STAFF REPORTS

NO. 1069
AUGUST 2023

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Federal Reserve Bank of New York Staff Reports, no. 1069
August 2023
https://doi.org/10.59576/sr.1069

Abstract

I build a tractable random search model with firm dynamics, on-the-job search, and aggregate shocks. Multi-worker firms make recruitment decisions, choose whether to enter or exit the market, and design wage contracts. Tractability is obtained by showing that, under a set of assumptions on the recruitment technology, the decisions of workers and firms can be expressed in terms of the firms' current productivity. I introduce a numerical solution method to accommodate aggregate shocks in this environment and show that the model can replicate salient features of both firm-level data on productivity and employment and aggregate time series describing the business cycle. I use this framework to quantify the drivers of worker reallocation over the recent business cycle in Britain.

JEL classification: E3
Keywords: firm dynamics, search, business cycle

Audoly: Federal Reserve Bank of New York (email: richard.audoly@ny.frb.org). This paper is a revised version of the first chapter of the author's doctoral dissertation (first circulated version: January 2017). He is thankful to his supervisors at University College London (UCL), Fabien Postel-Vinay, and Jan Eeckhout for their advice and support. He also thanks Wei Cui, Rasmus Lentz, three anonymous referees, and many others for helpful comments and suggestions.

1 Introduction

How do economic fluctuations affect workers? A large literature has documented that recessions coincide with substantial changes in worker flows. Recessions have been shown to markedly increase inflows into unemployment and to decrease the pace at which workers, both employed and unemployed, find new jobs.[1] Given the large degree of firm heterogeneity in the data in terms of productivity, wages, and employment, an important question arising from these regularities is: to what extent do fluctuations in worker flows reallocate workers to better firms?[2] The answer to this question matters for the design of economic policies, such as those that subsidize the search of jobless workers.

In this paper, I develop a rich quantitative framework to measure worker reallocation over the business cycle. The key features of this framework are firm dynamics, a frictional labor market where workers can search while employed, and aggregate shocks. Conceptually, the model implies that workers have well-defined preferences over firms. In equilibrium, more productive firms deliver better wage contracts, and workers gradually quit to move to these firms over time. Calibrated on detailed firm-level data on productivity, the model then gives a structural measure of worker reallocation to better firms over the business cycle. In the quantitative part of the paper, I compute this measure using data from Britain over the period 1997-2018 and quantify the drivers of worker reallocation within the calibrated model.

The main features of this framework are guided by the stylized facts on the evolution of worker flows over the business cycle and firm heterogeneity. Search frictions in the labor market give rise to transitions in and out of unemployment. With on-the-job search, the model has a counterpart to job-to-job transition flows. These flows are substantial in typical data sets: workers are at least twice as likely to directly transition to another job as to become unemployed.[3] Multi-worker firms are needed to lay a credible foundation for the job ladder at

[1] See, among others, Blanchard et al. [1990] for unemployment inflows and outflows and Fujita et al. [2020] for US evidence on job transitions. As shown later, these patterns also hold in the British data used to quantify the model (Section 5).

[2] See, for instance, Syverson [2011] for an analysis of productivity differences. Gibrat [1931] is an early reference on the distribution of firm size.

[3] Twice as likely in the US. In the British data I use in the paper, this ratio is around four.

the micro-level, since there is no measure of productivity at the job level in standard data sets. Aggregate shocks, finally, are a pre-requisite to studying business cycle patterns.

A key contribution of this paper is to identify a set of model primitives such that the model remains tractable with aggregate shocks. In the model, firms make hiring decisions, choose whether to enter or exit the market, and commit to a state-contingent wage contract. I show that, under specific assumptions on the cost of recruitment, the state-space relevant to the firm's decisions reduces to its current productivity, the realization of the aggregate shock, and the employment-weighted distribution of firm productivity. I also derive several characterization results. I show that the optimal wage contract is increasing in firm productivity and derive its closed-form solution. With this last result, wages are well-defined and straightforward to compute in this environment.

This set of results implies that the preferences of workers over firms map into firm productivity: the job ladder is a productivity ladder.[4] In the model, a natural statistic to summarize the location of workers along the job ladder at any point in time is therefore the employment-weighted distribution of firm productivity. This summary statistic can be further decomposed into a firm component, which sums up the productivity of active firms at each point in time, and an interaction term, which gives the location of workers on the productivity ladder relative to the set of active firms. This structural decomposition represents one of the key objects of interest, and each of its terms can be quantified within the calibrated model.

Given the close connection between the job ladder and labor productivity at the firm level, I use detailed data on the balance-sheet of British businesses for the period 1997-2018 to discipline the model. I gross up these data into a labor productivity index that is closely connected to the employment-weighted distribution of firm productivity used to summarize the state of the job ladder in the model. This index can similarly be decomposed into a firm component and an interaction component, where each term is also directly related to the structural decomposition derived from the model. The decomposition obtained from these time series represents a set

[4]This result is common to a large class of random search model with on-the-job search [Burdett and Mortensen, 1998, Moscarini and Postel-Vinay, 2013, Coles and Mortensen, 2016]. I discuss the notion of worker reallocation implied by the model further in Section 4.

of new moments, which I use to assess the worker relocation properties of the model with aggregate shocks. Specifically, these data show that 20 percent of the overall fall in the labor productivity index during the Great Recession in Britain is accounted for by the interaction term.

Another contribution of this paper is to propose a numerical solution method suitable to my random search environment with aggregate shocks. This solution method is required for two reasons. First, the model features an infinitely dimensional variable in the state-space (the full employment-weighted distribution of firm productivity). In addition, standard linearization techniques [Reiter, 2009] do not apply to my environment because endogenous firm entry-exit makes the firm's problem discontinuous. I therefore rely on a simulation-based approach in which the employment-weighted distribution of firm productivity is approximated by a set of its moments. I also check the accuracy of this procedure with several alternative tests. I believe that, beyond the model considered in this paper, this approach can potentially be useful in other models with aggregate shocks and a similar discontinuity.

In the quantitative part of the paper, I calibrate the model to match a set of moments related to worker mobility and firm dynamics. Some of these moments come from the cross-section of firms implied by the model, such as the firm productivity distribution and the firm size distribution. These cross-sectional moments provide a foundation for the job ladder implied by the model. Another set of moments is related to the business cycle properties of the model, such as the volatility of worker flows. I further compare model simulations to time series on labor productivity, wages, and labor flows. These simulations are obtained by fitting a sequence of aggregate shocks to replicate the cyclical evolution of labor productivity in Britain over the recent period. Overall, I find that the model performs well along these dimensions.

In the final part of the paper, I use the calibrated model to measure the reallocation of workers along the job ladder over the recent business cycle in Britain. Worker reallocation is summarized using the structural decomposition of the job ladder into its firm component and interaction component. Through the lens of the model, most of the fluctuations in labor productivity are driven by worker reallocation. I also find a key role for the firm component of worker reallocation, which accounts for 60 percent of the overall drop in worker reallocation after the Great Recession. The mechanism behind this potentially counter-intuitive result is that lower productivity firms have a lower rate of quits because in searching for new jobs, their employees compete with a larger pool of unemployed workers; therefore, since these firms benefit from unemployment, they are less likely to exit. This effect counteracts the negative effect of a productivity shock at the calibrated parameters. Lastly, I quantify worker reallocation in a counterfactual with countercyclical unemployment benefits. In this exercise, I find that countercyclical unemployment benefits impact the firm component and interaction component of worker reallocation along the job ladder in opposite directions. There are fewer firms at the bottom of the productivity distribution relative to the baseline model, but with more unemployment, the reallocation of workers up the job ladder is also slower.

Related literature This work is related to the growing literature that combines firm dynamics with frictional labor markets. This literature brings together firm dynamics models in the tradition of Hopenhayn and Rogerson [1993], which maintain the assumption that labor markets clear [Khan and Thomas, 2013, Clementi and Palazzo, 2016, Sedláček and Sterk, 2017], and search and matching models in the tradition of Mortensen and Pissarides [1994], which emphasize firm-workers' matches without a meaningful notion of a firm with multiple workers. Multi-worker firms are a pre-requisite to jointly studying firm-level concepts (productivity, employment, job flows) and worker flows (unemployment inflows and outflows). My work adds to the recent papers integrating firm dynamics and frictional labor markets by combining three features: (i) firm dynamics, (ii) random search with on-the-job search, and (iii) business cycle fluctuations.

In the existing literature, firm dynamics and search frictions have been integrated using two distinct approaches: directed search and random search. A first series of papers builds on the theoretical results in Menzio and Shi [2011] to introduce firm dynamics in an environment where workers can direct their search to specific jobs [Kaas and Kircher, 2015, Schaal, 2017]. This approach is highly tractable in the presence of aggregate shocks. With the appropriate free-entry condition, all distributions vanish from the state-space and the model can be solved numerically using standard recursive methods. While Kaas and Kircher [2015] abstract from on-the-job search, Schaal [2017] presents an elegant model combining firm dynamics, on-the-job search, and aggregate shocks. But, in a directed search environment, on-the-job search implies a very specific theory of worker reallocation because there is no job ladder: the theory does not specify which firms offer better jobs and systematically poach workers from other firms.[5]

A second strand of this literature introduces firm dynamics in a random search environment. Early contributions abstract from on-the-job search [Elsby and Michaels, 2013, Acemoglu and Hawkins, 2014]. A central feature of the handful of random search models with on-the-search and multi-worker firms is the existence of a job ladder: there is a clear theory of which firms offer better jobs. Implicitly, these models all imply a measure of worker reallocation along the job ladder in response to aggregate shocks. I make this mechanism explicit in the paper and quantify the drivers of worker reallocation in a model disciplined with detailed firm-level data. I return to this point in Section 4 where I compare worker reallocation in my environment relative to other random search models with on-the-job search.

The model developed in this paper expands on the random search frameworks with on-the-job search in Moscarini and Postel-Vinay [2013] and Coles and Mortensen [2016]. Similarly to the wage determination protocol in Moscarini and Postel-Vinay [2013], I assume that firms can commit to a state-contingent wage contract. I expand on their framework by allowing for firm dynamics (firm entry-exit, firm-specific shocks) while still retaining the tractability of the model. Similarly to Coles and Mortensen [2016], I assume that the cost of recruitment has a specific functional form and I obtain the same size-independence result. But with state-contingent wage contracts, I am able to relax their assumption of exogenous firm entry and exit and allow the set of active firms to evolve endogenously over the business cycle.

My environment maintains the assumption that firms operate a linear production technology. Two contemporaneous papers, Elsby and Gottfries [2022] and Bilal et al. [2022], consider

[5]The objective in Schaal [2017] is not to quantify worker reallocation but to study the role of uncertainty shocks as drivers of the US business cycle.

similar environments with decreasing returns to production. Elsby and Gottfries [2022] show under two wage-setting protocols that the job ladder can be characterized in terms of a single variable: the marginal product of labor. A key difference with my framework is that they do not allow for firm entry and exit; so all worker reallocation arises through the flows of workers among the same set of firms. Bilal et al. [2022] describe a model related to Elsby and Gottfries [2022] that allows for firm entry and exit. They build on the framework introduced in Lentz and Mortensen [2012] to characterize the job ladder in terms of the marginal joint value of a firm and its workers. An important difference with my framework is that the theory in Bilal et al. [2022] is agnostic about how the surplus is split within a firm, and wages are therefore not determined. By contrast, wages are well-defined and straightforward to compute in my environment, which I use as an additional check on the reallocation properties of the model.

Relative to these two important contributions, my focus is on quantifying worker reallocation along the job ladder over the business cycle. While their analysis is restricted to a comparison of steady states and to perfect-foresight shocks, I propose a solution method to simulate the full model with aggregate shocks. With this solution, I can benchmark the model to key time series and quantify the degree of worker reallocation along the job ladder associated with business cycle fluctuations.

Outline Section 2 introduces the model. Section 3 defines the equilibrium. Section 4 discusses the worker reallocation mechanism in the model. Section 5 describes the calibration and numerical solution. Section 6 quantifies the magnitude of worker reallocation within the calibrated model, and Section 7 concludes.

2 A model of firm dynamics with on-the-job search

This section introduces a rich framework to analyze worker reallocation over the business cycle. Firms with heterogeneous productivities make recruitment decisions, decide whether to enter and exit the market, and design wage contracts. Workers move between these firms by searching both when employed and when unemployed. These decisions are taken in an environment with aggregate shocks.

2.1 Environment

Time $t = 0, 1, 2, \ldots$ is discrete and the horizon is infinite. There is a unit measure of infinitely lived and ex-ante identical workers who are either (i) employed earning wage w_t, (ii) unemployed with home production b, or (iii) an entrepreneur attempting to start up a new firm. There is a measure of firms evolving endogenously due to entry and exit. Each firm operates a constant return technology with labor n as the only input and productivity factor $\omega_t p_t$. ω_t is an economy-wide component, which follows a first-order Markov process $\Gamma_\omega(.|\omega_{t-1})$ with realizations in some positive interval $\left[\underline{\omega}, \overline{\omega}\right]$. p_t is a firm-specific component, which follows a first-order Markov process $\Gamma_p(.|p_{t-1})$ with realizations in some positive interval $\left[\underline{p}, \overline{p}\right]$. Both workers and firms are risk-neutral and maximize their expected pay-offs discounted with factor $\beta \in (0, 1)$.

Labor market flows are constrained by search frictions. Unemployed workers sample job offers with some probability $\lambda_t \in (0, 1]$ at time t. Employed workers sample job offers with probability $s\lambda_t \in (0, 1]$, where the exogenous parameter s denotes the search intensity of employed relative to unemployed job seekers. Both unemployed and employed workers sample at most one offer per period t. Each employed worker is separated from her employer with probability δ_t. Workers also transition to unemployment when their current firm decides to exit. Newly unemployed workers do not search in the current period.

Workers, both unemployed and employed, become entrepreneurs in each period t with probability $\mu \in (0, 1]$. Employed workers quit their current job to become entrepreneurs with no opportunity to go back to their previous employer. Potential entrepreneurs draw an initial productivity p_0 from the exogenous distribution Γ_0 and decide whether to enter. They do not search in the current period. They become unemployed if they decide not to enter.

The job destruction rate is assumed to be an exogenous function of aggregate productivity $\delta_t = \delta(\omega_t)$. The probability λ_t that a worker makes contact with a potential employer is instead determined in equilibrium through a matching function. The matching function takes as inputs aggregate ads from firms and aggregate search effort from workers. In each period, a firm with employment n can hire a measure H of workers at a cost $C(H,n)$. The cost $C(H,n)$ corresponds to the production that is lost due to the process of searching for and training new hires. C is assumed to be homogeneous of degree one. The cost of hiring $H = hn$ new workers can then be written $C(H,n) = nC(h,1) = nc(h)$, where c is positive, continuously differentiable, increasing, convex, and $c(0) = 0$. To hire, the firm has to post $a \geq 0$ job ads before workers have a chance to search.

I introduce some additional notation to formally define the matching technology. Let $M_t(p,n)$ denote the cumulative measure of firms with current productivity at most p and employment size at most n at the onset of period t, before decisions take place. Let $a_t(p,n)$ denote the equilibrium number of ads posted by a firm with current productivity p and employment size n. Let $\chi_t(p,n)$ denote the equilibrium decision of the firm to continue ($\chi_t(p,n) = 1$) or exit ($\chi_t(p,n) = 0$). equilibrium decisions $a_t(p,n)$ and $\chi_t(p,n)$ depend on aggregate variables, such as aggregate productivity ω_t, which are subsumed in the time subscript t for now. The aggregation of all ads posted by continuing firms gives

$$A_t = \int \chi_t(p,n) a_t(p,n) dM_t(p,n), \tag{1}$$

Aggregate search effort is given by

$$Z_t = (1-\mu)\left[u_t + (1-\delta_t)s \int \chi_t(p,n) n dM_t(p,n)\right], \tag{2}$$

the measure of $(1-\mu)$ workers who are not entrepreneurs and are either unemployed at the start of the period (with the unemployment rate $u_t = 1 - \int n dM_t(p,n)$) or employed and not separated adjusting for their relative search intensity s. The total number of contacts in a period t is then given by

$$\lambda_t Z_t = \eta_t A_t = \min\{Z_t, A_t\}, \tag{3}$$

where η_t is the probability that an advert reaches a worker.

Each period t can be divided into the following six phases:

1. Productivity shocks. Aggregate productivity ω_t and firm-specific productivity p_t are realized.[6]

2. Entrepreneurial shock. With probability μ, workers become entrepreneurs. They draw an initial idea with productivity $p_0 \sim \Gamma_0$ and decide whether to enter.

3. Firm exit. Firms decide whether to stay on or discontinue their operations based on the realization of the productivity shocks. If they exit, all of their workers become unemployed.

4. Exogenous separations. Employees at continuing firms lose their jobs with exogenous probability δ_t.

5. Search. Firms post vacancies to hire. Both unemployed and employed workers search for jobs. Recruitment at incumbent firms takes place.

6. Production and payments. Unemployed workers have home production b. Firms produce with their employees after the search stage. Wages accrue to employed workers. Newly created businesses start producing.

A recursive formulation is used throughout the paper. All value functions in subsequent sections are written from the production and payment stage onward, taking expectation $\mathbf{E}_{t-1}\{.\}$ over the events occurring in period t, conditional on the information available at the end of period $t-1$.

[6]The notation for the cumulative measure of firms $M_t(p, n)$ in the productivity-size space (p, n) is recorded at this point in time in the within-period sequence of events.

2.2 Wage determination: Contract-posting

Each firm chooses and commits to an employment contract upon entry to maximize the present discounted value of profits, taking the contracts offered by other firms as given. This contract specifies a state-contingent wage payment $w_t(p,n)$. Firms are bound by an equal treatment constraint, which restricts them to offering the same contract to all their employees.[7] With full commitment, the discounted sum of future wage payments can be summarized by an equilibrium contract value $V_t(p,n)$.

I now introduce the notation needed to formally define the firm's problem. The cdf of offered wage contracts is denoted

$$F_t(W) = A_t^{-1} \int \mathbb{1}\left\{V_t(p,n) \leq W\right\} \chi_t(p,n) a_t(p,n) dM_t(p,n), \tag{4}$$

where $\mathbb{1}\{.\}$ is the indicator function. $F_t(W)$ is the fraction of ads posted by firms that offer less than contract value W. Job seekers draw offers from the distribution of values F_t. The value of unemployment is given by

$$U_{t-1} = b + \beta \mathbf{E}_{t-1}\Big\{\mu Q_t + (1-\mu)\Big[(1-\lambda_t)U_t + \lambda_t \int \max\left\{\tilde{W}, U_t\right\} dF_t(\tilde{W})\Big]\Big\}. \tag{5}$$

The unemployed worker has home production b in period $t-1$. Conditional on the realization of the shocks in the next period, she becomes an entrepreneur with chance μ, which gives her continuation value Q_t (defined explicitly below). With chance $(1-\mu)$, she is part of the pool of unemployed job seekers and draws an offer from the distribution of offered contracts F_t with probability λ_t. This offer is accepted if it delivers more than the continuation value of being unemployed U_t.

A firm offering contract value $W_t < U_t$ given the realization of the state variables at time t loses its entire workforce ($n_t = 0$): workers are better off being unemployed if $W_t < U_t$, and they are all offered the same contract due to the equal treatment constraint. The employment

[7] This constraint can be interpreted as a non-discrimination rule among ex-ante identical workers. The contract therefore cannot be contingent on outside offers, which are specific to each worker.

contract therefore specifies firm exit after some realizations of the state. The firm's decision to continue χ_t can be expressed as a function of the employment contract $\chi_t = \mathbb{1}\{W_t \geq U_t\}$. Exit is permanent. Workers are an input in the recruitment technology $nc(h)$, so hiring is not possible with $n = 0$. There is no realization of the state where the firm's present discounted value of profits is negative in equilibrium. The firm can always choose $W_t < U_t$ and get zero profits thereafter, so any contract where the firm's present discounted value of profits is negative cannot be optimal.

A firm offering any contract value $W_t \geq U_t$ given the realization of the state at time t sees a fraction μ of its workforce become entrepreneurs and a fraction $(1-\mu)\delta_t$ separated to unemployment, and the remainder quits to work at other firms at rate $q_t(W_t) = s\lambda_t\big(1-F_t(W_t)\big)$. The quit rate $q_t(W_t)$ is the probability $s\lambda_t$ that employed workers contact an alternative employer times the probability $\big(1-F_t(W_t)\big)$ that they draw a better offer. The firm chooses to hire new workers at rate h_t. Employment at a continuing firm with size n_{t-1} at the start of the period evolves according to

$$n_t = (1-\mu)(1-\delta_t)\big[1 - q_t(W_t) + h_t\big]n_{t-1}. \tag{6}$$

The firm posts a measure of ads a_t in accordance with its choice of gross hires in the current period $(1-\mu)(1-\delta_t)h_t n_{t-1}$, given aggregate search frictions. The number of posted ads a_t is implicitly defined by

$$(1-\mu)(1-\delta_t)h_t n_{t-1} = a_t \eta_t Y_t(W_t). \tag{7}$$

Total gross hires at the firm are equal to posted ads a_t times the chance these ads reach a worker η_t times the chance that the worker accepts the firm's employment contract W_t. The acceptance rate $Y_t(W_t)$ is the ratio between the measure of job seekers with current value less than W_t and all job seekers:

$$Y_t(W_t) = \frac{u_t + (1-\delta_t)s\int \mathbb{1}\{V_t(p,n) \leq W_t\}\chi_t(p,n)ndM_t(p,n)}{u_t + (1-\delta_t)s\int \chi_t(p,n)ndM_t(p,n)}. \tag{8}$$

Exiting firms do not hire and therefore do not post vacancies. They do not contribute to the distribution of offered contracts (4), so all offered contracts give at least value U_t. Unemployed workers accept all offers and transition to employment conditional on making a contact.

