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Abstract

This paper presents empirical evidence on the nature of idiosyncratic shocks to firms and discusses its role for firm behavior and aggregate fluctuations. We document that firm-level sales and productivity are hit by heavy-tailed shocks, mostly unexplained by observable factors, and follow a nonlinear stochastic process, thus departing from the canonical linear AR(1). We estimate a state-of-the-art model to flexibly capture the rich dynamics uncovered in the data and characterize the drivers of nonlinear persistence and non-Gaussian shocks. We show the role these features play in achieving empirically plausible volatility and persistence of micro-originated (granular) aggregate fluctuations.

JEL classification: D22, E23, E32
Keywords: firm dynamics, productivity, nonlinearities, non-Gaussian shocks, granular fluctuations

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1 Introduction

Firms are hit by idiosyncratic shocks of various nature and magnitude. The features of this firm-level variation are crucial for many economic questions that relate to firm dynamics.[1] Despite their importance, however, the statistical properties of firms' fundamentals remain elusive, especially when it comes to higher-order moments and rich time series processes.[2] The vast majority of firm dynamics models assume that firm-level productivity follows an AR(1) process with normal innovations, but potential departures from this benchmark are likely to have substantial consequences. Rare, nonlinear, idiosyncratic shocks can even become a source of aggregate fluctuations.

In this paper, we document that, even among relatively large, publicly listed, US firms, firm-level sales and productivity substantially depart from standard AR(1) models with Gaussian innovations. We find that heavy-tailed, extreme, shocks are a pervasive feature of firm-level growth rates. Changes to revenue-based productivity are even more heavy-tailed than sales, and most of the overall variation in tail growth remains unexplained by observable factors. In addition, we document substantial nonlinearities in persistence: firms can be hit by large shocks that erase the impact of previous shocks. We estimate a model of productivity that separates persistent and transitory shocks, is consistent with our empirical findings, and helps us characterize their drivers. Finally, we show analytically and quantitatively how non-Gaussianity and nonlinearities affect the importance of idiosyncratic (granular) shocks for aggregate fluctuations. Simultaneously accounting for these features is crucial to obtain empirically plausible volatility and autocorrelation of granular fluctuations.

Our starting point is an agnostic analysis of firm sales, using data from US Compustat. We document that firm-level sales growth is very concentrated – relative to a Gaussian benchmark – but exhibits heavy tails. Sales growth is fairly symmetric, but very large changes of either sign are likely to occur: kurtosis is much higher than what implied by a Gaussian distribution at all quantiles of firm sales, and especially high for small firms.

We then investigate the dynamics of firm sales. Following recent work on household earnings by Arellano et al. (2017), we estimate quantile autoregressions and document that the persistence of lagged firm sales is not constant in the size and sign of sales changes as

[1] For example, firm-level stochastic processes matter for the role played by input adjustment costs (e.g., Bloom (2009) and Roys (2016)), misallocation losses from financial frictions (see Moll (2014)), and in general are key inputs to macroeconomic models with firm heterogeneity (e.g., Khan and Thomas (2008)). Moreover, idiosyncratic shocks can even be a source of aggregate fluctuations (see Gabaix (2011), Di Giovanni, Levchenko and Mejean (2014) Carvalho and Grassi (2019)).

[2] Complementary to our analysis, Salgado, Guvenen and Bloom (2019) show that the skewness of sales and productivity growth is procyclical. A burgeoning literature has recently documented non-normality and nonlinearities in household earnings dynamics (see Arellano, Blundell and Bonhomme (2017) and Guvenen, Karahan, Ozkan and Song (2021)): our analysis is indeed inspired by this influential recent work.

would be implied by a standard AR(1) process. In particular, the approach introduced by Arellano et al. (2017) provides a measure of persistence of past shocks, often referred to as the persistence of history. We find that such persistence is heterogeneous, with small firms being hit by large positive shocks to their sales having the lowest persistence. This means that a large positive shock hitting a small firm can wipe out the history of past negative shocks. However, among large firms, good and bad past shocks exhibit similar persistence.

Firm sales clearly represent the combination of exogenous fundamental shocks and endogenous inputs. We find that our findings on sales are not solely driven by endogenous responses, as revenue-based labor productivity and revenue-based Total Factor Productivity (TFP) do not seem to be characterized by either a AR(1) process or to face Gaussian innovations. In fact, we find that TFP growth is even more concentrated and heavy-tailed than labor productivity and, in turn, sales. Nearly half of TFP growth rates are less than 5% in absolute value, 5 times more likely than if the data was normally distributed and more than twice as likely than for sales growth. Rare, large, TFP changes of either sign occur: kurtosis is larger than for sales. TFP growth exhibits mildly negative skewness among large firms, but these firms seem to absorb left-tailed shocks given what we find for sales. An illustrative exercise suggests that this is consistent with firms adjusting their inputs increasingly less as the productivity shock becomes larger.[3]

TFP also exhibits nonlinear persistence of history: in fact, one that is more pronounced than for sales, with double the range of estimated persistence values. In particular, we find evidence of microeconomic disasters hitting large firms. Since this feature holds for productivity but not for sales, it suggests that large firms have ways to absorb the impact of disastrous negative shocks. In summary, we conclude that firms face fundamental shocks that feature non-Gaussianity and nonlinearities.

In order to investigate further the drivers and the implications of our empirical results, we estimate a rich, state-of-the-art, model of productivity dynamics, following work by Arellano et al. (2017) and Arellano et al. (2021) on household earnings. The model features both transitory and persistent components, and allows for non-Gaussianity and nonlinear persistence of history, hence making it consistent with our empirical findings. The estimated model indicates that both persistent and, especially, transitory shocks are non-Gaussian. While a decomposition of this type is common in the household literature, transitory shocks are often overlooked in firm dynamics.

To dig deeper into the source of firm-level shocks, we employ machine learning techniques

[3] In complementary recent work, Fella, Galvez, González, Ruiz-Garcia and Senga (2021) find that Spanish firm-level productivity exhibits non-linearity and non-normality. They find that accounting for these productivity features alters the estimation of capital adjustment costs.

and exploit the richness of our dataset to identify the best predictors of firm-level tail events. Financial constraints, competitors' growth, and some firm-level characteristics weakly correlate with extreme growth. R&D expenses predict greater sales and TFP growth, especially productivity booms. However, most of the overall variation in tail growth remains unexplained by observable factors. Hence, our results are consistent with Berlingieri, De Ridder, Lashkari and Rigo (2024), who show that product innovation works through bursts, which are however not predictable due to the stochastic nature of the innovation process. We also use our model estimates to separately identify the drivers of persistent and transitory shocks. We find that observables explain close to no variation in extreme transitory growth.

Our findings therefore suggest that firms are hit by extreme and unpredictable shocks, of varying degree of persistence. The firm–level stochastic processes we estimate are thus likely to be important for various aspects of firm dynamics, such as input adjustment costs (Cooper and Haltiwanger (2006), Fella et al. (2021)), financial frictions (Moll (2014)), insurance provision (Guiso, Pistaferri and Schivardi (2005)). In the last part of the paper, we focus on two additional aspects. First, we investigate how the nature of idiosyncratic shocks affects the power law behavior of firm size distributions. This aspect has been overlooked by the literature, which typically considers proportional random growth (see Gabaix (2016) for a survey of the literature). Second, rare, large, idiosyncratic shocks as those we document can be a direct source of aggregate fluctuations. Indeed, Gabaix (2011) supports his granular hypothesis with many historical examples that are consistent with our empirical findings.[4] Conceptual frameworks studying the granular hypothesis are typically static and feature little or no discussion on the nature of the shocks. Two notable exceptions are Acemoglu, Ozdaglar and Tahbaz-Salehi (2017) and Carvalho and Grassi (2019). The former paper studies the conditions under which idiosyncratic shocks and sectoral heterogeneity result in macroeconomic tail risk. Carvalho and Grassi (2019) build a systematic framework linking firm dynamics and micro-originated fluctuations. Complementary to these papers, we study the role of specific features of idiosyncratic stochastic processes, namely non-Gaussian shocks and nonlinear persistence, and their quantitative relevance, for the granular hypothesis. We proceed in two steps.

First, we show analytically the role played by the heavy-tailed distribution of shocks in a stylized theoretical setting. Intuitively, non-Gaussian shocks can provide a reason for

[4] Among others, the major strike at General Motors in 1970, or natural disasters in the U.S. (Barrot and Sauvagnat (2016)), Japan (Boehm, Flaaen and Pandalai-Nayar (2019)), and more generally rare economic disasters (Barro (2006)). Other examples of large negative shocks are scandals that reduce firms' profitability. For example, in January 2016 55 E. Coli bacteria infections were linked to food served at Chipotle restaurants. Q1 2016 Chipotle revenues dropped 23%. Large shocks can also be positive, such as those at Walmart in 2002. Sterk, Sedláček and Pugsley (2021) study high-growth, "gazelles", startups, who make persistent contributions to aggregate productivity growth.

a heavy-tailed distribution of firm sizes, which is the typical cornerstone of the granular hypothesis. Moreover, we show that correlated shocks may amplify power law behavior. Extreme and nonlinearly persistent shocks, like those we uncover in the empirical analysis, can therefore substantially affect firms' concentration and its evolution over time. Furthermore, for a given size distribution, we discuss how non-Gaussian shocks play an ambiguous role for granular aggregate fluctuations, depending on whether heavier tails are coupled with larger idiosyncratic variance, and thus greater aggregate volatility.

Second, we use our estimated model to quantitatively assess the importance of non-Gaussianities and nonlinearities for granular fluctuations. To do so, we compute the volatility and autocorrelation of the growth rate in an aggregated measure of TFP, generated by simulating our baseline estimated model and four alternative models. Separating a transitory and a persistent component of productivity, even in an AR(1) setup, increases aggregate volatility towards the empirical counterpart, but at the expense of a counterfactually negative autocorrelation. In contrast, modeling a single component with nonlinear persistence, using estimates from empirical quantile autoregressions, makes the autocorrelation of aggregate TFP growth rates excessively large. Adding Gaussian transitory shocks to this framework results in excessively high volatility and negative autocorrelation. It is only with our full model, which features nonlinear persistence *and* non-Gaussian transitory and persistent shocks, that we can generate aggregate fluctuations that align with the data.

Our exercise thus suggests that non-Gaussian shocks as typically observed in the data are associated with less volatile micro-originated aggregate fluctuations, compared to a Gaussian counterpart. We therefore propose a new, empirically grounded, dampening channel, complementary to, but conceptually distinct from, the size-variance tradeoff documented by Stanley et al. (1996) and Yeh (2019). Our exercise also shows that nonlinearities and non-Gaussianities need to be simultaneously accounted for; failure to do so generates counterfactual behavior, not just at the firm level, but also in the aggregate.

Our paper is closely related to three recent studies investigating heavy–tailed and nonlinear firm-level shocks. Jaimovich, Terry and Vincent (2023) focus on the implications for the stationary distribution of firms and the effects of firm subsidies. Fella et al. (2021) discuss how non-normalities influence the quantitative importance of capital adjustment costs. Salgado et al. (2019) focus on the cyclicality of firm-level skewness. Complementary to these studies, we explore a novel dimension through which the nature of firm-level shocks matters: micro-originated aggregate fluctuations.

The paper is organized as follows. In Section 2.1 we present our empirical results on firm sales. Section 2.2 shows that TFP also follows a non-Gaussian and nonlinear stochastic process. Section 3 estimates a model consistent with the empirical findings. Section 4 investigates the drivers of extreme shocks. Section 5 shows, analytically and quantitatively, how non-Gaussianity and nonlinearities affect power law behavior and the granular hypothesis. Section 6 concludes.

2 Firm-level non-Gaussianity and nonlinearities

2.1 Firm sales

We start by documenting the empirical features of firm sales, using data from US Compustat over the period between 1987 and 2017. We keep firms that remain in operation for at least 2 consecutive years and apply standard selection criteria. We focus on the manufacturing sector but we show robustness to the whole economy in Appendix B.4. Appendix A provides details on the sample. We regress the log of firm sales on industry dummies and a linear time trend, and retain the residuals as our variable of interest. By doing so, we absorb observable permanent differences and low frequency aggregate fluctuations. As such, our measures can be thought as the combination of idiosyncratic shocks and heterogeneous sensitivities to aggregate shocks, if any.[5] For the remainder of the paper, when mentioning sales and productivity we always refer to their residualized counterparts.

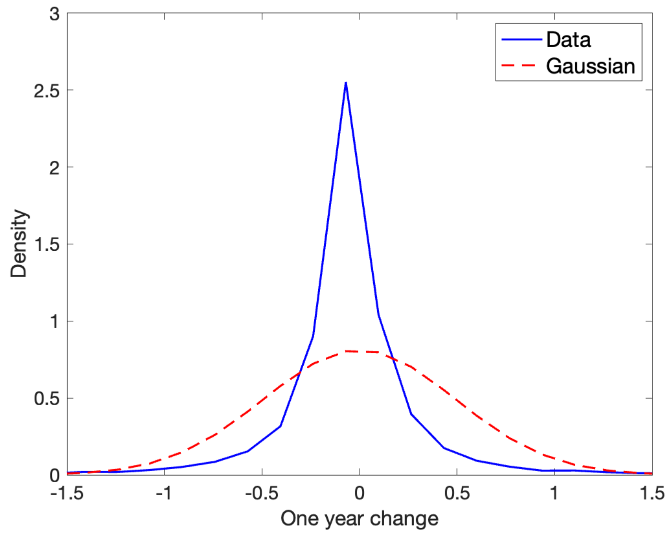
Figure 1 shows how the distribution of sales growth, measured as annual changes in residualized log sales, is non-Gaussian. In the data there are far more extreme sales changes than what is implied by a Gaussian distribution with the same standard deviation as in the data. Indeed, sales growth displays high concentration and heavy tails. In the data, 20% of the time sales growth is less than 5% in absolute value, compared to only 8% under the Gaussian distribution. Two thirds of firms experience an annual sales growth rate smaller than 20% in absolute terms; if the data was normally distributed, only one third of the observations would experience these growth rates.

What are the features of this non-Gaussianity? To investigate this further, we look at the skewness and kurtosis of sales growth, by quantiles of past sales.[6] First, shocks are roughly symmetric even for subsets of firms, as shown in Figure 2a. We report Kelley (1947) skewness, defined as $S_K = \frac{(P90-P50)-(P50-P10)}{P90-P10}$. A positive Kelley Skewness implies that the right part of the distribution $(P50-P10)$ accounts for a larger share of the overall dispersion $(P90-P10)$ than the lower part. This seems to be particularly true for firms with low sales in $t-1$,

[5] For some further discussion on this, see Section 5. In Appendix B.6 we show how our results change if we do not detrend our data.

[6] The population skewness and kurtosis might be infinite with heavy-tailed data, see for instance Sarpietro et al. (2022). Thus, we consider quantile-based measures of skewness and kurtosis, which are robust to the existence of moments, and refer to them simply as skewness and kurtosis throughout the rest of the paper.

Figure 1: Distribution of one-year log sales changes



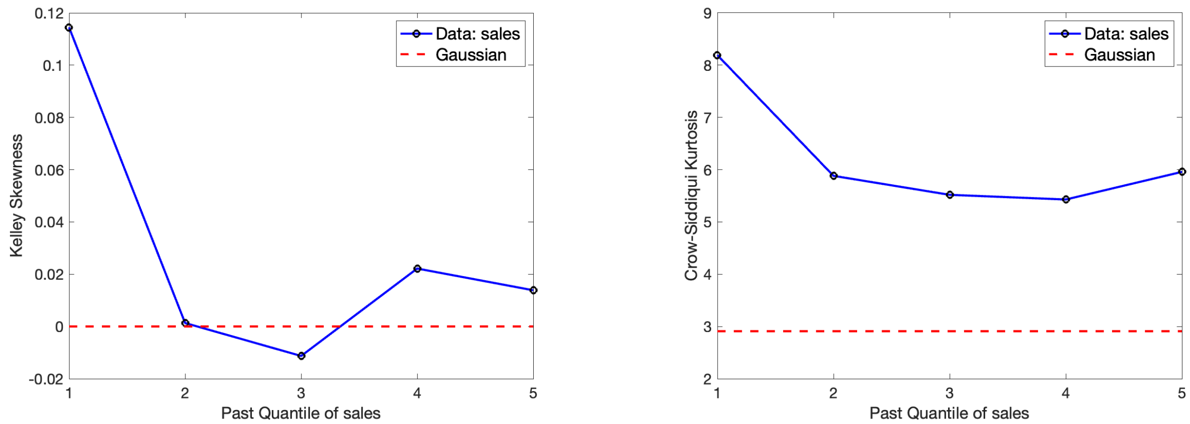
Notes: Empirical densities of residualized log sales changes between year t and $t-1$ depicted in solid blue. The standard deviation is 0.491, Kelley Skewness 0.053 and Crow-Siddiqui Kurtosis 7.207. Gaussian density with the same standard deviation of data shown in dashed red. See main text for details on the data.

although even for this group of firms the departure from symmetry is mild. These relatively small deviations from symmetry are in sharp contrast with substantially negative skewness of household earnings growth that has been documented, among others, by Guvenen et al. (2021). Our findings are instead consistent with the symmetric distribution of employment growth rates among mature firms documented by Decker et al. (2016).

Kurtosis, rather than skewness, is the main driver of non-Gaussianity in firm sales growth. As shown in Figure 2b, firms of different size all experience heavy tailedness of shocks. We measure this with the Crow-Siddiqui measure of kurtosis, defined as $K_K = \frac{P97.5-P2.5}{P75-P25}$, which purges the measure from outliers. Kurtosis is highest at the bottom of the sales distribution and is always higher than 5 for all other quintiles of past sales. These values are significantly higher than the benchmark kurtosis of 2.906 for the standard Normal distribution. Even the largest firms face sales growth that is twice as heavy-tailed as its Gaussian counterpart. Large values of conditional kurtosis are common to previous findings on household earnings. Our results are also consistent with German data on investment surprises, which display excess kurtosis but no significant skewness, as showed by Bachmann et al. (2017). This study interprets their findings as at least partly driven by fat-tailed underlying firm risk, a conclusion that is consistent with our later findings on productivity.

Finally, we focus on the dynamics of firm sales. Following Arellano et al. (2017), we

Figure 2: Skewness and kurtosis of one-year log sales changes



(a) Kelley Skewness (b) Crow-Siddiqui Kurtosis

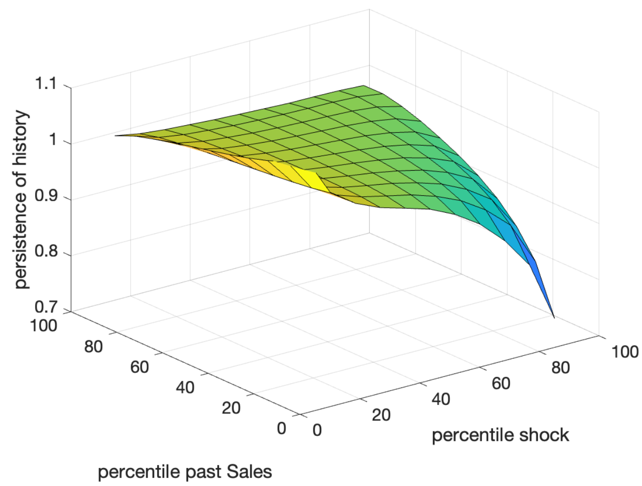
Notes: Observations are ordered by quantiles of the distribution of $t-1$ residualized log sales (x-axis). For each quantile, we show on the y-axis the Kelley Skewness of residualized log sales changes in Figure 2a and the Crow-Siddiqui Kurtosis in Figure 2b.

plot a measure of persistence in the data in Figure 3. The figure shows the estimated average derivative of the conditional quantile function of residualized log sales $y_{i,t}$ given $y_{i,t-1}$.[7] Derivatives are taken with respect to $y_{i,t-1}$ and evaluated at a given percentile of $y_{i,t-1}$, and at a given percentile of the shock. This derivative effect can be interpreted as a measure of persistence in an autoregressive process for log sales. As such, it provides a measure of persistence of past shocks, often referred to as the persistence of history. As discussed by Arellano et al. (2017), a standard AR(1) process would deliver the same level of history persistence for all firms, regardless of their past sales and/or shocks: in other words, a flat three-dimensional figure. Figure 3, instead, shows some nonlinearities, with persistence of firm sales history ranging from 0.73 to slightly above 1. History persistence is lowest for low-sales firms hit by a large positive shock. Thus, a large positive shock to firms at the bottom of the sales distribution can wipe out the history of past negative shocks (see the right end of the graph). The opposite does not hold true: for high-sales firms hit by the smallest (most negative) shocks persistence is around 1. Hence, negative shocks do not cancel out the good sales history of large firms.

