series properties of the direct effects of the pandemic and associated public health actions on economies. For the Euro Area, we use a GDP-weighted stringency index with 2019 GDP weights. As the OxCGRT project suspended data collection at the end of 2022, we assume each indicator variable declines linearly beginning in 2023:Q1, reaching zero in 2024:Q4. The COVID indicator is set equal to zero up to and including 2019:Q4 in all economies. It is assumed to be an exogenous variable.

We incorporate the COVID variable as an adjustment to the natural rate of output, within the output gap specification. In particular, the COVID-adjusted natural rate of output is given by

$$y^*_{t,COVID} = y^*_t + \frac{\phi}{100} d_t \tag{15}$$

where y^*_t is the standard natural rate of output, d_t is the COVID-19 indicator, and ϕ is an estimated parameter that translates the COVID variable d_t into effects on output. The output gap is correspondingly modified, with the COVID-adjusted natural rate of output replacing the standard natural rate of output:

$$\widetilde{y}_{t,COVID} = 100(y_t - y^*_{t,COVID}) = 100(y_t - y^*_t) - \phi d_t \tag{16}$$

[9]See Hale et al. (2021). We use the national weighted average of the stringency indices for vaccinated and unvaccinated populations, and our results are robust to using the index for vaccinated individuals only.

where y_t is the logarithm of real GDP. We estimate the parameter ϕ together with the remaining model parameters by maximum likelihood, including the variance-scaling parameters $\kappa_{2020}, \kappa_{2021}$, and κ_{2022}.

The COVID-adjusted output gap replaces the standard output gap in the measurement equations, which are the IS and Phillips curve equations. The state equations describing the law of motion for the unobserved state variables, including those for the natural rate of output and its trend growth rate, are unchanged.

4 Empirical Findings

In this section, we describe the main empirical findings using data through the end of the sample. We start with the estimation results and then analyze the behavior of key latent variables over the pandemic period. We estimate the COVID-adjusted HLW model for the United States, Canada, and the Euro Area. In previous work, we estimated the HLW model for the United Kingdom. Extending the sample to include pandemic-era data has weakened the estimated relationship between the output gap and the real interest rate gap in the UK data, making estimates of the natural rate of interest highly unreliable. For that reason, we no longer estimate the model for the United Kingdom. We present estimates from the COVID-adjusted LW model in Appendix A4.

4.1 Estimation Results

Table 1 reports the estimates of model parameters for the three economies. For comparison, the corresponding estimates from the model estimated through 2019:Q4 are reported in the Appendix Table A1. The parameter estimates are broadly similar to estimates from the pre-pandemic sample. Worth noting is that for Canada and the United States, the estimated values of λ_g (λ_z) are somewhat higher (lower) than in the pre-pandemic sample. The estimated values of the parameter c, linking the trend growth rate to the natural rate of interest, are close to unity (the value imposed in HLW 2017).

Figure 6 shows key estimated latent variables for the United States, Canada, and the Euro Area, respectively. The upper panel of each figure shows the time series of the estimated COVID-adjusted output gap and model-based real interest rate gap, defined to be the model definition of the real interest rate less the estimated value of the natural rate of interest. The lower panel of each figure shows the estimates of the natural rate of interest and the trend growth rate, g_t. All three economies display the secular decline in estimates of the trend growth rate and the natural rate of interest highlighted in HLW (2017).

The model parameters introduced to address COVID-related effects on the economy are generally statistically significant at the 5 percent level. As discussed below, the estimated values of the parameter on the COVID shock variable, ϕ, have economically large effects. Consistent with the observations of enormous outliers early in the pandemic, the estimated values of the parameter κ_{2020} are sizable for 2020:Q2-Q4 and statistically significant. For the Euro Area and the United States, the estimated values of κ_{2021} are near 2 and κ_{2022} are around 1-1/2, and are generally statistically significant. For Canada, the estimated κ_{2021} is close to unity and κ_{2022} is around 1-1/2, and both are less precisely estimated than for the other economies.

The effects of the model modifications can be seen by comparing the standardized auxiliary residuals from the unadjusted model to those from the COVID-adjusted model described in Section 3. The blue lines in Figure 3 shows the standardized auxiliary residuals, described in Section 2.2, to the output gap and inflation equations, with the horizontal lines indicating two standard deviations. The gold lines show residuals from the version of the model with parameters held fixed at 2019:Q4 values. In all three economies, IS equation residuals in 2020 indicate extreme outliers to the model. The magnitude of the estimated κ_{2020} parameters is consistent with these extreme outliers. The blue lines show auxiliary residuals from the modified HLW model. With the two modifications, these residuals are of similar magnitude to the pre-pandemic period, and no longer indicate the presence of outliers. A similar, albeit less extreme, pattern is seen in the comparison of standardized auxiliary residuals from the inflation equation. The blue lines in Figure 4 displays the standardized auxiliary residuals from the equation for the natural rate of output. As with the output gap equation, the massive outliers in the standard version of the model are no longer present in the COVID-adjusted model, despite the fact that the model specifications of the natural rate of output and of the associated stochastic processes are unchanged.

4.2 The Effects of the COVID-19 period on g, r^*, and y^*

We now trace the evolution of key latent variables since the onset of the pandemic. This analysis reveals three key findings. First, the modified estimation procedure yields results that are overall quite similar to those from the original model during the pre-pandemic period. Second, the current estimates of the natural rate of interest are similar to those estimated directly before the pandemic. Third, the estimates of the natural rate of output at the end of 2022 are much lower than predicted before the pandemic.

Over the pre-pandemic sample, the estimates of the output gap, trend growth, and the natural rate of interest from the modified model are close to the corresponding estimates from the model without the COVID modifications (i.e. $\phi = 0, \kappa_t = 1$ for all periods). Figure 5 compares the estimates for the United States, Canada, and the Euro Area, respectively, from the modified model estimated through 2022:Q4 with estimates from the model with parameter values fixed their 2019:Q4 values. While the two sets of estimates are very similar through 2019, they differ sharply during the acute period of the pandemic, when the estimates from the model without COVID adjustments exhibit large swings due to the presence of extreme outliers.

The estimates of the trend growth rate of output in all three economies are slightly lower in 2022 than in 2019. Table 2 reports the estimates of the trend growth rate, g_t, in selected years as well as changes over sub-samples. It also reports forecasts of trend growth rates from various sources, which are generally little changed from 2019 to 2022. This represents a continuation of the pattern of declining estimates of trend growth observed over the preceding three decades, as seen in the columns in the table that report the changes in estimates over 1990-2007 and 2007-2019. Interestingly, for each economy, the 2022 estimates from the HLW model are reasonably close to outside forecasts.

In all three economies, the estimates of the natural rate of interest in 2022 are within a few tenths of a percentage point of the corresponding estimates in 2019. Table 3 reports annual averages of the estimates for selected years, along with changes over sub-samples. For comparison, it reports selected forecasts and market-based measures of longer-run real interest rate expectations. Note that market-based measures include term premia that can change over time, so they are not directly comparable to estimates of natural rates. Market-based measures of longer-run real interest rates rose from 2019 to 2022, but in each case, this brought them closer to the HLW estimates.

The HLW estimates of r_t^* for Canada, the Euro Area, and the United States show no evidence of a quantitatively meaningful reversal of the decline in estimated natural rates of interest evident in prior decades. This result that the estimates of the natural rate of interest have not changed much since the start of the pandemic runs counter to some commentary that very large fiscal stimulus and rising levels of government debt, alongside evidence of large output gaps and high inflation, points to a higher level of the natural rate than before the pandemic. One reason why the HLW estimates of the natural rate of interest are little changed over the past three years is that lower estimates of trend growth work to offset any increase in the estimates of z.

The largest differences between model estimates pre- and post-pandemic relate to the level of each economy's natural rate of output. Figure 7 compares the model projections of the natural rate of output based on estimates using data through 2019:Q4 (the black lines) with estimates from 2022:Q4 (the blue lines). The figure also shows the effect of the COVID adjustment over this period (the difference between the blue and gold lines). As seen in the figure, the estimated shortfalls in the COVID-adjusted natural rate of output relative to pre-pandemic projections are economically sizable. For example, at the end of 2022, the COVID-adjusted level of the natural rate of output in the United States is about 4 percent below the pre-pandemic projection, with nearly half of that shortfall explained by the COVID shock measure and the remainder a permanent change in the natural rate of output.

5 Robustness

This section examines the sensitivity of the empirical findings to alternative specifications of the model. We consider two alternative specifications of time-varying volatility by adjusting the start and end dates over which we introduce scale factors to the variances of the stochastic innovations. Next, we estimate a version of the model without the COVID supply shock. Finally, we consider two alternative specifications for output. Our finding that estimates of the natural rate of interest remain low following the COVID-19 pandemic is robust to these alternative specifications.

5.1 Specification of Time-Varying Volatility

In order to account for the extreme GDP and inflation observations during the COVID-19 pandemic, we introduce time-varying volatility in the HLW model by applying a scale factor to the innovation variances of the model's measurement equations during the period of increased volatility associated with the pandemic, as described in Section 3. In our baseline specification, we allow the innovation variances to the output gap and inflation equations to increase during the years 2020, 2021, and 2022. The corresponding vector of variance scale parameters κ_t is given by Equation 13. Our baseline specification is informed by the observed data, the model one-step-ahead prediction errors, and the model auxiliary residuals. In this section, we consider two alternative specifications of κ_t. Because the onset of the pandemic occurred later within the first quarter of 2020 in the economies we study, the period of increased volatility begins in 2020:Q2 in our main specification. However, the auxiliary residuals to the IS equation, shown in Figure 3, indicate the presence of sizable outliers in 2020:Q1. Our first alternative specification includes 2020:Q1 when estimating and applying the variance scale parameters, with κ_t given by

$$\kappa_t = \begin{cases} \kappa_{2020} & \text{2020:Q1} \leq t \leq \text{2020:Q4} \\ \kappa_{2021} & \text{2021:Q1} \leq t \leq \text{2021:Q4} \\ \kappa_{2022} & \text{2022:Q1} \leq t \leq \text{2022:Q4} \\ 1 & \text{otherwise} \end{cases} \tag{17}$$

Although the timing of the pandemic onset is known, it is more difficult to determine the end date of the COVID-19 pandemic. In fact, many metrics indicate that the pandemic is ongoing at the time of publication. However, the GDP and inflation realizations as well as model auxiliary residuals in 2022 are more in line with the pre-pandemic sample. While our main specification allows for increased innovation variances in the year 2022 (and subsequently fixes $\kappa_t = 1$ in 2023 and beyond), our second alternative specification assumes that the innovation variances return to their non-pandemic levels in 2022, with κ_t given by

$$\kappa_t = \begin{cases} \kappa_{2020} & 2020\text{:Q2} \leq t \leq 2020\text{:Q4} \\ \kappa_{2021} & 2021\text{:Q1} \leq t \leq 2021\text{:Q4} \\ 1 & \text{otherwise} \end{cases} \tag{18}$$

In all three economies, estimates of the latent natural rate of output under the alternative specifications of time-varying volatility are close to the baseline specification.[10] Estimates of y_t^* at the end of 2022 lie below the projected level of the natural rate of output based on 2019:Q4 data, regardless of the κ_t specification, with the gap between projected y_t^* as of 2019:Q4 and current model estimates generally widening over the past three years. Our finding that the pandemic had a persistently negative effect on the estimated natural rates of output is robust across specifications.