2.3 The firm's problem

The firm's problem can be written recursively by introducing an additional state variable for the value that the firm has committed to deliver to its workers from period $t-1$ onward [Moscarini and Postel-Vinay, 2013]. Let $\overline{V}$ denote that value. The present discounted value of the firm's profits is given by

$$\begin{aligned} &\Pi_{t-1}\left(p_{t-1}, n_{t-1}, \overline{V}\right) = \\ &\max_{\substack{w,W,\\ h\geq 0}} (\omega_{t-1}p_{t-1} - w)n_{t-1} + \beta\mathbf{E}_{t-1}\Big[\chi\cdot\Big(-c(h)(1-\mu)(1-\delta_t)n_{t-1} + \Pi_t(p_t, n_t, W)\Big)\Big], \end{aligned} \tag{9}$$

subject to the law of motion for employment (6) and the promise-keeping constraint

$$\begin{aligned} \overline{V} = w + \beta\mathbf{E}_{t-1}\Bigg\{&\mu Q_t + (1-\mu)\Big[\Big((1-\chi) + \delta_t\chi\Big)U_t \\ &+ \chi\cdot(1-\delta_t)\Big(\big(1-q_t(W)\big)W + s\lambda_t\int \max\big\{\tilde{W}, W\big\}dF_t(\tilde{W})\Big)\Big]\Bigg\}. \end{aligned} \tag{10}$$

In the current period, flow revenues at the firm are $p_{t-1}\omega_{t-1}n_{t-1}$, and its wage bill is wn_{t-1}. In the next period, conditional on the realization of the states, the firm decides whether to remain active $\chi = \mathbb{1}\{W \geq U_t\}$. If it remains active, it chooses its number of gross hires $h(1-\mu)(1-\delta_t)n_{t-1}$ at a cost $c(h)(1-\mu)(1-\delta_t)n_{t-1}$ and posts the corresponding ads in accordance with (7).

The promise-keeping constraint (10) states that the firm's choices must deliver value $\overline{V}$ to its workers, since it is committed to its employment contract.[8] The firm can deliver value $\overline{V}$ by adjusting the wage w and the state-contingent continuation value of its employment contract W. The firm's choice takes into account workers' continuation value. With chance μ, workers become entrepreneurs which gives them value Q_t. Otherwise with probability $(1-\mu)$, they transition to unemployment either because the firm exits $(1-\chi)$ or because they are hit by an exogenous separation shock $(\chi\delta_t)$, which gives them value U_t. Workers who are not separated $\big(\chi \cdot (1-\delta_t)\big)$ draw from the contract offer distribution F_t at rate $s\lambda_t$ and compare their offer with the contract chosen by the firm W. Workers remain with the firm and get value W if they do not get a better offer $\big(1-q_t(W)\big)$.

With constant returns to the production and hiring technology, the firm's strategy can be shown to be independent of firm size n:

Result 1 (Size-independence) *Firm profits* (9) *have a linear solution* $\Pi_{t-1}(p_{t-1}, n_{t-1}, \overline{V}) = n_{t-1}J_{t-1}(p_{t-1}, \overline{V})$*, where* $J_{t-1}(p_{t-1}, \overline{V})$ *denotes profits per worker and is independent of firm-size* n_{t-1}*. The corresponding optimal choices for the continuation decision* $\chi_t(p_t)$*, contract* $V_t(p)$*, and hiring rate* $h_t(p)$ *are all independent of firm size* n_{t-1}*.*

The proof is in Appendix A.1.

This firm-size independence result is similar to the result in Coles and Mortensen [2016], but it is obtained under different assumptions on the wage-setting protocol. I assume that firms can commit to a full wage schedule after each realization of the aggregate state, which is reflected in the promise-keeping constraint. Coles and Mortensen [2016] assume that workers form beliefs about the firm's productivity based on the current wage it offers. I further relax some of the restrictions they impose on their environment and allow for endogenous firm entry and exit.

I stress that there are multi-worker firms in the model due to the equal treatment constraint (it offers the same contract to all employees) and the recruitment technology (the convex recruitment cost). The firm's policies $V_t(p)$ and $h_t(p)$ are independent of firm size, but they

[8] The firm must deliver at least value $\overline{V}$ to satisfy the promise-keeping constraint. This constraint binds at the optimum.

determine the firm's quit rate $q_t(V_t(p))$ and gross hiring rate $h_t(p)$. $q_t(V_t(p))$ and $h_t(p)$ pin down the growth rate of employment at the firm $(1-\mu)(1-\delta_t)\big[1-q_t(V_t(p_t))+h_t(p_t)\big]$. Given some initial employment n_0, the firm's law of motion for employment (6) then allows me to keep track of its size. At the estimated parameters, the accumulation of firm-specific shocks generates a realistic firm-size distribution in the cross-section (see Section 5).

2.4 Joint firm-workers' surplus

The firm's problem can equivalently be expressed in terms of the discounted production surplus implied by its current employment level n_{t-1}:

$$\Pi_{t-1}\left(p_{t-1}, n_{t-1}, \overline{V}\right) + n_{t-1}\overline{V} = n_{t-1}J_{t-1}\left(p_{t-1}, \overline{V}\right) + n_{t-1}\overline{V} = n_{t-1}S_{t-1}(p_{t-1}).$$

The joint surplus between the firm and its workers is the sum of discounted firm profits $\Pi_{t-1}\left(p_{t-1}, n_{t-1}, \overline{V}\right)$ and the contract value promised to its current workers $n_{t-1}\overline{V}$. Result 1 shows that firm profits are independent of firm-size. The joint firm-workers' surplus can then be expressed as the product between firm-size n_{t-1} and the surplus of the firm and each individual worker $S_{t-1}(p_{t-1}) = J_{t-1}\left(p_{t-1}, \overline{V}\right) + \overline{V}$.

The firm's decision to continue or exit in period t can be expressed as a function of the firm-workers' surplus $S_t(p_t)$. The firm's present discounted value of profits is never negative in equilibrium. The wage contract can always be designed to deliver less than the value of unemployment U_t for this realization of the state, in which case the current value of firm profits is zero. The contract value to which the firm commits from period t onward is therefore greater than U_t if and only if $S_t(p_t) \geq U_t$. The firm's decision to continue can then be written $\chi_t(p_t) = \mathbb{1}\{S_t(p_t) \geq U_t\}$.

Solving for the wage in the promise-keeping constraint (10) and inserting it in the expression for firm profits (9) gives the following Bellman equation for the surplus of the firm and each individual worker

$$
\begin{aligned}
S_{t-1}(p_{t-1}) &= p_{t-1}\omega_{t-1} \\
&+ \beta \mathbf{E}_{t-1}\left\{\mu Q_t + (1-\mu)\left[\Big(\big(1-\chi_t(p_t)\big) + \delta_t \chi_t(p_t)\Big)U_t + \chi_t(p_t)(1-\delta_t)\Psi_t(p_t)\right]\right\}.
\end{aligned} \tag{11}
$$

The firm-workers' surplus consists of the flow value of production in the current period $p_{t-1}\omega_{t-1}$. In the next period, the worker becomes an entrepreneur with chance μ and has continuation value Q_t. Otherwise, the worker makes a transition to unemployment due to either firm exit $\big(1-\chi_t(p_t)\big)$ or an exogenous shock δ_t and has continuation value U_t. The firm's continuation value from this employment relationship is zero when the worker transitions to entrepreneurship or unemployment. In the event $(1-\mu)\chi_t(p_t)(1-\delta_t)$ in which the firm-workers' relationship continues, the firm maximizes the joint continuation value

$$
\begin{aligned}
&\Psi_t(p_t) = \\
&\max_{W,h\geq 0} -c(h) + \big(S_t(p_t) - W\big)h + \big(1-q_t(W)\big)S_t(p_t) + s\lambda_t \int \max\big\{\tilde{W}, W\big\} dF_t(\tilde{W}).
\end{aligned} \tag{12}
$$

The firm chooses the hiring rate h to maximize the value of hiring at cost $c(h)$ to get additional discounted profits $\big(S_t(p_t) - W\big)h$. The firm chooses the employment contract value W to maximize the joint value of retention $\big(1-q_t(W)\big)S_t(p_t)$ taking into account the loss in profits for new hires $\big(S_t(p_t) - W\big)h$ and the value to the worker of finding a better job $s\lambda_t \int \max\big\{\tilde{W}, W\big\} dF_t(\tilde{W})$. Equations (11) and (12) are derived in Appendix A.2.

The joint firm-workers' surplus representation follows from two assumptions: (i) the firm fully commits to the employment contract, and (ii) utility is transferable (workers and firms are risk neutral). If the firm decides to continue, its state-contingent contract $V_t(p_t)$ and hiring policy $h_t(p_t)$ solve the unconstrained problem (12) and can be expressed as functions of the joint firm-workers' surplus $S_t(p_t)$. The resulting state-contingent contract is consistent with the promise-keeping constraint (10). Given the state-contingent contract $V_t(p)$ solving (12) and given the value $\overline{V}$ the firm is committed to deliver from period $t-1$ onward, wages adjust to satisfy the promise-keeping constraint. This pins down flow wages w paid to workers.

2.5 Firm entry

New firms are created by entrepreneurs (workers subject to a μ-shock). Entrepreneurs draw an initial firm productivity from the distribution of business ideas Γ_0 and decide whether to enter given the current aggregate state in period t. Entrepreneurs who decide to enter get the full firm-workers' surplus: their business is purchased by some outside investors and they become the first workers at the new firms. Entrepreneurs' outside option is the value of unemployment: a μ-shock forces employed workers out of their current job with no recall option. The value of being an entrepreneur is given by

$$Q_t = \int \max\left\{S_t(\tilde{p}), U_t\right\} d\Gamma_0(\tilde{p}). \tag{13}$$

The entry process gives an initial employment $n_0 > 0$ to newborn firms, since entering entrepreneurs become the first workers at the newborn firms. There is no meaningful notion of a firm with employment zero in the model because the hiring technology $nc(h)$ requires positive employment. I normalize n_0 to unity: the measure of entering firms is equal to the measure of entrepreneurs deciding to enter.

3 Rank-monotonic equilibria

This section formalizes the definition of equilibrium used in the remainder of the paper. It makes the state variables subsumed in the time index t explicit. It also shows that, under some conditions on the cost of hiring function c, the optimal wage contract is increasing in the current realization of firm productivity p. This result is central to the tractability of the model since it yields a closed-form formula for the optimal wage contract.

3.1 Stationary equilibria

There is a notion of an equilibrium independent of calendar time t in the model. I label these equilibria "stationary." The aggregate state is given by (ω, M). By assumption, the aggregate shock ω satisfies the Markov property. The measure of firms in the productivity and employment-size space is then sufficient to compute all labor market aggregates. Given the aggregate state (ω, M), the firm can compute the acceptance rate (8) conditional on the optimal choice to continue χ, the optimal hiring choice h, and the optimal employment contract value V at all other firms. Given the acceptance rate (8), the optimal posting of ads a follows from the accounting relationship (7). Given firms' optimal posting of ads, the offer distribution follows from the aggregation of ads posted by firms offering different employment contract values (4).

Result 1 establishes that firms' optimal policies are independent of firm employment n. The employment-weighted measure of firm productivity

$$L_t(p) = \int_{\tilde{p} \leq p} \int_n n dM_t(\tilde{p}, n) \tag{14}$$

is therefore sufficient to compute the acceptance rate and offer distribution. Intuitively, it is sufficient to compute the acceptance rate to know the measure of workers employed at firms with productivity p, since these firms offer the same employment contract by the size-independence result.[9] The aggregate state is then equivalently given by (ω, L), where the measure L in the state-space is uni-dimensional.

Formally:

Definition 1 (Stationary equilibrium) *A stationary equilibrium is a triple of policy functions (χ, h, V) and a pair of value functions (S, U) that depend on the current realization of firm-specific productivity p, the current realization of aggregate productivity ω, and the employment-weighted measure of productivity L. Conditional on all firms following the policies given by (χ, h, V), these functions are such that:*

[9] This is established formally for the acceptance rate and offer distribution in Appendix A.3.

1. *The equations for the acceptance rate* (8)*, ads posting* (7)*, and contract offer distribution* (4) *hold with firms' optimal choices* $\chi_t(p) = \chi(p, \omega, L)$*,* $h_t(p) = h(p, \omega, L)$*, and* $V_t(p) = V(p, \omega, L)$*.*

2. *The contract and hiring functions solve the maximization problem* (12)*. The continuation decision is given by* $\chi(p, \omega, L) = \mathbb{1}\{S(p, \omega, L) \geq U(\omega, L)\}$*.*

3. S *and* U *solve, respectively,* (11)-(12) *and* (5)*.*

In Definition 1, a stationary equilibrium is a fixed-point: the firm's policies solve the Bellman equation defined by (11)-(12) taking the policies of other firms as given, and these policies coincide in equilibrium. Characterizing this fixed-point analytically is challenging in general. Solving for the optimal employment contract in the firm's problem (12) requires knowing the distribution of offered contracts, which itself depends on the distribution of employment contracts at all firms.

An analytical characterization of the equilibrium is critical to the numerical solution with aggregate shocks, as the employment-weighted distribution of firm productivity (14) evolves over time. Without such a characterization, a full numerical solution would require solving the fixed point problem described in Definition 1 taking into account the evolution of employment across firm productivity.

3.2 Rank-monotonic equilibria

In what follows, I focus on a subset of stationary equilibria for which an analytical characterization of the equilibrium is possible. I label this subset of stationary equilibria "rank-monotonic." A rank-monotonic equilibrium (RME) adds the following requirement to the optimal contract in a Stationary equilibrium:

Definition 2 (rank-monotonic equilibrium) *A rank-monotonic equilibrium is a stationary equilibrium in which the optimal contract* $V(p, \omega, L)$ *is weakly increasing in the firm's current realization of productivity* p *for all* ω *and* L*.*

Result 2 below is the key characterization result. It provides sufficient conditions on the cost of hiring function that guarantee that any stationary equilibrium is rank-monotonic.

Result 2 (sufficient conditions for RME) *Assume that the Markov process for firm-specific productivity satisfies first-order stochastic dominance and that the conditional distribution* Γ_p *is everywhere differentiable.*[10] *Assume that the hiring cost function is twice differentiable, increasing, and convex. Assume that the distribution of offered contracts* $F(., \omega, L)$ *is everywhere differentiable with respect to* p*. Then, for any stationary equilibrium:*

1. *The firm-workers' surplus defined by Equation* (11) *is differentiable and weakly increasing in* p*;*

2. *The equilibrium is rank-monotonic provided* $hc''(h)/c'(h) \geq 1$ *at all* $h \geq 0$*.*

The proof is in Appendix A.4.

The condition on the Markov process for firm-specific productivity in Result 2 requires a form of persistence to guarantee that the firm-workers' surplus (11) is increasing in p. This condition is satisfied by most productivity processes commonly used in the firm dynamics literature.

The condition on the cost function $hc''(h)/c'(h) \geq 1$ in Result 2 requires that the cost function have a high degree of convexity. More productive firms offer better contracts to limit quits only to the extent that hiring is sufficiently costly. With identical workers and no learning on the job, the model could potentially generate a large amount of churning at the top of the productivity distribution if employers can easily hire new workers. $hc''(h)/c'(h) \geq 1$ makes hiring costly enough for large h that more productive firms find it optimal to use the retention margin when choosing the rate at which employment changes.

The requirement that $F(., \omega, L)$ be everywhere differentiable rules out stationary equilibria in which the equilibrium contract offer distribution has mass points. Mass points in $F(., \omega, L)$

[10] First-order stochastic requires $\Gamma_p(.|p') \leq \Gamma_p(.|p)$ for $p' > p$ with strict inequality for some productivity level. The differentiability of Γ_p (with respect to both arguments) is required to ensure that the distribution of firm-specific productivity is smooth.

imply that some or all firms offer the same contract irrespective of their current firm-specific productivity p. The standard argument in random search models with on-the-job search and wage posting to rule out mass points is that firms at a mass point can increase profits by offering jobs paying slightly more, thus poaching workers from other firms [Burdett and Mortensen, 1998]. This argument does not directly translate to this framework because the quit and hiring margins are separately controlled by the firm, respectively through wage contracts and hiring effort. For instance, if all firms offer the value of unemployment at all realizations of the aggregate state, a firm that promises a marginally larger wage contract would not increase its discounted profits. Its hiring costs are the same, and its quit rate is unchanged. A sufficient condition to rule out this equilibrium is to assume that workers move when they are indifferent between two firms. In the case where all firms offer the value of unemployment, a firm that would promise a marginally larger wage contract decreases its quit rate from $s\lambda(\omega, L)$ to zero, and the equilibrium unravels.

Result 2 is not an existence statement, but a property of the optimal contract conditional on the existence of such a stationary equilibrium. This result is nonetheless important, since rank-monotonic equilibria are markedly more tractable. I now show that, in a rank-monotonic equilibrium, there is a closed-form expression for the optimal employment contract V as a function of the firm-workers' surplus (11) and value of unemployment (5).

3.3 Additional characterization of rank-monotonic equilibria

In a rank-monotonic equilibrium, the optimal contract is increasing in firm-specific productivity p. The entry decision of entrepreneurs and the exit decision of existing firms can therefore be summarized as a single firm-specific productivity threshold $p_E(\omega, L)$. $p_E(\omega, L)$ is implicitly defined as the firm-specific productivity p solving $S(p, \omega, L) = U(\omega, L)$.

The optimal employment contract can be expressed as a function of the firm-workers' surplus (11) and the value of unemployment (5) only:

Result 3 (employment contract in RME) *In a rank-monotonic equilibrium, the equilib-*

rium contract is given by

$$V(p,\omega,L) = \frac{uU(\omega,L) + \big(1-\delta(\omega)\big)s\int_{p_E(\omega,L)}^{p} S(\tilde{p},\omega,L)dL(\tilde{p})}{u + \big(1-\delta(\omega)\big)s\Big[L(p) - L\big(p_E(\omega,L)\big)\Big]}. \tag{15}$$

The proof is in Appendix A.5.

Result 3 reveals that the optimal contract is a weighted average between the value of unemployment and the firm-workers' surplus. The optimal contract (15) is reminiscent of the Nash-Bargaining solution used in many standard search and matching models, where the firm-workers' surplus is split between each party conditional on a constant, exogenously given bargaining weight [e.g., Mortensen and Pissarides, 1994]. In this model, the weights are fully endogenous: they evolve with the employment-weighted measure of firm productivity L over the business cycle. The rents that workers can extract from the joint firm-workers' surplus (11) are directly linked to on-the-job search. As the search intensity of employed workers s goes to zero, the optimal contract (15) is given by $V(p,\omega,L) = U(\omega,L)$. Employed workers get the value of unemployment.

The optimal hiring rate $h(p,\omega,L)$ follows directly from solving the firm's joint maximization problem (12). Given optimal contract $V(p,\omega,L)$, the optimal hiring rate follows from inverting the derivative of the cost function in the firm's first-order condition associated with the maximization problem (12):

$$c'\big(h(p,\omega,L)\big) = S(p,\omega,L) - V(p,\omega,L). \tag{16}$$

In a rank-monotonic equilibrium, the acceptance rate for a firm with current productivity p can be expressed as

$$Y\big(V(p,\omega,L),\omega,L\big) = \frac{u + \big(1-\delta(\omega)\big)s\Big[L(p) - L\big(p_E(\omega,L)\big)\Big]}{u + \big(1-\delta(\omega)\big)s\Big[L(\overline{p}) - L\big(p_E(\omega,L)\big)\Big]}. \tag{17}$$

Because contracts are increasing in p, all searching workers employed at firms with current productivity less than p accept a firm-p employment contract. The distribution of offered contracts can then be shown to simplify to

$$\lambda(\omega, L)F\big(V(p,\omega,L),\omega,L\big) = \int_{p_E(p,L)}^{p} \frac{\big(1-\delta(\omega)\big)h(\tilde{p},\omega,L)}{u + \big(1-\delta(\omega)\big)s\Big[L(\tilde{p}) - L\big(p_E(\omega,L)\big)\Big]} dL_t(\tilde{p}). \tag{18}$$

Equations (17) and (18) are derived in Appendix A.5.

The distribution of offered contracts in an RME (18) fully summarizes the evolution of employment at each level of firm-specific productivity p. Let L^P denote the measure of workers employed at firms with productivity of at most p at the production stage (end of period). Given the aggregate state $\big(\omega, L\big)$, the production stage measure of workers at firms with productivity of at most p is related to the start of period measure L by

$$\begin{aligned} L^P(p) =& \mu\Big[\Gamma_0(p) - \Gamma_0\big(p_E(\omega,L)\big)\Big] \\ &+ (1-\mu)\Big[L(p) - L\big(p_E(\omega,L)\big)\Big]\big(1-\delta(\omega)\big)\Big(1 - q\big(V(p,\omega,L),\omega,L\big)\Big) \\ &+ (1-\mu)u\lambda(\omega,L)F\big(V(p,\omega,L),\omega,L\big). \end{aligned} \tag{19}$$

The first term is the measure of entering entrepreneurs with an initial draw less than p. The second term is the measure of workers still employed after the realization of the shocks who do not find a job at a firm with productivity greater than p: $\Big(1 - q\big(V(p,\omega,L),\omega,L\big)\Big)$. The third term is the measure of unemployed workers who find a job at a firm with productivity of at most p.

In a rank-monotonic equilibrium, knowledge of the value functions S and U for all values of the aggregate state is sufficient to simulate the model. The optimal contract has a closed-form solution (15). The optimal hiring rate can be solved directly from the first-order condition (16). Given the firm's choice of hiring rate, the offer distribution (18) can be readily computed. The employment-weighted measure of firm productivity can be simulated forward using the aggregate law of motion (19). Finally, the distribution of firm-specific shocks Γ gives the employment-weighted measure of firm productivity at the start of the next period.

4 Worker reallocation in the model

This section illustrates the connection between the firm productivity ladder identified in the previous section and worker reallocation. I use simulations to compare the steady states implied by different types of shocks. I then contrast the notion of worker reallocation implied by the model with other models proposed in the literature.