The one-sided nonlinearity that we document for firm sales differs from recent findings on household earnings by Arellano et al. (2017), with high-income households facing "micro disasters". It is possible that these micro disasters also hit large firms and make them exit the market, thus eluding our selected sample. However, we believe that this channel should play a limited role. First, publicly quoted firms are less likely to exit than all firms in the

[7] As done in the literature, quantile functions are set as third-order Hermite polynomials.

Figure 3: Quantile autoregressions of log sales



Notes: Estimated average derivative of the conditional quantile function of residualized log sales $y_{i,t}$ given $y_{i,t-1}$. Derivatives are taken with respect to $y_{i,t-1}$ and evaluated at a given percentile of $y_{i,t-1}$, and at a given percentile of the shock. Details on data and estimation of quantile functions in the text.

population, thus making Compustat less prone to this selection issue. Second, we are also keeping firms that stay in the sample for only two years, thus limiting selection as much as possible. Third, we find that the probability of firm exit monotonically decreases with past sales. As such, even if present, this selection issue is less likely to be relevant among high-sales firms. Fourth, as we show next, large firms do face micro disasters – in the form of fundamental shocks – but they seem to absorb them via input adjustment. As such, persistence of history is more nonlinear for productivity than for sales.

To conclude, we have shown several statistical properties of firm sales that are inconsistent with a simple random walk process with gaussian shocks, even in a fairly homogeneous sample as Compustat. Next, we investigate whether our findings are solely driven by endogenous responses turning to two alternative productivity measures: Labor Productivity (LP) and Total Factor Productivity (TFP).

2.2 Productivity

In this section we show that the documented non-Gaussianity and nonlinearities in firm sales likely reflect the features of the distributions of more exogenous measures of firm performance.

We look at two measures of productivity: revenue labor productivity and revenue-based total factor productivity. We assume that firms operate with a Cobb-Douglas production, using a composite measure of variable inputs, C, and capital stock, K. Then, for firm

$i = 1, ..., N$ at time $t = 1, ..., T$, we estimate TFP as a residual of the revenue function as follows:

$$z_{i,t} = y_{i,t} - \beta_c c_{i,t} - \beta_k k_{i,t} \quad (1)$$

where $y_{i,t}$ is the log sales of firm i at time t, $c_{i,t}$ is the log of the variable input and $k_{i,t}$ is the logged capital stock. Following the analysis by De Loecker et al. (2020), also with Compustat data, we use cost of goods sold (COGS) for $c_{i,t}$. This variable combines all expenses directly attributable to the production of the goods sold by the firm; as such, it includes not only labor costs, but also other inputs such as materials, intermediates, and energy. Compustat does not provide a detailed breakdown of these inputs.[8] We construct the capital stock following Ottonello and Winberry (2020), as we discuss in Appendix A. Since we do not observe firm-level prices, our TFP measure effectively combines idiosyncratic technical efficiency shocks with demand shocks. With this caveat in mind, this remains the relevant measure of fundamental shocks that affect firms' revenues, as discussed by Decker et al. (2020) and Blackwood et al. (2021).[9] In order to estimate the revenue elasticities, β_x, with $x = \{c, k\}$, we follow Decker et al. (2020) and note that the first-order condition on a given production input x from static profit maximization implies that β_x equates the share of that factor's costs in total sales. We construct these ratios using industry-level data from the KLEMS database constructed by the Bureau of Labor Statistics (BLS). Our revenue elasticities are year-specific and industry-specific. The industry granularity we use is 3-digit NAICS for the manufacturing sector and 2-digit for the whole economy. The first order condition pinning down revenue elasticities holds if there are no factor adjustment costs or wedges; as is typically assumed in the literature (see for instance Decker et al. (2020)), we require this condition to hold on average across firms within an industry. As suggested by Syverson (2011), we further strengthen this argument by averaging the revenue elasticities over 6-year windows. We choose this approach over some alternatives for TFP estimation to avoid imposing restrictive assumptions on the stochastic process. In the literature, our TFP measure has been called TFPS (for "TFP-shares"), as discussed by Decker et al. (2020).

As for sales, our baseline sample consists of firms operating in the manufacturing sector from US Compustat over the period between 1987 and 2017; we discuss various robustness exercises in Section 2.3. As done for sales in the previous section, we regress the log of firm productivity measures on industry dummies and a linear time trend and retain the residuals

[8] Keller and Yeaple (2009) impute a measure of intermediate inputs, but this requires additional assumptions.

[9] See De Loecker and Goldberg (2014) for a discussion of output and input price heterogeneity in the context of TFP estimation. DeVera (2023) use Portuguese data to disentangle productivity from demand shocks: they find nonlinear persistence in either component, consistent with our results.

Table 1: Concentration of sales, labor productivity and TFPS growth rates

$S:$	$[-0.05, 0.05]$			$[-0.10, 0.10]$			$[-0.20, 0.20]$		
	Data	$\mathcal{N}(\mu_x, \sigma_x)$	Ratio	Data	$\mathcal{N}(\mu_x, \sigma_x)$	Ratio	Data	$\mathcal{N}(\mu_x, \sigma_x)$	Ratio
$Prob\,(\lvert\Delta\text{Sales}\rvert \in S)$	19.8	8.1	2.4	38.7	16.2	2.4	64.5	31.4	2.1
$Prob\,(\lvert\Delta\text{LP}\rvert \in S)$	28.8	8.6	3.4	49.2	16.9	2.9	70.8	33.2	2.1
$Prob\,(\lvert\Delta\text{TFP}\rvert \in S)$	45.4	8.9	5.1	67.4	18.1	3.7	83.6	35.0	2.4

Note: One year changes in logged variables, using baseline sample as described in the text. Columns for $\mathcal{N}(\mu_x, \sigma_x)$ refer to draws, of size equal to the dataset, from a normal distribution with mean μ_x and standard deviation σ_x, where x denotes the variable considered in each row. Standard deviations are 0.491, 0.468, and 0.441 for changes in log sales, LP, and TFP, respectively.

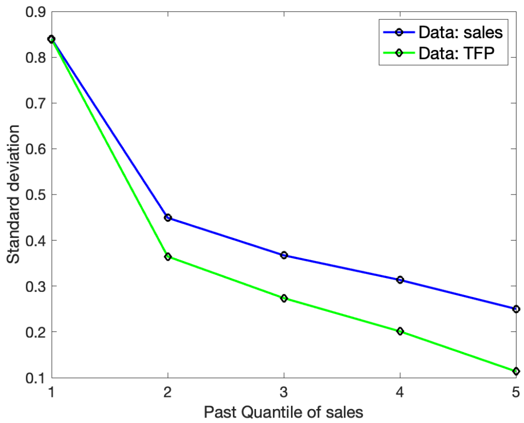
as our variable of interest.

We document that growth rates (i.e., one-year log changes) of both productivity measures display clear departures from non-Gaussianity, in the form of high concentration and heavy tails. Indeed, firms typically experience small or no changes in productivity, but they are occasionally hit by extreme, negative and positive, shocks. In fact, Table 1 reports how these features are more markedly pronounced for productivity than for sales. Sales growth less than 5% in absolute value has a 20% probability of occurring in the data, more than twice as likely as if the data was normally distributed, with the same empirical standard deviation. Changes in productivity are even more concentrated: nearly half of TFP growth rates are less than 5% in absolute value, 5 times more likely than if the data was normally distributed. In Appendix B.1 we show this concentration towards negligible growth rates is compensated by the presence of rare and heavy-tailed events.

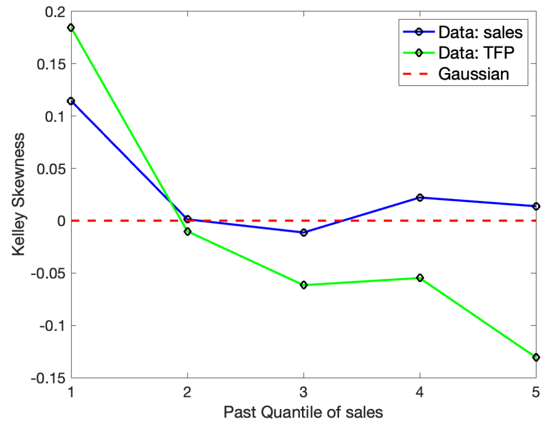
We further explore the features of this non-Gaussianity by investigating moments of firm-level changes. We focus on TFP, but report similar results for labor productivity in Appendix B.3. Figure 4a plots the standard deviation of firm-level one-year log-changes in sales and TFP, for 5 quintiles of past sales. Both measures decay with size; this behavior could be consistent with the fact that larger firms are more stable and established. Not only they face less volatile sales, but also less dispersed fundamental shocks. The standard deviation of sales growth is slightly higher than for productivity, a feature consistent with standard theoretical frameworks of firm decisions.

These results may suggest that sales inherit most of the underlying features of the productivity process. However, we show that the picture is more complicated when we look at higher order conditional moments of productivity changes, which lie at the core of our results on non-Gaussianity and nonlinearities. Specifically, we focus on quantile-based measures of skewness and kurtosis as done for sales in Section 2.1. Figure 4b plots Kelley Skewness.

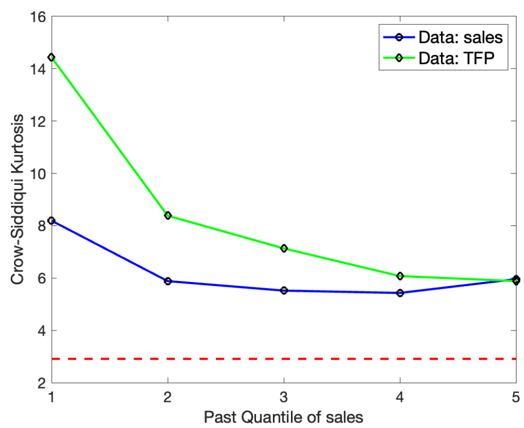
Figure 4: Conditional moments of firm-level one year log-changes



(a) Standard Deviation



(b) Kelley Skewness



(c) Crow-Siddiqui Kurtosis

Notes: Solid-circled blue lines refer to the log of firm sales, and solid-diamond green lines to log TFP. For each measure, observations are ordered by quantiles of the sales distribution of $t-1$ (x-axis). For each quantile, we show on the y-axis the Standard Deviation in Figure 4a, the Kelley Skewness in Figure 4b and Crow-Siddiqui Kurtosis in Figure 4c of log-changes.

Similar to what shown for sales, there is little departure from symmetry. However, conditional skewness of productivity growth falls more clearly with firm size, while the same is not true for sales. Most importantly, large firms display negative tail risk in productivity, but symmetric sales growth. This suggests that they are more likely to be hit by large negative productivity shocks, but manage to avoid large sale losses, likely through input adjustment.

We also find that the heavy tails in sales growth are most likely driven by heavy-tailed underlying exogenous shocks. Figure 4c displays Crow-Siddiqui kurtosis for our different measures, conditional on past quantiles of sales. Departures from gaussianity are even more pronounced for TFP than for sales. Firms at the bottom of the distribution are particularly likely to face very large shocks, of either sign.

In summary, we conclude that firms face shocks that are fundamentally non-Gaussian, and therefore what documented for firm sales is not solely the result of endogenous responses to these shocks. In fact, firms behave in a way that reduces the degree of heavy tailedness and asymmetry of their sales, relative to the shocks that they face. While investigating this is beyond the scope of this paper, in Appendix C we estimate a simple reduced form setup that can shed some light on these findings. We find that firms respond to shocks increasingly less as the productivity change becomes of larger magnitude. This behavior is broadly consistent with various economic models of firm behavior, such as those featuring hiring costs and financial frictions.

Finally, we study the dynamics of productivity. We have previously shown how firm sales depart from a standard AR(1) process and display lower persistence of history among small firms being hit by large positive shocks. In other words, large positive shocks erase part of the history of previous bad realizations for firms with low sales. We confirm that this behavior seems to be inherent to fundamental shocks, rather than firm behavior, as shown in the right panel of Figure 5. Moreover, we highlight the appearance of a nonlinear left tail. Such behavior implies that very negative productivity shocks cancel the history of past positive productivity shocks when hitting firms with highest productivity. An equivalent mechanism has been highlighted for household earnings by Arellano et al. (2017); we show it is also present among firms, albeit quantitatively more limited.[10] Persistence of history is more heterogeneous for TFP than for sales, ranging between 0.5 and 1.3.

2.3 Robustness

Our results are robust to different samples. In addition, we show how they change with the frequency of the data.

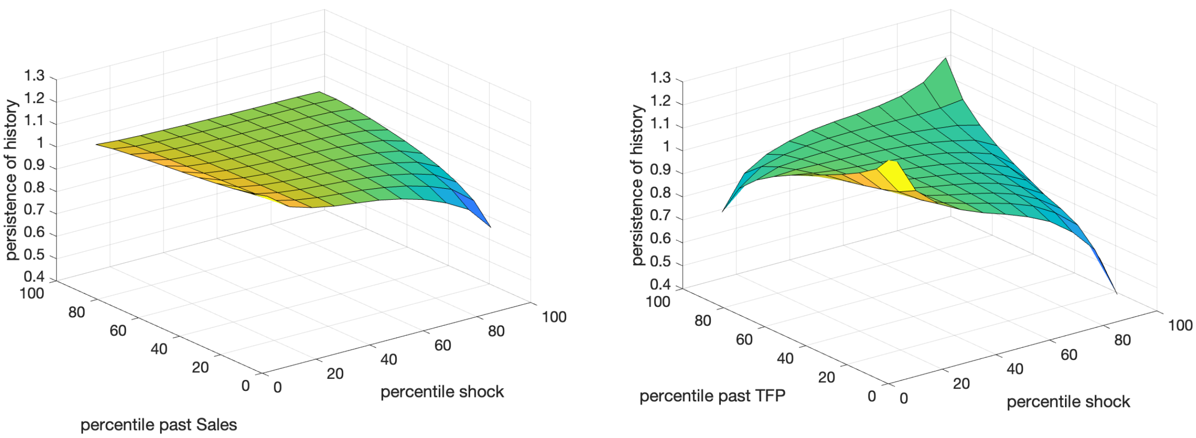
Industries, trimming, and recessions

All our empirical findings broadly apply to the whole economy, rather than just the manufacturing sample, as we show in Appendix B.4. In a second robustness check, we restrict the sample by trimming relatively small firms.[11] All the results are basically unaffected, suggesting that the differential behavior at the bottom of the distribution is not driven by outliers. We also compute confidence bands via bootstrapping. The tight confidence bands shown in Appendix B.5 confirm the precision and reliability of our findings. Finally, our findings

[10] All the results presented in this section are qualitatively unchanged when considering all industries.

[11] In particular, we drop firms with less than 10 employees, or either sales and total assets less than 100 thousand dollars. As such, we exclude about 3% of the firms relative to the baseline sample.

Figure 5: Quantile autoregressions



(a) Sales (b) TFP

Notes: Estimated average derivative of the conditional quantile function of residualized values $y_{i,t}$ given $y_{i,t-1}$. Derivatives are taken with respect to $y_{i,t-1}$ and evaluated at a given percentile of $y_{i,t-1}$, and at a given percentile of the shock. Details on data and estimation of quantile functions in the text.

on kurtosis do not hinge on specific aggregate conditions.[12] Indeed, sales growth kurtosis is above 5 for all years in our sample and high across all quintiles of past sales even during recessions. We thus conclude that the heavy tailedness is a key feature of the distribution of sales and productivity growth and represents a clear departure from normality.

Data frequency

First, we investigate whether the departures from normality are driven by transitory changes only. Following Guvenen et al. (2021), we consider a measure of persistent changes defined as the difference between two 3-year averages of residualized log sales (or productivity) as follows: $\Delta(\tilde{y}) = \bar{y}_{t-1,t+1} - \bar{y}_{t-2,t-4}$, where $\bar{y}_{t-j,t-k}$ is the average of log sales over the period $[t-j, t-k]$. Through averaging before differencing, this alternative measure allows us to purge the effects of the transitory component of sales changes. We repeat our analysis with this measure of persistent changes and obtain qualitatively similar results, as shown in Appendix B.2. We find that persistent changes are non-Gaussian, although they exhibit lower concentration and relatively less heavy-tailedness compared to the benchmark measure of overall sales and productivity changes.

Second, we leverage the Compustat data and look at firm-level dynamics at the quarterly frequency. We find that the distributions of quarter-on-quarter sales and productivity growth

[12]For an analysis on how higher order moments of sales growth evolve over the business cycle see Higson et al. (2002) and Bloom et al. (2018).

are also clearly non-Gaussian, especially due to high kurtosis. Consistent with what shown before for persistent changes, quarterly changes are more heavy-tailed than yearly growth. This suggests that some non-Gaussian shocks – transitory in nature – wash out over time, but some effects persist for longer: these mechanisms will be featured in the statistical model we estimate in the next section, which formally separates transitory and persistent components of firm-level productivity. Persistence of history is also nonlinear, in a similar fashion to what shown previously for one-year changes. We detail these results in Appendix B.7.

We have shown that firm-level productivity features non-Gaussian shocks and nonlinear persistence. What statistical process is consistent with our findings and how can we use an estimated model to learn about micro and macro fluctuations? This is what we investigate in the next section.

3 A nonlinear model of productivity

The empirical evidence presented so far suggests that the dynamics of firm productivity are characterized by a rich process. In this section, we estimate a model of productivity that can rationalize the empirically observed nonlinearities and non-Gaussian shocks. We use a state-of-the-art framework and techniques proposed by Arellano et al. (2017) (thereafter, ABB) and Arellano et al. (2021) (thereafter, ABBL) and leverage two main features. First, we assume that the stochastic process of productivity is a combination of a persistent and a transitory component, allowing us to disentangle the drivers of our empirical findings showed so far. Second, we introduce potential nonlinearities in the persistence of shocks using a quantile-based panel data framework.

Allowing for shocks of different degrees of persistence is not common in the study of firm dynamics, where an AR(1) process for the log of productivity is typically assumed.[13] This stands in sharp contrast with the literature on household earnings dynamics, in which transitory and persistent shocks are typically separated, even in a standard framework without nonlinearities.[14] Persistence of idiosyncratic productivity is, however, of first-order importance for firm dynamics too. Among others, Moll (2014) shows how the persistence of productivity shocks interacts with financial frictions and matters for the size and the speed of aggregate transitions. Moreover, transitory and persistent shocks are likely to affect firms' inputs differently, depending on the nature of their adjustment costs, and whether they face a static or forward-looking input demand (see seminal work by Cooper and Haltiwanger

[13] Guiso et al. (2005) and Roys (2016) are some notable exceptions, although both of these works do no allow for any nonlinearities.