We find that the estimated coefficient ϕ, which translates the COVID indicator variable into effects on the output gap, is sensitive to the specification of κ_t. As a result, estimates of the COVID-adjusted natural rate of output are more sensitive, but key results hold across all three specifications. In all cases, the COVID-adjusted natural rate of output is below the standard natural rate of output during the pandemic period. In the United States and the Euro Area, the COVID-adjusted natural rate of output is significantly lower than the estimated natural rate of output in 2020 and 2021 and remains below y_t^* at the end of 2022. In Canada, $y^*_{t,COVID}$ lies below y_t^* in 2020 and 2021 in all three specifications, but the gap relative to the baseline is narrower and closes in 2022 as the COVID indicator variable declines. In the Euro Area, the COVID-adjusted natural rate of output remained below its 2019 level at the end of the sample, while the United States and Canada had recovered to their pre-pandemic levels. As of the sample end, the COVID-adjusted natural rate of output had not recovered to the pre-pandemic trend in any of the three economies.

Accordingly, the COVID-adjusted output gap deviates substantially from the baseline estimates in some cases. Despite a wider range of output gap estimates across the κ_t specifications, estimates of r_t^* and g_t are relatively close to the baseline estimates. In the United States, estimates of r_t^* under alternative specifications are within 50 basis points of the baseline specification at the sample end, well within

[10] Note that the equation for the natural rate of output is unchanged from the standard HLW model; the variance scale parameter κ_t does not enter the y_t^* equation.

the estimated standard error bands. Estimates in Canada are very close across all three specifications. Estimates of r_t^* and g_t in the Euro Area are more sensitive to the specification of κ_t throughout the sample, but remain within 40 basis points of the baseline specification at the sample end. We find that the COVID pandemic has a large effect on the estimated natural rates of output, though the magnitude of the effect varies across specification, while estimates of trend growth and the natural rate of interest are generally close to their pre-pandemic levels and robust to alternative specifications of time-varying volatility.

5.2 Model without COVID Supply Shock

We now consider a version of the model that includes time-varying volatility but does not include the COVID supply shock. This is equivalent to imposing $\phi = 0$ in the COVID-adjusted model. In all economies, we find that the output gap is much deeper and rebounds much more slowly in the specification without the COVID supply shock. However, in all economies, the level of the (unadjusted) output gap under this alternative specification reaches that of the baseline COVID-adjusted output gap by the end of 2022. Estimates of the natural rate of interest are largely similar to, or slightly below, the baseline model at the end of the sample, as shown in Figure 10. In the United States, estimates of trend growth and the natural rate of interest from the model without the COVID supply shock are lower than the baseline model during the pandemic period, with an estimate of r_t^* that is 50 basis points below the baseline estimate in final quarter of 2022. When we impose $\phi = 0$ in the model, we find that estimates of κ_{2020} are larger in all three economies.

5.3 Alternative Measures of Output

Finally, we consider two alternative specifications of output for the United States. Figure 11 displays estimates of the natural rate of interest under these alternative specifications. The first is a version of the model in which potential output is an exogenous variable, using estimates from the Congressional Budget Office as data. While the HLW model jointly estimates the natural rates of output and interest and models the trend growth rate as a determinant of r_t^*, there is no explicit role for trend GDP growth in this specification. The natural rate of interest follows a random walk and is the only latent variable in the model; since we do not model trend growth as a determinant of r_t^* in this extension, there is no separate specification of other determinants z_t, and we estimate r_t^* directly via the Kalman filter. Because we take the CBO's measure of potential output as given, we do not introduce the COVID supply shock to the natural rate of output described in Section 3.2 (i.e. $\phi = 0$). Otherwise, we use the COVID-adjusted HLW model, which includes time-varying volatility during the pandemic period. The output gap based on estimates of potential output from the CBO lies below the estimated output gap from the HLW model, with substantial differences between the two measures emerging in 2008 and persisting through the end of the sample. Estimates of the natural rate of interest under this alternative specification, shown in Figure 11, are substantially negative and significantly lower than the baseline HLW estimates in the decade preceding the pandemic as well as during the pandemic period, with the gap widening at the end of the sample.

In the second alternative specification, output is defined as the average of real GDP and real Gross Domestic Income, the latter of which is an alternative measure of economic output constructed from income data and production costs by the Bureau of Economic Analysis. Switching to this alternative output measure has a minor effect on estimates of the latent variables, with r_t^* estimates that are about 50 basis points below the baseline HLW model (which uses real GDP data alone) from 2020 through 2022. In each of the alternative specifications we consider, estimates of the natural rate of interest lie below r_t^* estimates from the baseline HLW model for the entirety of the pandemic period. Across these alternative specifications, estimates of the natural rates of interest remain at historically low levels.

6 Conclusion

This paper develops and implements an approach to address the unusual effects of the pandemic on the economy and allows continued estimation of the natural rates of output and interest in the post-pandemic era using the Holston, Laubach, and Williams and Laubach and Williams models. These methods can be applied to a broad set of models that estimate latent variables over the pandemic period or other periods of large-scale disruptions to economic activity. According to the model estimates, the main longer-term consequence from the pandemic period is a reduction in the natural rate of output, but the imprint on the natural rate of interest appears to be relatively modest. We do not find evidence that the era of historically low estimated natural rates of interest has come to an end. These findings are shown to be robust to a range of alternative model specifications.

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Table 1: Parameter Estimates

Parameter	United States	Canada	Euro Area
Sample	1961-2022	1961-2022	1972-2022
λ_g	0.073	0.061	0.039
λ_z	0.021	0.011	0.023
$\sum a_y$	0.936	0.946	0.945
a_r	-0.079	-0.077	-0.037
	(4.215)	(2.947)	(1.757)
b_y	0.073	0.049	0.088
	(3.003)	(1.821)	(2.310)
c	1.128	1.109	0.953
	(3.574)	(2.794)	(1.155)
ϕ	-0.085	-0.062	-0.135
	(2.199)	(3.442)	(6.557)
$\kappa_{2020Q2-Q4}$	9.033	9.377	18.609
	(2.351)	(2.134)	(2.384)
κ_{2021}	1.791	1.087	1.960
	(2.941)	(1.835)	(2.150)
κ_{2022}	1.676	1.613	1.557
	(2.060)	(1.293)	(1.231)
$\sigma_{\tilde{y}}$	0.452	0.579	0.318
σ_π	0.787	1.343	0.965
σ_{y^*}	0.500	0.448	0.382
σ_g	0.145	0.109	0.060
σ_z	0.118	0.085	0.201
$\sigma_{r^*} = \sqrt{c^2\sigma_g^2 + \sigma_z^2}$	0.202	0.148	0.209
S.E. (sample ave.)			
r^*	1.140	1.603	2.739
g	0.428	0.413	0.264
y^*	1.612	2.474	1.662
S.E. (final obs.)			
r^*	1.565	1.858	3.805
g	0.689	0.639	0.405
y^*	2.914	3.679	2.878

Notes: t statistics are in parentheses; σ_g is expressed at an annual rate.

Table 2: Trend Growth Estimates

	1990	2007	2019	2022	Change 1990-2007	Change 2007-2019	Change 2019-2022
United States							
HLW g estimates	3.3	2.8	2.1	1.8	−0.5	−0.7	−0.2
Consensus Forecasts	2.4	2.8	2.0	1.9	0.4	−0.8	−0.1
IMF World Economic Outlook	2.6	3.0	1.6	1.8	0.4	−1.5	0.2
Blue Chip Financial Forecasts	2.5	2.8	2.0	1.9	0.4	−0.8	−0.1
Canada							
HLW g estimates	3.4	2.5	1.8	1.5	−0.9	−0.7	−0.3
Consensus Forecasts	2.8	2.4	1.8	1.8	−0.4	−0.7	0.0
IMF World Economic Outlook	3.2	2.7	1.7	1.7	−0.5	−1.0	0.0
Euro Area							
HLW g estimates	2.7	2.0	1.2	1.0	−0.7	−0.9	−0.1
Consensus Forecasts	*n/a*	1.9	1.3	1.2	*n/a*	−0.6	−0.1
IMF World Economic Outlook	*n/a*	2.0	1.3	1.4	*n/a*	−0.7	0.1

Notes: Consensus Forecasts data are the mean of panelists' trend estimates of expected GDP growth in the subsequent 6 to 10 years. IMF estimates are the 5-year-ahead forecast for real GDP growth. Blue Chip estimates are the mean long-run forecasts for real GDP growth. For these forecasts, reported numbers are averages of the Spring and Fall publications. HLW estimates are annual averages. Numbers may not sum due to rounding.
Sources: Consensus Economics Inc, London; IMF World Economic Outlook; Blue Chip Financial Forecasts.

Table 3: Natural Rate of Interest Estimates

	1990	2007	2019	2022	Change 1990-2007	Change 2007-2019	Change 2019-2022
United States							
HLW r^* estimates	3.6	2.5	0.9	0.8	−1.1	−1.5	−0.2
TIPS yields (5-10 years ahead)	*n/a*	2.5	0.6	0.8	*n/a*	−1.9	0.2
Blue Chip Financial Forecasts	3.1	2.4	0.5	0.5	−0.7	−1.9	0.0
Canada							
HLW r^* estimates	3.5	2.7	1.7	1.5	−0.8	−1.0	−0.2
Real Return Bond yields	*n/a*	2.0	0.4	1.0	*n/a*	−1.5	0.5
Euro Area							
HLW r^* estimates	2.6	2.3	0.6	0.7	−0.3	−1.6	0.0
Swap-implied real yields (5-10 years ahead)	*n/a*	2.1	−0.7	0.0	*n/a*	−2.8	0.7

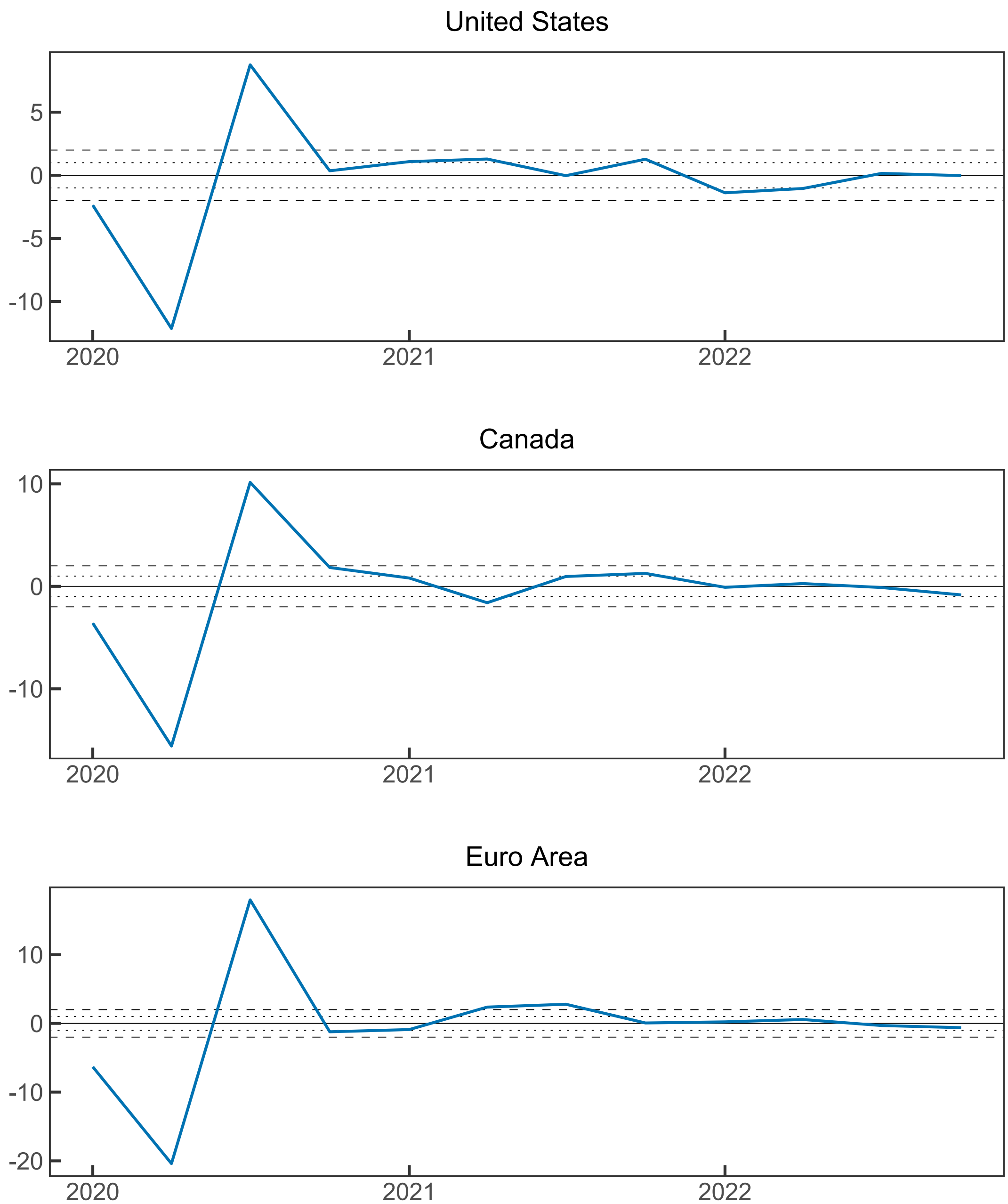
Notes: All numbers are annual averages. Blue Chip estimates are the difference in the mean long-years-ahead forecasts of the federal funds rate and CPI (average of Spring and Fall publications). Canadian Real return bond yields are long-term benchmark yields from the Bank of Canada. Euro Area yields are calculated using French nominal yields and data on zero-coupon inflation swaps linked to Euro Area HICP inflation. Numbers may not sum due to rounding.