4.1 Impact of aggregate shocks

In a rank-monotonic equilibrium, firms with a higher realization of productivity offer wage contracts with greater value, which implies that all job-to-job transitions are toward these firms. This is true for any realization of the aggregate shock provided the conditions in Result 2 hold. But the pace at which workers move to more productive firms varies with aggregate shocks. Similarly, the entry-exit threshold $p_E(\omega, L)$ also varies in response to these shocks.

I illustrate these mechanisms by analyzing the steady state of the model in response to several types of shocks in Figure 1. I consider two types of negative aggregate shocks commonly used in the macro labor literature: a decrease in aggregate productivity ω and an increase in the aggregate job destruction rate δ [Shimer, 2005, Moscarini and Postel-Vinay, 2016]. The figures are obtained for the model parameterized as described in Section 5.

A decrease in aggregate productivity ω has the following intuitive impact on the steady state of the model. It first increases the entry-exit productivity threshold ($p_E(\omega, L)$ moves to the right in Figure 1), since the surplus between a firm and its workers is lower at all p. For the same reason, the hiring rate $h(p)$ is lower at all p (Figure 1a). The rate at which workers are reallocated up the productivity ladder $q(p)$ also falls at all p (Figure 1b). There are fewer chances to make contact with a more productive firm and therefore to move up the ladder in this scenario. Figure 1c summarizes the impact of these changes in the hiring rate $h(p)$ and voluntary quit rate $q(p)$ on the employment-weighted distribution of productivity, where I introduce the notation $\bar{L}^P(p) = L^P(p)/L^P(\overline{p})$ for the normalized measure of workers below p. This distribution moves to the right with the higher entry-exit threshold, but the lower

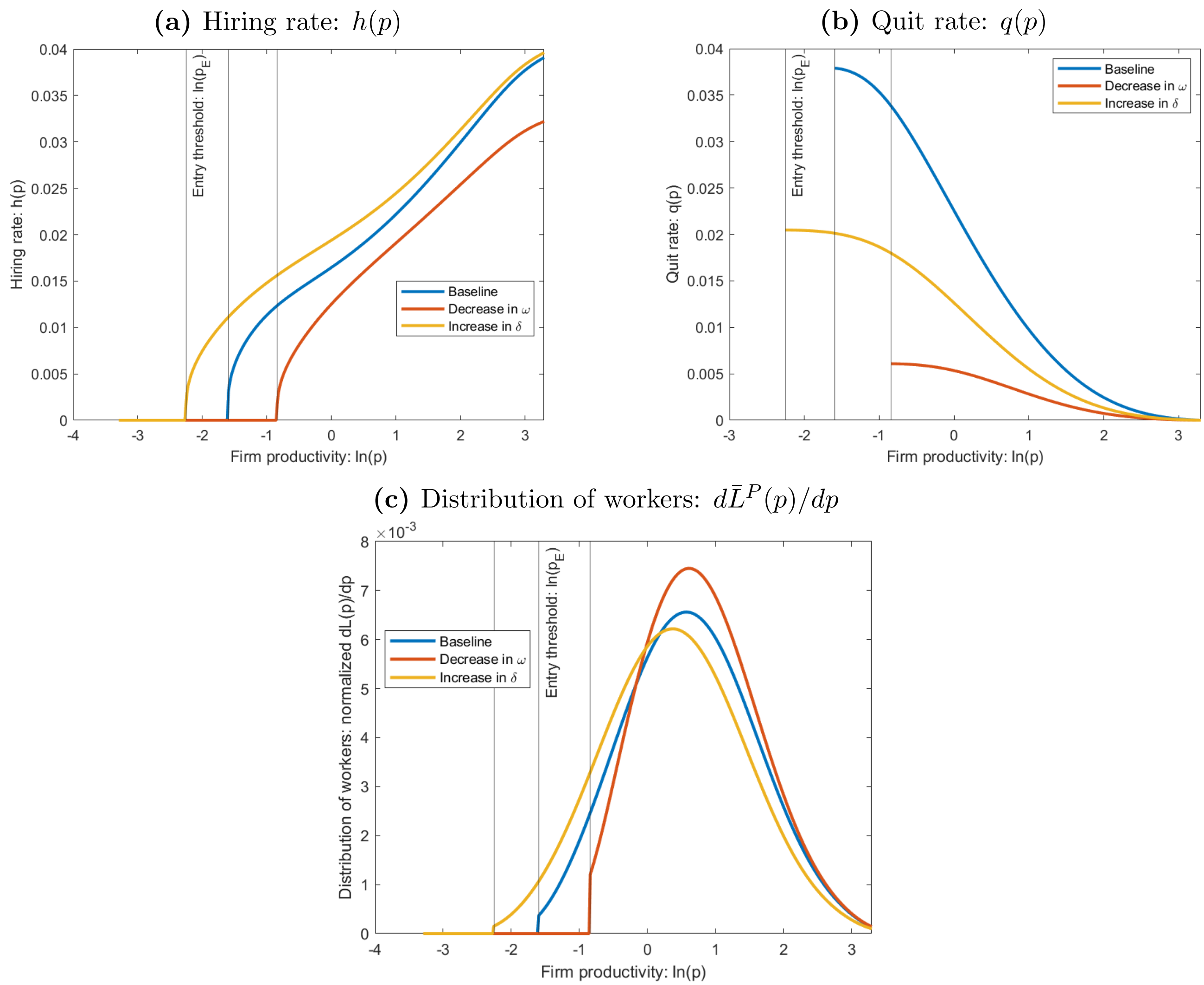


Figure 1: Equilibrium steady state in response to negative aggregate shocks: a decrease in aggregate productivity ω and an increase in the aggregate separation rate δ. Model simulations at the calibration described in Section 5. The vertical line denotes the entry-exit threshold in each scenario.

rate of reallocation up the ladder acts as a countervailing force. As a result, the mode of the distribution is roughly identical to the baseline steady state.

An increase in the aggregate rate of job destruction δ has the following less intuitive impact on the steady state of the model. A δ-shock directly affects how the surplus is split between the firm and its workers in the model. To gain some intuition, consider the direct impact of a change in δ on the optimal wage contract (15). It can be shown that $\partial V(p,\omega,L)/\partial\delta \leq 0$. The optimal wage contract specifies that the increased separation risk is, at least to some extent, passed on to workers. The overall impact on the firm-workers' surplus is ambiguous since a lower wage contract also potentially increases the discounted profits of the firm. In Figure 1, this lower wage contract decreases the entry-exit productivity threshold: less productive firms are viable relative to the baseline steady state equilibrium. Worker reallocation up the productivity ladder is lower than in the baseline ($q(p)$ is lower than in the baseline at all p in Figure 1b). The resulting distribution shifts to the left with both less productive firms surviving and less reallocation up the productivity ladder.

This comparison shows that alternative types of negative shocks potentially do not have the same impact on worker reallocation. In the quantitative part of the paper in Section 5, the calibration of these aggregate shocks is disciplined by the cyclical properties of key time series.

4.2 Summary statistics of the job ladder

The simulations in Figure 1 show that the employment-weighted distribution of firm productivity is useful to measure worker reallocation in response to shocks in the model. In what follows, I use the employment-weighted average of firm (log) productivity

$$\text{Worker Ladder}_t = \int_{p_{E,t}}^{\overline{p}} \ln(\tilde{p}) d\bar{L}_t^P(\tilde{p}), \tag{20}$$

to summarize the location of workers along the job ladder in period t (the mean of the distribution shown in Figure 1c).[11] I define the ladder climbed by workers in terms of the logarithm

[11] I switch back to subsuming the aggregate states in t for concision.

of productivity (a monotonic transformation) in line with the measure I construct in the data below.

Further insights into changes in the job ladder can be gained by decomposing the employment-weighted average (20) into an unweighted average term (Firm Ladder$_t$) and an interaction term (Ladder Interaction$_t$), a decomposition associated with Olley and Pakes [1996] in the productivity literature. In the model, the unweighted average term of this decomposition is given by

$$\text{Firm Ladder}_t = \int_{p_{E,t}}^{\overline{p}} \ln(\tilde{p}) d\bar{K}_t^P(\tilde{p}), \tag{21}$$

where I introduce the notation $\bar{K}_t^P$ for the distribution of firm productivity, the unweighted counterpart to $\bar{L}_t^P$. Equation (21) is the average (log) firm productivity in the economy and summarizes the support of the ladder on which workers move. The second term of this decomposition is given by

$$\text{Ladder Interaction}_t = \int_{p_{E,t}}^{\overline{p}} \ln(\tilde{p}) d\bar{L}_t^P(\tilde{p}) - \int_{p_{E,t}}^{\overline{p}} \ln(\tilde{p}) d\bar{K}_t^P(\tilde{p}). \tag{22}$$

It summarizes the relative location of the weighted and unweighted distributions of firm productivity, and therefore, how high up the ladder workers are, given the current support of the firm productivity ladder.

The summary of the location of workers on the productivity ladder Worker Ladder$_t$ and its decomposition

$$\text{Worker Ladder}_t = \text{Firm Ladder}_t + \text{Ladder Interaction}_t \tag{23}$$

are a key object of interest in the analysis. In Section 6, I quantify the evolution of each term over the business cycle within the fully calibrated model.

4.3 Relation to data on firm productivity

The summary measure of the job ladder (20) is closely related to the following index of labor productivity

$$\mathrm{LP}_t = \int_{p_{E,t}}^{\overline{p}} \ln\left(\frac{\omega_t p n}{n}\right) d\bar{L}_t^P(p) = \ln(\omega_t) + \int_{p_{E,t}}^{\overline{p}} \ln(p) d\bar{L}_t^P(p) = \ln(\omega_t) + \text{Worker Ladder}_t, \quad (24)$$

where $\ln\left(\frac{\omega_t p n}{n}\right)$ is labor productivity at the firm level in the model. I now provide empirical evidence on the evolution of this index during a large recession. I require data on labor productivity (a proxy to $\omega_t p$ in the model) and employment (a proxy to n) at the firm level for a representative sample of firms.

I combine several administrative data sets from the UK [Office for National Statistics, 2019, 2020, 2021] to obtain these measures for a large sample of British firms every year between 1997 and 2018. Details on the construction of this data set are relegated to Appendix B.1. Importantly for the paper's focus on business cycle fluctuations, these data cover several years before and after the Great Recession (2008q2-2009q3 in the UK).

I construct an empirical counterpart to the employment-weighted average of firm productivity in Equation (24) as

$$\mathrm{LP}_t = \sum_i \mathrm{ES}_{i,t} \cdot \mathrm{LP}_{i,t}, \quad \mathrm{ES}_{i,t} = \frac{\text{employment}_{i,t}}{\sum_i \text{employment}_{i,t}}, \quad \mathrm{LP}_{i,t} = \ln\left(\frac{\text{value added}_{i,t}}{\text{employment}_{i,t}}\right),$$

where i denotes a firm and t denotes a year.[12] Since this index is also a weighted average, it can similarly be decomposed into an unweighted average and an interaction term as

$$\mathrm{LP}_t = \sum_i \mathrm{ES}_{i,t} \cdot \mathrm{LP}_{i,t} = \overline{\mathrm{LP}}_t + \sum_i \left(\mathrm{ES}_{i,t} - \overline{\mathrm{ES}}_t\right)\left(\mathrm{LP}_{i,t} - \overline{\mathrm{LP}}_t\right), \quad (25)$$

where $\overline{\mathrm{ES}}_t$ and $\overline{\mathrm{LP}}_t$ denote the corresponding unweighted averages.[13]

The evolution of each of the terms in Equation (25) is depicted in Figure 2, expressed in

[12] This definition is similar to the definition of labor productivity in Bartelsman et al. [2013].

[13] This equality follows directly from expanding the second term and noting that $\sum_i \mathrm{ES}_{i,t} = 1$ by definition.

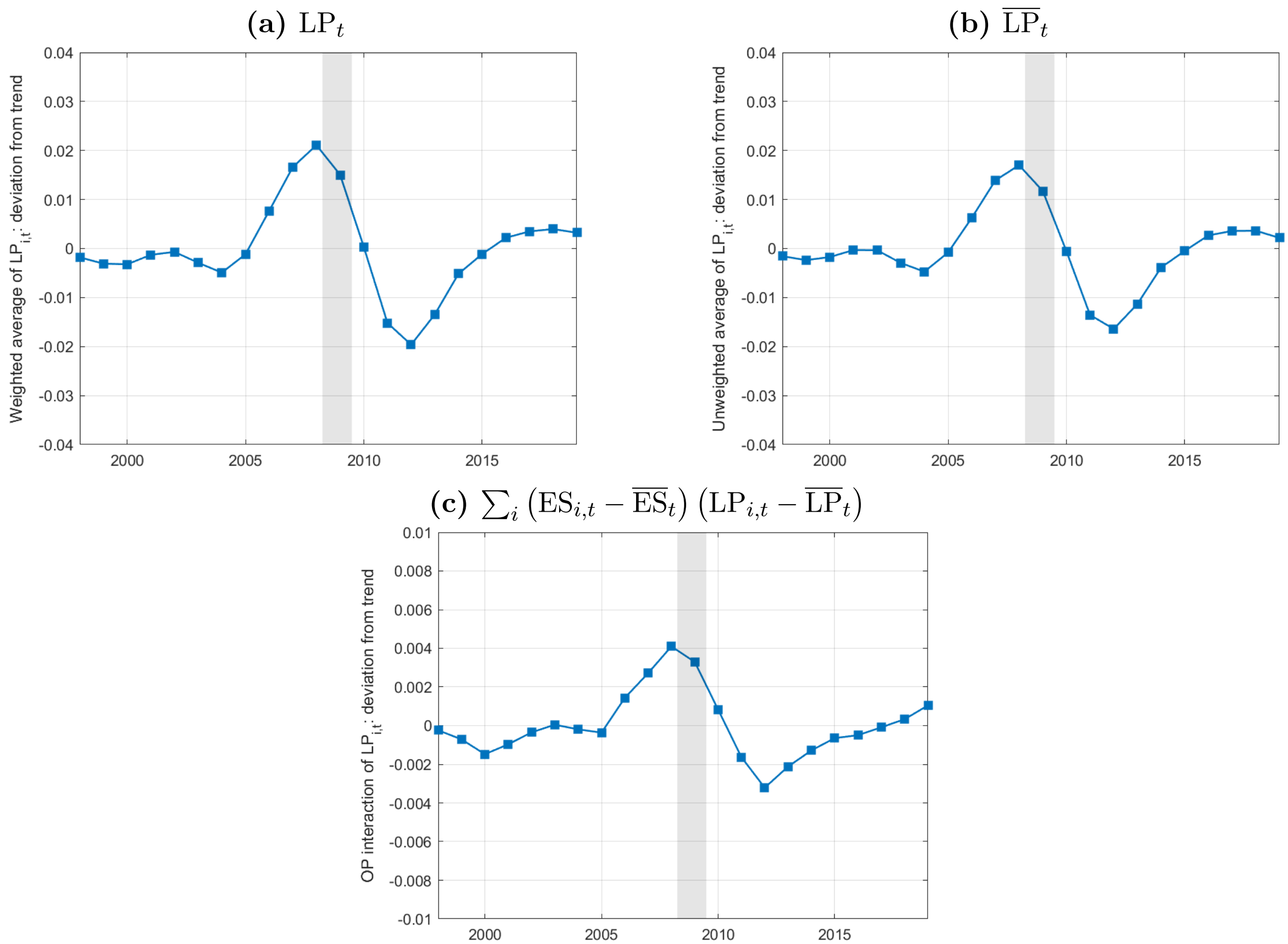


Figure 2: Labor productivity decomposition (25). Series are shown in deviation from a HP-filter trend with smoothing parameter 100. Gray band denotes the Great Recession period in the UK.

deviation from trend. The figure shows that both the employment-weighted average labor productivity (Figure 2a) and each of its components (Figures 2b-2c) decrease markedly during the recession and slowly recover. Around 20 percent of the overall decrease in employment-weighted productivity can be accounted for by the interaction term $\sum_i \left(\mathrm{ES}_{i,t} - \overline{\mathrm{ES}}_t\right)\left(\mathrm{LP}_{i,t} - \overline{\mathrm{LP}}_t\right)$.

Figure 2 is evidence that workers are reallocated along the firm-productivity ladder during an economic downturn. This is because the model implies the following mapping between the data decomposition in (25) and the structural decomposition of the job ladder implied by the model

$$
\begin{aligned}
\ln(\omega_t) + \text{Worker Ladder}_t &= \text{LP}_t, \\
\ln(\omega_t) + \text{Firm Ladder}_t &= \overline{\text{LP}}_t, \\
\text{Ladder Interaction}_t &= \sum_i \left(\text{ES}_{i,t} - \overline{\text{ES}}_t\right)\left(\text{LP}_{i,t} - \overline{\text{LP}}_t\right),
\end{aligned} \tag{26}
$$

which follows directly from the expression for LP_t in the model (24). Since the aggregate shock ω_t is unobserved, the degree of implied worker reallocation can only be quantified within the calibrated model, as done in Section 6.

4.4 Worker reallocation in alternative search models

Worker reallocation as defined in this paper is only meaningful in the subset of models of the labor market where (i) workers have preferences over alternative jobs, and (ii) workers move across these jobs over time. These models need to imply a well-defined notion of the share of workers in "good" jobs. The question of worker reallocation is moot in models where the labor market consists of a single type of job at each point of the business cycle, such as the standard search-and-matching model with homogeneous jobs [Shimer, 2005].

Within the class of models with a well-defined notion of worker reallocation, workers' preferences over jobs depend on the specific features of each framework. In the model introduced in this paper, workers prefer firms with larger realizations of productivity, since these firms offer wage contracts with larger value. This mapping from worker preferences over jobs to firm productivity is common to a large set of random search models with on-the-job search. The Burdett and Mortensen [1998] model in a steady state environment, its extension to an environment with aggregate shocks in Moscarini and Postel-Vinay [2013] and Coles and Mortensen [2016], and the bargaining framework with on-the-job search in Postel-Vinay and Robin [2002] and Cahuc et al. [2006], all imply a job ladder in terms of firm productivity.

As mentioned in the introduction, the contemporaneous contributions by Elsby and Gottfries [2022] and Bilal et al. [2022] propose frictional labor market environments with on-the-job search where firms operate a decreasing returns to scale production technology. These models imply a notion of worker reallocation conceptually similar to that described in this paper, since worker preferences over jobs can also be mapped into firm-specific characteristics. In Elsby and Gottfries [2022], firms with a higher marginal product of labor offer better jobs and are able to poach workers from other firms in equilibrium. In Bilal et al. [2022], the equilibrium job ladder is in terms of the marginal surplus of a firm and its workers. An expression similar to (20) could be adapted to the job ladders implied by each of these environments to study worker reallocation in the presence of aggregate shocks.

I stress that worker reallocation as defined in this paper is a positive description of the location of workers along the job ladder over the business cycle. It is related to, but separate from determining the degree of misallocation in this environment. Misallocation implies that an efficient allocation can be derived conditional on the primitives of the model.[14] This efficient allocation represents a tradeoff between the reallocation of workers along the productivity ladder and the recruitment cost incurred by firms. While a full characterization of the planner's problem in this model is beyond the scope of the paper, I explore this tradeoff using numerical simulations in Appendix D.2. Perhaps surprisingly, this analysis suggests that, from the planner's perspective, improving on the allocation implied by a rank-monotonic equilibrium is not straightforward.

5 Quantitative analysis

In this section, I calibrate the model using a mix of micro level data on workers and firms and aggregate time series. I begin by describing how the model is solved numerically in the presence of aggregate shocks. I then give details on the parameterization and the corresponding moment targets, before assessing the fit of the model to the business cycle.

[14]Bilal et al. [2022] analyze a notion of misallocation defined as the frictionless limit and derive the implications for the productivity of this economy. This is distinct from the notion of worker reallocation in response to aggregate shocks defined here. To the best of my knowledge, there are no theoretical results on the planner's problem for random search models with on-the-job search.

5.1 Solution method

The size-independence result (Result 1) and the rank-monotonic equilibrium result (Result 2) simplify the firm's problem. Given these results, the relevant state-space is $\big(p, \omega, L\big)$. But since the employment-weighted measure of firm productivity L is an infinitely dimensional object evolving with aggregate shocks, this state-space still represents a challenge for the numerical solution of the model. I then proceed in two steps.

The first step is trivial. I shut down aggregate shocks ($\omega_t = \omega$ for all t) and solve for the corresponding steady state RME. This equilibrium is such that the firm's policies imply a constant measure of workers L at all productivity levels given the law of motion for employment (19).[15] Details of the corresponding solution algorithm can be found in Appendix C.1.

In the second step, I reintroduce aggregate shocks into the model. An expanding literature building on Reiter [2009] proposes to solve heterogeneous agent models by linearizing around the steady state.[16] But my simulations suggest that taking a derivative around the steady state is highly inaccurate in the context of my model due to the discontinuity implied by the firm's endogenous entry and exit threshold. I therefore rely on a simulation-based approach that adapts ideas from Krusell and Smith [1998] to my setting. Since this solution method is distinct from the original paper, I briefly outline the key ideas here. The details are relegated to Appendix C.2.

The solution method relies on the following two approximations. First, the measure of workers L_t is summarized by a vector $\boldsymbol{m}_t$ of moments. This vector includes the unemployment rate, $m_t^0 = u_t = 1 - L_t(\overline{p})$, and N_m moments $\{m_t^1, \ldots, m_t^{N_m}\}$ from the distribution of workers $L_t/L_t(\overline{p})$. With this approximation of L_t, the aggregate state-space relevant to the firm is now $\big(\omega_t, \boldsymbol{m}_t\big)$. The second approximation is to parameterize the value of unemployment (5) and the firm-workers' surplus (11) out of the steady state with a flexible polynomial in $\big(\omega_t, \boldsymbol{m}_t\big)$.

Given these approximations, the procedure works as follows. Draw a sequence of aggregate shocks $\{\omega_t\}_{t=1}^{T}$. Conditional on a guess for the coefficients in the polynomial, the law of motion

[15] I do not have a proof that this equilibrium exists and is unique. Numerically, I have checked that the algorithm converges to the same solution with alternative initial conditions for a large number of parameters.