[14] In the earnings literature, the transitory-persistent time series structure can be dated back to Friedman (1957) and MaCurdy (1982).

(2006) and a specific discussion by Roys (2016)). A further example of the importance of this distinction can be found in the corporate finance literature: Décamps et al. (2016) show how permanent and transitory cash-flow shocks are associated with very different corporate policies. For instance, their relative importance substantially alters the cash–flow sensitivity of cash.

Motivated by these considerations, we estimate the model proposed by ABB. Let $o_{i,t}$ denote the residualized log of productivity, net of a linear time trend and industry effects, of firm i at time t, where $t = 1, ..., T$ and $i = 1, ..., N$. We model $o_{i,t}$ as the sum of a general Markovian persistent component and a transitory innovation, as follows:

$$o_{it} = \eta_{it} + \epsilon_{it} \tag{2}$$

We allow the persistent component η_{it} to follow a first-order Markov process:

$$\eta_{it} = Q(\eta_{it-1}, u_{it}) \qquad (u_{it}|\eta_{it-1}, \eta_{it-2}, ...) \sim Uniform(0, 1), \quad t = 2, ..., T \tag{3}$$

where $Q(\eta_{it-1}, u_{it})$ is the u_{it}-th conditional quantile of η_{it} given η_{it-1}. The transitory component, ϵ_{it}, has zero mean and is independent over time and of η_{is} for all $s, s = 1, ..., T$.[15]

We take from ABB the argument for identification, which relies on the literature on nonlinear models with latent variables (see Hu and Schennach (2008), Wilhelm (2015)). This quantile-based model allows us to capture nonlinear persistence and conditional heteroskedasticity. In particular, it highlights a role for the persistence of histories. Consider a shock u_{it} to η_{it-1}; its impact on TFP depends on the shock occurring at time $t + 1$, u_{it+1}, whose impact on TFP will in turn depend on the shock occurring at time $t + 2$, and so on. More formally, nonlinear persistence is:

$$\rho(\eta_{it-1}, \tau) = \frac{\delta Q(\eta_{it-1}, \tau)}{\delta \eta} \tag{4}$$

which is a measure of the impact of η_{it-1} on η_{it} and, as a result, on o_{it}, when firm i is hit by a shock u_{it} that has rank τ.

[15]Note that the process can be extended to allow for a higher order Markov process for η, a moving average for ϵ, or to allow for unobserved time-invariant firm-specific effects. As noted in the literature, it is not possible to disentangle measurement error from the transitory innovation without additional information. We believe, however, that this should be less of a concern for our dataset – audited balance sheets – than for survey data commonly used in the earnings literature.

Average nonlinear persistence across η is:

$$\rho(\tau) = E\Big[\frac{\delta Q(\eta_{it-1}, \tau)}{\delta \eta}\Big] \tag{5}$$

The ρ's are measures of persistence of productivity histories. Here one can see how this flexible model can allow the persistence of the shocks to the persistent component, η_{it-1}, to depend on the size and sign of current and future shocks u_{it},u_{it+1}... Within this framework it is possible to analyze how unusual shocks to firm productivity can significantly reduce or increase the persistence of past shocks. For instance, micro-disasters that wipe out the memory of all past productivity shocks.[16] We therefore study a framework in which unusual shocks can occur, which wipe out the productivity effect of a series of past shocks. Similarly, the persistence of current shocks is subject to the realization of future shocks.

We closely follow the estimation strategy proposed by ABBL. We specify the conditional quantile function describing the process for the persistent component η_{it} using Hermite polynomials. As a result, we write the expression in Equation (3) as:

$$\eta_{it} = Q(\eta_{it-1}, \tau) = \sum_{\kappa=0}^{K} a_{\kappa}^{Q}(\tau)\varphi_{\kappa}(\eta_{i,t-1}) \tag{6}$$

where φ_κ denotes the basis of the Hermite polynomials and a_κ^Q the corresponding coefficients. In implementation, we follow ABBL and set $K = 3$. We define $\tau_l = l/(L+1)$, with $L = 11$, and specify the functions a_κ^Q to be piecewise-linear in the interval $[\tau_1, \tau_L]$. In addition, we follow ABBL and treat a_0^Q as quantiles of an exponential distribution in the ranges $(0, \tau_1]$ and $[\tau_L, 1)$. This introduces two tail parameters, those defining the exponential distributions; we estimate each not only for $\eta_{i,t}$, but also for the transitory shock and for the persistent component in the first period, $\eta_{i,0}$. Computationally, estimation is implemented using the Sequential Monte Carlo methods proposed by ABBL. We defer to their paper for an extensive discussion of the computational details of the algorithm.[17]

In the remainder of this section, we analyze the drivers of the empirically documented nonlinearities in productivity through the lens of the estimated model. Figure 6 displays estimates of the average derivative of the conditional quantile function of residualized log-

[16]Note that this nonlinear model nests the canonical (linear) model, typically used in the literature, as a special case. The canonical model is obtained by restricting η to follow a random walk, i.e. $\eta_{i,t} = \eta_{i,t-1} + \nu_{i,t}$. In the canonical model, all shocks are associated with the same persistence, irrespective of the history of a firm's productivity. A shock ν_{it} enters η_{it} linearly with persistence $\rho = 1$, so that the shock transmits entirely to o_{it}, regardless of a firm's past sequence of shocks to TFP.

[17]The Laplace tails assumption is only imposed in the implementation and can be relaxed and replaced by a more heavy-type tails assumption. However, we do not pursue this idea here and follow instead the same implementation strategy as suggested by ABB and ABBL.

TFP. The top-left panel 6a reproduces the empirical persistence for the data, also showed previously in Figure 5b. Persistence of productivity is far from homogeneous, as it depends both on the initial level of productivity and the size of the productivity shock, and it ranges between 0.5 and 1.3. The nonlinear model does a good job at replicating the empirical patterns, as shown in Figure 6b. In Figure 6c, we show the derivatives of the conditional quantile function for the persistent component of the estimated productivity process. The nonlinear patterns of persistence are similar to those estimated for the entire productivity process, although slightly less pronounced, suggesting a marginal role of transitory shocks for nonlinear dynamics. Our estimates also shed light on the behavior of specific firms. In Appendix E we show how large positive and negative shocks shocks to the persistent component of productivity asymmetrically affected a large firm in the sample (Apple). This case study provides an example of how micro disasters can erase the persistence of history.

Another important feature of the nonlinear model is that it allows for non-Gaussianity of both components of the productivity process. As shown in Appendix E, we find that both shocks are non-Gaussian, even when sorting firms by their productivity.[18] Indeed, even the most productive firms face persistent and transitory shocks that feature significant departures from Gaussianity.

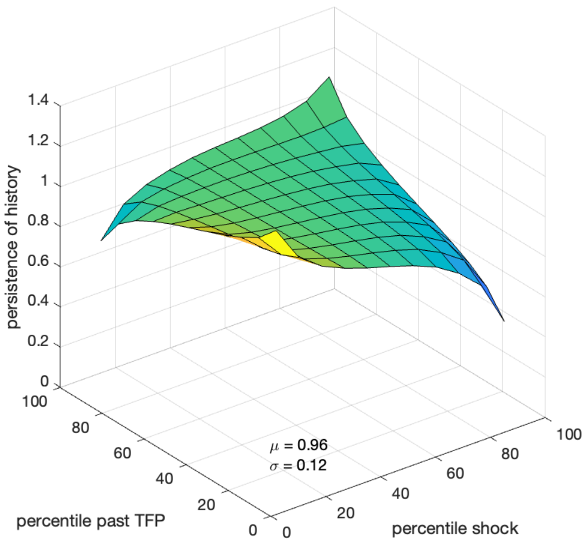
4 What is the source of firm-level shocks?

In this section we employ econometric and machine learning techniques to investigate what lies behind the non-Gaussianities and non-linearities that we have documented thus far. Although our analysis does not isolate causal links, we identify which economic factors are the most important predictors of firm-level extreme growth. Despite uncovering several correlations, we find that most of the overall variation in tail growth remains unexplained. We also use the estimates from our econometric model to separately investigate whether the source of persistent shocks differs from that of transitory shocks.

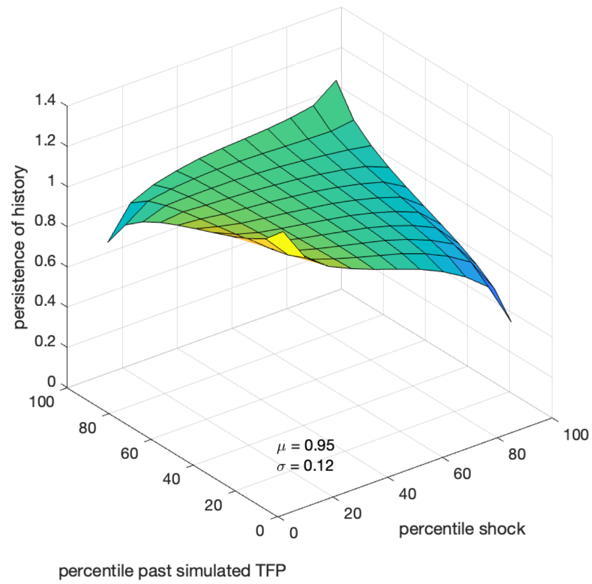
We consider the following drivers. First, we follow Farre-Mensa and Ljungqvist (2016) and construct various measures of financial constraints employed in the literature. Second, we consider some firm-level financial characteristics, such as the cash-to-asset ratio or the leverage ratio. Third, we look at other firm characteristics like their age, their R&D expenses (as a share of total assets), the industry in which they operate. Fourth, we construct industry-year measures of sales and TFP growth. These measures aim at capturing competitors' performance. Fifth, we use patent and publications data from Arora et al. (2019). Finally,

[18] Reassuringly, the model is able to match well the conditional moments of productivity growth presented in Section 2.2, as we show in Appendix E.

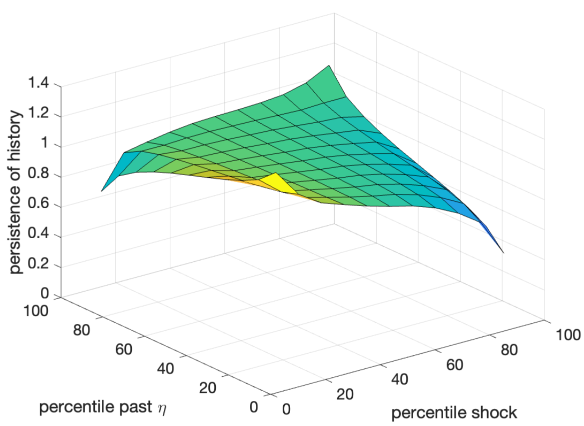
Figure 6: TFP persistence in the data and in the model



(a) Data



(b) Nonlinear model



(c) η_{it}, nonlinear model

Notes: Panels (a) and (b) plot the estimated average derivative of the conditional quantile function of $o_{i,t}$ given $o_{i,t-1}$. Derivatives are taken with respect to $o_{i,t-1}$ and evaluated at a given percentile of $o_{i,t-1}$, and at a given percentile of the shock. Panel (a) is based on the Compustat data, while panel (b) is based on data simulated according to the nonlinear model, with parameters set to their estimated values. Panel (c) shows the same object for the persistence component of the nonlinear model, η. μ and σ are the average and standard deviation of the persistence of history in each panel.

we consider production inputs potentially affecting firm sales, such as employment, cost of goods sold, capital, both in levels and in growth rates. The list of all the variables considered, and their construction, are detailed in Appendix F.1.

We begin by looking at what predicts sales growth. To identify the most relevant predictors, we run a LASSO estimation for different levels of the regularization parameter and at the optimal shrinkage parameter selected through cross-validation.[19] The most important predictors are average sales growth among the industry competitors, R&D expenses, acquisitions, and profitability, which are all positively correlated with sales growth.[20] However, all these variables (and many others) explain less than 10% of the overall variation in sales growth. Not surprisingly, sales growth is strongly correlated with changes in flexible production inputs, such as the cost of goods sold, and the R^2 increases substantially when we include them. We report all these results in detail in Appendix F.2.

We repeat the same analysis for TFP growth, finding similar predictors as for sales growth: profitability, R&D, as well as changes in production inputs such as employment growth. Financially constrained firms are typically associated with higher TFP growth in the optimal LASSO, but the coefficients are close to zero. The average TFP growth of competitors in the industry is strongly positively correlated with firm-level TFP growth, suggesting that productivity booms may reflect industry shocks. However, the adjusted R^2 remains low: typically below 3%, and less than 20% even when we include changes in firm's production factors.

We are particularly interested in what drives tail growth. To shed light on this, we perform a classification LASSO approach analogous to what discussed previously, but using as dependent variables whether TFP growth is in the bottom 1% of the distribution, in the top 1%, or in either tail. Interestingly, when the regularization parameter is set "optimally" through cross-validation, there are more predictors for extremely negative TFP growth than for positive shocks. The cash-to-assets ratio and profitability strongly predict tail TFP growth. R&D expenses is a strong, positive, predictor of large productivity booms, and the nonlinear relationship with tail TFP growth can also been seen via the quantile regression estimates reported in the Appendix.[21] The growth rate in the number of patents is positively correlated with TFP (and especially sales) growth at the top. Moreover, the number of new publications and new patents are a very important predictor of top 1% TFP growth when

[19] To start with, we exclude the variables that are mechanically associated with sales growth through the firm's production function.

[20] When considering a sub-sample matched with patent information, the growth rate in patents and publications are also "optimal" predictors, but the associated LASSO coefficients are close to zero.

[21] We also build a random forest, which allows for nonlinear effects. This approach highlights the importance of other predictors: for instance, investment opportunities (market-to-book ratio as in Frank and Goyal (2009)) predicting large TFP drops.

considering a random forest, which allows for nonlinear effects. Hence, our findings are consistent with recent work by Berlingieri et al. (2024), who show that product innovation works through bursts, which eventually lead to large revenue growth. However, most of the likelihood of experiencing tail growth remains unexplained by observable factors. As we show in Appendix F.2, R^2 is always less than 15%, and typically well below that value.[22]

Finally, we use the estimates of our econometric model outlined in Section 3 to conduct the same analysis separately for changes in the nonlinear, persistent, component of TFP – $\Delta\eta$ – and changes in the transitory component – $\Delta\epsilon$.[23] The predictors of the persistent component are similar to those of overall TFP growth. The R^2 remains low across specifications. Predicting transitory shocks appears to be even harder. Profitable firms are less likely to face extremely negative transitory shocks; interestingly, R&D does not only positively affect the likelihood of a persistent productivity shock, but also that of a transitory boom, although this effect is small. Nevertheless, transitory shocks remain mostly unexplained by observables, especially at the tails, with the R^2 reaching 2% at most.

In summary, our findings suggest that some economic drivers, a few of which have been explored in the literature, lie behind extreme firm shocks. These forces provide an economic micro-foundation for our statistical estimates. Nevertheless, most of the variation in firm (tail) growth remains unexplained. The firm–level stochastic processes we estimate can thus be used as inputs for structural models that take the source of the shocks as given. In the next sections we offer two examples of this approach. First, we discuss analytically how non-Gaussian shocks can affect the power law behavior of firm size distributions. Second, how a rich stochastic process with nonlinear persistence affects the aggregate implications of granular shocks.

5 Micro shocks and macro outcomes

In this last section, we investigate how the nature of idiosyncratic shocks affects the power law behavior of firm size distributions and the aggregate implications of granular shocks.

The literature has typically focused on how proportional random growth can lead to a power law in the firm size distribution (see Gabaix (2009) and Gabaix (2016) for surveys of the literature and Gibrat (1931)), thus restricting and, at the same time, overlooking the actual nature of idiosyncratic shocks. Similarly, the granular hypothesis (see Gabaix (2011) for a seminal paper) typically hinges on a heavy-tailed firm size distribution; in this setting,

[22] A similar finding holds for sales tail growth, as we show in Appendix F.2.

[23] Simulations are done in-sample via sequential MCMC for η. ϵ is defined as the difference between residualized log TFP in the data and model–simulated η.

idiosyncratic firm-level shocks – of any form – do not average out, and thus shocks to large firms can explain a significant part of aggregate movements.[24]

We shift the focus towards the features of the *shocks* that firms face, which have received much less attention. First, we highlight some key mechanisms in a stylized theoretical setting. Second, we discuss a quantitative exercise that showcases the role of non-Gaussianity and nonlinearities for the volatility and persistence of micro–originated aggregate fluctuations.

Our approach and interpretation of firm-level shocks are similar in spirit to Salgado et al. (2019), who argue that business cycles can originate from fluctuating skewness of idiosyncratic shocks. As in their framework, one might wonder whether the features of the distribution of the idiosyncratic shocks derive endogenously from macro shocks rather than being the sources of aggregate fluctuations. This decomposition is beyond the scope of our paper. Conceptually, we effectively consider a world without aggregate shocks, where aggregate fluctuations in productivity are solely generated by idiosyncratic shocks. Within this setting, we study the role that the features of the distribution of firm-level shocks and their dynamics play for aggregate fluctuations.

5.1 Analytical insights

We consider the following general setting. We assume that there are N firms and each firm i produces a quantity S_{it} of consumption good at time t. The firm growth rate is:

$$\frac{\Delta S_{it+1}}{S_{it}} = \frac{S_{it+1} - S_{it}}{S_{it}} = U_{it+1} \tag{7}$$

where U_{it+1} are random variables with unrestricted distribution except for zero mean. To draw a parallel with the model introduced in Section 3, note that the difference $o_{it+1} - o_{it}$, where o_{it} is defined in Equation (2), approximates $\frac{\Delta S_{it+1}}{S_{it}}$ in Equation (7) for small $\frac{\Delta S_{it+1}}{S_{it}}$. Hence, U_{it+1} approximates the innovations to the components $\eta_{it} + \epsilon_{it}$.[25]

This setting can be used to study the origins of the power law behavior in firm size distribution and its evolution over time. In the literature, proportional random growth is a popular explanation for power laws (Champernowne (1953); Simon (1955)). In the general model described above, let us assume the existence of an initial distribution of firm sizes, S_{i0}, and that U_{it+1} is independent of firm sizes and over time. Then, firm sizes satisfy Gibrat's law (Gibrat (1931)), which posits that all firms have the same expected growth rate and the same variance of growth rates, i.e. $\text{Var}(U_{it+1}) = \sigma^2$. Over time, this leads to a distribution

[24] There have been several recent empirical applications of this idea. Among others, see Amiti and Weinstein (2018) on granular bank supply shocks and aggregate investment.

[25] To make this parallel we have obviously assumed that S is a measure of productivity.

that becomes lognormal with larger and larger variance; this, coupled with some friction such as a lower bound on size, results in a stationary distribution which is a power law (see Gabaix (2016)).

We explore two properties of the distribution of shocks that may influence the rate at which power law behavior emerges and the heaviness of the tails of the firm size distribution. First, we introduce heavy-tailed shock distributions. Second, we depart from proportional random growth and explore a dynamic framework where shocks may exhibit persistence and correlate with size.