Table 4: Robustness to κ_t Specification

	ϕ	κ_{2020}	κ_{2021}	κ_{2022}
United States				
Baseline	-0.09	9.03	1.79	1.68
With 2020:Q1	-0.16	5.37	1.76	2.28
$\kappa_{2022} = 1$	-0.05	10.64	1.83	1.00
Canada				
Baseline	-0.06	9.38	1.09	1.61
With 2020:Q1	-0.03	8.99	1.13	1.32
$\kappa_{2022} = 1$	-0.02	8.99	1.14	1.00
Euro Area				
Baseline	-0.13	18.61	1.96	1.56
With 2020:Q1	-0.11	17.60	1.99	1.32
$\kappa_{2022} = 1$	-0.13	18.09	1.97	1.00

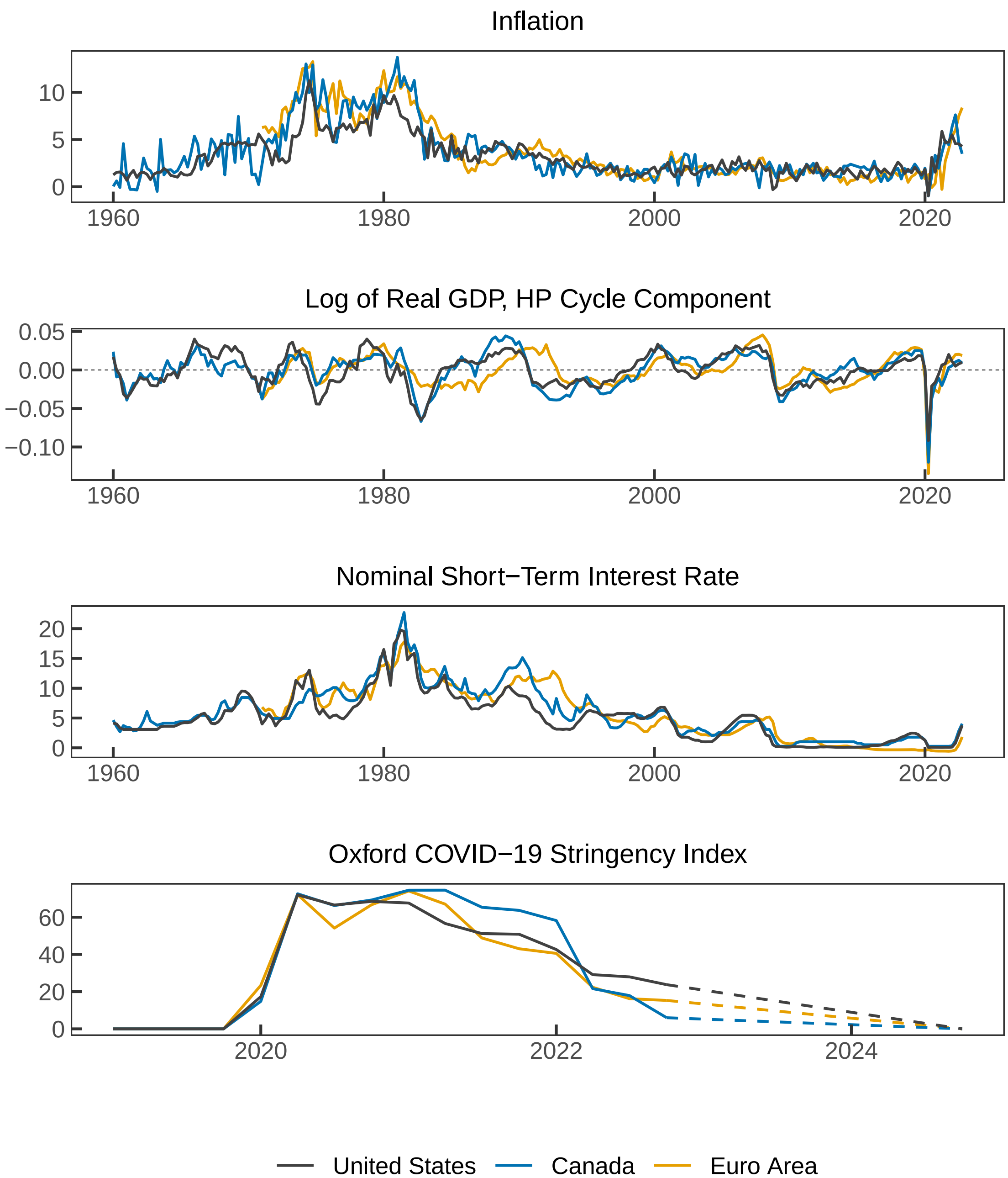
Notes: The rows labeled "with 2020:Q1" report parameter estimates from an alternative specification of the model in which κ_{2020} includes 2020:Q1. Rows labeled "$\kappa_{2022} = 1$" report parameter estimates from a specification of the model with variance scale parameters estimated for 2020 and 2021 only. "Baseline" parameter estimates are from the COVID-adjusted HLW model.

Figure 1: Normalized GDP Growth



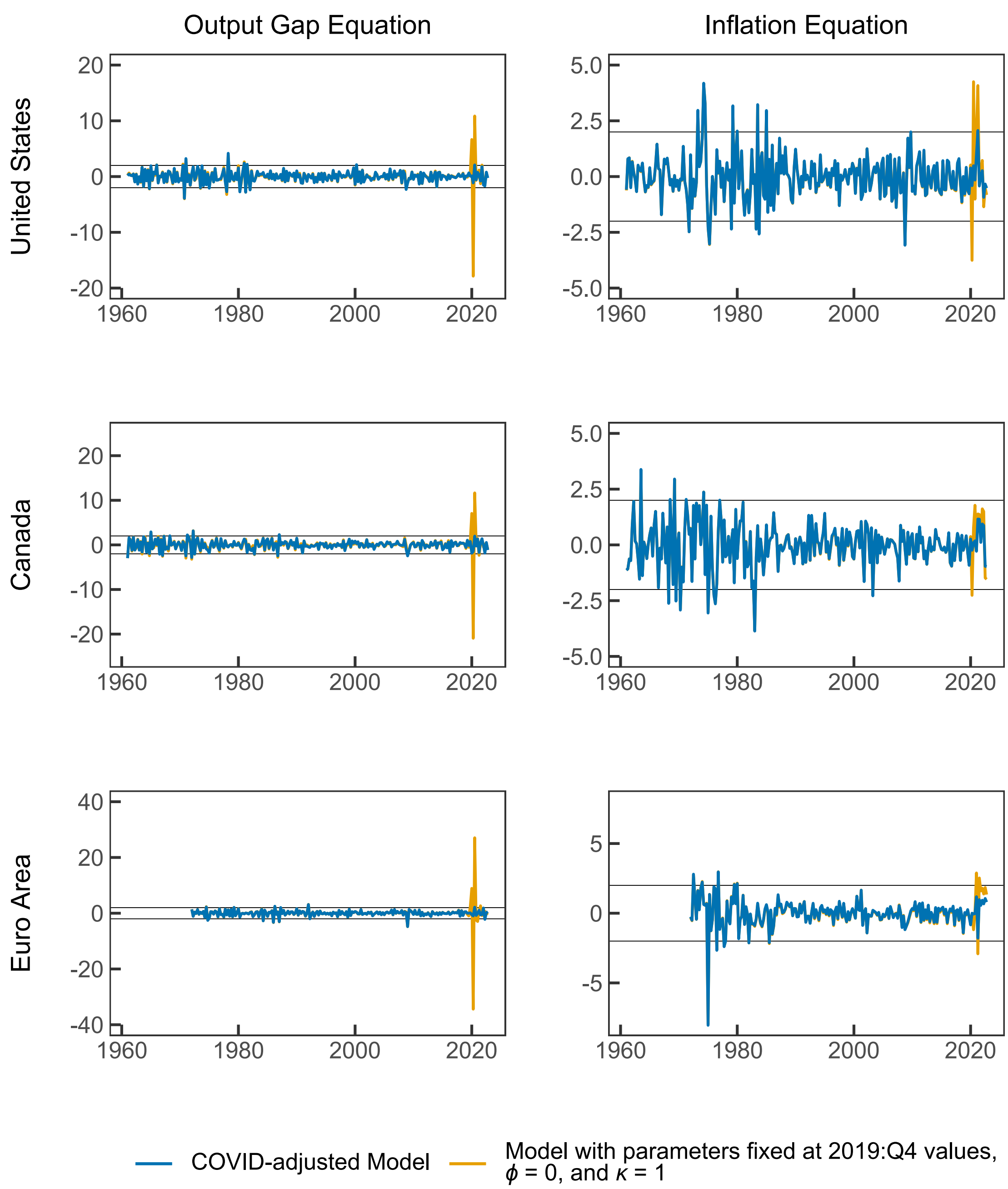
Notes: The chart shows the log-difference of quarterly real GDP less the pre-pandemic mean, divided by the pre-pandemic standard deviation. The pre-pandemic period refers to 1970-2019 for the United States and Canada, and 1972-2019 for the Euro Area. The solid horizontal line indicates zero, the dotted lines indicate the value one, and the dashed lines indicate the value two.