[16] See, among others, Sedláček and Sterk [2017] and Winberry [2021] for applications to firm dynamics models.

for employment (6) can be solved forward in time for this sequence of shocks. Conditional on the law of employment along the sequence of shocks, the value of unemployment (5) and the firm-workers' surplus (11) can be solved backward in time. The coefficients in the polynomial can then be updated by running a regression of the simulated value functions on the simulated aggregate states. The solution algorithm proceeds by iterating on these steps until the coefficients in the polynomial converge.

Relative to the solution method proposed in Krusell and Smith [1998], my approach forecasts the agents' value function conditional on the realization of the states. Solving for the agents' value functions in my framework requires knowledge of the full employment-weighted measure of firm productivity, not just the forecast of an aggregate variable, such as the capital stock. This measure is needed to compute the offer distribution aggregating up the recruitment of all firms (18), and it is simulated directly as part of the iterations forward in time. Relative to linearization-based solutions, the main advantage of this approach is that it is robust to the kink in the agents' value functions implied by endogenous entry and exit. Its main disadvantage is that it is computationally more intensive.

I report several robustness checks for the proposed solution algorithm in the appendix. First, I implement a version of the accuracy test described in den Haan [2010] and show that the procedure performs well according to this metric (Appendix C.3). Second, I experiment with alternative numbers of moments N_m to summarize the measure of workers L_t (Appendix C.4) and justify my choice of $N_m = 2$ used for the results reported in the paper. Third, I benchmark my proposed solution algorithm to a linearization approach in a version of the model without endogenous entry and exit, and therefore without a kink in the agents' decision problem and find that the two solutions yield very similar results (Appendix D.3).[17]

5.2 Calibration strategy

The model is solved under the following parametric assumptions. The functional form for the cost of hiring function is guided by the conditions derived in Result 2. I assume that the

[17] I thank an anonymous referee for suggesting this test.

per-worker cost to the firm of hiring at rate h is given by

$$c(h) = \frac{c_0 h^{c_1+1}}{c_1+1}, \tag{27}$$

which satisfies the condition in Result 2 provided $c_1 \geq 1$. I ensure that this condition is satisfied when searching over the parameter space. I specify the Markov process for the firm-specific productivity shock (p_t) as

$$\ln p_t = \rho_p \ln p_{t-1} + \sigma_p \varepsilon_t^p, \qquad \varepsilon_t^p \sim \mathcal{N}(0,1). \tag{28}$$

The process for firm-specific productivity shocks (28) satisfies first-order stochastic dominance conditional on its past realizations, which is required for the equilibrium to be rank-monotonic (Result 2).[18] I also assume that draws from Γ_0, the productivity distribution of new entrants, come from the stationary distribution implied by (28). The Markov process for the aggregate shock (ω_t) is specified as

$$\ln \omega_t = \rho_\omega \ln \omega_{t-1} + \sigma_\omega \varepsilon_t^\omega, \qquad \varepsilon_t^\omega \sim \mathcal{N}(0,1). \tag{29}$$

To improve on the business cycle properties of the model, I allow some parameters to co-move with the aggregate shock. Specifically, I let the separation rate δ and the scale of the hiring cost function c_0 depend on ω_t. I specify this dependence as

$$\ln \delta(\omega_t) - \ln \delta = \epsilon_{\delta,\omega} \ln(\omega_t),$$
$$\ln c_0(\omega_t) - \ln c_0 = \epsilon_{c_0,\omega} \ln(\omega_t),$$

where the parameters $\left(\epsilon_{\delta,\omega}, \epsilon_{c_0,\omega}\right)$ control the response to aggregate shocks.

I calibrate the model by targeting a number of moments from the data.[19] All parameters

[18] I assume in Section 2 that $p_t \in \left[\underline{p}, \overline{p}\right]$ and $\omega_t \in \left[\underline{\omega}, \overline{\omega}\right]$. This is the case in practice given that (28) and (29) are discretized.

[19] See Appendix B for a full list of data sources and details on the construction of the variables underlying these moments.

Parameter		Value	Moment (Data source)	Data	Model
A. Externally set					
β	Discount factor	0.996	5 percent annually	—	—
n_0	Size of entrants	1	Normalization	—	—
B. Steady-state parameters					
δ	Separation rate	4.1E-04	Avg. EU_t (BHPS)	0.003	0.005
c_0	Hiring cost:	4.9E+06	Avg. UE_t (BHPS)	0.058	0.050
c_1	$c(h) = (c_1 + 1)^{-1} c_0 h^{c_1+1}$	3.026	Reg. $\Delta \ln n_{i,t+1}$ on $\text{LP}_{i,t}$ (mBSD)	0.136	0.121
s	Relative search effort	0.787	Avg. EE_t (BHPS)	0.016	0.016
μ	Prob. of start-up shock	6.91E-04	Avg. firm $n_{i,t}$ (ARD)	12.1	10.3
b	Unemployment flow value	1.182	Firm exit rate (mBSD)	0.130	0.087
ρ_p	Autocor. AR1 $\ln p_t$	0.962	Autocor. of $\ln n_{i,t}$ (mBSD)	0.949	0.989
σ_p	Std. of AR1 $\ln p_t$	0.301	IQR of $\text{LP}_{i,t}$ (ARD)	1.129	1.070
			Pareto tail of empl. size (ARD)	1.066	1.029
			Cov($\text{ES}_{i,t}$,$\text{LP}_{i,t}$) (ARD)	1.3E-07	1.6E-07
C. Employment cost moments					
			IQR of $\text{EC}_{i,t}$ (ARD)	1.352	0.778
			Reg. $\text{EC}_{i,t}$ on $\text{LP}_{i,t}$ (mBSD)	0.704	0.619
			Reg. $\Delta \ln n_{i,t+1}$ on $\text{EC}_{i,t}$ (mBSD)	0.131	0.154
D. Aggregate shock parameters					
ρ_ω	Autocor. of AR1 $\ln \omega_t$	0.843	Autocor. of $\widehat{\text{LP}}_t$ (ARD)	0.998	0.996
σ_ω	Std. of AR1 $\ln \omega_t$	7.7E-04	Std. of $\widehat{\text{LP}}_t$ (ARD)	0.010	0.008
$\epsilon_{\delta,\omega}$	Response c_0 to ω	-450.5	Std. of $\widehat{\text{EU}}_t$ (BHPS)	2.7E-04	2.9E-04
$\epsilon_{c_0,\omega}$	Response δ to ω	-51.3	Std. of $\widehat{\text{UE}}_t$ (BHPS)	2.3E-03	1.6E-03

Table 1: Calibrated parameters and targeted moments.

and targeted moments are reported in Table 1. I distinguish three groups of parameters in the calibration: externally set parameters, steady state parameters, and aggregate shock parameters. In discussing these parameters and my choice of calibration targets, I heuristically link each parameter to a specific moment, but all parameters and moments are related in the actual model.

Externally set parameters A small subset of parameters are set externally. A period t is set to a month. The discount factor β is set in line with a 5 percent annual discount rate. The size of new entrants is normalized to $n_0 = 1$.

Steady-state parameters A second set of parameters are calibrated based on simulations from the model at the steady state targeting long-run moments. Long-run moments are derived from the pooled data for the period 1997-2018. My choice of moment targets reflects both the worker mobility component and the firm dynamics component of the model.

On the worker side, I compute the unemployment to employment (UE_t), employment to unemployment (EU_t), and job-to-job (EE_t) monthly transition rates in the UK from the British Household Panel Survey (BHPS). This data set is available from 1992. These series are derived following the methodology described in Postel-Vinay and Sepahsalari [2019].[20] I use the average of these series over the period 1997-2018 to calibrate, respectively, the scale of the hiring cost c_0, the exogenous separation rate δ, and the search intensity of employed workers $s \leq 1$ relative to unemployed workers. The calibrated relative search intensity parameter is higher than the values typically obtained for this class of models based on US data. This result reflects the large average value of the EE_t transition rate relative to the UE_t transition rate in the BHPS (respectively 0.016 and 0.058 on average) compared to the US (respectively 0.02 and 0.21 in the Survey of Income and Program Participation).

On the firm side, I use the firm-level administrative data introduced in Section 4 to compute several cross-sectional moments on firm employment, labor productivity, and labor cost [Office for National Statistics, 2019, 2021, 2020]. I have access to these data for the period 1997-2018. The Annual Respondents Database (ARD) and its successor the Annual Business Survey (ABS) provide yearly firm balance-sheet data for a large repeated cross-section of firms. To compute dynamics at the firm-level, I merge these data sets with the Business Structure Database (BSD), which is a yearly panel of the near-universe of firms in Britain but does not have information on their balance-sheets. I refer to this merged sample as "mBSD" in Table 1. The model counterpart to these firm-level data is obtained by simulating a large panel of firms at the steady state.

The firm-level data are used to discipline the remaining steady state parameters. The exponent in the cost of hiring function c_1, which can be interpreted as the elasticity of the hiring rate to the firm's discounted profits in the model, is pinned down by the coefficient of a regression of firm-level employment growth $\Delta_{12} \ln n_{i,t+12}$ on $\text{LP}_{i,t}$, where the timing reflects the

[20] Additional details can be found in Appendix B.2.

yearly frequency of the data. The probability of starting a firm μ is calibrated to target the average employment of firms $n_{i,t}$ in the data, since μ controls the relative measure of workers to firms conditional on the normalization for n_0. The flow value of unemployment b is disciplined by the annual rate of firm exit in the data. This parameter shifts the value of unemployment, so it is related to endogenous exits in the model. The process for firm-specific productivity (28) is calibrated to match the autocorrelation of firm employment $\ln n_{i,t}$ (12 months apart because the data are yearly) and the inter-quartile range (IQR) of labor productivity $\text{LP}_{i,t}$.[21] All data targets are computed by pooling the data across available years in line with the steady state model.

Since the index of aggregate productivity $\text{LP}_t = \sum_i \text{ES}_{i,t} \cdot \text{LP}_{i,t}$ is a key object of interest to quantify the dynamics of the job ladder, I add more moment targets to ensure that the model provides a consistent micro-foundation for this summary statistic. The dispersion of $\text{LP}_{i,t}$ is already targeted through the inter-quartile range. I also target the Pareto tail of the employment-size distribution to make sure that the model implies a realistic distribution of firm sizes and therefore of the employment shares $\text{ES}_{i,t}$. A long tail in the firm-size distribution arises in the model because the process describing the evolution of firm employment is (i) independent of the firm's current employment (Result 1), and (ii) a birth-death process because of firm entry and exit. These two conditions on the process underlying firm employment are precisely those identified by the literature on the emergence of power law distributions in economics [Gabaix, 1999, Reed, 2001].[22] Finally, I include the covariance between $\text{ES}_{i,t}$ and $\text{LP}_{i,t}$ to summarize the joint distribution between employment shares and labor productivity.

In Table 1, I also report on several non-targeted moments on the cost of employment (Part C in Table 1). I use the payroll expenditure variable in the firm balance-sheet data (ABS/ARD) and define a firm-level measure of employment cost $\text{EC}_{i,t}$ similarly to labor productivity $\text{LP}_{i,t}$

[21] I target the autocorrelation of employment because I can only construct labor productivity in the balance-sheet data, which is a repeated cross-section.

[22] See Gouin-Bonenfant [2019] for a similar discussion in the context of a related search model.

as

$$\text{EC}_{i,t} = \ln\left(\frac{\text{payroll expenditures}_{i,t}}{\text{employment}_{i,t}}\right).$$

I use this measure to benchmark the firm-level wages implied by the contract-posting assumption in the model. With contract-posting, wages are backed out from the promise-keeping constraint (10) evaluated at the optimal RME contract (15). I find that the model implies a realistic degree of association between EC_{it} and $\text{LP}_{i,t}$, as measured by a univariate regression. Similarly, it also matches the slope of a regression of employment growth $\Delta_{12} \ln n_{i,t+12}$ on EC_{it}. Finally, the model captures about 60 percent of the IQR of EC_{it} found in the data.

Aggregate shock parameters The last set of parameters is calibrated based on simulations of the model with aggregate shocks targeting moments in deviation from trend. Since solving the model with aggregate shocks is computationally intensive, I calibrate these parameters conditional on the steady state parameters obtained in the previous step.

I use the same combination of worker flow and productivity data to construct time-series in deviation from trend. These time series are de-trended in three steps to harmonize data with different frequencies and isolate business cycle fluctuations. I first average all series to yearly frequency. I then de-trend these series using an HP filter with smoothing parameter 100. I finally interpolate the resulting cyclical components to a monthly frequency.[23] This last interpolation step allows me to fit aggregate shocks month-by-month to replicate some key time series within the calibrated model, as I describe below. Another advantage of this approach is to isolate business cycle fluctuations from short run monthly and quarterly changes, which are likely due to measurement error.

I pin down the parameters of the aggregate shock process (29) by targeting the autocorrelation and standard deviation of $\widehat{\text{LP}}_t$, where a hat denotes a variable in deviation from trend. The parameters indexing the response of δ and c_0 to the aggregate shock ω are calibrated to target, respectively, the standard deviations of $\widehat{\text{EU}}_t$ and $\widehat{\text{UE}}_t$.

[23] I use spline interpolation to obtain smooth series.

5.3 Fit to business cycle

To assess the business cycle properties of the model, I obtain two sequences of ω-shocks to fit two key time series. I first target $\widehat{\mathrm{LP}}_t$. This series is aggregated up from the micro data used to calibrate the model. As such, it represents my preferred option for finding a sequence of ω-shocks matching the UK business cycle. But a limitation of this time series is that the underlying firm-level data are not available before 1997. I therefore also fit an alternative sequence of shocks going back to 1955 by instead targeting the de-trended cyclical component of log-GDP. In what follows, I refer to the corresponding sequences of shocks as the "LP shocks" and "GDP shocks."

Figure 3 shows the fit to the labor productivity index decomposition given in Equation (25) (left column). By construction, the model exactly replicates the evolution of employment-weighted firm productivity $\mathrm{LP}_{i,t}$, since it is targeted to find the shocks (Figure 3a). The model also matches the evolution of the unweighted average term (Figure 3c) and the interaction term (Figure 3e) of the decomposition, which are not targeted directly as part of the calibration above. Because of the tight connection between the job ladder implied by the model and the decomposition of LP_t (see Equation 26), it is reassuring that the model does well along this specific dimension. The right column of Figure 3 also shows the fit of the model to the same decomposition as in Equation (25), but for the employment cost $\mathrm{EC}_{i,t}$:

$$\mathrm{EC}_t = \sum_i \mathrm{ES}_{i,t} \cdot \mathrm{EC}_{i,t} = \overline{\mathrm{EC}}_t + \sum_i \left(\mathrm{ES}_{i,t} - \overline{\mathrm{ES}}_t\right)\left(\mathrm{EC}_{i,t} - \overline{\mathrm{EC}}_t\right).$$

The model also does well at matching this decomposition, although it tends to slightly lead the data. Overall, the fit reported in Figure 3 suggests that the reallocation of workers along the productivity ladder implied by the model during the Great Recession is consistent with the firm-level data, using both a measure of labor productivity and a measure of employment cost per worker.

Figure 4 shows the model fit to post-war recessions in Britain. For this exercise, I use the "GDP shocks" constructed by fitting the de-trended series for log-GDP going back to the

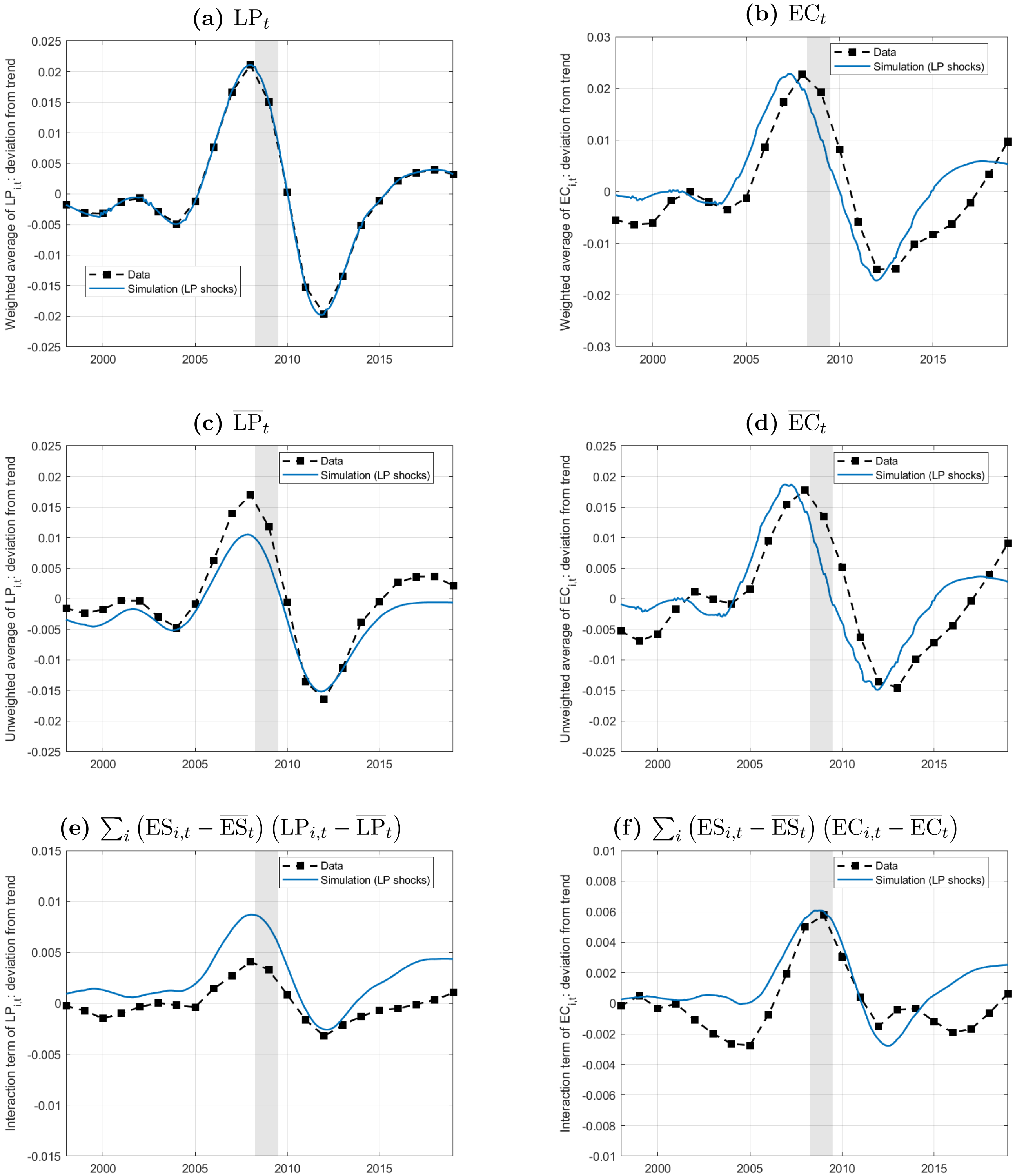


Figure 3: Fit to LP_t and EC_t decompositions. Gray bands denote the Great Recession period in Britain.

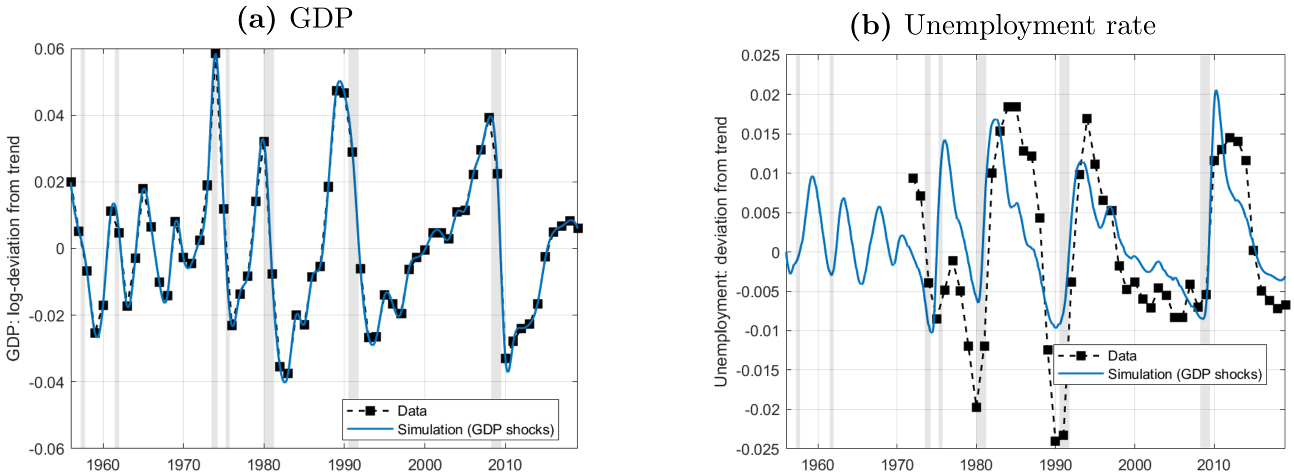


Figure 4: Post-war UK business cycle. Gray bands denote recession periods in Britain.

1950s, as shown in Figure 4a. The model matches the de-trended unemployment rate relatively well (Figure 4b), though it does not fully replicate the size of the recession spikes in the 1980s and 1990s. It does well at replicating the path of de-trended unemployment during the Great Recession, the period for which the steady state is calibrated.

Figure 5 shows the fit to de-trended worker flows, which are only available from the 1990s. For completeness, I report model simulations for both the shocks fitted to $\widehat{\mathrm{LP}}_t$ ("LP shocks") and the shocks fitted to $\widehat{\ln \mathrm{GDP}}_t$ ("GDP shocks"). As shown in Figure 5a, the two series of shocks imply a different timing and size of the drop in GDP.[24] Since the labor market flows come from another data set (BHPS), I report both simulated series. The model does well at replicating the de-trended path of UE_t and EU_t (Figures 5b and 5c). Up to the difference in timing, the "LP shocks" in particular replicate the magnitude of the change in these series around the time of the Great Recession. The model does less well at replicating the path of EE_t (Figure 5d). The simulated drop in the EE transition rate around the time of the Great Recession is around 50 percent smaller than in the data. This is not perfect, but there is also some uncertainty about the exact evolution of the worker flow rates in this period. Unfortunately, the recession period corresponds with a redesign of the worker panel used to compute these transition rates

[24]The recession periods are defined in terms of consecutive quarters of negative output, so they match fluctuations in GDP by construction. See Appendix B.4.