Heavy-tailed shocks can provide a reason for heavy-tailed size distributions

Let us start with the static case. We analyze how the distribution of U_{it+1} may alter the distribution of S_{it+1}.[26] From Equation (7), we can rewrite next-period sales as:

$$S_{it+1} = S_{it}(1 + U_{it+1}) \tag{8}$$

The tail behavior of next-period sales depends on the tail behavior of the two factors on the right hand side of Equation (8). Let us assume that $S_{i,t}$, which has positive support, and U_{it+1} are independent. To fix ideas, we consider different cases. If both S_{it} and U_{it+1} are Pareto distributed, then S_{it+1} will have a power tail behavior, i.e., S_{it+1} will have a Pareto-type tail, being regularly varying (RV) at ∞ in the upper tail.[27] If, instead, $S_{i,t}$ has an exponential upper tail, then the power law-behavior of the tail of U_{it+1} is enough to make S_{it+1} have a power tail behavior. As such, non-Gaussian shocks can provide a reason for heavy-tailed size distributions. Finally, if $S_{i,t}$ is Pareto, then $S_{i,t+1}$ will have a power law behavior regardless of whether the distribution of U_{it+1} has an upper heavy tail or not. To establish these results, see, for instance, Proposition 1.3.9 in Mikosch (1999).

While non-Gaussianity of U_{it+1} maintains that S_{it+1} will have a power law behavior, it is not obvious that S_{it} will dominate the power law behavior of S_{it+1}. The heaviness of the tails and variance of U_{it+1} might alter the features of the distribution of S_{it+1}. Thus, the distribution of the error term might be relevant in shaping the distribution of S_{it+1}.

[26] For simplicity, consider $o_{it} = \epsilon_{it}$. Then, $U_{it+1} = \epsilon_{it+1} - \epsilon_{it}$. If ϵ_{it} and ϵ_{it+1} are uncorrelated and feature heavy tails, then U_{it+1} will also feature heavy tails by Remark 1.3.5 in Mikosch (1999).

[27] A random variable X is said to be regularly varying at ∞ with Pareto exponent $\alpha(X^+)$ if, for any $x > 0$ as $t \to \infty$:

$$\frac{\text{Prob}(X > tx)}{\text{Prob}(X > t)} \to x^{-\alpha(X^+)}.$$

Correlated shocks may amplify power law behavior

Next, we consider a dynamic framework with potentially correlated shocks. To take into account the full history of shocks, we express S_{it+1} in terms of initial sales, S_{i0}, and the sequence of growth rate shocks over time as follows:

$$\begin{aligned}
S_{it+1} &= S_{it}(1+U_{it+1}) \\
S_{it+1} &= S_{it-1}(1+U_{it})(1+U_{it+1}) \\
&\dots \\
S_{it+1} &= S_{i0}\prod_{j=0}^{t}(1+U_{ij+1})
\end{aligned}$$

Assuming independence between S_{i0} and $\Pi_{j=0}^{t}(1+U_{ij+1})$, the behavior of the upper tail of S_{it+1} will be dominated by that of the distribution with the heavier tail, either that of S_{i0} or of $\Pi_{j=0}^{t}(1+U_{ij+1})$. The tail behavior of the latter will depend on the heaviness of the tails of the shock distribution.

If U is uncorrelated over time, the tail index of their product is determined by the minimum of the tail indices of the individual factors. If instead U exhibits persistence, as in the model estimated in Section 3, then the tail behavior of their product depends on the nature of their dependence.[28] For instance, for a squared random variable with symmetric tails, the tail index of the square is half that of the original variable. If the two variables are positively correlated, extreme positive shocks tend to co-occur, and their product will amplify these extremes, thereby reinforcing the heavy-tailed behavior. This can result in even more pronounced power-law behavior in the product's distribution. As such, extreme and nonlinearly persistent shocks, like those we uncover in the empirical analysis, can be the source of power law firm size distributions.[29] In general, heavy-tailed shocks can impact the distribution of firm sizes, regardless of whether the initial distribution follows a power law.[30]

Heavy-tailed shocks can affect the granular hypothesis

Next, we investigate how the nature of shock affects micro-originated aggregate fluctuations. Gabaix (2011) considers the setting described above and examines whether firm-level shocks impact aggregate output by extending the model to include aggregate GDP, defined as

[28] See Appendix G.1 for an example and related derivations.

[29] For a similar argument specifically on innovation bursts, see Berlingieri et al. (2024).

[30] With data that track all firms from inception to death, it may be possible to quantify the relative importance of initial conditions and shocks on the development of the power law distribution of firms sizes.

$Y_t = \sum_{i=1}^{N} S_{it}$, and GDP growth:

$$\frac{\Delta Y_{t+1}}{Y_t} = \sum_{i=1}^{N} U_{it+1} \frac{S_{it}}{Y_t}$$

Under the assumption that the shocks U_{it+1} are uncorrelated random variables, then the standard deviation of GDP growth is:

$$\sigma_{GDP} = \left(\sum_{i=1}^{N} \sigma^2 \left(\frac{S_{it}}{Y_t} \right)^2 \right)^{1/2} \tag{9}$$

Gabaix (2011) proposes the granular hypothesis by deriving the asymptotic behavior of σ_{GDP} as $N \to \infty$. He shows that, as $N \to \infty$, if the size of firms, S_{it}, has a distribution with finite variance, then the CLT applies, and the decay rate of the shocks is $\sqrt{N}$. Otherwise, if the distribution of firms sizes has an infinite variance and potentially even infinite mean, then the rate of decay is $N^{1-1/\zeta}$ or $\ln N$, respectively (a Pareto law distribution where the Pareto exponent ζ is ≤ 2 (≤ 1) has infinite variance (mean)). Hence, granular shocks matter for the aggregate when the distribution of sales has a power law behavior.

Non-Gaussian shocks may increase the volatility of GDP in two ways. First, indirectly, as we have previously showed that heavy-tailed shocks can provide a reason for heavy-tailed size distributions, affecting S_{it} in Equation (9). Second, directly affecting σ. As an extreme case, if the distribution of the error term is such that the variance does not exist (infinite σ), then also aggregate GDP growth volatility in Equation (9) will be infinite. However, larger kurtosis is not necessarily coupled with larger variance. Heavy tailed-ness of a distribution is not informative of its variance, and the latter is the crucial statistic in Equation (9). This consideration is reminiscent of findings by Acemoglu et al. (2017), who show that, conditional on a degree of sectoral heterogeneity, microeconomic shocks that display exponential tails can result in macroeconomic tail risk without necessarily increasing the volatility of aggregate output.

This section illustrated how the features of the cross-sectional distribution of idiosyncratic shocks affect the granular hypothesis. Even in this stylized setting, non-Gaussian shocks play an ambiguous role for granular aggregate fluctuations. Several components have been assumed away but may be important: among others, the correlation between firms' size and the error term, its variance (Yeh (2019)), and heterogeneous dynamics in the process of firms' productivity. Given the complexity of these layers, in the next section we use numerical simulations to quantitatively assess the role of non-Gaussianities and nonlinearities for aggregate volatility and autocorrelation.

5.2 Quantitative exploration

In this section we use the estimates from our model – and other model alternatives – to quantitatively highlight how the different nature of firm-level shocks matters for micro–originated aggregate fluctuations.[31] In particular, we construct the growth rate of aggregated TFP as $G_t = \log \frac{1}{N_t} \sum_i \exp\{o_{i,t}\} - \log \frac{1}{N_{t-1}} \sum_i \exp\{o_{i,t-1}\}$, where $o_{i,t}$ is the simulated log of TFP, for a firm i at time t, using model-estimated parameters. By dividing by the number of firms N, we abstract from fluctuations along the extensive margin, which are not relevant for our analysis.[32] We consider TFP for continuity with the analysis in the paper, but our conclusions are qualitatively similar if we repeat the exercise for sales. We consider five different models. First, a plain vanilla AR(1) such that $o_{i,t+1} = \rho o_{i,t} + \nu_{i,t+1}$, where innovations ν are normally distributed with variance σ^2_ν. We estimate ρ and σ^2_ν with a standard minimum distance estimation technique proposed by Chamberlain (1984).[33] We obtain the following estimates: $\widehat{\rho} = 0.925$ and $\widehat{\sigma}^2_\nu = 0.095$. Second, we add Gaussian transitory shocks ε to the AR(1). This slightly increases the estimated persistence to 0.934, and lowers $\widehat{\sigma}^2_\nu$ to 0.081, with an estimated variance of transitory innovations $\widehat{\sigma}^2_\epsilon = 0.171$. Third, we estimate a model that only features a persistent component with nonlinear persistence. To do so, we estimate quantile autoregressions as done for Figure 5, and then use the estimated coefficients to simulate the sample of firms. Fourth, we estimate the flexible model of Section 3, but assume Gaussian transitory shocks. In this setup, the variance of transitory shocks is estimated to be much higher than for the AR(1), at 0.635. Finally, we consider our full model.

For each of these models, we simulate a sample of firms starting from the empirical distribution, and maintaining the structure of the data panel (i.e.: firm-specific spells). We provide details of estimation and simulation in Appendix D. For each model, we compute two objects of interest, which we report in Table 2. First, the volatility of G_t. This is the typical measure in most research on the granular hypothesis, as we have discussed in the previous section. Second, we also compute the one year autocorrelation of G_t. In recent work, Carvalho and Grassi (2019) show that if there is a finite number of firms, rather than a continuum, then the distribution of firms across productivity levels varies stochastically

[31] Strictly speaking, our TFP measures are not solely idiosyncratic shocks but also combine heterogeneous sensitivities to aggregate shocks, as previously discussed. Regardless of their origin, however, this simulation exercise treats these fluctuations as happening at the level of individual firms and then we aggregate them up. We show that if the stochastic process of these firm-level fluctuations features nonlinearities and non-Gaussianities, the implications for the resulting aggregate fluctuations are substantially different.

[32] This *aggregated* measure of TFP is likely different from time series of aggregate TFP, and indeed our exercise has nothing to say about the latter.

[33] We choose the estimator that minimizes the distance between the theoretical autocovariances generated by the model and the empirical autocovariances of productivity. We follow the recommendation of Altonji and Segal (1996) of using the identity matrix to reduce the substantial small sample bias that can result from choosing the optimal weighting matrix suggested by Chamberlain (1984).

Table 2: Micro–originated fluctuations with different idiosyncratic processes

	Data	(I)	(II)	(III)	(IV)	(V)
$\sigma(G_t)$	0.086	0.017	0.085	0.020	0.251	0.038
$\text{CORR}(G_t, G_{t-1})$	0.051	0.087	-0.150	0.351	-0.408	0.068

Note: We estimate each model on the baseline sample and simulate out of sample, maintaining the structure of the data panel. We simulate each model 100 times, computing the variance and the autocorrelation of aggregate TFP growth rate for each simulation, and then reporting medians across simulations. Details can be found in Appendix D.

over time. As a result, aggregate productivity also follows a stochastic process, even when all shocks are idiosyncratic.

The standard AR(1) process generates non-zero volatility of aggregate TFP growth and positive persistence, due to the finite number of firms and their fat-tailed productivity distribution. However, the volatility is lower than in the data. Results for this model are shown in column (I). Adding a transitory, normally distributed, component, as done in column (II), raises the volatility substantially, but introduces negative autocorrelation as this component dominates the properties of the overall stochastic process.[34] A similar pattern is observed in column (IV), which embeds Gaussian transitory shocks in the fully flexible model: volatility overshoots the data counterpart and the autocorrelation becomes even more negative. In column (III), instead, we extend the standard AR(1) with nonlinear persistence. Volatility is little affected. Aggregate growth rates are very persistent in this setup, as witnessed by the autocorrelation. Note that nonlinearities do not necessarily imply higher persistence, especially for the growth rate of aggregate TFP. Indeed, heterogeneity and aggregation can, in principle, work in both directions.

Our full model, showed in column (V), delivers plausible aggregate moments. Aggregate volatility undershoots what observed in the data but is closest to all other models except (II). However, model (V) generates a positive autocorrelation that is closer to what seen in the data. It should be stressed that these aggregate moments are not explicitly targeted; as such, we see the good performance of model (V) as an additional success of this modeling framework, previously overlooked in the literature.

By comparing different models, our analysis shows how the nature of firm-level shocks affects micro-originated aggregate fluctuations. Our findings underscore that *both* nonlinearities in persistence and non-Gaussian shocks should be accounted for. In addition, transitory and persistent components should be modeled separately. Although the dispersion and kurtosis of transitory shocks are substantially smaller than those of changes in the persistent

[34] In Appendix G.2 we show these results analytically, in a simplified setting.

component, the results of this section still suggest that it is crucial to account for non-gaussianities in both components. Failure to do so generates counterfactual behavior, not just at the firm level, but also in the aggregate.[35] We find that non-Gaussian shocks as typically observed in the data are associated with less volatile micro-originated aggregate fluctuations, relative to a Gaussian counterpart. Importantly, this dampening channel is separate from, but complementary to, the size-variance tradeoff recently put forward by Yeh (2019).[36] Moreover, nonlinear persistence, as estimated by our model, raises the persistence of micro-originated aggregate fluctuations.

6 Conclusion

We have documented that sales and productivity of U.S. publicly listed firms follow a stochastic process likely different from the canonical linear AR(1). Firms face heavy-tailed fundamental shocks and they partly absorb them through input adjustment. In addition, they face nonlinear and heterogeneous persistence. We estimate a quantile-based panel data framework borrowed from the literature on household earnings and show that it is a good representation of firm-level dynamics.

Nonlinear dynamics can affect firm behavior in multiple ways. We have focused on one: micro-originated (granular) aggregate fluctuations. While the literature on the granular hypothesis has typically focused on the firm size distribution, we have investigated the role played by the features of the shocks. Our analysis suggests that transitory and persistent productivity shocks need to be modeled separately, and nonlinearities and non-Gaussianities simultaneously accounted for; failure to do so generates counterfactual behavior, not just at the firm level, but also in the aggregate. We hope that the findings in this paper spur further research in the literature on semi-structural estimation approaches applied to firms.

[35] The model of column (IV) tries to fit rare shocks by estimating an excessively large standard deviation of the Gaussian distribution. This ends up dwarfing the persistent component, even when the latter is allowed to be non-gaussian and with nonlinear persistence.

[36] Note that in our data the variance of TFP growth rates also falls with firm size, as shown previously. As such, this channel is incorporated in our simulations.

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Supplemental Appendix to "Nonlinear Firm Dynamics"

by Davide Melcangi[†] and Silvia Sarpietro[‡]

A Data Appendix

Our data consists of yearly firm-level information from US Compustat, over the period between 1987 and 2017. We exclude firms that are not incorporated in the US. Moreover, we keep firms with positive sales and cost of goods sold.[1] We exclude firms without an industry classification ("unclassifiable firms"), as well firms whose Standard Industrial Classification (SIC) is financial and real estate activities (6021 – 6799) or utilites (4900 – 4991). We keep firms that remain in operation for at least 2 consecutive years. If a firm is in the sample for multiple spells, we keep the longest spell. Finally, we keep drop missing observations for sales, labor productivity, and TFP. For the baseline sample of manufacturing firms, this leaves us with an unbalanced panel of 6,018 firms and 57,882 firm-year observations. For the whole economy, we have 12,303 firms and 111,478 observations. We deflate sales, cost of goods sold, and the capital stock by the GDP deflator. The capital stock is constructed following Ottonello and Winberry (2020). For each firm, the first value of end of period capital is set to the level of gross plant, property, and equipment (ppegt in Compustat). We then compute the evolution of capital, from this period onwards, using changes of net plant, property, and equipment (ppent). If a firm has a missing observation for ppent, we impute its value with linear interpolation only if the missing observation has nonmissing values of ppent in the previous and following year. Capital stock is the lag of the end-of-period capital constructed as just described.

Revenue elasticities used in the calculation of TFPS are constructed using industry-level data from the KLEMS database constructed by the BLS. The revenue elasticity for capital is the ratio of capital cost and value of production, while for the variable input (cost of goods sold) we divide the sum of costs of materials, labor, and energy, by the value of production. Revenue elasticities are year-specific and industry-specific – 3-digit NAICS level

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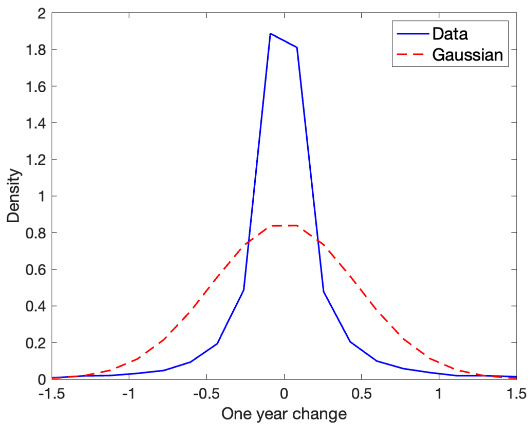
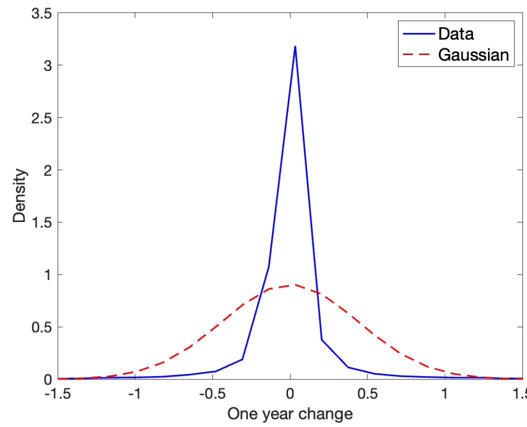
[1]In unreported results we find that our results are little sensitive to the exclusion of firms with sales or total assets less $100,000, or with less than 10 employees.

Figure A.1: Distribution of one-year log changes



(a) Labor productivity



(b) TFP

Notes: Empirical densities of log changes in labor productivity (left panel (a)) and TFP (right panel (b)) between year t and $t-1$ depicted in solid blue. In panel (a), the standard deviation is 0.468, Kelley Skewness 0.026 and Crow-Siddiqui Kurtosis 7.948. In panel (b), the standard deviation is 0.441, Kelley Skewness 0.006 and Crow-Siddiqui Kurtosis 10.713. Gaussian density with the same standard deviation of data shown in dashed red. Baseline sample of manufacturing firms.

for the manufacturing sector, 2-digit when looking at the whole economy. The first order condition pinning down revenue elasticities holds if there are no factor adjustment costs or wedges; as is typically assumed in the literature (see for instance Decker, Haltiwanger, Jarmin and Miranda (2020)), we require this condition to hold on average across firms within an industry. As suggested by Syverson (2011), we further strengthen this argument by averaging the revenue elasticities over 6-year windows.

After constructing the log of TFPS as the revenue function residual – using sales, cost of goods sold, the capital stock, and the revenue elasticities – we residualize it by regressing TFPS on industry dummies and a linear time trend.

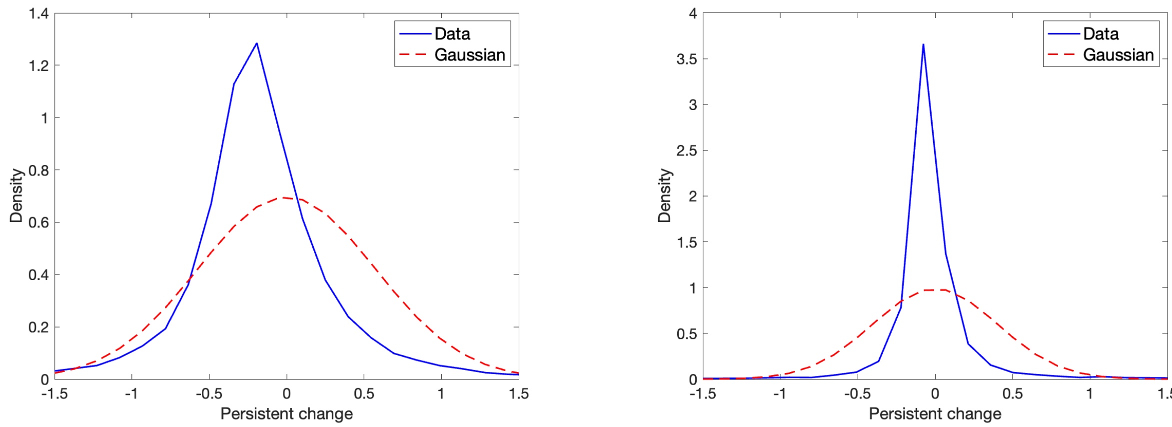
The quarterly dataset described in Section and Appendix follows similar steps and variable creation as in Ottonello and Winberry (2020). For this sample of manufacturing firms we have 6,840 firms and 234,105 firm-quarter-year observations.