Figure 2: Comparison of Data Across Economies



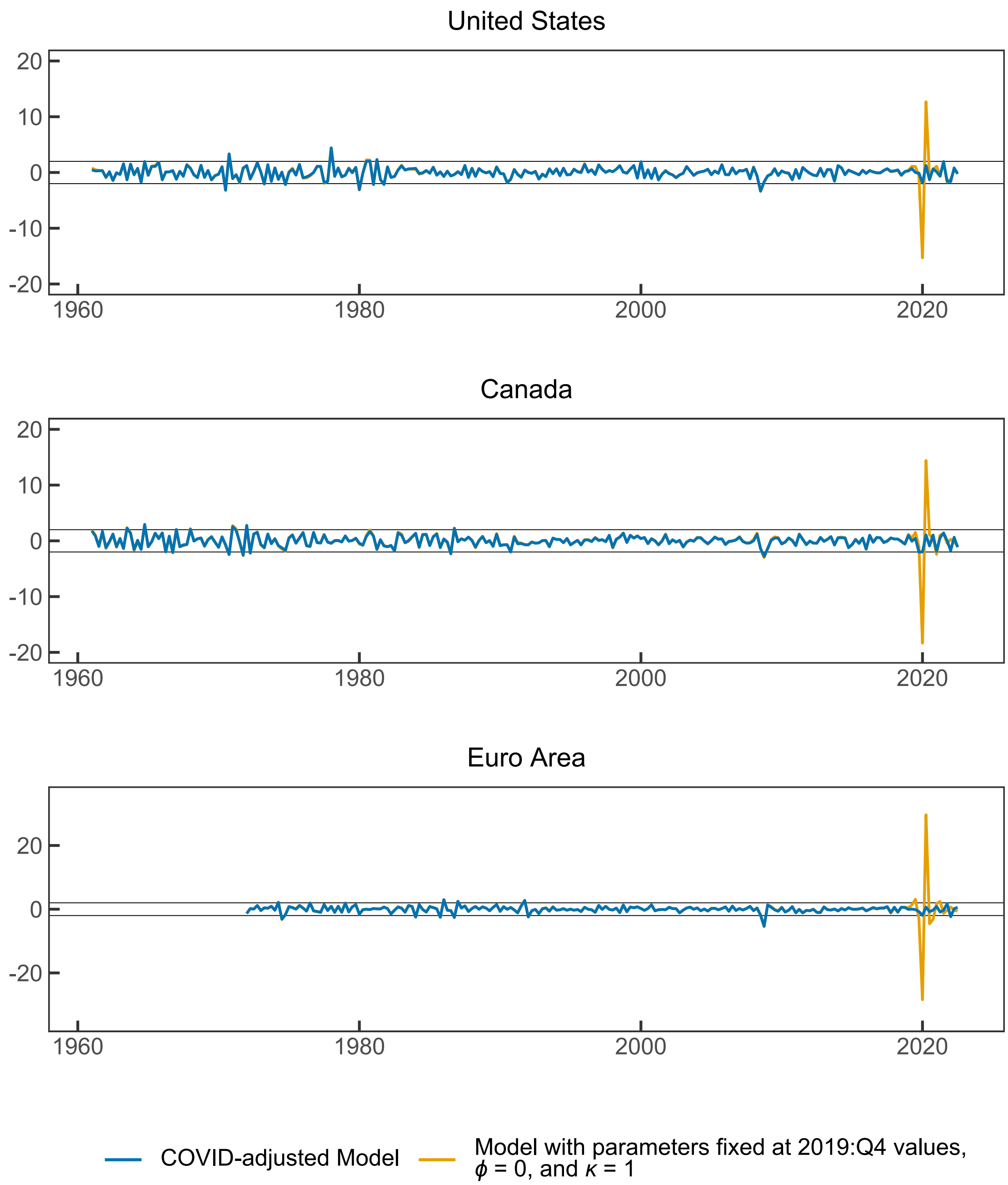
Notes: The upper three panels show the published data used in estimating the model. To ease comparison across economies, the second panel shows the cycle component of real GDP using an HP filter. The bottom panel shows the COVID index: the solid lines show the published values and the dashed lines show the assumed path over 2023-2024 when published numbers are no longer available.

Figure 3: Standardized Auxiliary Residuals: Output Gap and Inflation



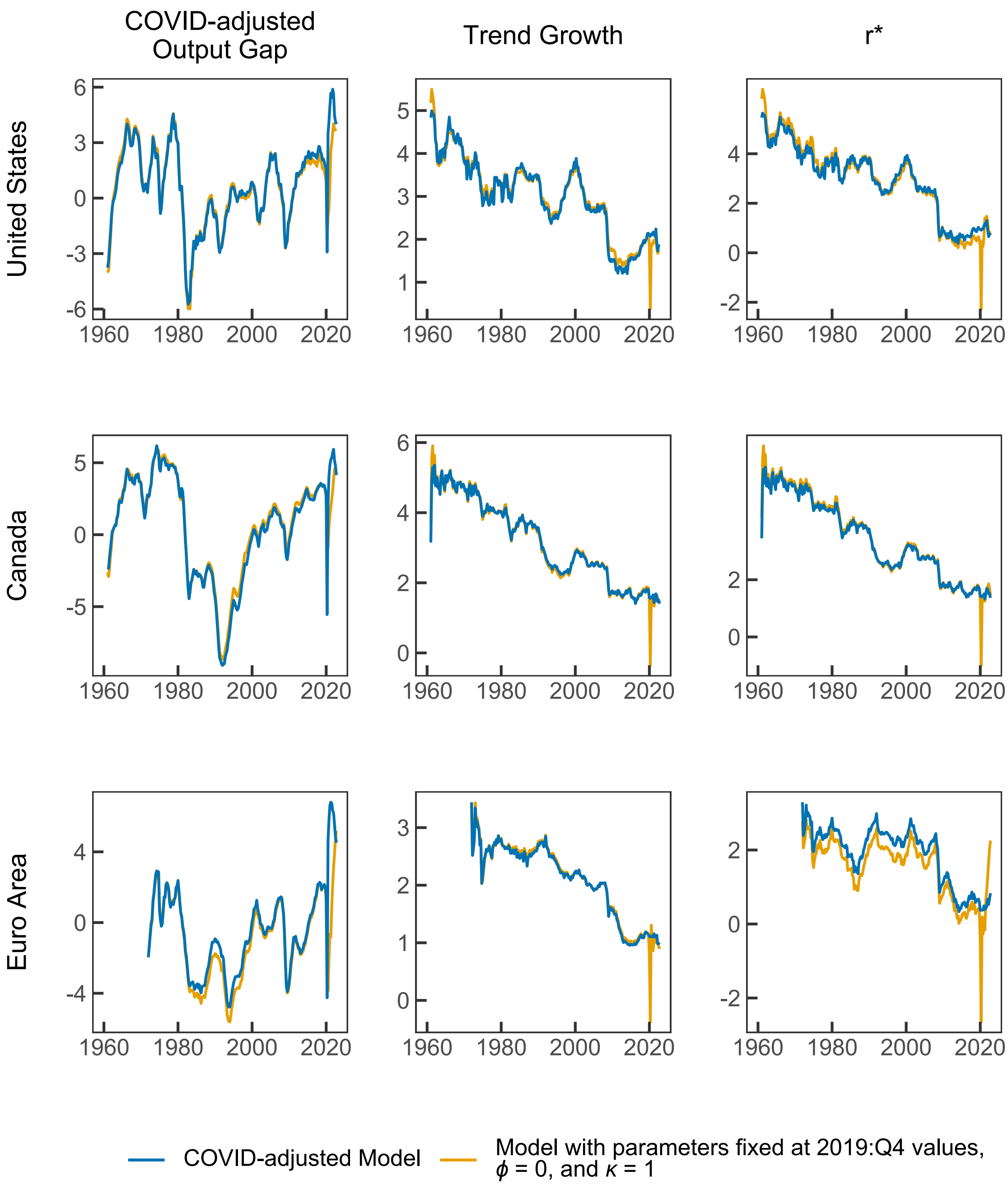
Notes: The left (right) panels show the standardized auxiliary residuals to the model IS equation (Phillips curve equation). The auxiliary residuals are smoothed estimators of the disturbances to the measurement equations, normalized by their standard deviations. The gold lines labeled "Model with parameters fixed at 2019:Q4 values" show the residuals using data through 2022:Q4, but where the model parameters are estimated on data through 2019:Q4. The blue lines labeled "COVID-adjusted model" show residuals from the full-sample estimation of the modified HLW model.

Figure 4: Standardized Auxiliary Residuals: Natural Rate of Output



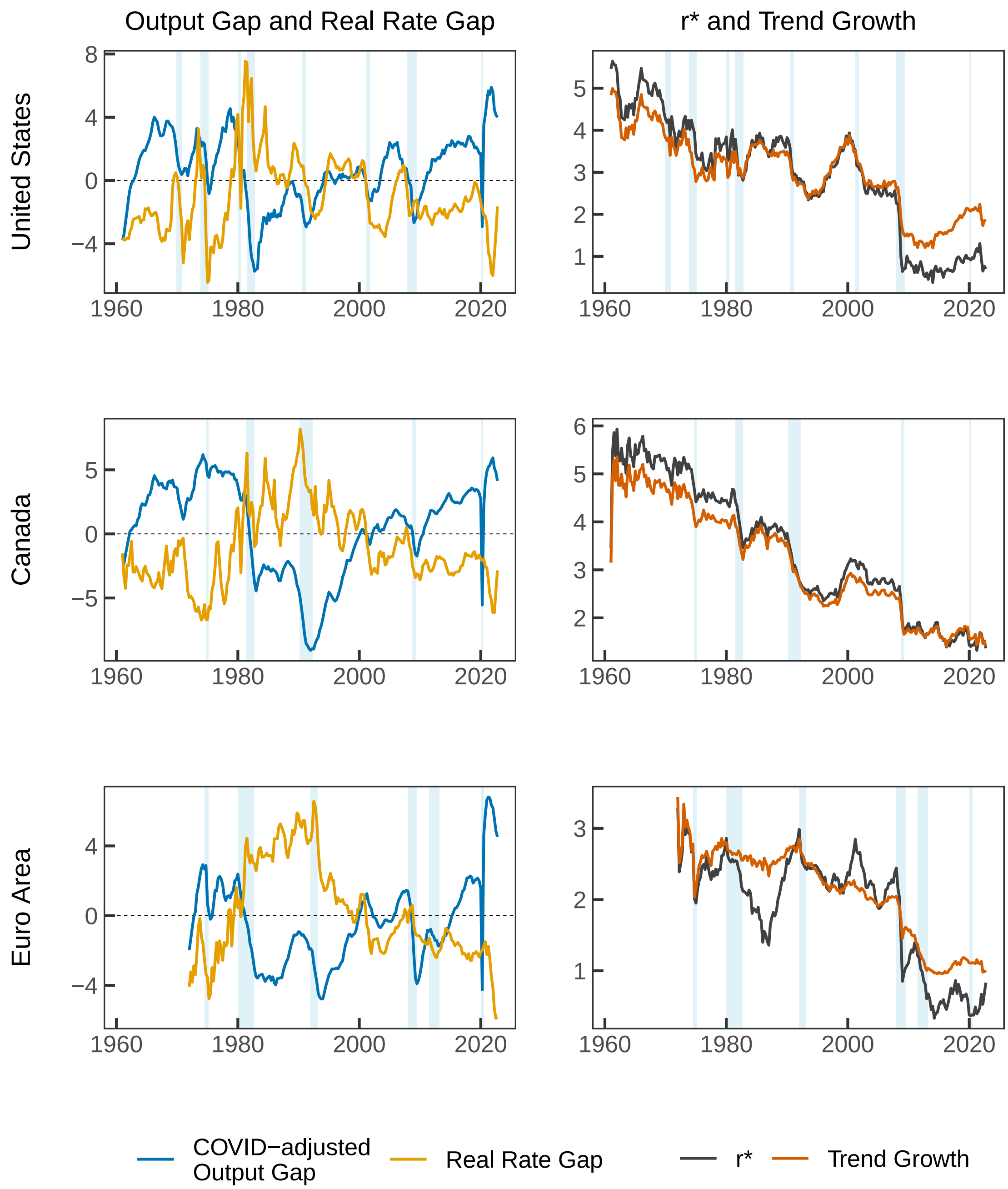
Notes: The panels show the standardized auxiliary residuals to the natural rate of output equation. The auxiliary residuals are smoothed estimators of the disturbances to the state equations, normalized by their standard deviations. The gold lines labeled "Model with parameters fixed at 2019:Q4 values" show the residuals using data through 2022:Q4, but where the model parameters are estimated on data through 2019:Q4. The blue lines labeled "COVID-adjusted model" show residuals from the full-sample estimation of the modified HLW model.