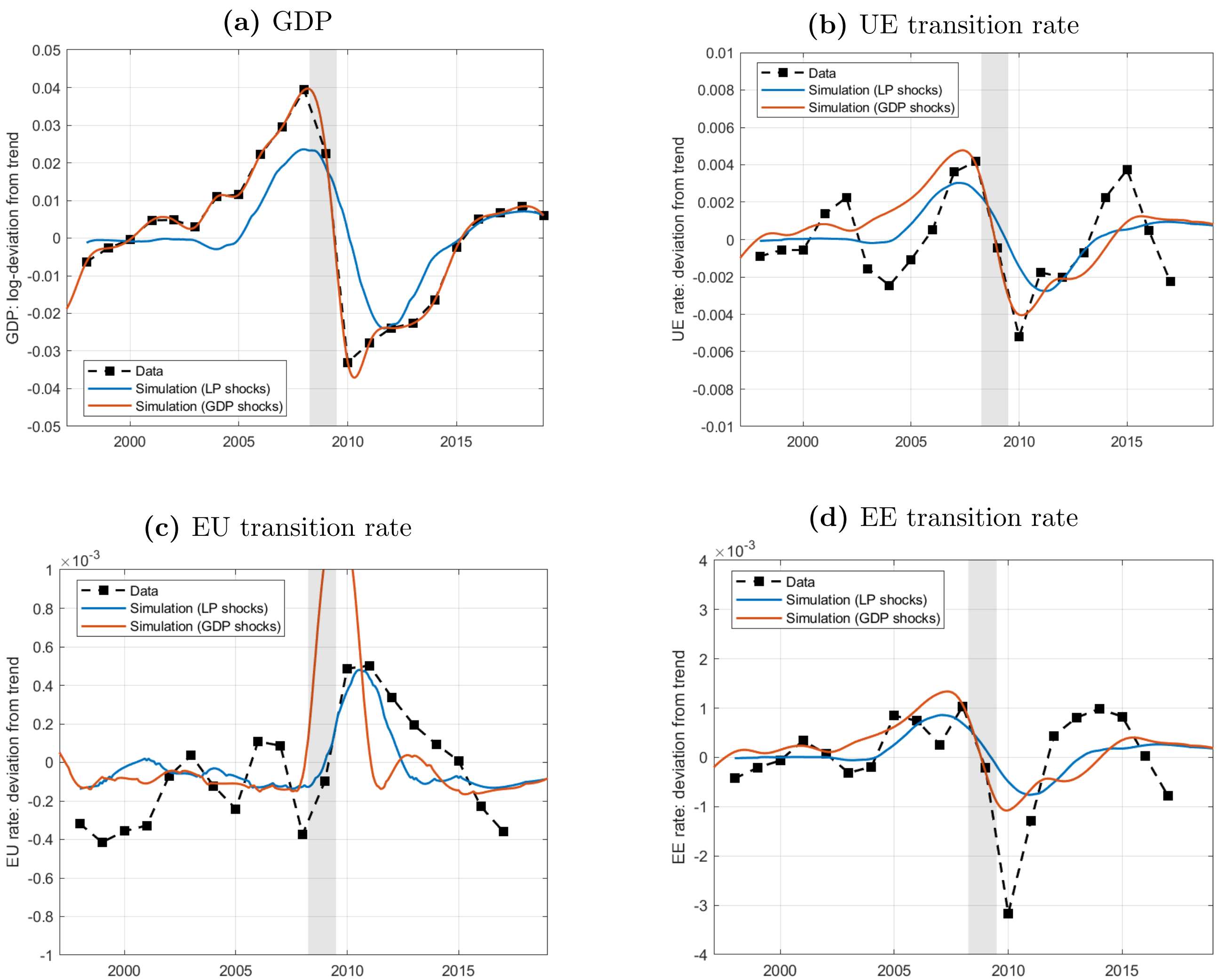


Figure 5: UK business cycle around the Great Recession.

and the data are missing between 2008m8 and 2009m12, so the exact size of the drop at this point in time is not well measured.[25]

6 Worker reallocation over the business cycle

Having shown that the calibrated model does well along a number of dimensions relevant to worker reallocation along the job ladder, I now use the model to quantify the drivers of worker reallocation over the business cycle. In a second step, I study this decomposition in a

[25]See Appendix B.2 for additional details on the construction of these series.

counterfactual environment with countercyclical unemployment benefits.

6.1 Structural decomposition

Figure 6 plots the terms in the structural decomposition (23) obtained by simulating the model around the time of the Great Recession. Recall that the Worker Ladder summary statistic of the location of workers along the firm productivity ladder can be decomposed into two components: a Firm Ladder component, which sums up the current productivity distribution of active firms, and a Ladder Interaction component, which captures the position of workers along the productivity ladder relative to the current productivity of firms.

Figure 6a first shows the evolution of the Worker Ladder term relative to the shocks obtained by fitting the time series $\widehat{\mathrm{LP}}_t$ (Figure 3a). The shocks are an order of magnitude smaller than the Worker Ladder measure. Through the lens of the calibrated model, most of the cyclical variation in LP_t during this recessionary episode can be interpreted as worker reallocation along the productivity ladder.

Figure 6b shows the evolution of each term separately. The Ladder Interaction term drops following the recession, as suggested by the evolution of its direct empirical counterpart $\sum_i \left(\mathrm{ES}_{i,t} - \overline{\mathrm{ES}}_t\right)\left(\mathrm{LP}_{i,t} - \overline{\mathrm{LP}}_t\right)$. In the model, there is less progression up the productivity ladder during a recession, since there is a larger pool of unemployed workers, which in turn implies that the opportunities to make contact with more productive firms drop for workers on the lower productivity rungs of the ladder. The Firm Ladder term also decreases, since the simulated recession moves the entry threshold down at the calibrated parameters through the mechanisms discussed in Section 4. In short, the optimal contract (15) internalizes the impact of an increase in the aggregate job destruction rate. This decrease in the optimal contract acts as a countervailing force to the fall in the discounted firm-workers' surplus implied by negative productivity shocks. At the calibrated parameters, this pushes the distribution of firm productivity down, which accounts for the fall in the Firm Ladder component. The reduction in this component accounts for around two-thirds of the fall in the Worker Ladder term.

(a) Shock vs Worker Ladder **(b)** Firm Ladder vs Ladder Interaction

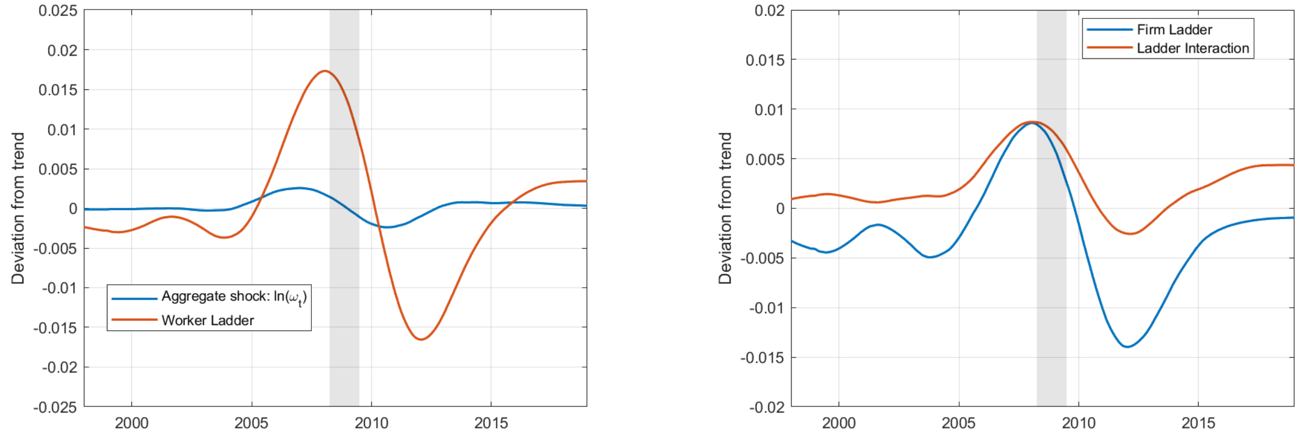


Figure 6: Decomposition of Worker Ladder simulated from the LP shocks (Figure 3a). The gray band denotes the Great Recession period in Britain.

Var(ln(ω_t) + Worker Ladder$_t$)		Var(Firm Ladder$_t$ + Ladder Interact.$_t$)	
Component	Share	Component	Share
Var(ln(ω_t))	0.029	Var(Firm Ladder$_t$)	0.384
Var(Worker Ladder$_t$)	0.820	Var(Ladder Interaction$_t$)	0.163
2 · Cov(ln(ω_t), Worker Ladder$_t$)	0.151	2 · Cov(Firm Ladder$_t$, Ladder Interaction$_t$)	0.453

Table 2: Variance decomposition of the worker ladder over the post-war business cycle. The model simulation is obtained using the GDP shocks (Figure 4a).

I also study the same structural decomposition over the post-war business cycle. In this exercise, I use the aggregate shocks obtained from fitting the evolution of log-GDP in deviation from trend (Figure 4a), since the shocks fitted to $\widehat{\text{LP}}_t$ only start in 1998.

Table 2 reports several variance decompositions. I first confirm that the model implies a small role for shocks in accounting for the fluctuations in LP$_t$. More than 80 percent of these fluctuations are ascribed to worker reallocation along the job ladder, as measured by the Worker Ladder term (left panel). In turn, the variance in the Worker Ladder term is primarily accounted for by changes in the Firm Ladder term and the co-movement between the Firm Ladder term and the Ladder Interaction term. The variance of the Ladder Interaction term on its own only makes up for around 15 percent of the overall variance of worker reallocation.

Finally, I use the calibrated model to quantify the evolution of each term in the structural

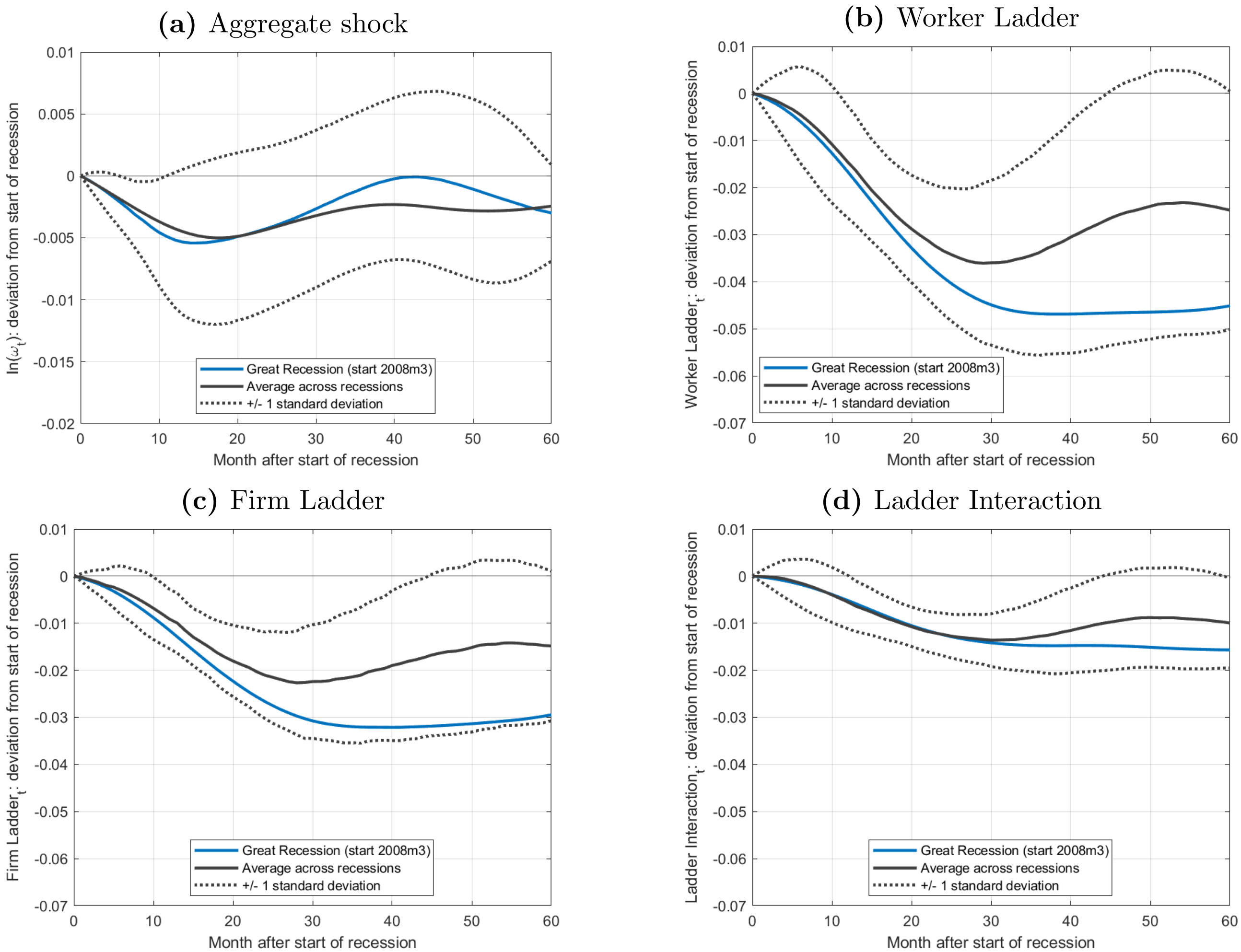


Figure 7: Worker Ladder decomposition across post-war recessions in Britain.

decomposition across post-war recessions. In Figure 7, I use the start date of each recession as a starting point and compute the evolution of the Worker Ladder, Firm Ladder, and Ladder Interaction terms for five years after that date. All series are given in deviation from their value at the start of the recession, and I report the average and a one standard-deviation band across recessions. For comparison, I also show the series specifically for the Great Recession. The takeaway from Figure 7 is that the impact of recessions on the job ladder is persistent relative to the shock. This exercise also confirms the importance of the Firm Ladder relative to the Ladder Interaction term in driving worker reallocation along the job ladder after a recession, similarly to the Great Recession pattern shown in Figure 6. In terms of worker reallocation along the job ladder, the Great Recession is interpreted as large, but not unusually large.

6.2 Counterfactual experiment: Contingent unemployment benefits

I consider a counterfactual where I make the flow value of unemployment countercyclical. I implement this counterfactual by assuming that the flow value of unemployment is given by

$$\ln b(\omega) - \ln b = \epsilon_{b,\omega} \ln(\omega),$$

where the parameter $\epsilon_{b,\omega} \leq 0$ indexes the dependence of b to the shocks ω. The baseline calibration with constant b corresponds to the case $\epsilon_{b,\omega} = 0$.

The rationale behind this extension is twofold. First, this counterfactual is in the spirit of the countercyclical unemployment insurance extensions that are implemented in the US when the labor market deteriorates.[26] Second, it introduces an additional degree of rigidity in the optimal wage contract (15), since this contract depends on b through the value of unemployment. While a micro-foundation for these rigidities in this environment is beyond the scope of this paper, this exercise gives some preliminary insights into the effect of additional wage rigidities on the reallocation of workers along the job ladder.

I experiment with two alternative values of $\epsilon_{b,\omega} \in \{-100, -50\}$. These values are chosen to yield an increase in the flow value of unemployment commensurate with the increase in the maximum duration of unemployment benefits following a typical US recession.[27] I solve the model again for these values of $\epsilon_{b,\omega}$ and study the Worker Ladder decomposition in response to the "GDP shocks."

Table 3 reports the variance decomposition of the Worker Ladder term in each counterfactual scenario. Countercyclical unemployment benefits appear to reduce the overall variance of the Worker Ladder over the business cycle. This variance is lower in the two counterfactual scenarios considered relative to the baseline. This overall decrease masks an increase in the variance

[26]The actual policy indexes the duration of unemployment benefits contingent on the current unemployment rate. I focus on a reduced-form implementation to avoid the need to introduce additional state variables to formally capture the eligibility of workers for unemployment benefits. See Rujiwattanapong [2019] for a model that fully captures the unemployment insurance extension program.

[27]The maximum duration of unemployment benefits typically doubles after a recession in the US over the post-war period [Rujiwattanapong, 2019, Figure 2].

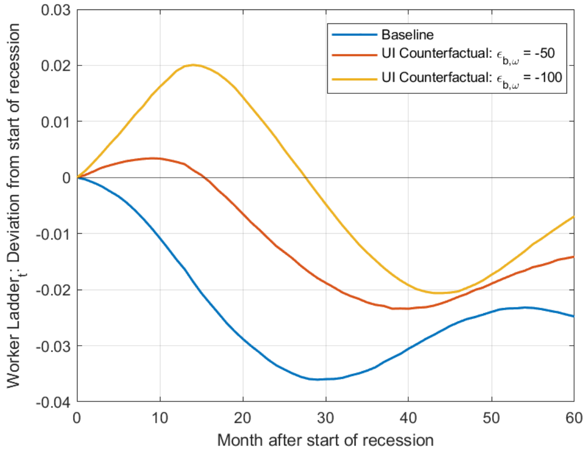
	Baseline	$\epsilon_{b,\omega} = -50$	$\epsilon_{b,\omega} = -100$
Var(Worker Ladder$_t$)	2.5E-04	1.2E-04	2.1E-04
Share of Var(Worker Ladder$_t$):			
Var(Firm Ladder$_t$)	0.384	1.041	3.130
Var(Ladder Interaction$_t$)	0.163	1.339	1.622
$2 \cdot$ Cov(Firm Ladder$_t$, Ladder Interaction$_t$)	0.453	-1.380	-3.752

Table 3: Variance decomposition of the Worker Ladder over the post-war business cycle in counterfactual with countercyclical unemployment benefits. Model simulations are obtained using the GDP shocks (Figure 4a).

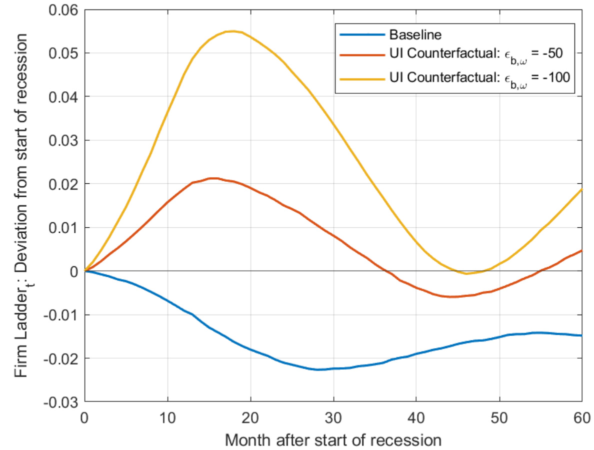
of the Firm Ladder term and Ladder Interaction terms, which are negatively correlated with countercyclical benefits instead of positively correlated in the baseline. This pattern is confirmed in Figure 8, which displays the average of these terms across post-war recessions in the baseline and in the countercyclical unemployment benefit scenarios. In contrast with the baseline model, the Firm Ladder term responds positively during a recession in these scenarios. There are fewer low productivity firms as the value of unemployment (and therefore the outside option of workers) increases with larger unemployment benefits. The drop in the ladder interaction term is also larger than in the baseline model. The slowdown in the relocation of workers up the productivity ladder is amplified with countercyclical unemployment benefits. Job-to-job transitions are lower at low-productivity firms in this counterfactual relative to the baseline, since (i) there are more unemployed workers and therefore the job-finding rate λ_t decreases, and (ii) all firms cut recruitment efforts since the optimal wage contract is larger.

Overall, this counterfactual analysis suggests that countercyclical unemployment benefits involve a tradeoff in terms of worker reallocation along the job ladder. They both increase the Firm Ladder term by driving out firms at the lower end of the productivity distribution and decrease the Ladder Interaction term by further slowing down the pace of worker reallocation. The reduction in the Ladder Interaction term following a recession remains a feature of both the baseline model and the model with countercyclical benefits.

(a) Worker Ladder



(b) Firm Ladder



(c) Ladder Interaction

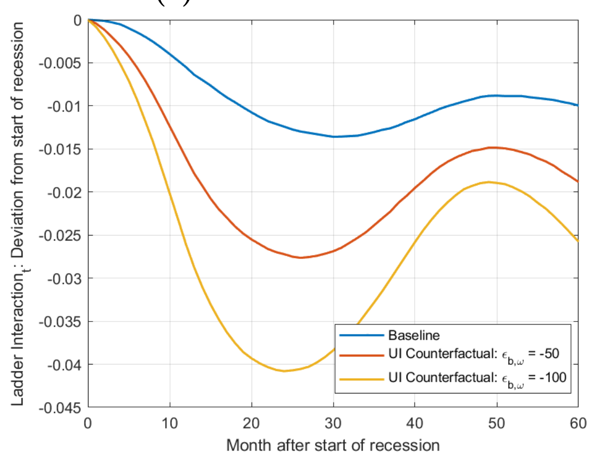


Figure 8: Decomposition of Worker Ladder in counterfactual scenario across post-war recessions in Britain. Each line gives the average across recessions.

7 Conclusion

I develop a random search model with three key features: (i) on-the-job search, (ii) firm dynamics, and (iii) aggregate shocks. Tractability is retained in this rich environment by identifying a set of conditions on the cost of hiring function such that agents' decisions can be expressed as a function of firm-specific productivity, aggregate productivity, and the employment-weighted distribution of firm productivity. The optimal wage contract offered by firms admits a closed-form solution, so wages are straightforward to compute. Using these results, I propose a numerical solution method suitable to this environment with endogenous firm entry-exit and aggregate shocks. In the quantitative part of the paper, I fit the model to data on the cross-section of firms and the cyclicality of labor productivity and labor flows. I then use the calibrated model to quantify the drivers of workers reallocation along the job ladder over the recent business cycle in Britain.