B Additional empirical results

B.1 Productivity distributions

Figure A.1 shows the empirical densities of one-year log changes in labor productivity and TFP. We show how both measures are clearly not distributed in a gaussian way.

Figure A.2: Distribution of persistent changes



(a) Sales

(b) TFP

Notes: Empirical densities of persistent changes ($\Delta(\tilde{y}) = \bar{y}_{t-1,t+1} - \bar{y}_{t-2,t-4}$) in sales (left panel (a)) and TFP (right panel (b)) depicted in solid blue. In panel (a), the standard deviation is 0.572, Kelley Skewness 0.092 and Crow-Siddiqui Kurtosis 5.120. In panel (b), the standard deviation is 0.404, Kelley Skewness 0.131 and Crow-Siddiqui Kurtosis 8.013. Gaussian density with the same standard deviation of data shown in dashed red. Baseline sample of manufacturing firms.

B.2 Persistent changes

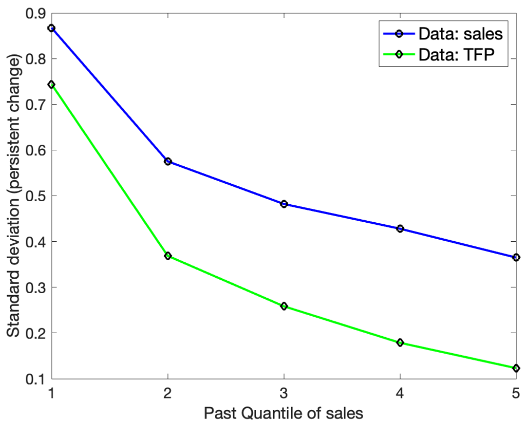
In this section we report the results for persistent changes, as described in Section 2.3. In particular, we consider the difference between two 3-year averages of our logged variables. First, we show the distributions. In Figure A.2a we show how the empirical density of these persistent changes in sales is more concentrated than its gaussian counterpart, constructed with the same standard deviation. Compared to the distribution of one-year log changes, shown in Figure 1, the departures from non-Gaussianity are less pronounced, suggesting that transitory shocks are also non-Gaussian.

A similar story applies to TFP, as shown in Figure A.2b. Figure A.3 confirms that the main source of non-Gaussianity is heavy tailedness (kurtosis). In addition, small firms display a much larger Crow-Siddiqui kurtosis of persistent changes in TFP, compared to persistent changes in sales, in line with what shown for one-year changes.

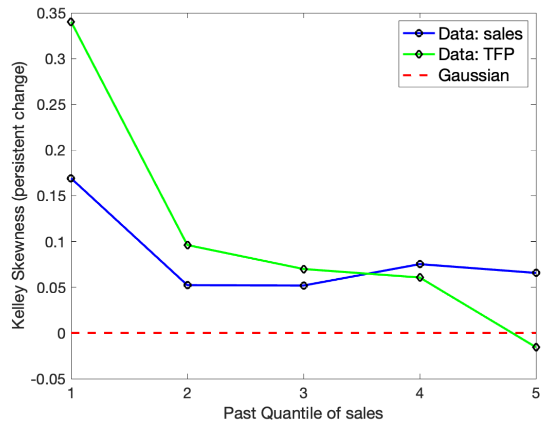
B.3 Labor productivity

In this section we report higher order moments of log changes in labor productivity, as well as quantile autoregressions, for the baseline sample discussed in Section 2. In Figure A.4 we show that the dispersion of changes in labor productivity falls as firm sales increase. There is little asymmetry of labor productivity changes, except for a mild positive skewness among smallest firms. Finally, labor productivity shocks are very heavy-tailed, with marked

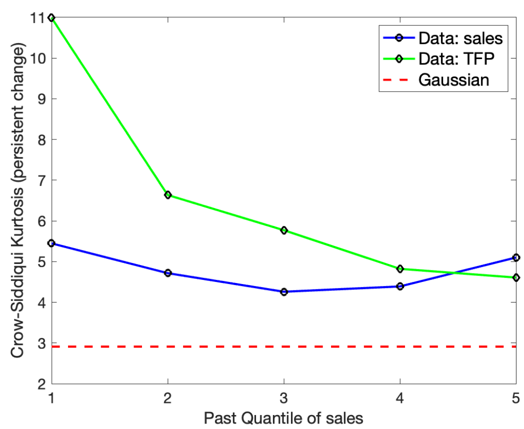
Figure A.3: Conditional moments of firm-level persistent changes



(a) Standard Deviation



(b) Kelley Skewness



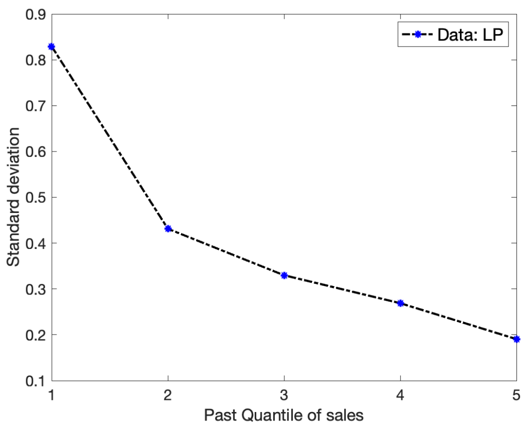
(c) Crow-Siddiqui Kurtosis

Notes: Solid-circled blue lines refer to the log of firm sales, while solid-diamond green line to log TFP. For each measure, observations ordered by quantiles of the distribution of $\bar{y}_{t-2,t-4}$ as defined in Figure A.2 (x-axis), where y is sales. For each quantile, we show on the y-axis the Standard deviation of persistent changes ($\Delta(\tilde{y}) = \bar{y}_{t-1,t+1} - \bar{y}_{t-2,t-4}$) in Figure A.3a, the Kelley Skewness in Figure A.3b and Crow-Siddiqui Kurtosis in Figure A.3c.

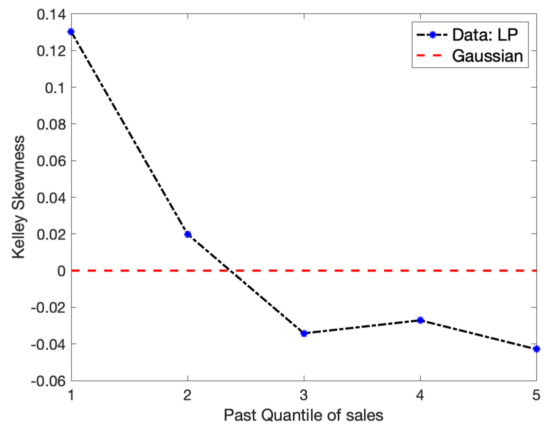
departures from gaussianity.

In Figure A.5 we show that quantile autoregressions of log labor productivity suggest that also this alternative measure of firm profitability has a markedly nonlinear persistence of history.

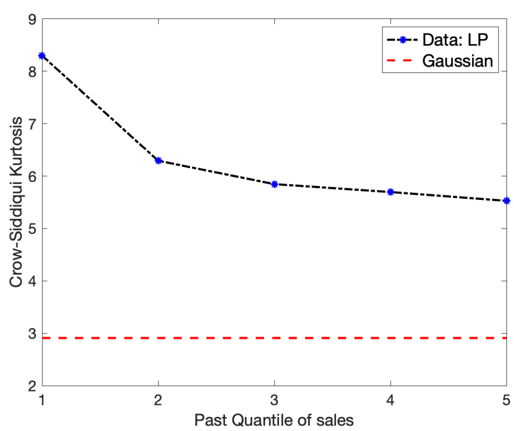
Figure A.4: Conditional moments of firm-level one year log-changes: labor productivity



(a) Standard Deviation



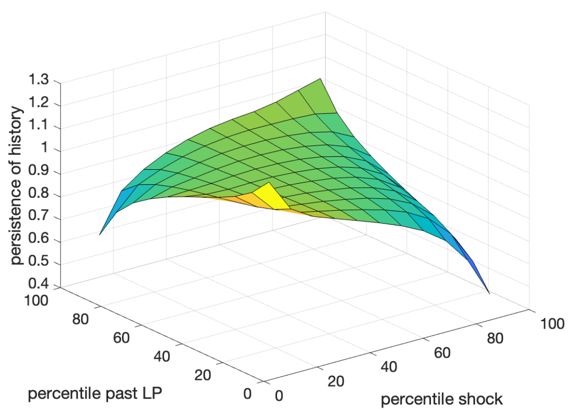
(b) Kelley Skewness



(c) Crow-Siddiqui Kurtosis

Notes: Observations are ordered by quantiles of the sales distribution in $t-1$ (x-axis). For each quantile, we show on the y-axis the Standard Deviation in Figure A.4a, the Kelley Skewness in Figure A.4b and Crow-Siddiqui Kurtosis in Figure A.4c of log-changes in labor productivity.

Figure A.5: Quantile autoregressions of log labor productivity



Notes: Estimated average derivative of the conditional quantile function of residualized log labor productivity $y_{i,t}$ given $y_{i,t-1}$. Derivatives are taken with respect to $y_{i,t-1}$ and evaluated at a given percentile of $y_{i,t-1}$, and at a given percentile of the shock, labelled τ_{shock}. Details on data and estimation of quantile functions in the text.

Table A.1: Concentration of sales, labor productivity and TFPS growth rates: all industries

S :	$[-0.05, 0.05]$			$[-0.10, 0.10]$			$[-0.20, 0.20]$		
	Data	$\mathcal{N}(\mu_x, \sigma_x)$	Ratio	Data	$\mathcal{N}(\mu_x, \sigma_x)$	Ratio	Data	$\mathcal{N}(\mu_x, \sigma_x)$	Ratio
$Prob\,(\lvert\Delta\text{Sales}\rvert \in S)$	19.6	8.3	2.4	39.0	16.6	2.4	65.3	32.4	2.0
$Prob\,(\lvert\Delta\text{LP}\rvert \in S)$	29.9	8.7	3.4	49.9	17.1	2.9	70.8	33.6	2.1
$Prob\,(\lvert\Delta\text{TFP}\rvert \in S)$	42.5	9.6	4.4	64.0	19.3	3.3	81.4	37.6	2.2

Note: One year changes in logged variables, sample of firms using all industries. Columns for $\mathcal{N}(\mu_x, \sigma_x)$ refer to draws, of size equal to the dataset, from a normal distribution with mean μ_x and standard deviation σ_x, where x denotes the variable considered in each row. Standard deviations are 0.478, 0.463, and 0.409 for changes in log sales, LP, and TFP, respectively.

B.4 All industries

In this section we repeat the analysis of Section 2.1 and 2.2 for the whole economy. Table A.1 confirms that the distributions of sales, labor productivity, and TFP growth rates are much more concentrated than a Normal distribution with the same empirical variance. In addition, concentration is typically highest for TFP, which is arguably the more exogenous measure of fundamental shocks.

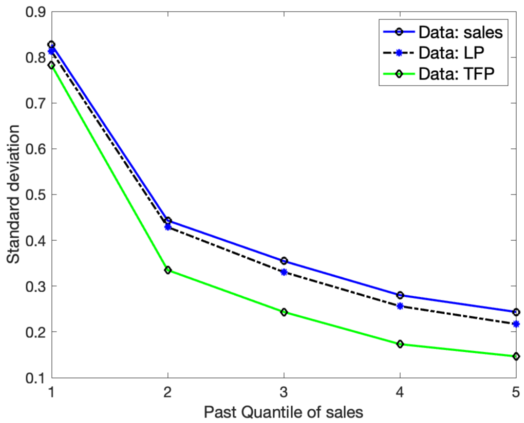
We then turn, in Figure A.6, to conditional moments of those distributions. Our results are similar to what shown for the manufacturing sector in Figure 4. Smaller firms face more dispersion in TFP growth rates, and even more so in sales growth rates. Small firms face shocks that are slightly positively skewed; in contrast, large firms display negative skewness in TFP log changes, but this does not translate into similar asymmetries for sales. We confirm that heavy tails in sales shocks are most likely driven by heavy-tailed underling exogenous shocks. This is particularly true for firms at the bottom of the sales distribution.

Finally, we compute quantile autoregressions for this larger sample and confirm our main results, as shown in Figure A.7. Firm sales depart from a standard AR(1) process and display lower persistence of history among small firms hit by large positive shocks. This nonlinear persistence is even more pronounced when looking at productivity, and a nonlinear left tail appears. Quantitatively, nonlinearities are almost identical in this broader sample to what shown for the manufacturing sector.

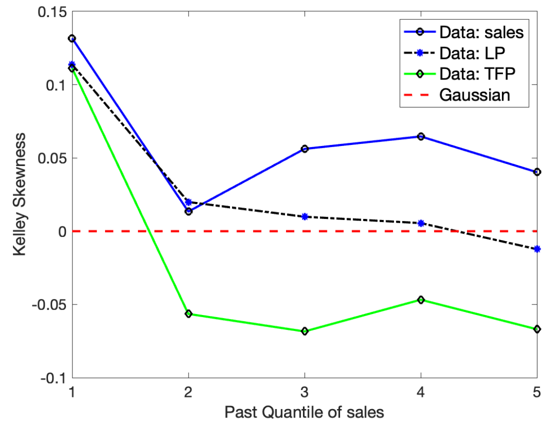
B.5 Confidence intervals

We bootstrap the baseline empirical sample, at the firm level, with replacement, and compute for each bootstrapped sample the quantile autoregressions, conditional Kelley skewness and conditional Crow-Siddiqui Kurtosis. The results showed in Figures A.8 and A.9 point to tight 95% pointwise confidence bands.

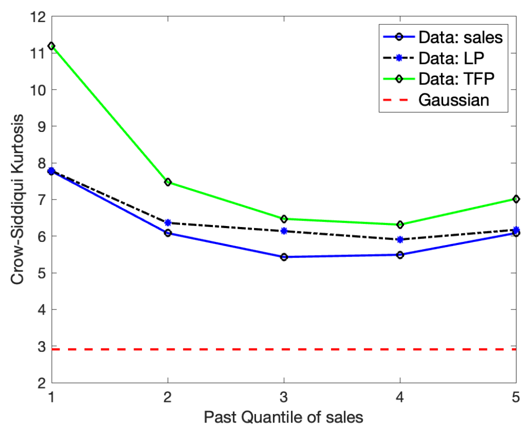
Figure A.6: Conditional moments of firm-level one year log-changes: all industries



(a) Standard Deviation



(b) Kelley Skewness



(c) Crow-Siddiqui Kurtosis

Notes: Solid-circled blue lines refer to the log of firm sales, dash-dotted black line to the log of labor productivity, solid-diamond green line to log TFP. For each measure, observations are ordered by quantiles of the sales distribution in $t-1$ (x-axis). For each quantile, we show on the y-axis the Standard deviation in Figure A.6a Kelley Skewness in Figure A.6b and Crow-Siddiqui Kurtosis in Figure A.6c of log-changes. Firm-level data on all industries in the sample.

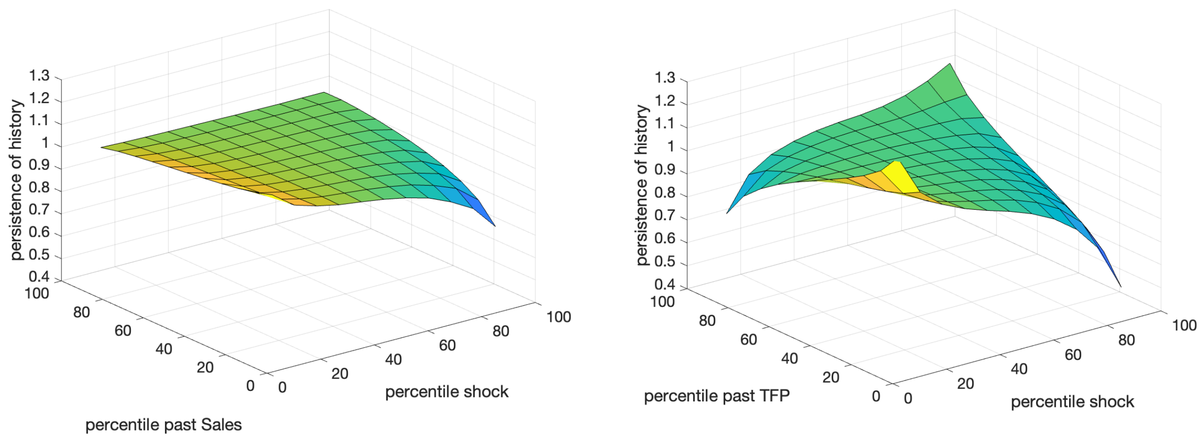
B.6 No Detrending

In section we repeat the analysis of Section 2.1 and 2.2 for data that has not been detrended – only industry fixed effects are subtracted from the variable of interest. As shown in Figure A.10 and A.11, the patterns are virtually identical to our baseline results.

B.7 Quarterly data

In this section we report the results for quarterly data, as described in Section 2.3. In Figure A.12a we show how the empirical density of these quarterly changes in sales is more

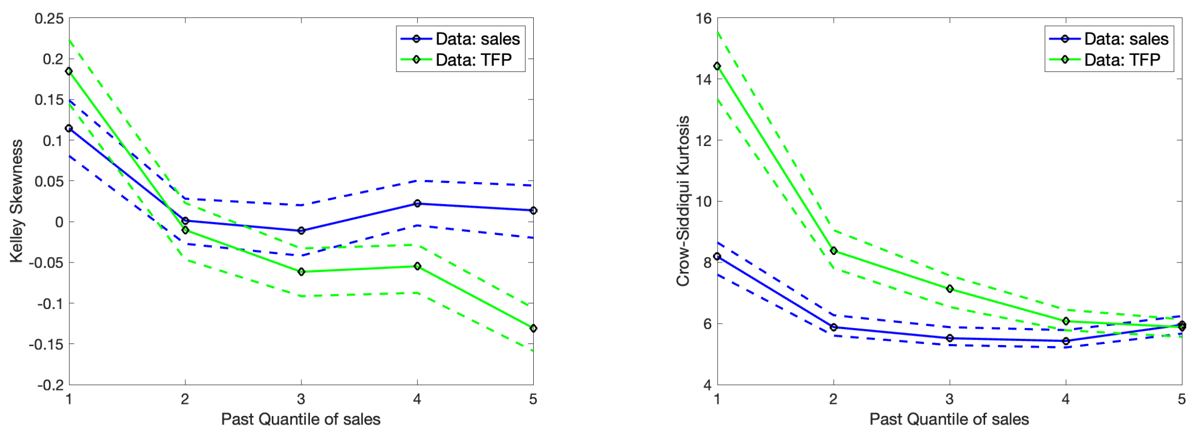
Figure A.7: Quantile autoregressions: all industries



(a) Sales (b) TFP

Notes: Estimated average derivative of the conditional quantile function of residualized values $y_{i,t}$ given $y_{i,t-1}$. Derivatives are taken with respect to $y_{i,t-1}$ and evaluated at a given percentile of $y_{i,t-1}$, and at a given percentile of the shock. Details on data and estimation of quantile functions in the text.