Figure 5: HLW Estimates of Key Latent Variables



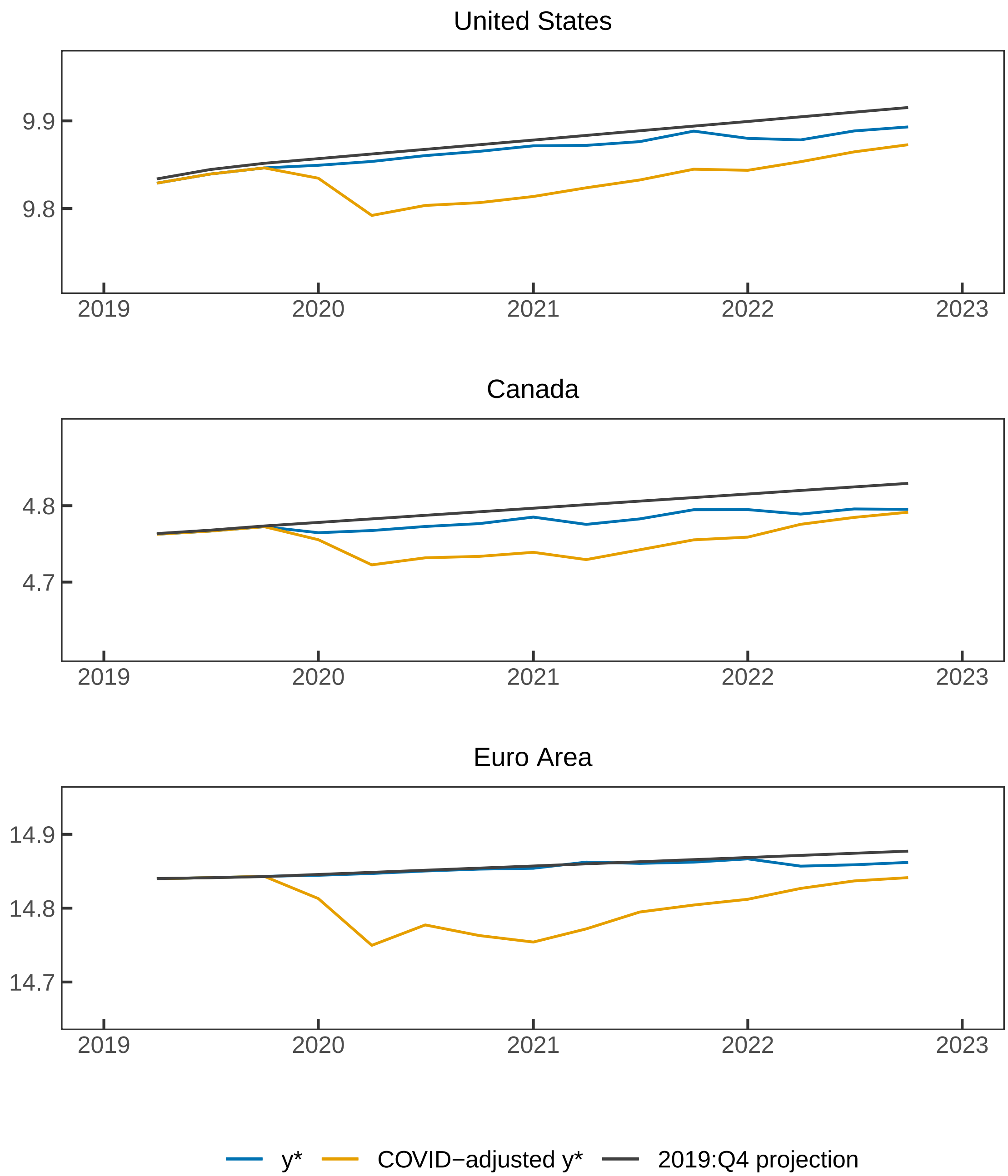
Notes: The gold lines labeled "Model with parameters fixed at 2019:Q4 values" show the estimates of the indicated variables using data through 2022:Q4, but where the model parameters are estimated on data through 2019:Q4. The blue lines labeled "COVID-adjusted model" show the full-sample estimates of the modified HLW model.

Figure 6: HLW Full-sample Estimation Results



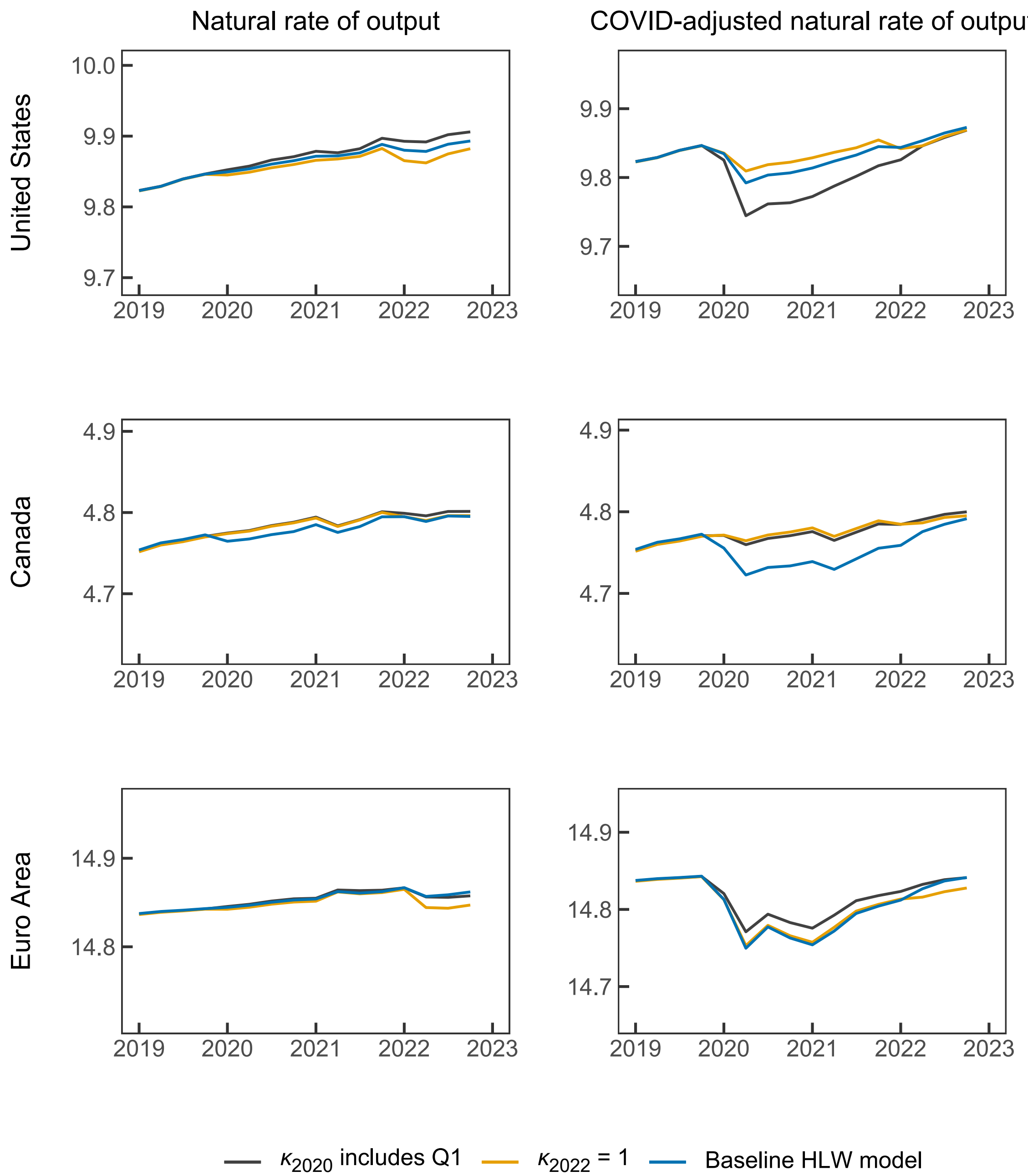
Notes: The lines show the full-sample estimates from the modified HLW model. The vertical shaded regions indicate periods when the economy was in a recession.

Figure 7: HLW Estimates of the Natural Rate of Output



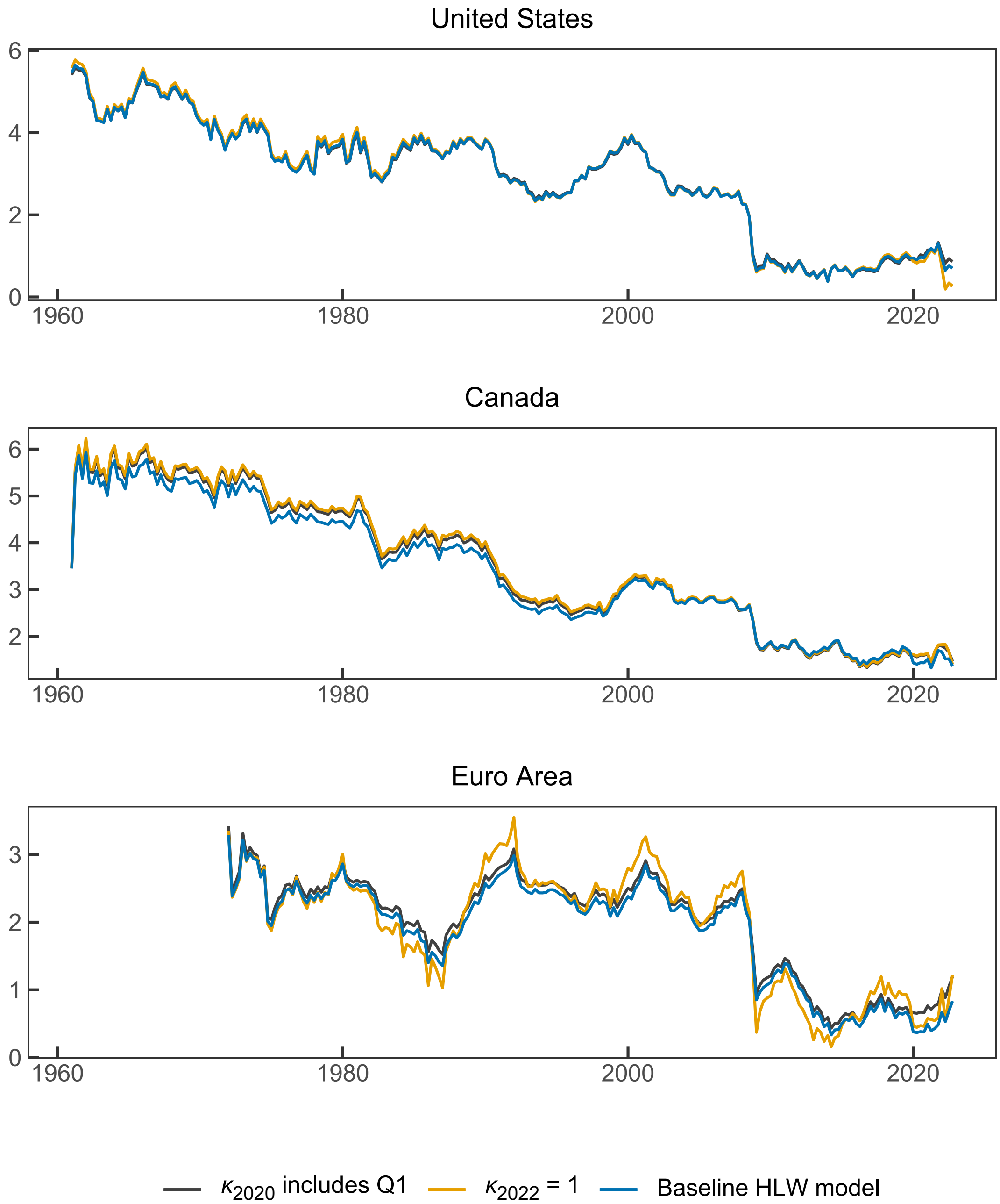
Notes: The black lines show the projected path of the natural rate of output based on model estimates using data through 2019:Q4. The blue lines show the estimates of the natural rate of output from the modified HLW model using data through 2022:Q4. The gold lines show the estimates of the COVID-adjusted natural rate of output from the modified HLW model using data through 2022:Q4.