The model presented in this paper has maintained the common assumption in the literature that wages flexibly adjust in response to shocks. An interesting direction for future research would be to study environments with a similarly rich degree of firm heterogeneity, but where some form of rigidity is imposed on wages. Though retaining tractability in such an environment with aggregate shocks is likely to be challenging, it would further enlarge the set of empirical regularities that can be studied within this class of models.

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Appendix

A Omitted derivations and proofs

A.1 Size-independent firm profits

I guess and verify that a solution to the Bellman equation for firm profits (9) has the form

$$\Pi_{t-1}\left(p_{t-1}, n_{t-1}, \overline{V}\right) = n_{t-1} J_{t-1}\left(p_{t-1}, \overline{V}\right).$$

The objective is to show that

$$\Pi_t(p_t, n_t, W) = n_t J_t(p_t, W) \implies \Pi_{t-1}\left(p_{t-1}, n_{t-1}, \overline{V}\right) = n_{t-1} J_{t-1}\left(p_{t-1}, \overline{V}\right).$$

Starting from the equation for firm profits (9), still subject to the law of motion for employment (6) and the promise-keeping constraint (10), the term inside the expectation on the right-hand side rewrites

$$\begin{aligned}
& -c(h)(1-\mu)(1-\delta_t)n_{t-1} + J_t(p_t, n_t, W) \\
&\quad = -c(h)(1-\mu)(1-\delta_t)n_{t-1} + n_t J_t(p_t, W) \\
&\quad = n_{t-1}(1-\mu)(1-\delta_t)\Big[-c(h) + (1 - q_t(W) + h)J_t(p_t, W)\Big]
\end{aligned}$$

for some decision to continue χ, choice of contract value W, and hiring rate h. The second line substitutes in the guess $n_t J_t(p_t, W)$. The last line uses the law of motion for the firm's workforce. Using this last expression in firm profits (9) gives

$$\begin{aligned}
\Pi_{t-1}\left(p_{t-1}, n_{t-1}, \overline{V}\right) = n_{t-1} \max_{w, W, \chi, h \geq 0} & \Bigg\{ \omega_{t-1} p_{t-1} - w \\
& + \beta \mathbf{E}_{t-1}\Big[\chi \cdot \Big((1-\mu)(1-\delta_t)\big[-c(h) + (1 - q_t(W) + h)J_t(p_t, W)\big]\Big)\Big]\Bigg\}.
\end{aligned}$$

It follows that

$$\Pi_{t-1}\left(p_{t-1}, n_{t-1}, \overline{V}\right) = n_{t-1} J_{t-1}\left(p_{t-1}, \overline{V}\right)$$

with

$$J_{t-1}\left(p_{t-1}, \overline{V}\right) = \max_{w,W,\chi,h\geq 0}\Bigg\{\omega_{t-1}p_{t-1} - w \\ + \beta\mathbf{E}_{t-1}\Bigg[\chi\cdot\Big((1-\mu)(1-\delta_t)\big[-c(h) + (1-q_t(W)+h)J_t(p_t,W)\big]\Big)\Bigg]\Bigg\}. \quad (30)$$

This last expression represents a per worker formulation to the firm's problem, still subject to the promise-keeping constraint (10). The corresponding optimal choices for the continuation decision $\chi_t(p_t)$, contract $V_t(p)$, and hiring rate $h_t(p)$ are all independent of the firm's employment size n_{t-1}.

A.2 Firm-workers' match surplus

Solving the promise-keeping constraint (10) for w conditional on some state-contingent contract W and the decision to continue χ gives

$$w = \overline{V} - \beta\mathbf{E}_{t-1}\Bigg\{\mu Q_t + (1-\mu)\Bigg[\Big((1-\chi)+\delta_t\chi\Big)U_t \\ + \chi\cdot(1-\delta_t)\Big(\big(1-q_t(W)\big)W + s\lambda_t\int\max\big\{\tilde{W}, W\big\}dF_t(\tilde{W})\Big)\Bigg]\Bigg\}.$$

Starting from the definition of $S_{t-1}(p_{t-1})$ and substituting w above in the expression for firm profit per worker (30) gives

$$
\begin{aligned}
S_{t-1}(p_{t-1}) &= J_{t-1}(p_{t-1}, \overline{V}) + \overline{V} \\
&= -\overline{V} + \max_{W, h \geq 0, \chi} \Bigg\{ p_{t-1}\omega_{t-1} \\
&\quad + \beta \mathbf{E}_{t-1}\Bigg[\mu Q_t + (1-\mu)\Bigg(\big(1-\chi\big) U_t + \chi \delta_t U_t \\
&\qquad + \chi \cdot (1-\delta_t)\Big[\big(1-q_t(W)\big)W + s\lambda_t \int \max\big\{\tilde{W}, W\big\} dF_t(\tilde{W}) \\
&\qquad - c(h) + \big(1 - q_t(W) + h\big)\pi_t(p_t, W)\Big]\Bigg)\Bigg]\Bigg\} + \overline{V}.
\end{aligned}
$$

Using the fact that the continuation decision can be expressed as a function of the joint firm-workers' surplus $\chi_t(p_t) = \mathbb{1}\{S_t(p_t) \geq U_t\}$, taking the max operator inside the expectation, and grouping terms gives

$$
\begin{aligned}
S_{t-1}(p_{t-1}) = p_{t-1}\omega_{t-1} + \beta \mathbf{E}_{t-1}\Bigg[\mu Q_t + (1-\mu)\Bigg(&\big(1 - \chi_t(p_t)\big)U_t + \chi_t(p_t)\delta_t U_t \\
+ \chi_t(p_t)(1-\delta_t) \max_{W, h \geq 0} \Big\{ &- c(h) + \big(1 - q_t(W)\big)S_t(p_t) + \big(S_t(p_t) - W\big)h \\
&+ (1-\delta_t)s\lambda_t \int \max\big\{\tilde{W}, W\big\} dF_t(\tilde{W}) \Big\}\Bigg)\Bigg].
\end{aligned}
$$

A.3 Labor market aggregates with size independence

I give the formal steps to confirm the intuition that the employment-weighted distribution (14) allows firms to compute the acceptance rate (8) and offer distribution (4). With a slight abuse of notation, the employment-weighted distribution of firm productivity (14) can be written

$$
L_t(p) = \int_{\tilde{p} \leq p} \int_n n dM_t(\tilde{p}, n), \qquad dL_t(p) = \int_n n dM_t(p, n),
$$

and the unemployment rate

$$u_t = 1 - \int_p \int_n n dM_t(p,n) = 1 - \int_p dL_t(p).$$

Given Result 1, we can immediately check that the numerator of the acceptance rate (8) can be written

$$\begin{aligned} u_t + (1-\delta_t)s \int_p \int_n & \mathbb{1}\{V_t(p) \leq W_t\}\chi_t(p) n dM_t(p,n) \\ &= u_t + (1-\delta_t)s \int_p \mathbb{1}\{V_t(p) \leq W_t\}\chi_t(p) \int_n n dM_t(p,n) \\ &= u_t + (1-\delta_t)s \int_p \mathbb{1}\{V_t(p) \leq W_t\}\chi_t(p) dL_t(p). \end{aligned}$$

A similar derivation can be used for the denominator of the acceptance (8) to show that

$$Y_t(W_t) = \frac{u_t + (1-\delta_t)s \int \mathbb{1}\{V_t(p) \leq W_t\}\chi_t(p) dL_t(p)}{u_t + (1-\delta_t)s \int \chi_t(p) dL_t(p)}. \tag{31}$$

Turning to the contract offer distribution, it can be checked from the ads posting condition (7) that posted ads $a_t(p,n)$ are linear in n:

$$a_t(p,n) = \frac{h_t(p)}{\eta_t Y_t(V_t(p))}(1-\mu)(1-\delta_t)n.$$

We can then immediately check that the numerator of the offer distribution (4) can be written

$$\begin{aligned} &\int_p \int_n \mathbb{1}\left\{V_t(p) \leq W\right\} \chi_t(p) a_t(p,n) dM_t(p,n) \\ &\qquad = \frac{(1-\mu)(1-\delta_t)}{\eta_t} \int_p \mathbb{1}\left\{V_t(p) \leq W\right\} \chi_t(p) \frac{h_t(p)}{Y_t(V_t(p))} dL_t(p). \end{aligned}$$

Because the constant terms in front cancel out, a similar derivation for the denominator of the offer distribution (4) gives

$$F_t(W) = \frac{\int \mathbb{1}\left\{V_t(p) \leq W\right\} \chi_t(p) \cdot h_t(p)/Y_t\big(V_t(p)\big) dL_t(p)}{\int \chi_t(p) \cdot h_t(p)/Y_t\big(V_t(p)\big) dL_t(p)}.$$

A.4 Proof of rank-monotonic equilibrium

The proof is similar in spirit to the ones in Moscarini and Postel-Vinay [2013, 2016]. The goal is to show that the optimal contract is increasing in the firm's current realization of productivity p assuming the existence of a stationary equilibrium (Definition 1). Throughout, I subsume the aggregate state $\big(\omega, L\big)$ in the subscript t for concision.

The key difference with Moscarini and Postel-Vinay [2013, 2016] is that the firm's problem can be considered separately for each worker. By Result 1, profits are independent of firm-size, and the firm's problem can be expressed solely in terms of the individual firm-workers' surplus (11)-(12). There is therefore no need to account for the effect of contract value on the firm's own size. Moscarini and Postel-Vinay [2013, 2016] establish super-modularity so that more productive firms are offering larger employment contracts and are larger in size on the equilibrium path. In this model, the firm's optimal policies are independent of size, so the argument in the proof is simpler.

I establish the two following statements:

1. Conditional on S_t being increasing in p, $hc''(h)/c'(h) \geq 1$ for all $h \geq 0$ is sufficient to guarantee that the optimal contract V_t is increasing in p;

2. The operator implicitly defined by the firm-workers' surplus (11) maps differentiable and increasing functions of p into differentiable and increasing functions of p.

Sufficient conditions on c for an RME Assume that S_t is increasing in p. Conditional on the firm surviving, the firm's unconstrained problem (12) is given by

$$\Psi_t(p) = \max_{\substack{W \\ h \geq 0}} -c(h) + \big(1 - q_t(W)\big)S_t(p) + h\big(S_t(p) - W\big) + s\lambda_t \int_W^\infty \tilde{W} dF_t(\tilde{W}).$$

At any interior maximum, the firm's optimal choice $\big(h_t, V_t\big)$ must satisfy the following first-order conditions

$$\begin{aligned} 0 &= -c'\big(h_t(p)\big) + S_t(p) - V_t(p) \\ 0 &= -q_t'\big(V_t(p)\big)\big(S_t(p) - V_t(p)\big) - h_t(p) \end{aligned}$$

where I have implicitly used the assumption that F_t is everywhere differentiable to write $q_t' = -s\lambda_t F_t'$. In addition, the associated Hessian matrix H_t^Ψ must be negative-definite, which requires

$$\det\big(H_t^\Psi\big) = -c''\big(h_t(p)\big)\Big[q_t''\big(V_t(p)\big)\Big(S_t(p) - V_t(p)\Big) + q_t'\big(V_t(p)\big)\Big] - 1 > 0.$$

The two first-order conditions can be combined to give the following expression in $V_t(p)$

$$-c'\Big(- q_t'\big(V_t(p)\big)\big(S_t(p) - V_t(p)\big)\Big) + S_t(p) - V_t(p) = 0$$

and totally differentiating that last expression with respect to p gives

$$\frac{dV_t(p)}{dp} = \frac{\partial S_t(p)}{\partial p} \cdot \frac{-q_t'\big(V_t(p)\big)c''\big(h_t(p)\big) - 1}{\det(H_t^\Psi)}.$$

In this last expression, $\det\big(H_t^\Psi\big)$ is positive at any maximum. By assumption, the firm-workers' surplus (11) is increasing in p, so $\partial S_t(p)/\partial p \geq 0$. Noting that the two FOCs can be combined to give $-q_t'(V_t(p))c'(h_t(p)) = h_t(p)$, it follows that

$$\frac{dV_t(p)}{dp} \geq 0 \iff -q_t'\big(V_t(p)\big)c''\big(h_t(p)\big) - 1 \geq 0 \iff \frac{h_t(p)c''\big(h_t(p)\big)}{c'\big(h_t(p)\big)} \geq 1.$$

Firm-workers' surplus increasing in p Assume that S_t is differentiable and increasing in p_t. We want to show that

$$\begin{aligned} S_{t-1}(p_{t-1}) &= p_{t-1}\omega_{t-1} \\ &\quad + \beta \mathbf{E}_{t-1}\Bigg\{\mu Q_t + (1-\mu)\bigg[\Big(\big(1-\chi_t(p_t)\big) + \delta_t \chi_t(p_t)\Big)U_t + \chi_t(p_t)(1-\delta_t)\Psi_t(p_t)\bigg]\Bigg\}. \end{aligned}$$

is differentiable and increasing in p_{t-1}, where again Ψ_t denotes the firm's maximization problem

$$\Psi_t(p_t) = \max_{\substack{W \\ h\geq 0}} -c(h) + \big(1 - q_t(W)\big)S_t(p_t) + h\big(S_t(p_t) - W\big) + s\lambda_t \int_W^\infty \tilde{W} dF_t(\tilde{W}).$$

Differentiability of S_{t-1} in p_{t-1} follows directly from noting that the expectation in this last expression is differentiable in p_{t-1} as long as the conditional probability density of future productivity is. This is true by assumption.

To show that S_{t-1} is increasing conditional on $\partial S_t(p_t)/\partial p_t \geq 0$, first note that the envelope condition of the firm's optimization problem (12) gives

$$\frac{d\Psi_t(p_t)}{dp_t} = \frac{\partial \Psi_t(p_t)}{\partial p_t} = \Big(1 - q_t\big(V_t(p_t)\big) + h_t(p_t)\Big)\frac{\partial S_t(p_t)}{\partial p_t} \geq 0, \tag{32}$$

where $\big(V_t(p_t), h_t(p_t)\big)$ denote the optimal policies in the firm's optimization problem. The term inside the expectation in the firm-workers' surplus is then weakly increasing in p. It is weakly increasing in p by the envelope condition (32) on the part of the support of p where the firm continues. It is constant on the part of the support of p where the firm exits. Because the decision to continue can be written in terms of the firm-workers' surplus $\chi_t(p_t) = \mathbb{1}\big\{S_t(p_t) \geq U_t\big\}$, there exists a unique exit productivity threshold in the support of $\big[\underline{p}, \overline{p}\big]$.

To complete the proof, the assumption that the Markov process for firm-specific productivity satisfies first-order stochastic dominance is required. With this assumption, conditional on any two distinct previous realizations of p, the conditional densities of future idiosyncratic productivity satisfy a single-crossing property. Let $\hat{p}$ denote this crossing point. Let p_1 and p_2 be two productivity levels such that $p_2 > p_1$. The difference $S_{t-1}(p_2) - S_{t-1}(p_1)$ is then given by

$$S_{t-1}(p_2) - S_{t-1}(p_1) = \omega_{t-1}(p_2 - p_1) + \beta(1-\mu)\Big(\mathbf{E}_{t-1}\big[\kappa_t(p_t)\ |p_2\big] - \mathbf{E}_{t-1}\big[\kappa_t(p_t)\ |p_1\big]\Big),$$

where $\kappa_t(p_t)$ is a notation for the terms inside the expectation

$$\kappa_t(p_t) = \mu Q_t + (1-\mu)\Bigg[\Big(\big(1-\chi_t(p_t)\big) + \delta_t\chi_t(p_t)\Big)U_t + \chi_t(p_t)(1-\delta_t)\Psi_t(p_t)\Bigg].$$

I now explicitly condition on the current realization of productivity in the expectation operator. Showing that S_{t-1} is increasing in p amounts to showing that the difference in expectation in the last expression is non-negative. This difference can be rewritten

$$\int_{\underline{p}}^{\overline{p}} \mathbf{E}_{t-1}\big[\kappa_t(p_t)\big] \cdot \big(\gamma(p_t|p_2) - \gamma(p_t|p_1)\big)dp_t,$$

denoting $\gamma(p_t|p_{t-1})$ the density of p_t conditional on p_{t-1} and the expectation is now taken over the aggregate states. Now, given the crossing-point $\hat{p}$, we can rewrite

$$\begin{aligned}&\int_{\underline{p}}^{\overline{p}} \mathbf{E}_{t-1}\big[\kappa_t(p_t)\big] \cdot \big(\gamma(p_t|p_2) - \gamma(p_t|p_1)\big)dp_t \\ &= \int_{\underline{p}}^{\hat{p}} \mathbf{E}_{t-1}\big[\kappa_t(p_t)\big] \cdot \big(\gamma(p_t|p_2) - \gamma(p_t|p_1)\big)dp_t + \int_{\hat{p}}^{\overline{p}} \mathbf{E}_{t-1}\big[\kappa_t(p_t)\big] \cdot \big(\gamma(p_t|p_2) - \gamma(p_t|p_1)\big)dp_t\end{aligned}$$

and, since $\mathbf{E}_{t-1}\big[\kappa_t(p)\big]$ is weakly increasing in p, we can bound the terms in this last expression as

$$\int_{\underline{p}}^{\hat{p}} \mathbf{E}_{t-1}\big[\kappa_t(p_t)\big] \cdot \big(\gamma(p_t|p_2) - \gamma(p_t|p_1)\big)dp_t \geq \mathbf{E}_{t-1}\big[\kappa_t(\hat{p})\big] \int_{\underline{p}}^{\hat{p}} \big(\gamma(p_t|p_2) - \gamma(p_t|p_1)\big)dp_t$$

and

$$\int_{\hat{p}}^{\overline{p}} \mathbf{E}_{t-1}\big[\kappa_t(p_t)\big] \cdot \big(\gamma(p_t|p_2) - \gamma(p_t|p_1)\big) dp_t \geq \mathbf{E}_{t-1}\big[\kappa_t(\hat{p})\big] \int_{\hat{p}}^{\overline{p}} \big(\gamma(p_t|p_2) - \gamma(p_t|p_1)\big) dp_t,$$

where I use that, by the single-crossing property, $\gamma(p_t|p_2) - \gamma(p_t|p_1) \leq 0$ for $p \in [\underline{p}, \hat{p}]$ and $\gamma(p_t|p_2) - \gamma(p_t|p_1) \geq 0$ for $p \in [\hat{p}, \overline{p}]$. Finally, summing up the last two inequalities, we get

$$\mathbf{E}_{t-1}\big[\kappa_t(p_t)\ |p_2\big] - \mathbf{E}_{t-1}\big[\kappa_t(p_t)\ |p_1\big] = \int_{\underline{p}}^{\overline{p}} \mathbf{E}_{t-1}\big[\kappa_t(p_t)\big] \cdot \big(\gamma(p_t|p_2) - \gamma(p_t|p_1)\big) dp_t \geq 0.$$

This last inequality shows that $S_{t-1}(p_2) \geq S_{t-1}(p_1)$ for $p_2 > p_1$.

An extra step is required to formally establish that the recursion defining the firm-workers' surplus (11) implies that the value function S_t has the property. The argument is identical to Moscarini and Postel-Vinay [2013, Appendix A, pp.1571-6], and I summarize it intuitively here. Fix an arbitrary measure of employment-weighted firm productivity L^*. Given L^*, the rest of the state space is made of firm-specific productivity $(p, \omega) \in \big[\underline{p}, \overline{p}\big] \times \big[\underline{\omega}, \overline{\omega}\big]$, so Blackwell's sufficient conditions for a contraction mapping can be applied. For any fix L^*, the operator implicitly defined by the firm-workers' surplus recursion (11) has a unique solution S_{L^*}. In the general case where L is part of the state-space, Blackwell's conditions do not apply, since these conditions are restricted to functions defined on closed intervals of real numbers. But, if a solution exists in the general case, it must also solve the problem for any fixed measure of employment-weighted firm productivity L^*. By the uniqueness of the solution in the restricted problem, the solution S_t in the general case must have the property.

A.5 Proof of employment contract in an RME

This appendix contains the proof of Result 3. The proof proceeds in two steps: (i) it derives the distribution of offered contracts in an RME (18), and (ii) it derives the optimal RME contract (15). Throughout, I subsume the aggregate state $\big(\omega, L\big)$ in the subscript t for concision.