Figure A.8: Conditional moments of firm-level one year log-changes, 95% pointwise confidence bands



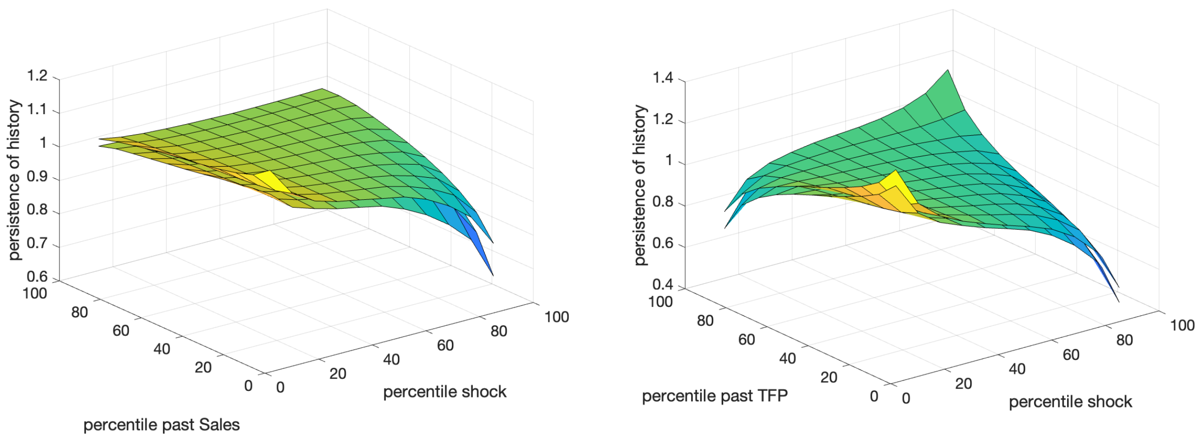
(a) Kelley Skewness (b) Crow-Siddiqui Kurtosis

Notes: See notes to Figure 4. Pointwise 95% confidence bands computed over 500 bootstrap replications are showed with dashed lines. Nonparametric bootstrap with replacement, clustered at the firm level. Solid lines report results for the empirical sample, as in Figure 4.

concentrated than its gaussian counterpart, constructed with the same standard deviation. Compared to the distribution of one-year log changes, shown in Figure 1, the departures from non-Gaussianity are even more pronounced, confirming that transitory shocks, even at quarterly frequency, are also non-Gaussian.

A similar story applies to TFP, as shown in Figure A.12b. Figure A.13 confirms that

Figure A.9: Quantile autoregressions, 95% pointwise confidence bands



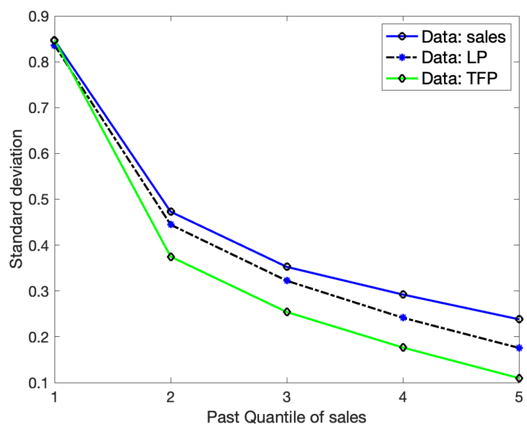
(a) Sales (b) TFP

Notes: See notes to Figure 5. Pointwise 95% confidence bands computed over 500 bootstrap replications. Nonparametric bootstrap with replacement, clustered at the firm level.

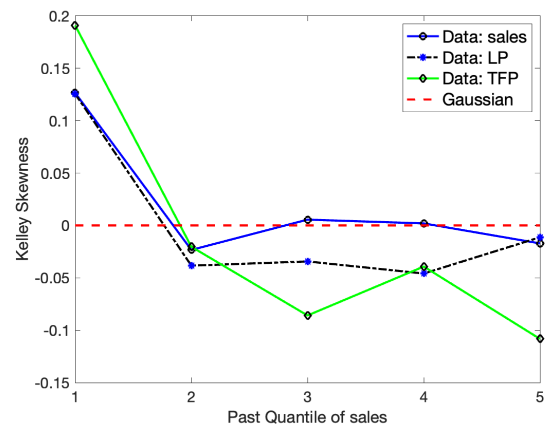
the main source of non-Gaussianity is heavy tailedness (kurtosis). In addition, small firms display a much larger Crow-Siddiqui kurtosis of quarterly changes in TFP, compared to quarterly changes in sales, in line with what shown for one-year changes.

Finally, in Figure A.14 we show that persistence of history is nonlinear even in the quarterly dataset.

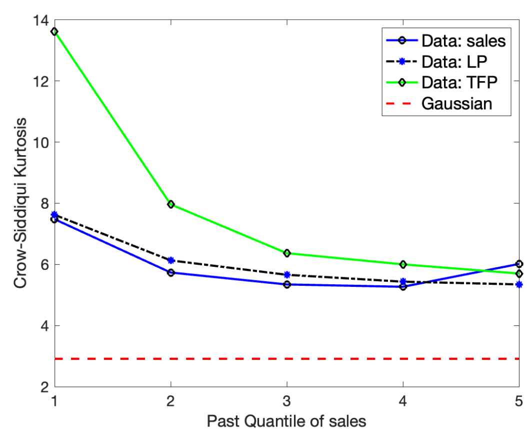
Figure A.10: Conditional moments of firm-level one year log-changes: no detrending



(a) Standard Deviation



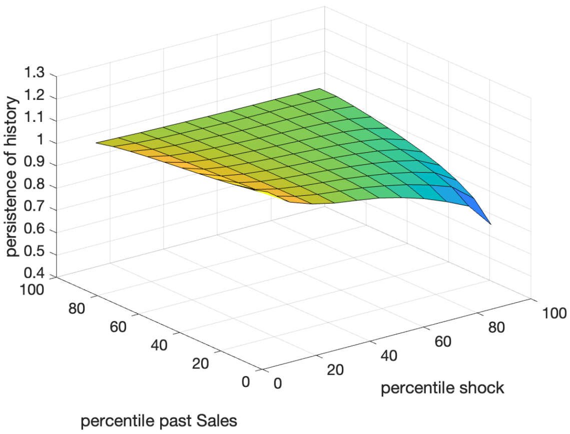
(b) Kelley Skewness



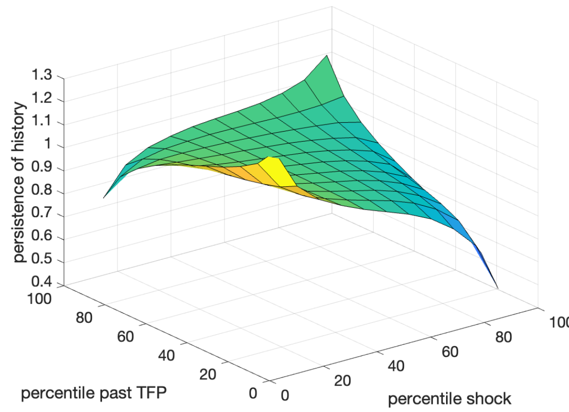
(c) Crow-Siddiqui Kurtosis

Notes: Solid-circled blue lines refer to the log of firm sales, dash-dotted black line to the log of labor productivity, solid-diamond green line to log TFP. For each measure, observations are ordered by quantiles of the sales distribution in $t-1$ (x-axis). For each quantile, we show on the y-axis the Standard deviation in Figure A.6a Kelley Skewness in Figure A.6b and Crow-Siddiqui Kurtosis in Figure A.6c of log-changes.

Figure A.11: Quantile autoregressions: no detrending



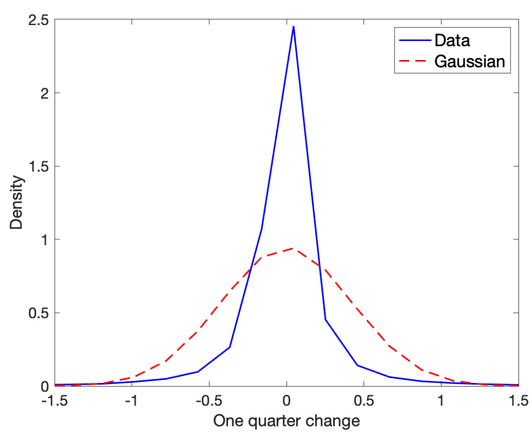
(a) Sales



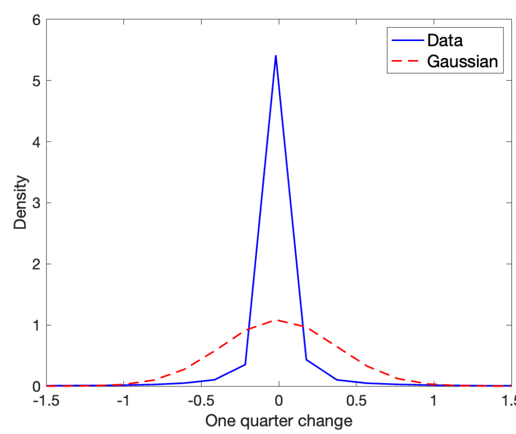
(b) TFP

Notes: Estimated average derivative of the conditional quantile function of residualized values $y_{i,t}$ given $y_{i,t-1}$. Derivatives are taken with respect to $y_{i,t-1}$ and evaluated at a given percentile of $y_{i,t-1}$, and at a given percentile of the shock. Details on data and estimation of quantile functions in the text.

Figure A.12: Distribution of quarterly changes



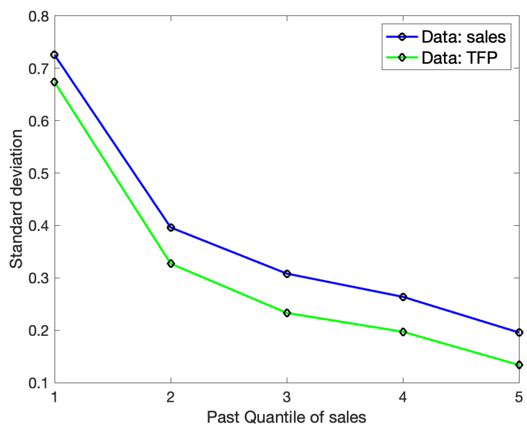
(a) Sales



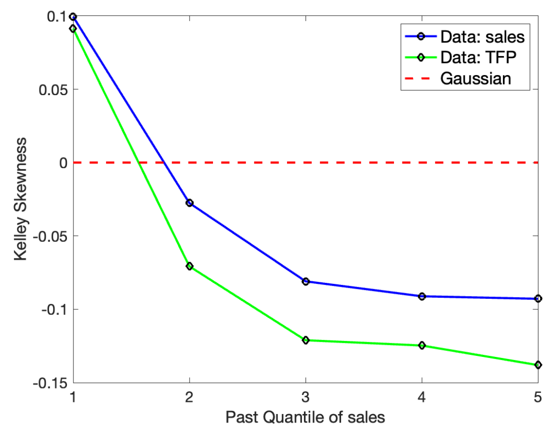
(b) TFP

Notes: Empirical densities of quarterly changes of residualized log sales (left panel (a)) and TFP (right panel (b)) depicted in solid blue. In panel (a), the standard deviation is 0.422, Kelley Skewness -0.028 and Crow-Siddiqui Kurtosis 7.297. In panel (b), the standard deviation is 0.367, Kelley Skewness -0.0594 and Crow-Siddiqui Kurtosis 11.148. Gaussian density with the same standard deviation of data shown in dashed red. Baseline sample of manufacturing firms.

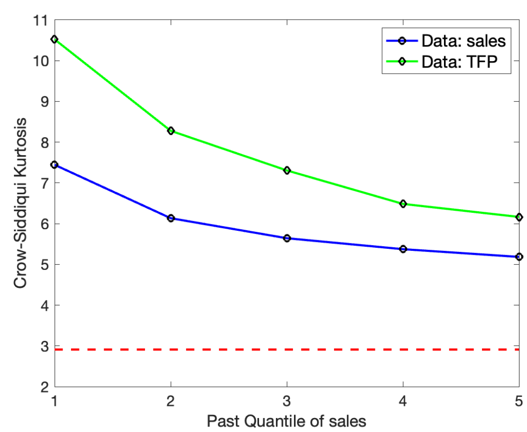
Figure A.13: Conditional moments of firm-level quarterly changes



(a) Standard Deviation



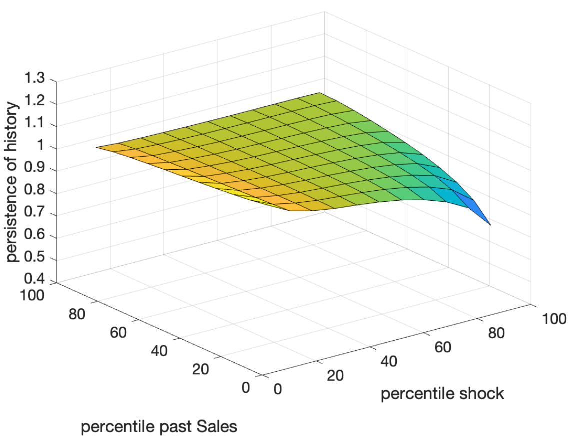
(b) Kelley Skewness



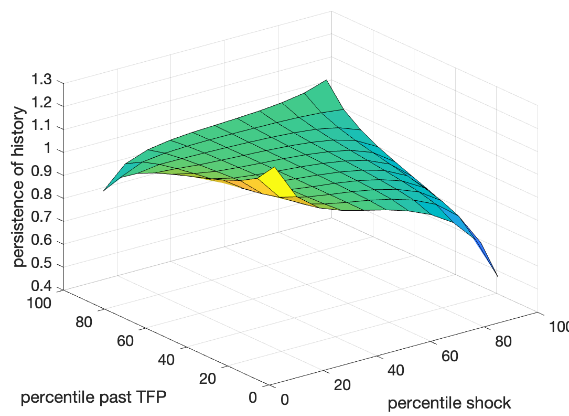
(c) Crow-Siddiqui Kurtosis

Notes: Solid-circled blue lines refer to the log of firm sales, while solid-diamond green line to log TFP. For each measure, observations are ordered by quantiles of the sales distribution of $t-1$ (x-axis). For each quantile, we show on the y-axis the Standard deviation in Figure A.13a, the Kelley Skewness in Figure A.13b and Crow-Siddiqui Kurtosis in Figure A.13c of log changes.

Figure A.14: Quantile autoregressions: quarterly data



(a) Sales



(b) TFP

Notes: Estimated average derivative of the conditional quantile function of residualized values $y_{i,t}$ given $y_{i,t-1}$, where t is a quarter-year. Derivatives are taken with respect to $y_{i,t-1}$ and evaluated at a given percentile of $y_{i,t-1}$, and at a given percentile of the shock. Details on data and estimation of quantile functions in the text.

C Decaying responsiveness to productivity shocks

What type of firm responsiveness is consistent with the findings shown in Section 2.2? To shed light on this, we come back to our simple production function in logs presented in Equation (1). Then we take first differences, and assume that changes in the two inputs of production respond to changes in TFP, such that $\Delta c_{i,t} = \phi_c \Delta z_{i,t}$ and $\Delta k_{i,t} = \phi_k \Delta z_{i,t}$. Then:

$$\Delta y_{i,t} = \Delta z_{i,t} + \beta_c \phi_c \Delta z_{i,t} + \beta_k \phi_k \Delta z_{i,t} \tag{10}$$

where y and z denote the log of sales and TFPS, respectively. For simplicity, we set $\phi_k = 0$. We take $\Delta z_{i,t}$ from the data and β_c as the average estimate from our baseline sample. With this we can back out $\Delta y_{i,t}$ from Equation (10) for three examples for ϕ_c and report the relevant moments, with their empirical counterparts, in Table A.2.[2]

In the first model, ϕ_c is the same for all firms and equal to 1, regardless of the sign and the size of Δz. This framework is consistent with the fact that the dispersion of sales growth is higher than for productivity growth, as long as $\beta_c \phi_c > 0$. With constant responsiveness, however, the Crow-Siddiqui measure of TFP growth is exactly the same as the one for sales growth. This is at odds with our empirical findings, not just in the entire sample, but also conditional on each quintile of sales.

In order to qualitatively match the fact that productivity growth displays heavy-tailed events more often than sales growth, we consider two alternative structures for ϕ_c. We assume a flexible dependence of this sensitivity to productivity shocks, such that $\phi_c = \gamma_1 e^{\gamma_2 |\Delta z|}$. Since the conditional skewness of sales changes is close to zero for nearly all quantiles of firm sales, we maintain a symmetric structure of ϕ_c, but our reasoning can easily be extended to loadings γ_2 that differ by the sign of productivity shocks. When $\gamma_2 < 0$, firms adjust their variable input c increasingly less as the productivity change becomes of larger magnitude; we label this case decaying response. As shown in Table A.2, choosing $\gamma_1 = 1$ and $\gamma_2 = -1$ allows us to qualitatively match what we observe in the data. Not only sales growth remains more dispersed than productivity growth, but it also has a smaller Crow-Siddiqui kurtosis. Not surprisingly, the increasing response features the opposite, and counterfactual, pattern.

This simple exercise suggests that not only firms face non-Gaussian and heavy-tailed fundamental shocks, but they also most likely respond to them by adjusting inputs with double exponential responsiveness. Our suggestive evidence is inconsistent with a simple, frictionless, model in which firms choose labor by setting the marginal product equal to an aggregate wage. That framework would be akin the "constant response" model presented

[2] In this example we consider all firms, but the intuition can be tailored to each of the quintile-specific result previously shown.

Table A.2: Shocks vs responsiveness

	Standard Deviation			Kurtosis		
	Sales	TFP	ratio	Sales	TFP	ratio
Data	0.49	0.44	1.11	7.21	10.71	0.67
Constant response	0.74	0.44	1.68	10.71	10.71	1.00
Decaying response	0.45	0.44	1.02	8.94	10.71	0.83
Increasing response	31.80	0.44	72.18	17.64	10.71	1.65

Notes: Baseline sample of manufacturing firms as described in Section 2. In the first row, Sales and TFP refer to log changes in residualized sales and TFP, as described in the text. The last three rows use empirical TFP growth – which is hence the same across models – and obtain sales growth by fitting Equation (10) with different parameter values as discussed in the text. Kurtosis is quantile-based (Crow-Siddiqui).

in Table A.2. Moreover, that model predicts that employment is a convex function of TFP. In contrast, our findings suggest that not only employment is likely a concave function of fundamental shocks, but also that the log of employment is concave in the log of TFP.[3] The investigation of the exact source of these relationships is beyond the scope of this paper and the findings of Table A.2 should be treated as suggestive. However, we speculate that this behavior can be seen as broadly consistent with various economic models of firm behavior. For example, hiring costs may limit the extent to which firms are able to reap the benefits of large productivity shocks, reducing their ability to hire more workers. Financial frictions, especially of the type directly constraining labor inputs, could also generate a similar pattern. These frictions are likely to be relevant for small firms: in line with this, these firms display the largest kurtosis gap. On the way down, firing costs would also be consistent with this behavior. Looking back at earlier results, this would rationalize why large firms are more likely to face negative tail risk in productivity, but are able to absorb those and thus face symmetric changes in sales.