Figure 8: Estimated Natural Rate of Output under Alternative Specifications of κ_t



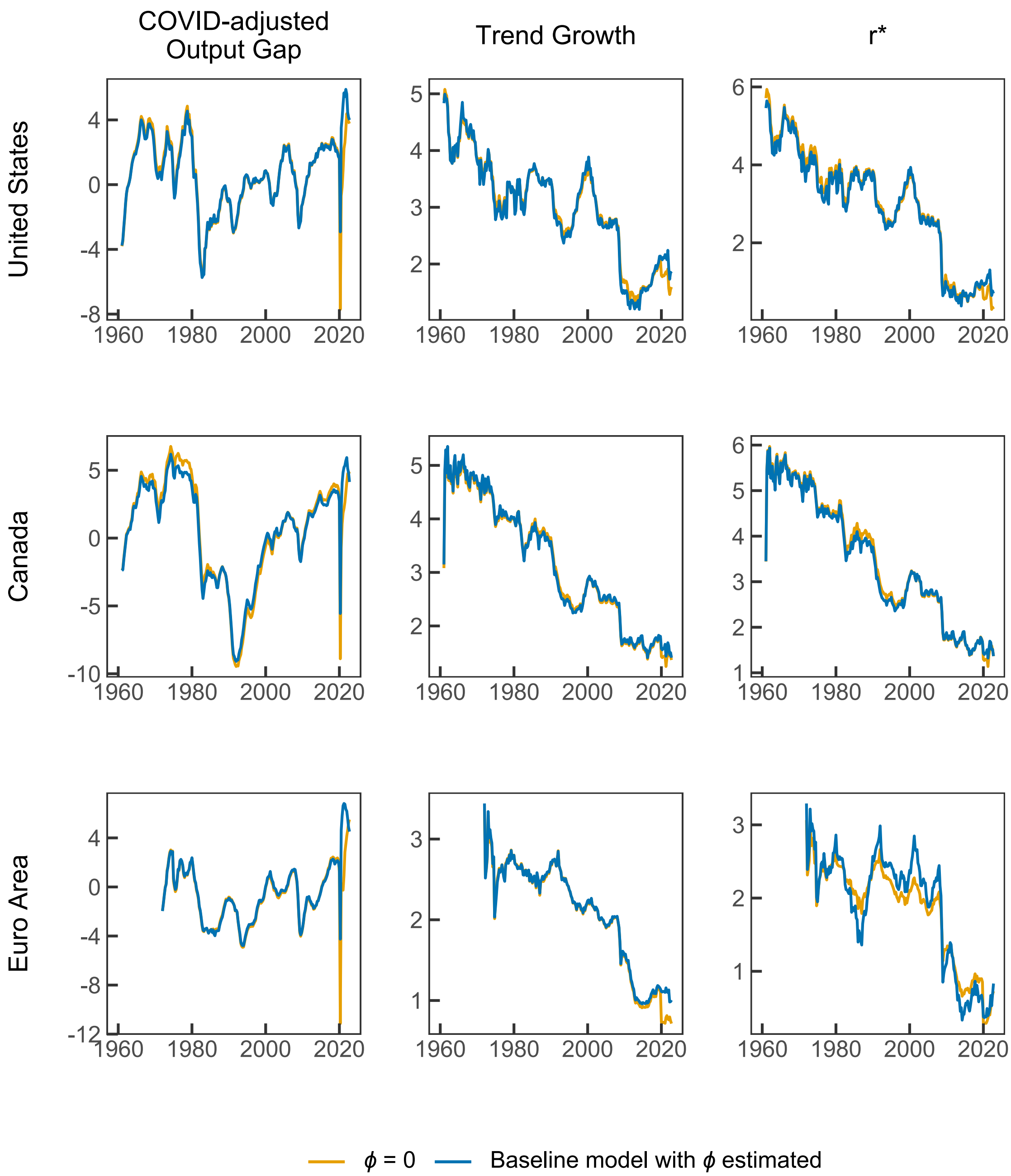
Notes: The left panels show estimates of the natural rate of output from the modified HLW model using data through 2022:Q4, while the right panels show estimates of the COVID-adjusted natural rate of output. The black lines labeled "κ_{2020} includes Q1" show estimates from a version of the model in which the variance scale parameter κ_{2020} includes 2020:Q1. The gold lines labeled "$\kappa_{2022} = 1$" show estimates from a specification of the model with variance scale parameters estimated for 2020 and 2021 only.

Figure 9: Natural Rate of Interest under Alternative Specifications of κ_t



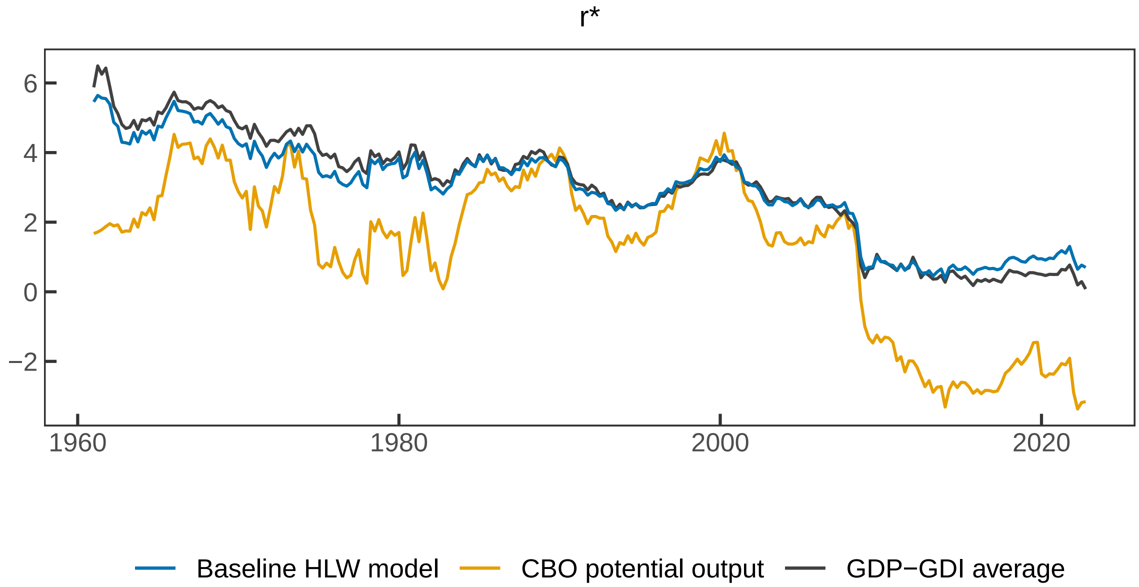
Notes: The panels show estimates of the natural rate of interest from the modified HLW model using data through 2022:Q4. The black lines labeled "κ_{2020} includes Q1" show estimates from a version of the model in which the variance scale parameter κ_{2020} includes 2020:Q1. The gold lines labeled "$\kappa_{2022} = 1$" show estimates from a specification of the model with variance scale parameters estimated for 2020 and 2021 only.

Figure 10: Model without COVID Supply Shock ($\phi = 0$)



Notes: The gold lines labeled "$\phi = 0$" show the estimates of the indicated variables from the COVID-adjusted model using data through 2022:Q4, but without the COVID supply shock in the model. The blue lines labeled "Baseline model with ϕ estimated" show the full-sample estimates of the modified HLW model, including the COVID supply shock.

Figure 11: Alternative Measures of Output



Notes: The gold line labeled "CBO potential output" shows the estimated natural rate of interest for the United States from an alternative specification of the model in which potential output is an exogenous variable, using data from the Congressional Budget Office. The black line labeled "GDP-GDI average" shows the estimated natural rate of interest from a version of the model in which output is specified as the average of GDP and Gross Domestic Income, using data from the Bureau of Economic Analysis. The blue line shows the estimates from the modified HLW model.

Appendix A1: State-Space Models

This section presents the COVID-adjusted and original HLW models in state-space form.[11] See HLW (2017) for a full description of our estimation procedure, which includes three stages. The first and second stage models represent versions of the final stage model, and each of the models can be cast in state-space form:

$$\mathbf{y}_t = \mathbf{A}' \cdot \mathbf{x}_t + \mathbf{H}' \cdot \xi_t + \epsilon_t \tag{19}$$

$$\xi_t = \mathbf{F} \cdot \xi_{t-1} + \eta_t \tag{20}$$

Here, $\mathbf{y}_t$ is a vector of contemporaneous endogenous variables, while $\mathbf{x}_t$ is a vector of exogenous and lagged endogenous variables. ξ_t the vector of unobserved state variables. In the HLW (2017) and Laubach and Williams (2003) models, the vectors of stochastic disturbances ϵ_t and η_t are assumed to be Gaussian and mutually uncorrelated, with mean zero and covariance matrices $\mathbf{R}$ and $\mathbf{Q}$, respectively. The covariance matrix $\mathbf{R}$ is assumed to be diagonal. In the COVID-adjusted model, we modify the covariance matrix $\mathbf{R}_t$ to be time-varying.

Each model has a corresponding vector of parameters to be estimated by maximum likelihood. Because maximum likelihood estimates of σ_g and σ_z, which are the standard deviations of the innovations to the g_t and z_t equations, are likely to be biased towards zero due to the pile-up problem (see Section 2.2 of HLW), we use Stock and Watson's (1998) median unbiased estimator to obtain estimates of two ratios, $\lambda_g \equiv \frac{\sigma_g}{\sigma_{y^*}}$ and $\lambda_z \equiv \frac{a_r \sigma_z}{\sigma_{\tilde{y}}}$. We estimate λ_g following the first stage model and λ_z following the second stage model, and impose these ratios in subsequent stages of the estimation, including when estimating the remaining model parameters by maximum likelihood. The COVID-adjusted model includes five additional parameters: the coefficient ϕ that translates the COVID indicator variable d_t into effects on output; the three variance scale parameters κ_{2020}, κ_{2021}, and κ_{2022}; and the coefficient c on trend growth g_t in the r_t^* equation, which appears only in the stage 3 model and is estimated in LW (2003) but fixed at unity in HLW (2017).

In addition to estimating the relationship between trend growth in the natural rate of output and r_t^*, we make two minor technical changes to the model that are not related to the COVID-19 pandemic.[12] First, we include a second lag of trend

[11] Notation follows Hamilton (1994) and is consistent with the corresponding R programs.