A.5.1 Offer distribution in an RME

From the equation for the acceptance rate rewritten using size-independence (31), we can immediately check that

$$Y_t\big(V_t(p)\big) = \frac{u_t + (1-\delta_t)s\int_{p_{E,t}}^{p} dL_t(\tilde{p})}{u_t + (1-\delta_t)s\int_{p_{E,t}}^{\overline{p}} dL_t(p)} = \frac{u_t + (1-\delta_t)s\Big[L(p) - L\big(p_{E,t}\big)\Big]}{u_t + (1-\delta_t)s\Big[L(\overline{p}) - L\big(p_{E,t}\big)\Big]}.$$

Using the firm's ads posting (7) and the equality $\eta_t A_t = \lambda_t Z_t$, the offer distribution (4) can be rewritten

$$\begin{aligned}
F_t(W) &= A_t^{-1}\int_p\int_n \mathbb{1}\left\{V_t(p) \leq W\right\}\chi_t(p)a_t(p,n)dM_t(p,n) \\
&= \int_p \mathbb{1}\left\{V_t(p) \leq W\right\}\frac{\chi_t(p)h_t(p)(1-\mu)(1-\delta_t)}{Z_t\lambda_t Y_t(W)}dL_t(p).
\end{aligned}$$

Evaluating this last expression at the optimal contract $V_t(p)$ and using the expression for the acceptance rate $Y_t\big(V_t(p)\big)$ derived above gives

$$\lambda_t F_t\big(V_t(p)\big) = \int_{p_E}^{p}\frac{(1-\delta_t)h_t(\tilde{p})}{u_t + (1-\delta_t)s\big[L_t(\tilde{p}) - L_t(p_E)\big]}dL_t(\tilde{p}).$$

A.5.2 Equilibrium contract in an RME

Start from the first-order condition with respect to the value of contracts in the firm's unconstrained problem (12) for some firm-productivity level p

$$-q_t'(V_t(p))\big[S_t(p) - V_t(p)\big] = h_t(p).$$

The derivative of the quit rate is given by $q_t'(W) = -s\lambda_t F_t'(W)$. In a rank-monotonic equilibrium, the derivative of the offer distribution (18) is given by

$$\lambda_t F_t'\big(V_t(p)\big)\frac{dV_t(p)}{dp} = \frac{(1-\delta_t)h_t(p)l_t(p)}{u_t + (1-\delta_t)s\big[L_t(p) - L_t(p_{E,t})\big]},$$

with $l_t(p) = dL_t(p)/dp$. Combining the last three expressions yields the following first-order differential equation in V_t

$$\frac{dV_t(p)}{dp} + \frac{s(1-\delta_t)l_t(p)}{u_t + (1-\delta_t)s\big[L_t(p) - L_t(p_{E,t})\big]} V_t(p) = \frac{s(1-\delta_t)l_t(p)}{u_t + (1-\delta_t)s\big[L_t(p) - L_t(p_{E,t})\big]} S_t(p)$$

with boundary condition $V_t(p_{E,t}) = U_t$. Noting that

$$\frac{d\ln\Big(u_t + (1-\delta_t)s\big[L_t(p) - L_t(p_{E,t})\big]\Big)}{dp} = \frac{s(1-\delta_t)l_t(p)}{u_t + (1-\delta_t)s\big[L_t(p) - L_t(p_{E,t})\big]},$$

the corresponding integrating factor is then

$$\exp\int \frac{s(1-\delta_t)l_t(p)}{u_t + (1-\delta_t)s\big[L_t(p) - L_t(p_{E,t})\big]} dp = u_t + (1-\delta_t)s\big[L_t(p) - L_t(p_{E,t})\big].$$

Along with the boundary condition, $V_t(p_{E,t}) = U_t$, this yields the expression for the optimal contract (15) in the main text

$$V_t(p) = \frac{u_t U_t + (1-\delta_t)s\int_{p_{E,t}}^{p} S_t(\tilde{p})dL_t(\tilde{p})}{u_t + (1-\delta_t)s\big[L_t(p) - L_t(p_{E,t})\big]}.$$

A.6 Alternative formulation: Firm-workers' net surplus

This Appendix shows that the model solution can be expressed in terms of a single value function. Subtracting the value of unemployment (5) from the firm-workers' surplus (11), the firm's problem can equivalently be expressed in terms of *net* firm-workers' surplus. I omit this notation from the main text so as not to clutter the description of the model. This more compact formulation is used when solving the model.

A.6.1 Firm-workers' net surplus

The net firm-workers' surplus is defined as the firm-workers' surplus net of the value of unemployment:

$$\Sigma_{t-1}(p) = J_{t-1}\big(p, \overline{V}\big) + \overline{V} - U_{t-1} = S_{t-1}(p) - U_{t-1}.$$

Adding and subtracting U_t within the expectation, the firm-workers' surplus (11) can be rearranged as

$$S_{t-1}(p) \;=\; p_{t-1}\omega_{t-1} \;+\; \beta\mathbf{E}_{t-1}\Big[U_t \;+\; \mu\big[Q_t \;-\; U_t\big] \;+\; (1 \;-\; \mu)\chi_t(p_t)(1 \;-\; \delta_t)\big[\Psi_t(p_t) \;-\; U_t\big]\Big].$$

Using the same strategy, the unemployed worker's value can similarly be rearranged as

$$U_{t-1} = b + \beta\mathbf{E}_{t-1}\Big[U_t + \mu\big[Q_t - U_t\big] + (1-\mu)\lambda_t \int \max\{\tilde{W} - U_t, 0\} dF_t(\tilde{W})\Big].$$

Let $W_\Sigma = W - U_t$ denote the value of the offered contract net of the value of unemployment. Let $F_{\Sigma,t}$ be the corresponding distribution of contracts, so $F_t(W) = F_t(W_\Sigma + U_t) = F_{\Sigma,t}(W_\Sigma)$. The net firm-workers' surplus $\Sigma_{t-1}(p) = S_{t-1}(p) - U_{t-1}$ can then be expressed as

$$\begin{aligned} \Sigma_{t-1}(p) &= p_{t-1}\omega_{t-1} - b \\ &\quad + \beta(1-\mu)\mathbf{E}_{t-1}\Big[\chi_t(p_t)\Big\{(1-\delta_t)\tilde{\Psi}_t(p) - \lambda_t \int \max\{\tilde{W}_\Sigma, 0\} dF_{\Sigma,t}(\tilde{W}_\Sigma)\Big\}\Big] \end{aligned} \tag{33}$$

where $\tilde{\Psi}_t(p)$ is the firm's optimization problem in net surplus form

$$\tilde{\Psi}_t(p) = \max_{W_\Sigma, h \geq 0} \Big\{ - c(h) + \big[1 - q_{\Sigma,t}(W_\Sigma)\big]\Sigma_t(p) + h\big[\Sigma_t(p) - W_\Sigma\big] + s\lambda_t \int_{W_\Sigma}^{\infty} \tilde{W}_\Sigma dF_t(\tilde{W}_\Sigma)\Big\}.$$

A.6.2 Firm policies as a function of the firm-workers' net surplus in an RME

Since $\Sigma = S - U$ and U does not depend on p, Σ is also increasing in p for every candidate equilibrium. In a RME, the corresponding net contract follows by subtracting $U(\omega, L)$ in the expression for the optimal contract (15), which gives

$$V(p,\omega,L) - U(\omega,L) = V_\Sigma(p,\omega,L) = \frac{\big(1-\delta(\omega)\big)s\int_{p_E(\omega,L)}^{p}\Sigma(\tilde{p},\omega,L)dL(\tilde{p})}{u + \big(1-\delta(\omega)\big)s\Big[L(p) - L\big(p_E(\omega,L)\big)\Big]}. \tag{34}$$

The optimal hiring rate can also be expressed as

$$c'\big(h(p,\omega,L)\big) = \Sigma(p,\omega,L) - V_\Sigma(p,\omega,L),$$

and the entry/exit decision as

$$\chi(p,\omega,L) = \mathbb{1}\big\{\Sigma(p,\omega,L) \geq 0\big\}.$$

B Data

B.1 Firm-level data

I use two main sources of firm-level administrative data from Britain:

1. The Annual Respondents Database [Office for National Statistics, 2020] and its successor the Annual Business Survey [Office for National Statistics, 2021] give detailed yearly balance-sheet information from the universe of large firms (with more than 250 employees) and a stratified random sample of smaller businesses (with fewer than 250 employees). The Annual Respondents Database (ARD) has data from 1997 to 2008. The Annual Business Survey has data from 2009 onward.

2. The Business Structure Database [Office for National Statistics, 2019] is a snapshot from the registry of all British businesses, but it only has data on some basic variables (em-

ployment, estimated turnover, industry). Businesses must satisfy one of two conditions to be included in the Business Structure Database. They must have either a sales turnover above the VAT registration threshold or at least one employee. In practice, these restrictions imply that all but the smallest businesses and the self-employed are included in these data.

Since the Business Structure Database (BSD) does not have information on value added or employment costs, I follow the procedure in Riley et al. [2015] to obtain meaningful aggregates from the Annual Respondents Database (ARD)/Annual Business Survey (ABS). I use the "gross value added at factor costs" and "total employment costs" variables, which are harmonized across survey year by the data provider, as the relevant concepts for a firm's value added and wage bill. I deflate these measures using industry-level deflators provided by the Office for National Statistics. The employment variable is directly taken from the Business Structure Database.

To gross up the data, I construct survey weights directly from the Business Structure Database, which represents the (near) universe of private sector employment. I define industry×firm-size cells and use the BSD employment counts as weights for the ARD/ABS. In constructing the analysis sample, I drop a few problematic sectors in the ARS/ABS: farming (A), mining & quarrying (B), energy supply (D), water (E), and real estate (L). All sectors dominated by public employment in the UK (education, health care, and social work) are also excluded. Finally, I also trim the top and bottom 2 percent of firms in the distribution of labor productivity, $LP_{i,t}$, in each industry×firm-size cell.

Note that these data sets are not publicly available. Access can be obtained through the UK Data Service.

B.2 Worker transition rates

The monthly time series for the worker transition rates (UE_t, EU_t, EE_t) are from Postel-Vinay and Sepahsalari [2019]. These series are derived from the British Household Panel Survey

(BHPS) and its successor Understanding Society (UKHLS), a monthly survey of British households. This paper uses data from 1992m1 to 2016m12. Because of the transition from the BHPS to the UKHLS, there is a gap in the series between 2008m8 and 2009m12, which is smoothed over using moving averages. Additional details on the construction of these series can be found in Postel-Vinay and Sepahsalari [2019].[28]

B.3 Additional macro time series

I also use the following aggregate time series, which are publicly available on the Office for National Statistics (ONS) website.

- ABMI: Gross Domestic Product (chained volume, seasonally adjusted), quarterly starting in 1955q1.
- A4YM: Output per Worker (seasonally adjusted), quarterly starting in 1959q3.
- MGSX: Unemployment rate (aged 16 and over, seasonally adjusted), monthly starting in 1971m1.

B.4 UK recession dates

The UK recession dates are defined as successive quarters of economic growth, as measured by the quarter-on-quarter growth in seasonally adjusted real GDP.[29] Table 4 lists the recessions used in the paper. This definition is used because there is no British equivalent to the "official" list of recessions defined by the National Bureau of Economic Research for the United States.

B.5 Robustness checks on productivity decomposition

I report two robustness checks on the productivity decomposition obtained from firm-level data. Figure 9 benchmarks the labor productivity index obtained by grossing up the British

[28] I am grateful to the authors for sharing these series and to Pete Spittal for explaining how they are affected by the transition from the BHPS to the UKHLS.

[29] The full list is available here: `https://en.wikipedia.org/wiki/List_of_recessions_in_the_United_Kingdom`

Recession start date	Recession end date	Duration (quarters)
31-Mar-57	30-Sep-57	2
30-Jun-61	31-Dec-61	2
30-Jun-73	31-Mar-74	3
31-Mar-75	30-Sep-75	2
31-Dec-79	31-Mar-81	5
30-Jun-90	30-Sep-91	5
31-Mar-08	30-Jun-09	5

Table 4: List of UK recessions implied by two-quarter rule.

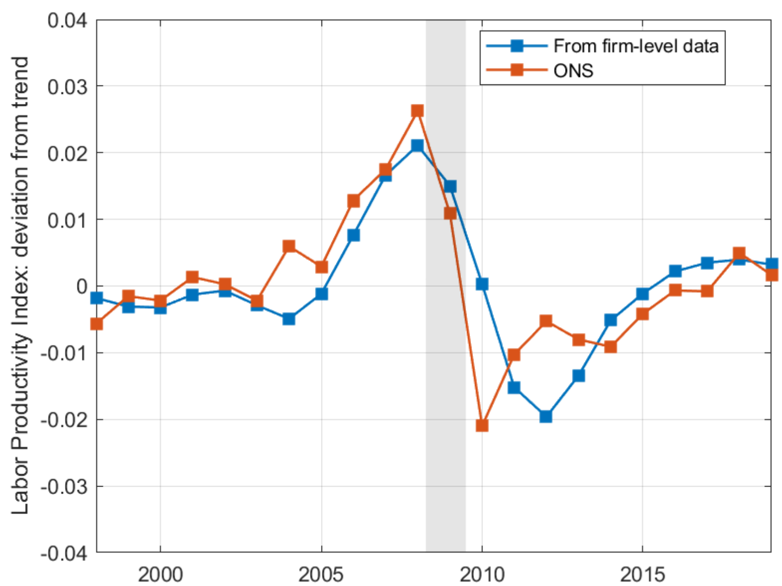


Figure 9: Benchmark of productivity index from micro data to official series.

micro data to the official labor productivity series (A4 YM) from the ONS. Though the drop and recovery in the ONS series are quicker, overall the two series exhibit a similar pattern in deviation from an HP trend.

Figure 10 shows the labor productivity decomposition (25) using alternative de-trending methods. These alternative methods (the HP filter and band-pass filter in particular) give very similar results.

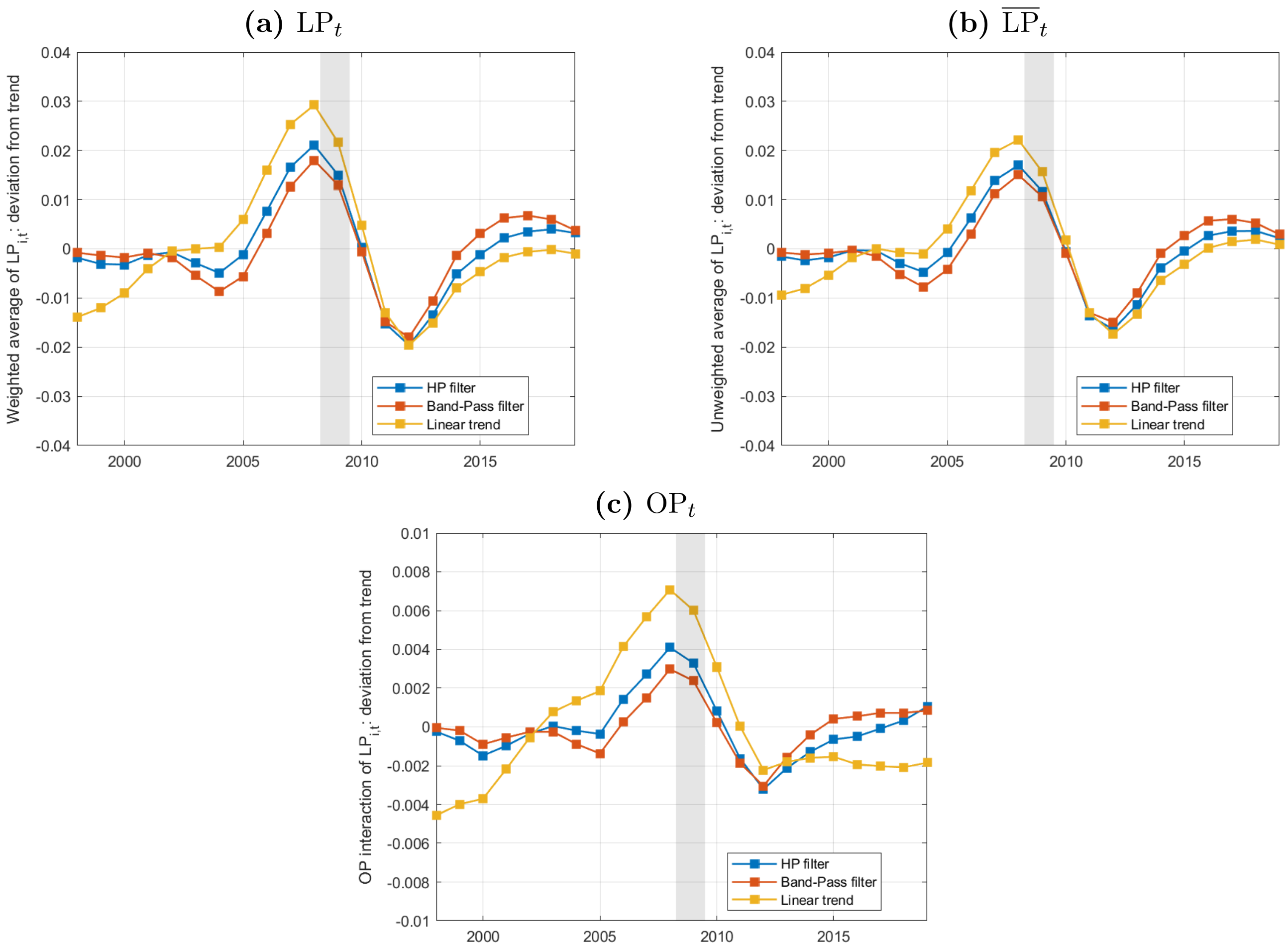


Figure 10: Labor productivity decomposition (25). Series are shown in deviation from several alternative trends: HP filter (smoothing parameter = 100), band-pass filter (fluctuations restricted to the range of 2 to 14 years), and a linear trend fitted over the sample period. Gray band denotes the Great Recession period in the UK.

C Quantitative analysis

C.1 Steady-state solution

As shown in Appendix A.6, the firm's policies can be expressed in terms of a single value function, the net surplus given in Equation (33). The algorithm below is expressed in terms of the net firm-workers' surplus formulation for concision.

Discretization At the steady state, p is the variable in the state-space. I discretize the process for idiosyncratic log-productivity using Tauchen's procedure with $N_p = 401$ points. This gives a grid $\{p_1, \ldots, p_{N_p}\}$ and the associated transition matrix.

This discretization on a thin grid allows me to approximate the relevant policy or value function as being constant on some (small) half-open interval. This gives an intuitive way to integrate against the measure of workers, L, by replacing the integral with the appropriate employment share weighted sum. As an example, the net optimal contract (34) at some productivity node p_k can be approximated as

$$V_\Sigma(p_k) = \frac{s(1-\delta)\int_{p_1}^{p_k} \chi(p')\Sigma(p')dL(p')}{u + s(1-\delta)\left(L(p_k) - L(p_E)\right)} = \frac{s(1-\delta)\sum_{i=2}^{k}\int_{p_{i-1}}^{p_i} \chi(p')\Sigma(p')dL(p')}{u + s(1-\delta)\left(L(p_k) - L(p_E)\right)}$$
$$\approx \frac{s(1-\delta)\sum_{i=2}^{k} \chi(p_{i-1})\Sigma(p_{i-1})\int_{p_{i-1}}^{p_i} dL(p')}{u + s(1-\delta)\left(L(p_k) - L(p_E)\right)},$$

where the integral in the last expression is simply the fraction of workers employed at firms in the interval between p_{i-1} and p_i.

Algorithm Given the discretization, the algorithm unfolds as follows.

1. Guess initial values for Σ and L on the grid $\{p_1, \ldots, p_{N_p}\}$. In line with the RME result, I start with some increasing function of p for the net surplus. I also initialize $L = 0$ (all workers initially unemployed).

2. Conditional on values for Σ and L, the agents' optimal policies can be computed. For example, the activity threshold, p_E, is the point at which Σ becomes positive. The optimal

contract can be computed from Equation (34) and the firm's choice of hiring intensity from the corresponding first-order condition.

3. The net surplus equation and the law of motion for the measure of employed workers imply new values for Σ and L on the grid. The net surplus equation gives an update for Σ in the previous period, while the law of motion for employment yields next period's employment at each productivity level. This does not matter since the algorithm solves for a steady state RME.

4. The final step consists of checking the convergence of L and Σ. If this is the case, the tuple (Σ, L) is the steady state RME. Otherwise, go back to point 2 with the updated values and iterate.

C.2 Aggregate shocks solution

Approximations As explained in the main text, the solution method with aggregate shocks relies on two approximations. First, the measure of employment at firms of different productivity is summarized by a set of $N_m + 1$ moments

$$\begin{aligned} m_t^0 &= u_t = 1 - \int dL_t(p) \\ m_t^1 &= \int p d\bar{L}_t(p) \\ &\dots \\ m_t^{N_m} &= \int p^{N_m} d\bar{L}_t(p), \end{aligned} \tag{35}$$

where $\bar{L}_t = L_t(p)/L_t(\overline{p})$ denotes the cumulative density associated with the cumulative measure of workers on p.

Second, I parameterize the value functions for the firm-workers' surplus S_t and the unemployed worker U_t with a polynomial. I choose to parameterize these value functions separately since they are positive by definition, so they can be expressed in log-deviation from the steady state. Because preserving the monotonicity of S_t (especially around the entry threshold) is key to the procedure, I use a separate polynomial for each productivity node p_i. The value functions are approximated outside of the steady state as

$$\ln S(p_i, \omega_t, L_t) - \ln \bar{S}(p_i) \approx \hat{S}(p_i, \omega_t, \hat{\boldsymbol{m}}_t; \theta_{p_i}), \quad p_i \in \{p_1, ..., p_{N_p}\},$$

and

$$\ln U(\omega_t, L_t) - \ln \bar{U} \approx \hat{U}(\omega_t, \hat{\boldsymbol{m}}_t; \theta_U),$$

where $\hat{\boldsymbol{m}}_t$ denotes the vector gathering all moments in (35) in log-deviation from steady state, while $\bar{S}$ and $\bar{U}$ stand, respectively, for the firm-workers' surplus and the value of unemployment at the steady state.

Algorithm The algorithm for the model solution with aggregate shocks then consists of the four following steps.

1. Draw a sequence of aggregate productivity shocks and guess an initial value for the coefficients of $\hat{S}$ and $\hat{U}$. I initialize them at zero.

2. Simulate the measure of employment forward, starting from the stationary solution. Conditional on the current values of $\{\theta_U, \theta_{p_1}, \ldots, \theta_{p_{N_p}}\}$, agents make optimal decisions about hiring and contract offers given the current states, which induces a law of motion for employment at each productivity level. The simulated measure of workers is approximated by a set of moments to compute the value functions and policy functions in each period.

3. Update $\hat{S}$ and $\hat{U}$, conditional on the simulation of L_t obtained in the previous step. This requires taking an expectation over future realizations of the aggregate shock. The aggregate shock is discretized using Tauchen's procedure with $N_\omega = 15$ nodes in practice.

4. Run a regression of $\hat{S}$ and $\hat{U}$ on the state variables to update the coefficients. Go back to step 2 and iterate on the coefficients $\{\theta_U, \theta_{p_1}, \ldots, \theta_{p_{N_p}}\}$ until convergence.