D Computational Appendix

We follow Arellano et al. (2017) and Arellano et al. (2021) to estimate the nonlinear hidden quantile model. To do so, we use a stochastic EM algorithm that consists of two steps: 1) drawing the latent components of the model (η_{it} and ϵ_{it}), 2) updating the model parameters given the latent draws. To perform step 1), we follow Arellano et al. (2021) and use a

[3]In many models of household behavior, such as Carroll and Kimball (1996), consumption is a concave function of wealth, implying that dollar-for-dollar MPCs fall with the size of a transitory income shock. Our findings suggest that a similar pattern may apply to firms too, and even to growth rates of employment.

Sequential Monte-Carlo (SMC) sampling method.[4] The simulation step is run in parallel across firms to simplify the estimation of the model with unbalanced data. For step 2), as in Arellano et al. (2017), we update the parameters using quantile regressions, after modelling all conditional quantile functions as linear quantile specifications at a grid of 11 equidistant percentiles. For additional details, see Arellano et al. (2017) and Arellano et al. (2021).

We estimate the 5 models described in Section 5.2 on the baseline sample of manufacturing firms, using residualized and detrended log TFP as discussed in the main body. The parameters for the AR(1) models – with and without transitory shocks – are estimated by minimizing the distance between the theoretical autocovariances generated by the model and the empirical autocovariances of productivity. We use an identity matrix. We repeat the estimation for 100 initial conditions and keep the estimated parameters associated with the best goodness of fit (the sum of squared deviations). For model (III), the one that only features nonlinear persistence, we follow Arellano et al. (2017) and estimate quantile autoregressions of residualized log TFP, specifying quantile functions as third-order Hermite polynomials. Model (V) is estimated as described above; model (IV) follows the same structure, adjusting the log-likelihood to assume that transitory shocks follow a Gaussian distribution.

The simulations in Section 5.2 work as follows, using the estimated parameters. We start from the firm's first empirical observation. We also follow the data in the firm-specific spells – i.e., how long a firm remains in operation. Each model is simulated 100 times. Out-of-sample simulations of models III–V can result in explosive behavior of some firms.[5] To deal with this, we restrict the support of model-simulated log TFP to be the same as in the data: any firm that hits the bounds in the simulation is dropped. This amounts to 0.7% for model V and 12.3% for model IV of all firms in the sample. In terms of firm–year observations, it accounts for 1.9% for model IV but less than 0.1% of all observations for the other models.[6] We compute the volatility over time and the one-year autocorrelation of aggregate TFP growth rates in each of the simulated panel dataset, for each model, and report the medians across simulations in Table 2 of Section 5.2.

[4]The use of this sampler instead of the Metropolis-Hastings employed by Arellano et al. (2017) is justified by numerical stability.

[5]An alternative to out-of-sample simulations is to use the SMC sampling method in the simulations, as in ABBL. This latter approach delivers a qualitative similar ordering across models, and a much better fit to the data for model (V). As such, our out-of-sample simulations can be seen as conservative.

[6]These restrictions can find justification through a procedure that accounts for path dependence in simulating firms' growth. This procedure would assign a history-dependent probability of converging towards a long-run firm-value. Given our data, we expect the probability of diverging to decrease, with history-dependent rates, as firms keep growing.

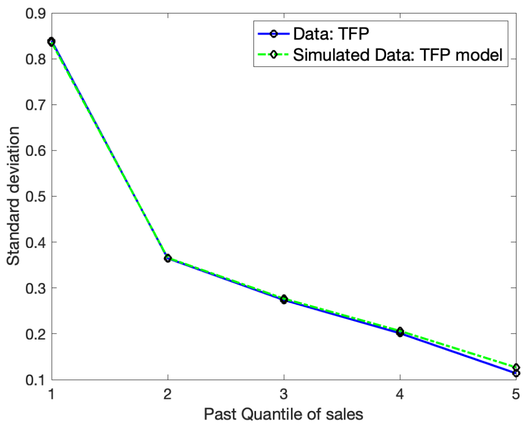
E Additional estimation results

Departures from normality and the ability of the model to match the data can be visualized looking at how conditional moments of the distribution of changes of log productivity residuals vary with firm sales. Figures A.15a, A.15b, and A.15c report the patterns of conditional dispersion, skewness, and kurtosis in the data, as documented in Section 2.2, and in the model. The model does a very good job at matching conditional dispersion and kurtosis, and a fair job with regard to skewness too, although it misses the mildly negative skewness for productive firms. Figure A.16 plots the histogram of the persistent and transitory components of log productivity as estimated by the model discussed in Section 3. We then obtain the histograms of the estimated components of log productivity by past quantiles of the distribution of simulated TFP and report them for the top and bottom quantiles in Figure A.17.

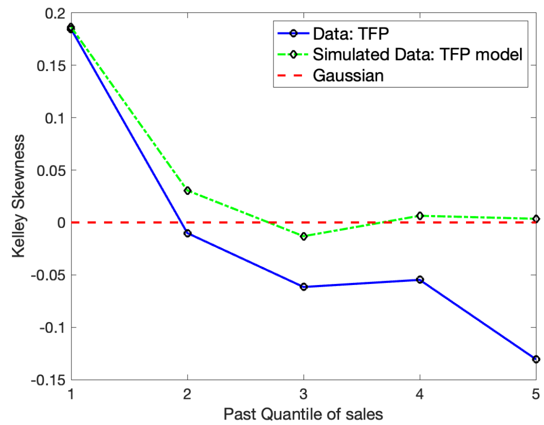
In Figure A.18, we show the evolution of TFP and its estimated persistent component for a specific large and productive firm, Apple. In 2006, Apple's productivity grew as much as in the top 5% of all firms in the sample for that year. The model estimates that this productivity boost was mostly driven by the persistent component and led to a subsequent sustained productivity growth. In 2012, Apple's productivity dropped sharply.[7] Again, the model assigns that even to the persistent component. In line with the results showed in Figure 6c of Section 3, the model suggests that negative shocks hitting productive firms will erase part of the persistence of histories. Indeed, Apple's productivity growth has been sluggish thereafter.

[7] That productivity change was in the bottom 5% of the distribution for that year.

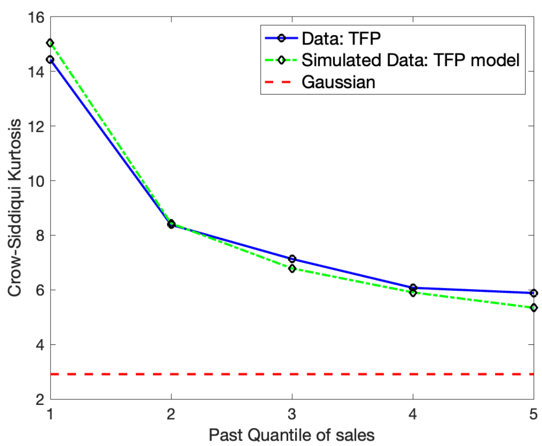
Figure A.15: Conditional moments of residualized log TFP: model vs data



(a) Conditional standard deviation



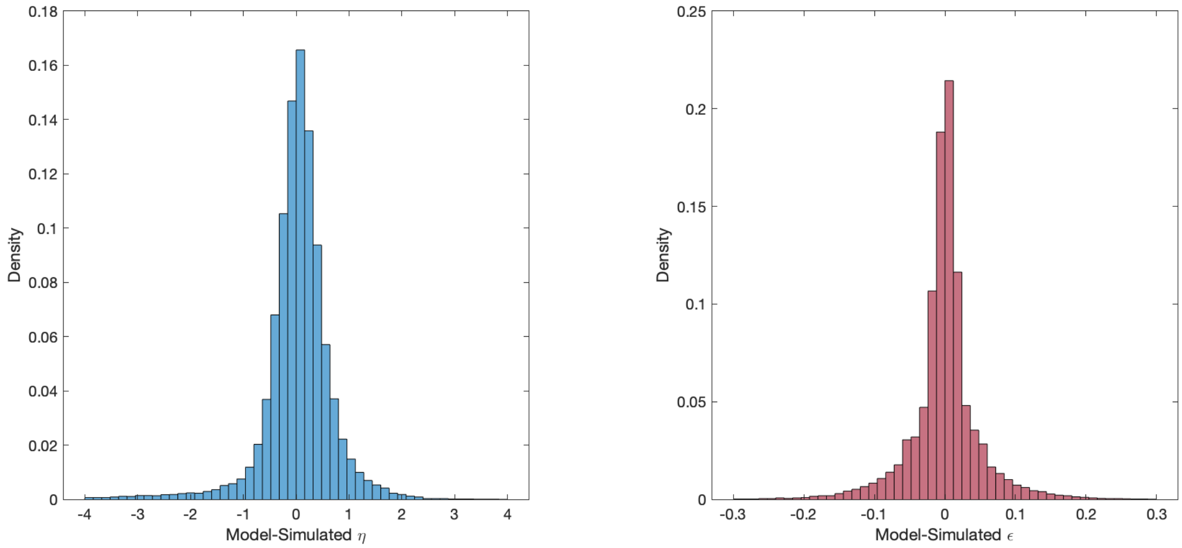
(b) Conditional skewness



(c) Conditional kurtosis

Notes: Blue lines refer to the Compustat data, while dashed-dotted green lines to data simulated according to the nonlinear model, with parameters set to their estimated values. Dashed red lines show the moments if the process was Gaussian.

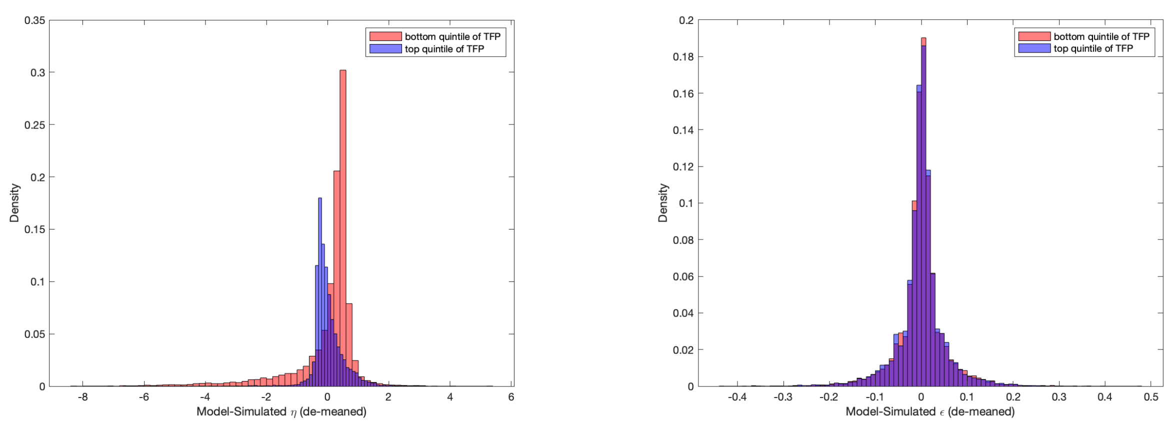
Figure A.16: Distribution of productivity components in the model



(a) Persistent component η (b) Transitory component ϵ

Notes: Model estimated on the baseline sample of manufacturing firms, as discussed in Section 3. Histogram of model-simulated persistent – left panel – and transitory – right panel – components of log productivity.

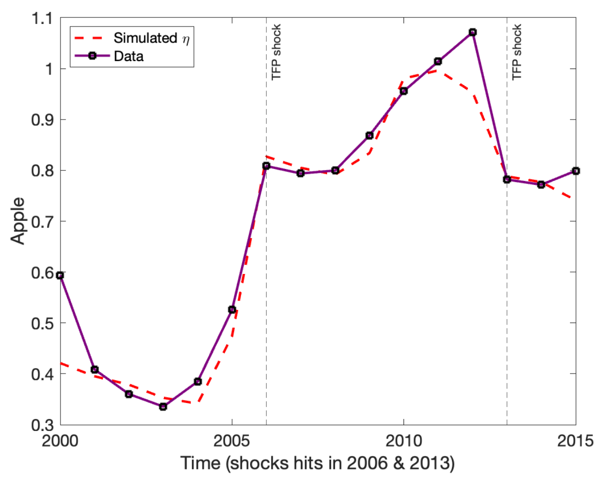
Figure A.17: Distribution of productivity components in the model by top and bottom quantiles of TFP



(a) Persistent component η (b) Transitory component ϵ

Notes: Model estimated on the baseline sample of manufacturing firms, as discussed in Section 3. Histogram of model-simulated persistent – left panel – and transitory – right panel – components of log productivity, by top and bottom past quantiles of the simulated TFP distribution.

Figure A.18: Evolution of productivity components for Apple



F Source of firm-level shocks

In this section we report details on variable construction and definition, as well as all the empirical results described in Section 4.

F.1 Data construction

1. Financial constraint measures (all defined as in Farre-Mensa and Ljungqvist (2016)). We consider these measures in contemporaneous "levels", as well as in changes (e.g., whether a firm categorized as financially unconstrained according to a metric in the previous year becomes constrained this year).

   - Nondividend payers.
   - KZ index, constructed following Lamont et al. (2001).
   - WW index, constructed following Whited and Wu (2006) and Hennessy and Whited (2007).
   - HP index, constructed following Hadlock and Pierce (2010).

2. Firm-level financial characteristics are the following, also constructed as in Farre-Mensa and Ljungqvist (2016): (i) the ratio of cash and short-term investments (*che*) to total assets, (ii) an indicator for any long-term debt, (ii) share of short-term debt, (iv) long-term book leverage, (v) total book leverage, (vi) return on assets (ROA), (vi) a profitable dummy taking 1 if ROA is positive and zero otherwise, (vi) investment opportunities as in Frank and Goyal (2009). Given extreme values, we winsorize (iv), (v), and (vi) at the top 99.5 percentile.

3. Firm-level characteristics are: (i) age (measured as the years in sample), (ii) Research and Development Expense over total assets (winsorized at the top 99.5 percentile), (iii) NAICS 3-digit industry, (iv) acquisitions over total assets (winsorized at the top 99.5 percentile).

4. Competitors' performance. For each year and NAICS 4-digit industry, we construct average (residualized) sales and TFP growth rates, excluding the own firm. We also construct growth rates of averages, for the same cells and also excluding own firms.

5. We use the following variables from Arora et al. (2019): (i) growth rates in the number of patents, (ii) growth rates in the number of publications, (iii) the log of 1 + the number of new publications in a given year, (iv) the log of 1 + the number of new

patents in a given year, (v) an indicator for whether it is the first year in which the firm obtained a patent. Growth rates in (i) and (ii) are symmetric as introduced by Davis et al. (2006), thus dividing the change between t and $t-1$ between the average level across both periods.

6. Finally, we consider the log of employment, total assets, cost of goods sold, materials (defined as in Keller and Yeaple (2009)), and capital, and their changes over time.

F.2 Drivers of firm-level shocks

Tables A.3-A.6 show the adjusted R^2 for the various dependent variables considered in Section 4, for different sets of predictors described in the table notes. Since matching patent variables results in a loss of observations, the analysis with patent variables is done separately in columns (9)-(11).

Table A.3: R^2 of predictors: sales growth

		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Financial constraints		✓				✓	✓	✓			✓	✓
Financial variables			✓			✓	✓	✓			✓	✓
Firm characteristics				✓		✓	✓	✓			✓	✓
Competitors					✓	✓	✓	✓			✓	✓
Production inputs							✓	✓				✓
Production inputs (changes)								✓				✓
Optimal Lasso									✓			
Patent variables										✓	✓	✓
N		47651	44496	41850	56946	28402	25535	25293	24336	24496	17387	15238
adj. R^2	Sales growth	0.005	0.037	0.019	0.011	0.068	0.246	0.601	0.596	0.002	0.059	0.543
	Bottom 1%	0.011	0.040	0.022	0.000	0.044	0.059	0.127	0.114	0.005	0.050	0.100
	Top 1%	0.014	0.023	0.038	0.001	0.037	0.035	0.075	0.065	0.005	0.039	0.064
	Either tail	0.023	0.057	0.059	0.001	0.073	0.052	0.050	0.041	0.010	0.084	0.054

Notes: Baseline sample of manufacturing firms. Dependent variables: residualized sales growth, indicators for residualized sales growth in the bottom 1%, top 1%, either top or bottom 1%. Estimation run with standard errors robust to heteroskedasticity. "Financial constraints" are defined in item 1 of Appendix Section F.1. We consider these measures in contemporaneous "levels", as well as in changes (e.g., whether a firm categorized as financially unconstrained according to a metric in the previous year becomes constrained this year). "Financial variables" are defined in item 2 of the same appendix. "Firm characteristics" are defined in item 3. "Competitors" in item 4. "Production inputs" in item 6. Column (5) also contains year dummies."Optimal Lasso" are the variables chosen by the Lasso estimation shown in Table A.8, at the optimal regularization parameter identified by cross-validation. "Patent variables" are defined in item 5 of Appendix Section F.1.

Figure A.20 shows the LASSO coefficients for a classification exercise predicting whether an observation is in the top 1% of the TFP growth distribution, at different levels of the regularization parameter λ. As λ increases, a heavier shrinkage is applied to the coefficient estimates, which are progressively forced to zero. Hence, LASSO performs variable selection. A variable is a better predictor the higher λ is required to shrink its coefficient to zero. The

Table A.4: R^2 of predictors: TFP growth

		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Financial constraints		✓				✓	✓	✓			✓	✓
Financial variables			✓			✓	✓	✓			✓	✓
Firm characteristics				✓		✓	✓	✓			✓	✓
Competitors					✓	✓	✓	✓			✓	✓
Production inputs							✓	✓				✓
Production inputs (changes)								✓				✓
Optimal Lasso									✓			
Patent variables										✓	✓	✓
N		47651	44496	41850	56946	28402	25535	25293	24336	24496	17387	15238
adj. R^2	TFP growth	0.003	0.009	0.005	0.002	0.020	0.116	0.194	0.126	0.000	0.023	0.165
	Bottom 1%	0.017	0.054	0.046	0.001	0.071	0.036	0.065	0.043	0.008	0.082	0.078
	Top 1%	0.018	0.040	0.050	0.002	0.059	0.115	0.028	0.013	0.009	0.067	0.029
	Either tail	0.035	0.092	0.098	0.002	0.128	0.137	0.071	0.036	0.017	0.147	0.094

Notes: Dependent variables: residualized TFP growth, indicators for residualized TFP growth in the bottom 1%, top 1%, either top or bottom 1%. Same notes of Table A.3 for variable definition.

Table A.5: R^2 of predictors: changes in persistent component ($\Delta\eta$)

		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Financial constraints		✓				✓	✓	✓			✓	✓
Financial variables			✓			✓	✓	✓			✓	✓
Firm characteristics				✓		✓	✓	✓			✓	✓
Competitors					✓	✓	✓	✓			✓	✓
Production inputs							✓	✓				✓
Production inputs (changes)								✓				✓
Optimal Lasso									✓			
Patent variables										✓	✓	✓
N		47651	44496	41850	56946	28402	25535	25293	24336	24496	17387	15238
adj. R^2	$\Delta\eta$	0.003	0.006	0.004	0.002	0.015	0.100	0.138	0.095	0.000	0.018	0.108
	Bottom 1%	0.017	0.054	0.046	0.001	0.072	0.039	0.066	0.048	0.009	0.085	0.095
	Top 1%	0.018	0.040	0.052	0.002	0.059	0.113	0.026	0.013	0.009	0.068	0.029
	Either tail	0.035	0.092	0.099	0.002	0.129	0.133	0.071	0.041	0.018	0.150	0.104

Notes: Dependent variables: first differences in η simulated from model estimates, indicators for the bottom 1%, top 1%, either top or bottom 1%. Same notes of Table A.3 for variable definition.

vertical dashed line denotes the optimal λ selected via cross-validation.