[12] We also explicitly include g_t and z_t in the vector of unobserved state variables in addition to

growth, g_{t-2}, in the stage 2 IS equation, consistent with the IS equation specification in the stage 3 model. Second, we correct the stage 2 state-space model so that the y_t^* equation is $y_t^* = y_{t-1}^* + g_{t-1} + \epsilon_{y^*,t}$ as expressed in the paper; previously, the second stage y_t^* equation included the second lag of trend growth, g_{t-2}, rather than the first lag in error (Buncic, 2021, 2022). These modifications to the model are highlighted in blue text in the following sections, while changes in response to the COVID-19 pandemic are highlighted in red. These technical changes have minor effects on our estimates of the unobserved state variables, including r_t^*.

7.1 The COVID-Adjusted State-Space Models

7.1.1 The COVID-adjusted Stage 1 Model

The first-stage model, which corresponds to the *rstar.stage1.R* program, can be represented by the following matrices:

$$\mathbf{y}_t = [y_t, \pi_t]' \tag{21}$$

$$\mathbf{x}_t = [y_{t-1}, y_{t-2}, \pi_{t-1}, \pi_{t-2,4}, d_t, d_{t-1}, d_{t-2}]' \tag{22}$$

$$\xi_t = [y_t^*, y_{t-1}^*, y_{t-2}^*]' \tag{23}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} \\ 0 & -b_y & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & 0 & 0 & \phi & -\phi a_{y,1} & -\phi a_{y,2} \\ b_y & 0 & b_\pi & 1-b_\pi & 0 & -\phi b_y & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y^*}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{R}_t = \begin{bmatrix} (\kappa_t \sigma_{\tilde{y}})^2 & 0 \\ 0 & (\kappa_t \sigma_\pi)^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_1 = [a_{y,1}, a_{y,2}, b_\pi, b_y, g, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*}, \phi, \kappa_{2020Q2-Q4}, \kappa_{2021}, \kappa_{2022}]$$

two lags of each variable as in HLW. This is purely an accounting change and has no effect on the estimates.

7.1.2 The COVID-adjusted Stage 2 Model

The second-stage model, which corresponds to the *rstar.stage2.R* program, can be represented by the following matrices:

$$\mathbf{y}_t = [y_t, \pi_t]' \tag{24}$$

$$\mathbf{x}_t = [y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}, 1, d_t, d_{t-1}, d_{t-2}]' \tag{25}$$

$$\xi_t = \left[y_t^*, y_{t-1}^*, y_{t-2}^*, g_t, g_{t-1}, g_{t-2}\right]' \tag{26}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & 0 & \frac{a_g}{2} & \frac{a_g}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 & a_0 & \phi & -\phi a_{y,1} & -\phi a_{y,2} \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi & 0 & 0 & -\phi b_y & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y^*}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\lambda_g \sigma_{y^*})^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{R}_t = \begin{bmatrix} (\kappa_t \sigma_{\tilde{y}})^2 & 0 \\ 0 & (\kappa_t \sigma_\pi)^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_2 = [a_{y,1}, a_{y,2}, a_r, a_0, a_g, b_\pi, b_y, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*}, \phi, \kappa_{2020Q2-Q4}, \kappa_{2021}, \kappa_{2022}]$$

7.1.3 The COVID-adjusted Stage 3 Model

The third-stage model, which corresponds to the *rstar.stage3.R* program, can be represented by the following matrices:

$$\mathbf{y}_t = [y_t, \pi_t]' \tag{27}$$

$$\mathbf{x}_t = [y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}, d_t, d_{t-1}, d_{t-2}]' \tag{28}$$

$$\xi_t = \left[y_t^*, y_{t-1}^*, y_{t-2}^*, g_t, g_{t-1}, g_{t-2}, z_t, z_{t-1}, z_{t-2}\right]' \tag{29}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & 0 & -4c\cdot\frac{a_r}{2} & -4c\cdot\frac{a_r}{2} & 0 & \frac{-a_r}{2} & \frac{-a_r}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 & \phi & -\phi a_{y,1} & -\phi a_{y,2} \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi & 0 & -\phi b_y & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1&0&0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y^*}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0 & 0 & 0 & (\lambda_g\sigma_{y^*})^2 & 0 & 0 & 0 & 0 & 0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{\lambda_z\sigma_{\tilde{y}}}{a_r}\right)^2 & 0 & 0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$\mathbf{R}_t = \begin{bmatrix} (\kappa_t\sigma_{\tilde{y}})^2 & 0 \\ 0 & (\kappa_t\sigma_\pi)^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_3 = [a_{y,1}, a_{y,2}, a_r, b_\pi, b_y, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*}, \phi, c, \kappa_{2020Q2-Q4}, \kappa_{2021}, \kappa_{2022}]$$

The law of motion for the natural rate of interest is $r_t^* = c \cdot g_t + z_t$.

7.2 The HLW (2017) State-Space Models

7.2.1 The Stage 1 Model

The first-stage model, which corresponds to the *rstar.stage1.R* program, can be represented by the following matrices:

$$\mathbf{y}_t = [y_t, \pi_t]' \tag{30}$$

$$\mathbf{x}_t = [y_{t-1}, y_{t-2}, \pi_{t-1}, \pi_{t-2,4}]' \tag{31}$$

$$\xi_t = [y_t^*, y_{t-1}^*, y_{t-2}^*]' \tag{32}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} \\ 0 & -b_y & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & 0 & 0 \\ b_y & 0 & b_\pi & 1-b_\pi \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y^*}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \sigma_{\tilde{y}}^2 & 0 \\ 0 & \sigma_\pi^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_1 = [a_{y,1}, a_{y,2}, b_\pi, b_y, g, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*}]$$

7.2.2 The Stage 2 Model

The second-stage model, which corresponds to the *rstar.stage2.R* program, can be represented by the following matrices:

$$\mathbf{y}_t = [y_t, \pi_t]' \tag{33}$$

$$\mathbf{x}_t = [y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}, 1]' \tag{34}$$

$$\xi_t = \left[y_t^*, y_{t-1}^*, y_{t-2}^*, g_{t-1}\right]' \tag{35}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & a_g \\ 0 & -b_y & 0 & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 & a_0 \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y^*}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\lambda_g \sigma_{y^*})^2 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \sigma_{\tilde{y}}^2 & 0 \\ 0 & \sigma_\pi^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_2 = [a_{y,1}, a_{y,2}, a_r, a_0, a_g, b_\pi, b_y, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*}]$$

7.2.3 The Stage 3 Model

The third-stage model, which corresponds to the *rstar.stage3.R* program, can be represented by the following matrices:

$$\mathbf{y}_t = [y_t, \pi_t]' \tag{36}$$

$$\mathbf{x}_t = [y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}]' \tag{37}$$

$$\xi_t = \left[y_t^*, y_{t-1}^*, y_{t-2}^*, g_{t-1}, g_{t-2}, z_{t-1}, z_{t-2}\right]' \tag{38}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & \frac{-a_r}{2} & \frac{-a_r}{2} & \frac{-a_r}{2} & \frac{-a_r}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \left(1+\lambda_g^2\right)\sigma_{y^*}^2 & 0 & 0 & (\lambda_g \sigma_{y^*})^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\lambda_g \sigma_{y^*})^2 & 0 & 0 & (\lambda_g \sigma_{y^*})^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{\lambda_z \sigma_{\widetilde{y}}}{a_r}\right)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{R} = \begin{bmatrix} \sigma_{\widetilde{y}}^2 & 0 \\ 0 & \sigma_\pi^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_3 = [a_{y,1}, a_{y,2}, a_r, b_\pi, b_y, \sigma_{\widetilde{y}}, \sigma_\pi, \sigma_{y*}]$$

The law of motion for the natural rate of interest is $r_t^* = g_t + z_t$.

Appendix A2: Data

For each economy, we require data for real GDP, inflation, and the short-term nominal interest rate, as well as a procedure to compute inflation expectations to calculate the ex ante real short-term interest rate r_t.[13] The variable y_t refers to the logarithm of real GDP. The inflation measure is the annualized quarterly growth rate of the specified consumer price series. With the exception of the United States, for which core personal consumption expenditure (PCE) price data are available over the entire sample, the inflation series is constructed by splicing the core price index with an all-items price index. We use a four-quarter moving average of past inflation as a proxy for inflation expectations in constructing the ex ante real interest rate. Short-term interest rates are expressed on a 365-day annualized basis.

For the United States, we use real GDP and core PCE data published by the Bureau of Economic Analysis. Inflation is constructed using the price index for PCE excluding food and energy, referred to as core PCE inflation. The short-term interest rate is the annualized nominal federal funds rate, available from the Board of Governors. Because the federal funds rate frequently fell below the discount rate prior to 1965, we use the Federal Reserve Bank of New York's discount rate, part of the IMF's International Financial Statistics Yearbooks (IFS), prior to 1965. All U.S. data can be downloaded from the St. Louis Fed's Federal Reserve Economic Data (FRED) website.[14]

Canadian real GDP data is taken from the IMF's IFS. The short-term nominal interest rate is the Bank of Canada's target for the overnight rate, taken as the end-of-period value for each month and aggregated to quarterly frequency. Since the Bank of Canada began treating the target rate as its key interest rate in May 2001, we use the bank rate as the short-term interest rate prior to that date. We use the Bank of Canada's core Consumer Price Index to construct our inflation series. Prior to 1984, we use CPI containing all items. With the exception of real GDP, all data is from

[13] A detailed description of our data and programs, as well as replication materials for the standard HLW model, is available on the Federal Reserve Bank of New York's website.

[14] Mnemonics are as follows. Real GDP: GDPC1; Core PCE: PCEPILFE; Federal Funds Rate: FEDFUNDS; FRBNY Discount Rate: INTDSRUSM193N.

Statistics Canada.[15]

Euro Area data is from the Area Wide Model (AWM), available from the Euro Area Business Cycle Network (Fagan et al., 2001). The inflation measure is based on the core price index, HICP excluding energy (series HEX) beginning in 1988; prior to 1988 we use the overall price index HICP. The nominal short-term interest rate is the three-month rate (series STN) and the real GDP mnemonic is YER. At the time of publication, the final update to the Area Wide Model was in 2017; we update the three series from the ECB's Statistical Data Warehouse.[16]

[15]Mnemonics from Statistics Canada are as follows. Core CPI: v41690926 (Table 326-0022); CPI: v41690914 (Table 326-0022); v41690973 (Table 326-0020); Bank Rate: v122530 (Table 176-0043); Target Rate: v39079 (Table 176-0048). Real GDP is IFS series "Gross Domestic Product, Real, Seasonally adjusted, Index".

[16]Mnemonics for the SDW are as follows.
Real GDP: MNA.Q.Y.I8.W2.S1.S1.B.B1GQ._Z._Z._Z.EUR.LR.N;
Core HICP: ICP.M.U2.N.XE0000.4.INX;
Nominal Short-term Rate: FM.Q.U2.EUR.RT.MM.EURIBOR3MD_.HSTA.
Because data availability is longer for the non-seasonally adjusted price series, we use those and seasonally adjust them.