I find the coefficients by running separate regressions for the firm-workers' surplus at each p-node on the variables in the state-space. I omit the constant in these regressions, which is equivalent to imposing the constraint that the steady state holds exactly at each node. Since these regressors are at times close to collinear in some iterations, I use a penalized (ridge) regression to regularize the problem. The coefficients for the unemployed worker's value function are found by solving

$$\min_{\theta_U} \sum_t \left[(\ln U_t - \ln \bar{U}) - \hat{U}(\omega_t, \hat{\boldsymbol{m}}_t; \theta_U) \right]^2 + \zeta \theta_U^T \theta_U,$$

where θ_{U_i} denotes individual elements of θ_U, $\zeta > 0$ is the associated regularization parameter, and

$$\hat{U}(\omega_t, \hat{\boldsymbol{m}}_t; \theta_U) = \theta_U^\omega \ln \omega_t + \sum_{k=0}^{N_m} \theta_U^{m_k} \left[\ln m_t^k - \ln \bar{m}^k \right].$$

The regularization parameter, $\zeta > 0$, ensures that the matrix of regressors is invertible by adding to it a ζ-diagonal matrix. I proceed similarly to find the coefficients in $\hat{S}(p_i, \omega_t, \hat{\boldsymbol{m}}_t; \theta_{p_i})$ at each productivity node p_i.

I finally allow for less than full updating between each step. With these parametric assumptions, the coefficients $\{\theta_U, \theta_{p_1}, \ldots, \theta_{p_{N_p}}\}$ are elasticities of the value functions with respect to the regressors, which gives some intuition about the appropriate convergence condition.

C.3 Accuracy test

I assess the accuracy of the procedure by adapting ideas from den Haan [2010]. The goal of this test is to check that the error implied by the polynomial approximation does not build up over time. The test proceeds as follows.

1. Draw a new sequence of shocks $\{\omega_t'\}_{t=1}^T$, separate from the sequence used to find the coefficients $\{\theta_U, \theta_{p_1}, \ldots, \theta_{p_{N_p}}\}$.

2. Compute the model solution along $\{\omega_t'\}_{t=1}^T$ in two different ways.

(a) Simulate the model forward in time using the polynomial approximations. This directly gives the value function $\{\hat{S}_t, \hat{U}_t\}_{t=1}^T$ and moments $\{\hat{\boldsymbol{m}}_t\}_{t=1}^T$.

(b) Construct the value functions by solving the model back in time. This gives the value functions (in log-deviation from the steady state) $\{\check{S}_t, \check{U}_t\}_{t=1}^T$. Using these value functions to solve for the firm's decisions, the model can be simulated once more forward to obtain the moments $\{\check{\boldsymbol{m}}_t\}_{t=1}^T$.

3. Compute the distance between the two model solutions at each point in time. For each alternative time series $\{X_t\}_{t=1}^T$ obtained from step 2, this distance can be expressed as

$$d(\hat{X}_t, \check{X}_t) = 100 \cdot \left|\hat{X}_t - \check{X}_t\right|, \tag{36}$$

which is (approximately) in percent given that all time series are in log-deviations from the steady state.

Table 5 reports summary statistics of the accuracy metric (36) for the two model solutions. This measure suggests that the accuracy of the procedure for the two value functions is very good, with a distance of at most 0.025 percent across model solutions. The least accurate part of the simulation procedure comes from differences in the entry-exit threshold, with a distance between the simulated unemployment rates (m_t^0) of at most 1 percent (so 8.9 percent vs 8.8 percent at the calibrated steady state). This suggests that the overall accuracy of the procedure is good.

C.4 Number of moments in approximation

I assess the sensitivity of this solution method to the number of moments used in approximating L_t with the following test. I incrementally introduce up to $N_m = 9$ moments to summarize L_t, and solve the model using the same sequence of aggregate shocks $\{\omega_t'\}_{t=1}^T$ using more moments. I can then compute a solution $\hat{S}^{N_m}(p, \omega_t, \hat{\boldsymbol{m}}_t; \theta_{p_i})$ and $\hat{U}^{N_m}(\omega_t, \hat{\boldsymbol{m}}_t; \theta_U)$ along the same sequence of aggregate shocks, where N_m indexes the number of moments included in the approximation.

	Accuracy measure: $100 \cdot \left|\hat{X}_t - \check{X}_t\right|$				
Variable	Mean	p75	p90	p95	Max
Value Functions					
S_t	0.001	0.002	0.002	0.003	0.025
U_t	0.001	0.001	0.001	0.001	0.014
Moments L_t ($\boldsymbol{m}_t$)					
$m_t^0 (:= u_t)$	0.242	0.328	0.486	0.604	1.003
m_t^1	0.014	0.020	0.034	0.043	0.079
m_t^2	0.017	0.025	0.040	0.051	0.092

Table 5: Accuracy test results

I proceed by defining the following measure of the solution's sensitivity to the inclusion of an additional moment k

$$\Delta_t^{N_m}(\hat{S}_t(p)) = \left|\hat{S}_t^{N_m}(p) - \hat{S}_t^{N_m - 1}(p)\right| = \left|\ln S_t^{N_m}(p) - \ln S_t^{N_m - 1}(p)\right|.$$

Figure 11 reports the average and maximum $\Delta_t^{N_m}$ across simulation periods $t = 1, \ldots, T$ and value functions $\{\hat{U}_t, \hat{S}_t(p_1), \ldots, \hat{S}_t(p_{N_p})\}$. The figure shows that after $N_m = 2$, changes in the approximated value functions become smaller than 0.01 percent. All results in the paper are obtained with $N_m = 2$.

D Model with exogenous entry and exit

This appendix reports a series of results for a simplified version of the model where entry and exit is exogenous. With respect to the full model introduced in the main text, this model has no endogenous entry and exit margin. After briefly describing this model, I use this framework in two exercises where the absence of a firm entry and exit margin is helpful.

I first provide insights into the planner's problem by computing welfare under different constraints on worker reallocation. Here the fact that firms are always active at all productivity level lets me isolate the worker reallocation margin from the firm entry and exit margin.

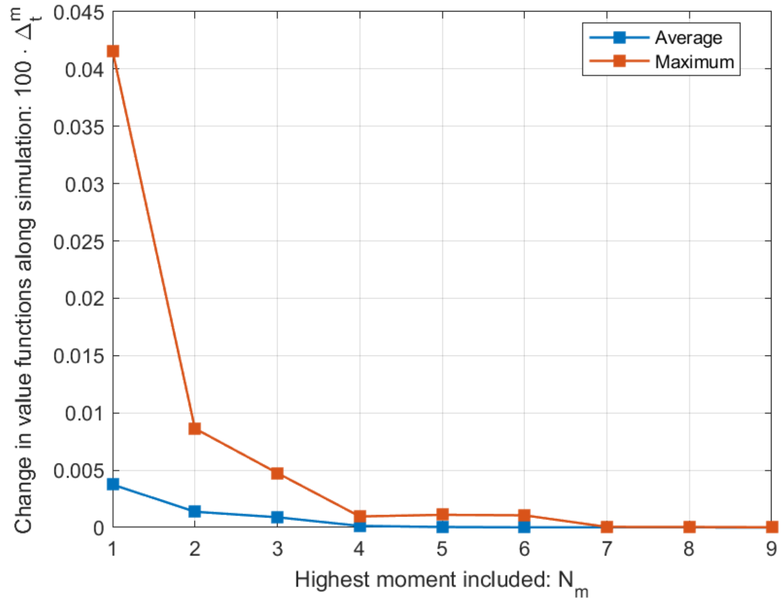


Figure 11: Robustness to number of moments N_m included in $\hat{\boldsymbol{m}}_t$

Second, I use this model as an additional check for the solution algorithm with aggregate shocks. Here the absence of an endogenous firm entry/exit decision allows me to benchmark the modified KS algorithm introduced in the main text to a linearization approach, where the discontinuity associated with the endogenous entry/exit margin is problematic.

D.1 Model summary

The model with exogenous entry and exit requires two modifications with respect to the full model introduced in the main text: (i) it must never be optimal for workers to quit to unemployment, and (ii) firms have to exit exogenously for a steady state equilibrium to be defined.

I implement the first condition with the parameter restriction $b = 0$ and $s = 1$. With the flow value of unemployment b equal to zero and the search intensity of workers on the job s equal to the search intensity of workers in unemployment, there is no motive for an employed worker to quit into unemployment. I redefine the job destruction shock δ as a firm destruction shock to satisfy the second condition. There is now a chance δ that a firm exits in every period. In this event, all workers transition to unemployment and start searching in the next period.

This model is essentially nested within the framework introduced in the body of the paper.[30] It can be shown that the key results on the characterization of rank-monotonic equilibria (Result 2) and the optimal wage contract (Result 3) apply to this simplified environment.

D.2 Exploration of the planner's problem

Assume that the economy without exogenous entry and exit is in the steady state. The planner's problem can be written as follows

$$\begin{aligned} &\max_{\chi(p),h(p),q(p)} -\int_{\underline{p}}^{\overline{p}} c(h(p))\chi(p)(1-\delta)(1-\mu)dL(p) + \int_{\underline{p}}^{\overline{p}} p dL^P(p) + bu \\ \text{s.t} \quad & l^P(p) = \chi(p)\Big[(1-q(p)+h(p))(1-\delta)(1-\mu)l(p) + \mu\gamma_0(p)\Big] \\ & l(p) = \int_{\underline{p}}^{\overline{p}} \gamma(p|p')l^P(p')dp' \\ & u = 1 - \int_{\underline{p}}^{\overline{p}} dL^P(p) \\ & h(p) \geq 0 \\ & \chi(p) \in \{0,1\} \\ & 0 \leq q(p) \leq s\lambda \end{aligned} \tag{37}$$

The planner maximizes output net of the hiring cost $c(h(p))$ subject to the law of motion for employment, the implied unemployment rate u, and the non-negativity constraint on the hiring rate at each productivity level $h(p)$. The constraint on $\chi(p)$ states that the planner chooses which firms are active given the realization of the idiosyncratic productivity shocks p.

The last constraint is the key constraint in a random search environment with on-the-job search. It captures the worker reallocation problem faced by the planner. Conditional on the matching technology (a primitive of the model that determines λ), the planner decides which contacts between an employer and a worker translate into a move. I formalize this decision as the planner choosing the quit rate at each productivity level. As an example, the planner could

[30] Essentially nested because there are no exogenous firm exit shocks in the main model.

Parameter	Value	Rationale/Target
c_0	2.17e+04	UE rate
c_1	2.00e+00	Cubic hiring cost
δ	3.50e-03	EU rate
s	1.00e+00	No endogenous entry/exit
μ	2.75e-04	Average firm size
ρ_p	9.80e-01	Exogenously set
σ_p	7.00e-02	Exogenously set
b	0.00e+00	No endogenous entry/exit

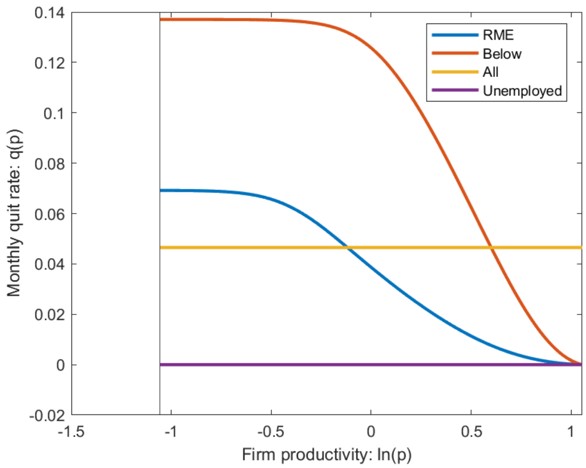
Table 6: Parameters in model with exogenous entry and exit.

decide that all employed workers move to an alternative firm if they are contacted, in which case $q(p) = s\lambda$ at all productivity level.

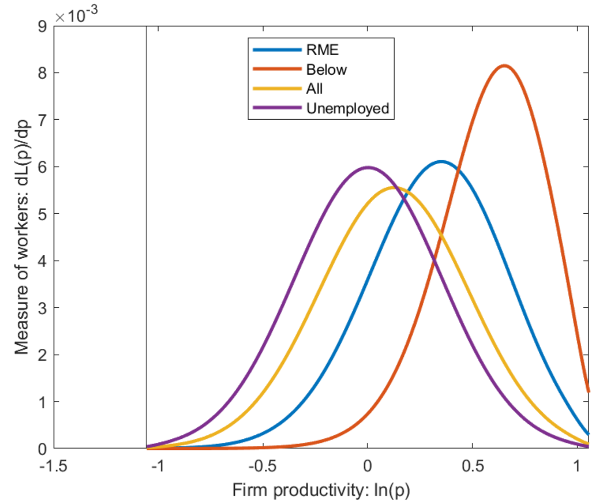
The planner's problem set out in (37) does not lend itself to an analytical characterization. Here, I explore the key tradeoffs using numerical simulations for the following subset of allocations. I focus on the case $\chi(p) = 1$ where the planner does not choose which firms survive. Given the parameter restrictions $s = 1$ and $b = 0$, all firms add to output over and above home production, since $b = 0 < \underline{p}$. So abstracting from this margin does not mechanically lower output. I stress that $\chi(p) = 1$ is still a simplification because it cannot be ruled out that a higher level of output can be achieved by selecting which firms are recruiting at each productivity level p.

The parameters in the model with no endogenous entry and exit are chosen to match a subset of the moments targeted in the main model, where the simulated moments are obtained in the (decentralized) RME steady state equilibrium. Specifically, I set several parameters exogenously and calibrate the remaining ones to match these data moments. These parameters are summarized in Table 6. s and b are set to guarantee that there is no endogenous entry and exit in the model. c_1 is set to give a cubic hiring cost, a formulation previously used in the literature [Merz and Yashiv, 2007]. (ρ_p, σ_p) are set to yield a reasonable degree of dispersion in firm productivity. c_0, δ, and μ are calibrated to match, respectively, the UE rate, the EU rate, and the average firm size.

I study four allocations to illustrate the tradeoffs faced by the planner in (37). These



(a) Quit rate



(b) Worker measure

Figure 12: Selected allocations in the planner's problem (37). The vertical black lines denote the minimum (log) productivity levels.

allocations are depicted in Figure 12. The first allocation is simply the RME (decentralized) equilibrium in the model without entry and exit. I then consider three candidate quit rules in the planner's problem.

1. $q(p) = s\lambda(1 - F(p))$ ("Below"). Workers below the productivity of the potential employer quit.

2. $q(p) = s\lambda$ ("All"). All workers quit when contacted by an alternative potential employer, irrespective of its position in the productivity distribution.

3. $q(p) = 0$ ("Unemployed"). No quits. There is no on-the-job search, and firms can only recruit from unemployment.

I solve for these allocations numerically. To find the corresponding hiring rate in each scenario, I assume that there is no wage transfer to workers, so the planner maximizes the net present value of the firm-workers' surplus given each of the quit rules and no wage payments to workers. As a result, the "Below" allocation differs from the RME allocations, since they correspond to different wage contracts.

As shown in Figure 12, the distribution of workers across firm productivity differs markedly across quit rules. The "Unemployed" allocation corresponds to the least workers at high-

	Allocation			
	RME	Below	Unemployed	All
Labor market				
Unemployment rate	0.048	0.025	0.003	0.070
UE rate	0.069	0.137	1.000	0.046
EU rate	0.003	0.003	0.003	0.003
EE rate	0.022	0.050	0.000	0.046
Production				
Welfare	1.239	0.276	1.060	0.219
Output	1.382	1.821	1.060	1.113
Ladder Interaction term	0.323	0.595	0.002	0.120

Table 7: Summary statistics for selected allocations.

productivity firms, while the "'Below" allocation has the most workers at high-productivity firms. The "All" and "RME" allocations correspond to the distribution of workers in-between.

Table 7 reports summary statistics on output and the labor market in each of these allocations. The quit rule implemented by the planner affects the level of hiring, which in turn affects the level of unemployment. With no quits ("Unemployed" allocation), there is only residual unemployment left (due to the δ-shock). With all workers quitting ("All" allocation), unemployment is largest because there is a lot of inefficient turnover. In terms of welfare, some degree of worker relocation in the RME decentralized allocation dominates the other allocations at this specific parameterization. Interestingly, the allocation without any worker relocation ("Unemployed") comes next in terms of welfare, since it implies that fewer resources are allocated to hiring. This specific parameterization highlights the stark tradeoff between worker relocation and recruitment costs faced by the planner in a random search environment where workers can search on-the-job.

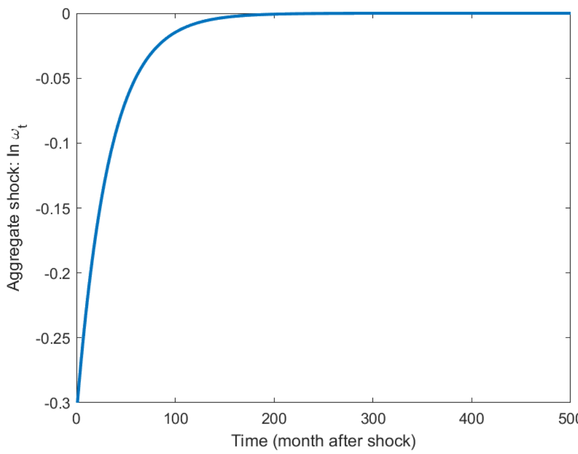
D.3 Additional test for aggregate shock solution

A key difficulty with the numerical solution of the main model is the discontinuity implied by the endogenous firm entry and exit threshold. This difficulty is not present in the model with exogenous entry and exit. I can then benchmark the proposed modified KS algorithm with two alternative solutions methods.

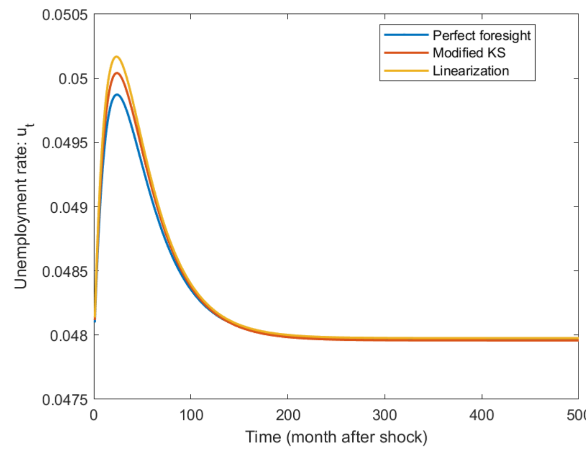
Perfect foresight Given a path of aggregate shocks $\{\omega_t\}_{t=1}^{T}$ and starting from the steady state, the model is solved by iterating forward on the law of motion for employment (6) and solving the firm-workers' surplus (11) backward. The algorithm stops when the measure of worker $L_t(p)$ in each period t and productivity level p is sufficiently close to that simulated in the previous iteration.

Linearization I follow the approach described in Reiter [2009] and linearize the model around its steady state. The "state" variables are: the discretized measure of workers $L_t(p)$, the unemployment rate u_t, and the aggregate shock ω_t. The "jump" variables are: $S_t(p)$ and U_t.

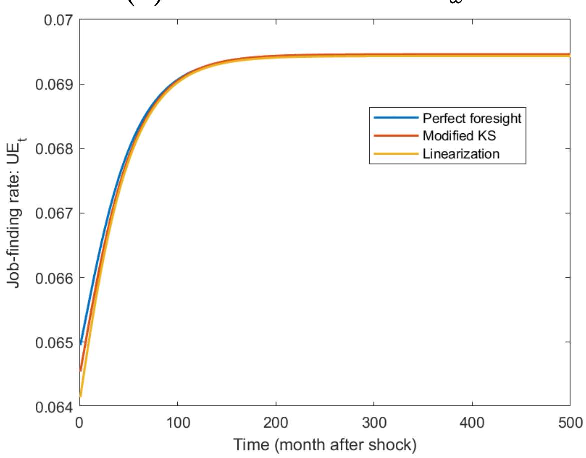
Figure 13 shows a selection of simulated aggregate variables obtained using each solution method given a one-time aggregate shock slowly reverting to its steady state according to the process in (29). The figure shows that the modified KS solution method is accurate in the sense of yielding aggregates very similar to the perfect foresight and linearization solution method in the model with exogenous firm entry and exit. I have experimented with different shock sizes and signs and obtained similar accuracy.



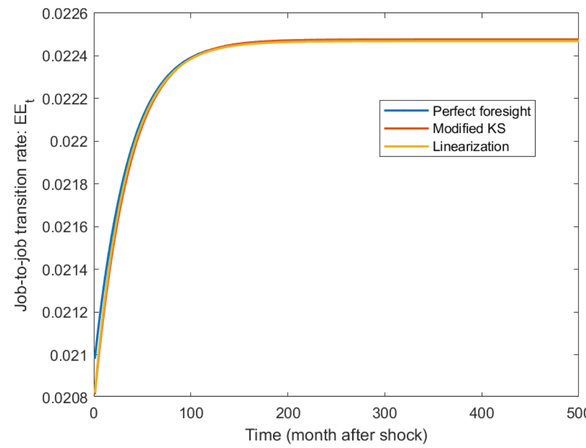
(a) One-time shock: $-2\sigma_\omega$



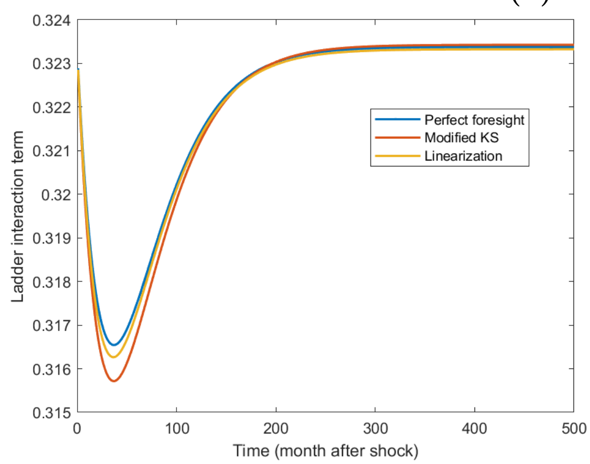
(b) Unemployment rate



(c) UE transition rate



(d) EE transition rate



(e) Ladder Interaction term

Figure 13: Impulse response using alternative solution methods.