Appendix A3: Sensitivity of Pre-Pandemic Estimates

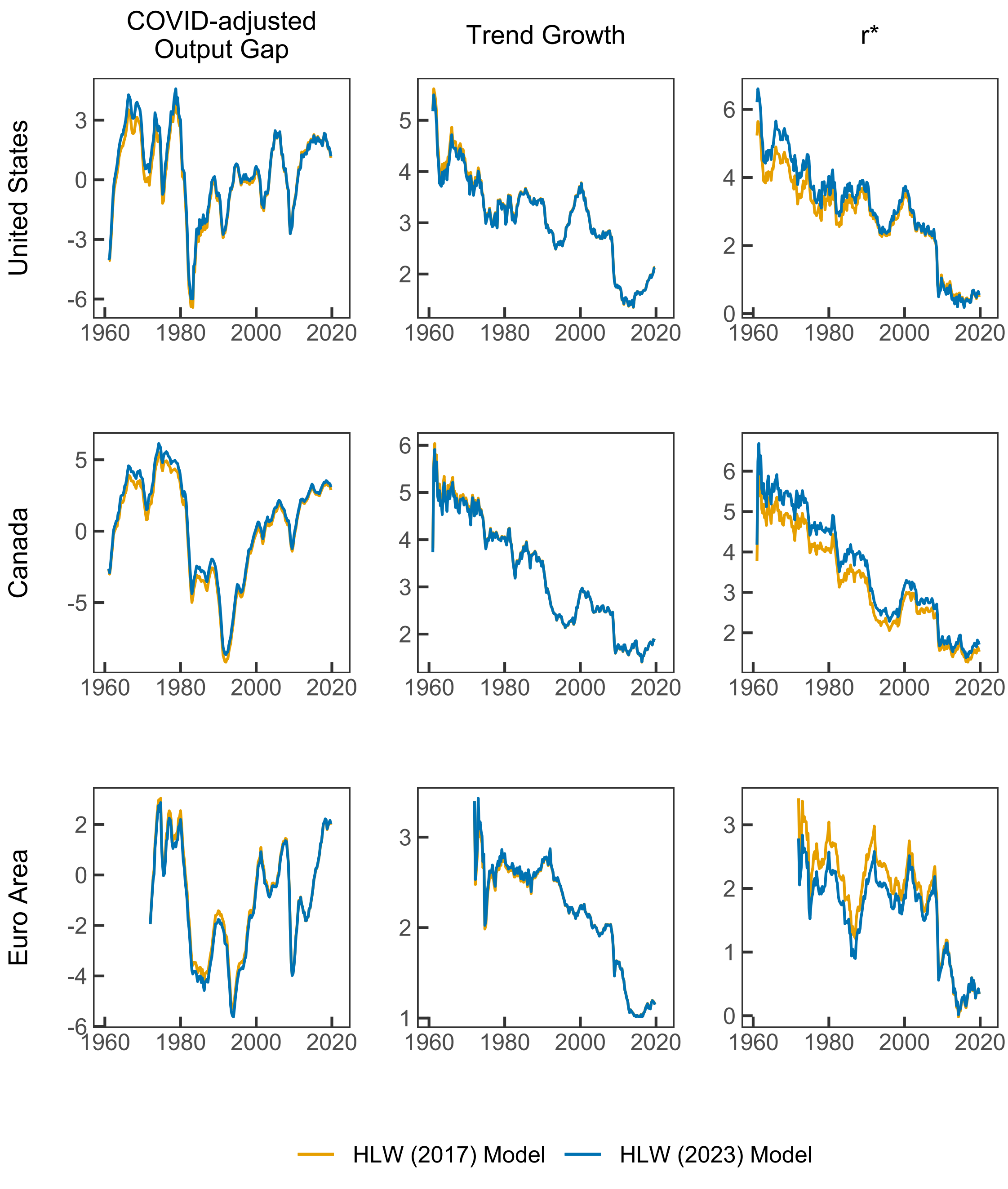
Table A1 reports parameter estimates from the model estimated through 2019:Q4. Figure A1 compares estimates from the original HLW (2017) model and the modified model through 2019:Q4. Both use the current data vintage at the time of publication.

Table A1: Parameter Estimates, Sample Ending 2019:Q4

Parameter	United States	Canada	Euro Area
Sample	1961-2019	1961-2019	1972-2019
λ_g	0.053	0.052	0.036
λ_z	0.031	0.016	0.032
$\sum a_y$	0.941	0.951	0.950
a_r	-0.067	-0.065	-0.037
	(3.973)	(2.969)	(1.798)
b_y	0.076	0.047	0.062
	(3.077)	(1.697)	(1.713)
c	1.198	1.130	0.816
	(3.484)	(2.716)	(0.888)
$\sigma_{\tilde{y}}$	0.344	0.395	0.289
σ_π	0.794	1.359	0.972
σ_{y^*}	0.568	0.581	0.397
σ_g	0.121	0.121	0.058
σ_z	0.157	0.096	0.246
$\sigma_{r^*} = \sqrt{c^2\sigma_g^2 + \sigma_z^2}$	0.213	0.167	0.251
S.E. (sample ave.)			
r^*	1.236	1.625	3.480
g	0.395	0.422	0.260
y^*	1.559	2.470	1.875
S.E. (final obs.)			
r^*	1.656	1.844	4.822
g	0.546	0.574	0.357
y^*	1.971	2.761	2.429

Notes: t statistics are in parentheses; σ_g is expressed at an annual rate. Sample through 2019:Q4, using current data vintage at time of publication.

Figure A1: Estimates through 2019:Q4, HLW (2023) vs. HLW (2017) Models



Notes: The gold lines labeled "HLW (2017) Model" show the estimates of the indicated variables using data through 2019:Q4 in the standard HLW model. The blue lines labeled "HLW (2023) Model" show the estimates of the modified HLW model on data through 2019:Q4. Both sets of estimates use the current data vintage at the time of publication.

Appendix A4: Laubach and Williams (2003) Model

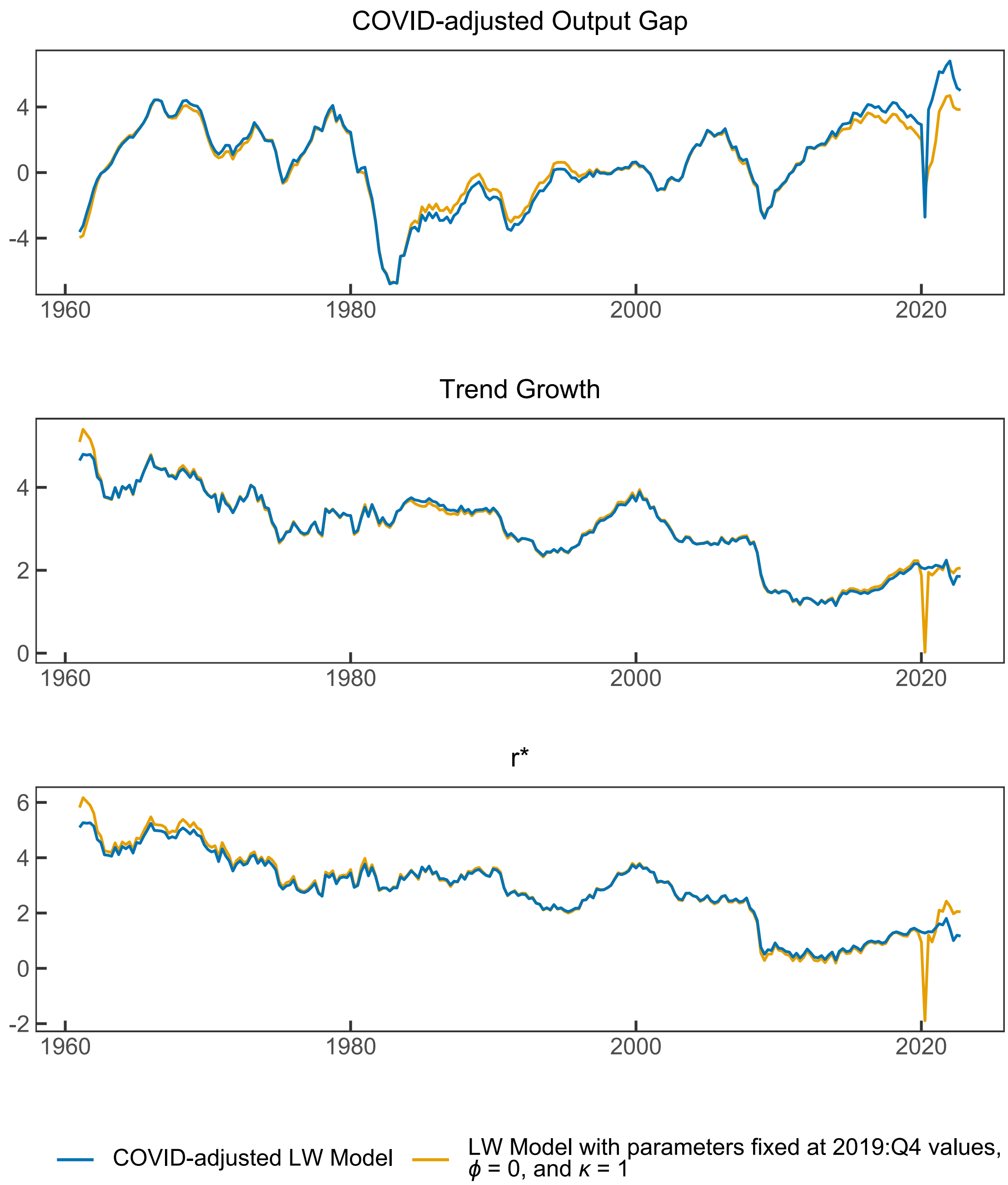
We apply the same model adjustments described in the paper to estimate the LW model for the United States. In addition to the pandemic-related modifications, we make the two technical adjustments described in Appendix A1. We make two additional changes that are already implemented in the HLW model: we initialize the state vector and its covariance matrix as described in HLW (2017), footnote 6, and impose the constraints that the slope of the IS equation (a_r) is negative and the slope of the Phillips curve (b_y) is positive in all three stages of the estimation. Table A2 reports parameter estimates from the LW model estimated through 2022:Q4. Figure A2 displays estimates from the COVID-adjusted LW model and from a version of the model with parameters fixed at 2019:Q4 values, analogous to Section 2.2.

Table A2: Parameter Estimates from LW Model

Parameter	LW	HLW
Sample	1961-2022	1961-2022
λ_g	0.070	0.073
λ_z	0.027	0.021
$\sum a_y$	0.946	0.936
a_r	-0.090	-0.079
	(4.718)	(4.215)
b_y	0.049	0.073
	(2.328)	(3.003)
c	1.097	1.128
	(3.256)	(3.574)
ϕ	-0.067	-0.085
	(2.212)	(2.199)
$\kappa_{2020Q2-Q4}$	9.572	9.033
	(2.182)	(2.351)
κ_{2021}	1.786	1.791
	(3.052)	(2.941)
κ_{2022}	1.000	1.676
	(1.766)	(2.060)
$\sigma_{\tilde{y}}$	0.457	0.452
σ_{π}	0.754	0.787
σ_{y^*}	0.503	0.500
σ_g	0.142	0.145
σ_z	0.139	0.118
$\sigma_{r^*} = \sqrt{c^2\sigma_g^2 + \sigma_z^2}$	0.208	0.202
S.E. (sample ave.)		
r^*	1.193	1.140
g	0.433	0.428
y^*	1.928	1.612
S.E. (final obs.)		
r^*	1.679	1.565
g	0.663	0.689
y^*	3.256	2.914

Notes: t statistics are in parentheses; σ_g is expressed at an annual rate.

Figure A2: Estimates from LW Model



Notes: The gold lines labeled "LW model with parameters fixed at 2019:Q4 values" show the estimates of the indicated variables using data through 2022:Q4, but where the model parameters are estimated on data through 2019:Q4. The blue lines labeled "COVID-adjusted LW model" show the full-sample estimates of the modified LW model.