Figure A.21 shows the results of a random forest, with the same dependent variable as in Figure A.20, but this time in the sample matched with patent data. We report two metrics that determine the variables' relative importance in predicting the dependent variable: increase in accuracy and Gini index. We show the 10 best predictors according to the increase in accuracy.

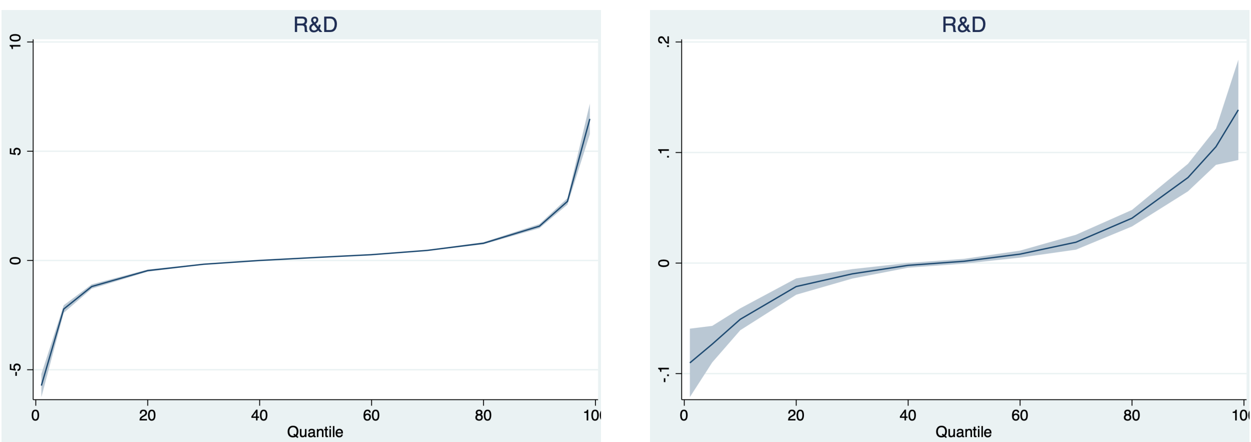
Table A.8 shows, for each dependent variable, the predictors selected at the optimal λ and the associated LASSO coefficients. When the coefficient is not reported, it means that the variable has not been selected at the optimal λ. Note that the LASSO estimation we run standardizes all variables so that they are on the same scale.

Table A.6: R^2 of predictors: changes in transitory component ($\Delta\epsilon$)

		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Financial constraints		✓				✓	✓	✓			✓	✓
Financial variables			✓			✓	✓	✓			✓	✓
Firm characteristics				✓		✓	✓	✓			✓	✓
Competitors					✓	✓	✓	✓			✓	✓
Production inputs							✓	✓				✓
Production inputs (changes)								✓				✓
Optimal Lasso									✓			
Patent variables										✓	✓	✓
N		47651	44496	41850	56946	28402	25535	25293	24336	24496	17387	15238
adj. R^2	$\Delta\epsilon$	0.003	0.012	0.002	0.006	0.030	0.046	0.083	0.052	0.000	0.035	0.099
	Bottom 1%	0.005	0.008	0.003	-0.000	0.009	0.012	0.020	0.012	0.001	0.007	0.020
	Top 1%	0.005	0.003	0.003	0.001	0.008	0.015	0.020	0.017	0.001	0.008	0.021
	Either tail	0.009	0.009	0.006	0.000	0.012	0.019	0.019	0.018	0.001	0.010	0.019

Notes: Dependent variables: first differences in ϵ simulated from model estimates, indicators for the bottom 1%, top 1%, either top or bottom 1%. Same notes of Table A.3 for variable definition.

Figure A.19: Quantile regression: TFP growth on R&D

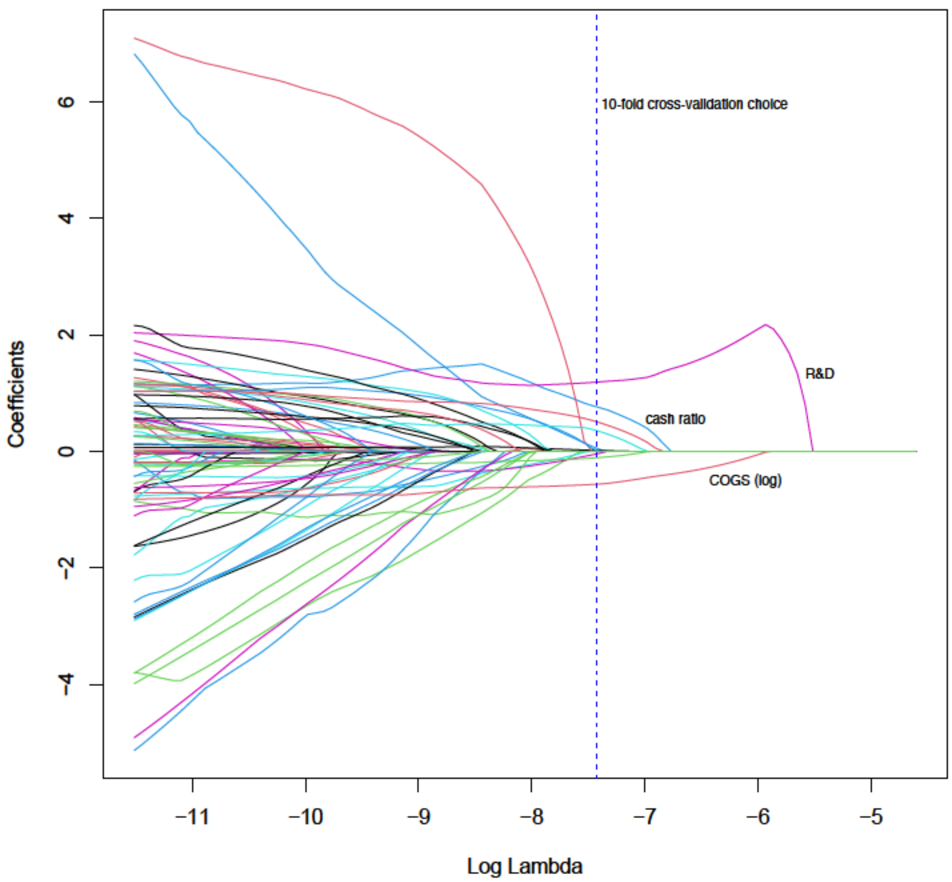


(a) TFP growth

(b) Transitory component ($\Delta\epsilon$)

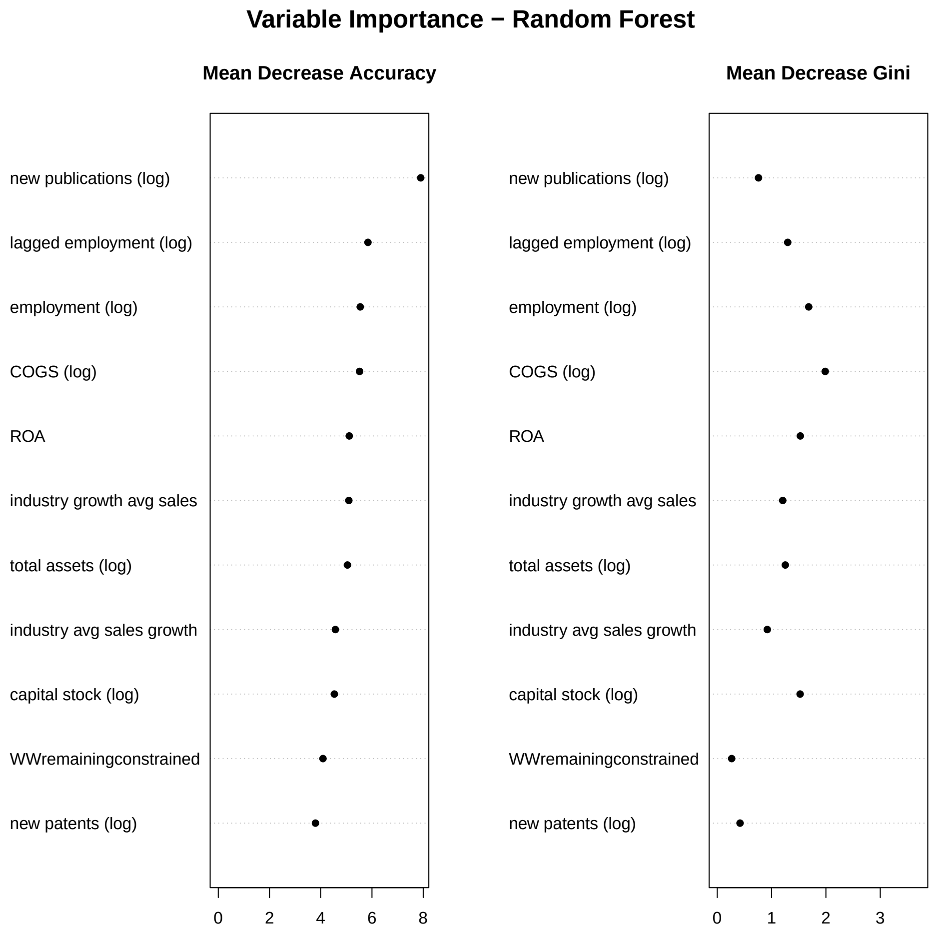
Notes: Baseline sample of manufacturing firms. Quantile regressions of residualized TFP growth (left panel) and change in its transitory component (right panel) on share of R&D expenses as defined in the text. Bootstrapped standard errors.

Figure A.20: LASSO estimates: indicator for top 1% TFP growth



Notes: Dependent variable: indicator for TFP growth in the top 1% of the distribution. Same notes of Table A.3 for variable definition.

Figure A.21: Random forest variable importance: indicator for top 1% TFP growth



Notes: Dependent variable: indicator for TFP growth in the bottom 1% of the distribution. Same notes of Table A.3 for variable definition. Sample with patents. We report only the 10 most important variables ranked according to the mean decreases in accuracy.

Table A.7: LASSO predictors: sales and TFP

	Sales growth				TFP growth			
	All	Bottom 1%	Top 1%	Either tail	All	Bottom 1%	Top 1%	Either tail
Financial constraints:								
HPbecomingconstrained						-0.3823		-0.1218
HPconstrained	0.0044		0.1701		0.0038			
HPremainingconstrained	-0.001				-0.0005	0.0375		
KZbecomingconstrained	0.0176				0.0143			
KZbecomingunconstrained	-0.0072				-0.0073			
KZconstrained	0.0162		0.0416	0.0774	0.0121			
KZremainingunconstrained				-0.2096				
KZunconstrained	-0.0074				-0.0045			
WWbecomingunconstrained				0.0004		0.6824		
WWconstrained	0.0133				0.0144			
WWremainingconstrained				-0.1782				
WWunconstrained				0.2096				
Financial variables:								
ROA	0.0117	-0.0289		-0.0204	0.0095	-0.0069	0.0012	
any long-term debt				-0.2136	-0.0006	-0.016		-0.0469
cashratio	0.0082	0.4273	0.5646	1.117	0.011	2.9157	0.5565	2.2422
investment opportunities	0.0062	0.0089	0.0057	0.0011	0.0054	0.0062		
leverage		0.1295		0.1106	-0.0008	0.0022	-0.0567	
profitable	0.0531	-0.5394	-0.3441	-0.9373	0.0504	-1.1987		-0.8061
Firm characteristics:								
acquisitions	-0.0011			2.9846				
age	-0.0002			-0.0049				
R & D	0.0159				0.0359		1.23	0.3021
any industry dummy	✓	✓	✓	✓	✓	✓		✓
Competitors:								
industry avg sales growth	0.0267					-0.4035		
industry avg TFP growth	0.0119				0.0718	-0.064		-0.0634
industry growth avg sales				-0.1989				
industry growth avg TFP	0.0175				0.0666			
Production inputs:								
total assets growth	0.0897	-0.0802		0.0187	0.0838	-0.6644		-0.2155
COGS growth	0.5197	-2.2842	1.6088	-0.2412	-0.0832	0.6327		0.4462
employment growth	0.1234	-0.5568	0.7009	-0.1646	0.1052	-0.7276	0.1477	-0.2577
capital stock (log)					-0.0021			
COGS (log)				-0.2949		-0.5151	-0.5019	-0.5969
lagged employment (log)			-0.0687					
materials growth	0.2012	-0.7434	0.9287		0.1638		0.308	
any year dummy	✓	✓		✓	✓	✓		

Notes: For each column, the table reports the LASSO coefficients when the variable is selected at the optimal regularization parameter λ. Same notes of Table A.3 for variable definition. For compactness, we do not report the coefficients for the specific year and industry dummies that are selected. Sample that is not merged with patents ($N = 25,293$). When we do the merge, the following variables currently in the table are not selected: HPremainingconstrained, WWremainingconstrained, and WWunconstrained. In contrast, long-term leverage is selected.

Table A.8: LASSO predictors: transitory and persistent shocks

	$\Delta\eta$				$\Delta\epsilon$			
	All	Bottom 1%	Top 1%	Either tail	All	Bottom 1%	Top 1%	Either tail
Financial constraints:								
HPbecomingconstrained		-0.4473						
HPconstrained	0.0051							
HPremainingconstrained	-0.0062	0.0101					-0.0006	
KZbecomingconstrained	0.0076				0.0058		0.7789	0.3891
KZbecomingunconstrained	-0.0078				-0.0007		-0.0614	
KZconstrained	0.0086				0.0043		0.0043	
KZremainingconstrained		0.012				-0.0136		-0.0687
KZremainingunconstrained	-0.0008							
KZunconstrained	-0.0053				-0.0001	0.1633		0.1827
WWbecomingconstrained	-0.0082				0.0014			
WWbecomingunconstrained		0.6481			-0.005	0.087		
WWconstrained	0.009		0.067		0.0056		0.2593	
WWremainingunconstrained	-0.0011							
WWunconstrained					-0.001			
nondividend payer						0.1442	0.7453	0.7412
Financial variables:								
ROA	0.0118	-0.0305	0.0239	-0.0236				
any long-term debt	-0.0036	-0.0494	-0.0889	-0.0201				
cashratio	0.0028	2.8243	0.7351	1.9517	0.0025	0.2316	1.1801	0.6679
investment opportunities	0.0053	0.0028			0.0008		0.0244	0.0081
leverage	-0.0034	0.03	-0.0711					-0.0074
profitable	0.0365	-1.3985	-0.0073	-0.7241	0.0166	-1.3385		-0.7299
Firm characteristics:								
acquisitions	-0.0461				0.0237		1.8569	1.3577
age	-0.0002				0		-0.0061	
R & D	0.0371		1.3278	0.4885	0.0002			
any industry dummy	✓	✓				✓	✓	✓
Competitors:								
industry avg sales growth	-0.0004	-0.6577			0.0072			
industry avg TFP growth	0.0565				0.0079		0.237	
industry growth avg sales	-0.0023				0.0042		0.6079	0.2308
industry growth avg TFP	0.0303				0.0276			
Production inputs:								
total assets growth	0.0714	-0.9041		-0.332	0.0128		0.5302	0.2798
COGS growth	-0.0873	0.6369		0.4611	-0.0007		-0.292	-0.0296
employment growth	0.0938	-0.6725	0.3649	-0.1615	0.0136	-0.0548	0.4654	
capital stock (log)	-0.0029							
materials (log)	0.0008							
COGS (log)		-0.4686	-0.5211	-0.5324				
lagged employment (log)					0		-0.0898	-0.092
materials growth	0.1485		0.539		0.0181	-0.0708	0.6975	0.0807
any year dummy	✓	✓			✓	✓	✓	✓

Notes: For each column, the table reports the LASSO coefficients when the variable is selected at the optimal regularization parameter λ. Same notes of Table A.3 for variable definition. Dependent variables: first differences in η and ϵ simulated from model estimates, indicators for the bottom 1%, top 1%, either top or bottom 1%. Sample that is not merged with patents ($N = 25,293$). When we do the merge, the following variables currently in the table are not selected: Kzremainingunconstrained, WWbecomingconstrained, WWremainingunconstrained. In contrast, the growth rate in patents and in publications, and dummy for first patent, are selected.

G Analytical appendix to Section 5

G.1 Example of how granular shocks affect the firm-size distribution

In this section, we derive the $t+1$ firm-size distribution as a function of initial sales, S_{i0} and the sequence of shocks over time specifying a model for sales that decompose transitory and persistent shocks with linear persistence. Specifically, we consider a model for log sales of firm i at time t, o_{it}, as follows:

$$o_{it} = \epsilon_{it} + \eta_{it}$$
$$\eta_{it} = \rho\eta_{it-1} + v_{it}$$

where ϵ_{it} is the transitory shock and η_{it} is the persistent shock, which follows an AR(1) process with linear persistence. We then get that $\frac{S_{it+1}-S_{it}}{S_{it}} = U_{it+1}$ is well approximated by $o_{it+1} - o_{it} = \epsilon_{it+1} - \epsilon_{it} + \eta_{it}(\rho-1) + v_{it+1}$ and, thus, $S_{it+1} = S_{i0}\prod_{j=0}^{t}(1+U_{ij+1})$ is well approximated by:

$$S_{i0}\prod_{j=0}^{t}(1+\epsilon_{ij+1} - \epsilon_{ij} + \eta_{ij}(\rho-1) + v_{ij+1}) \tag{11}$$

$$= S_{i0}\prod_{j=0}^{t}\left(1+\epsilon_{ij+1} - \epsilon_{ij} + \eta_{i0}(\rho-1)\rho^{j} + (\rho-1)\prod_{k=1}^{j}\rho^{j-k}v_{ik} + v_{ij+1}\right)$$

Assuming $t=0$,

$$S_{i1} \approx S_{i0}(1+\epsilon_{i1} - \epsilon_{i0} + \eta_{i0}(\rho-1) + v_{i1})$$

If ϵ_{i1}, ϵ_{i0}, η_{i0}, v_{i1}, S_{i0} are all independent random variables, then S_{i1} will be as heavy tailed as the heavier-tailed distribution among those of all the above variables.[8]

For $t \geq 1$, then square and product of the innovations appear in the factor $\prod_{j=0}^{t}(1+$

[8]For the negative terms appearing in the expression in parentheses, notice that multiplying a random variable X, regularly varying at infinity with index α, by a negative constant c preserves the regular variation with the same index α but shifts the focus from the upper tail of X to its lower tail.

$\epsilon_{ij+1} - \epsilon_{ij} + \eta_{ij}(\rho - 1) + v_{ij+1})$ in Equation (11). For instance, assuming $t = 1$,

$$
\begin{aligned}
S_{i2} &\approx S_{i0}(1 + \epsilon_{i1} - \epsilon_{i0} + \eta_{i0}(\rho - 1) + v_{i1})(1 + \epsilon_{i2} - \epsilon_{i1} + \eta_{i0}\rho(\rho - 1) + (\rho - 1)v_{i1} + v_{i2}) \\
&= S_{i0}(1 + \epsilon_{i2} + \eta_{i0}(\rho - 1)\rho + (\rho - 1)v_{i1} + v_{i2} + \epsilon_{i1}\epsilon_{i2} - \epsilon_{i1}^2 + \epsilon_{i1}\eta_{i0}(\rho - 1)\rho + \epsilon_{i1}(\rho - 1)v_{i1} + v_{i2}\epsilon_{i1} \\
&- \epsilon_{i0} - \epsilon_{i0}\epsilon_{i2} + \epsilon_{i0}\epsilon_{i1} - \epsilon_{i0}\eta_{i0}(\rho - 1)\rho - \epsilon_{i0}(\rho - 1)v_{i1} - \epsilon_{i0}v_{i2} \\
&+ \eta_{i0}(\rho - 1) + \epsilon_{i2}\eta_{i0}(\rho - 1) - \epsilon_{i1}\eta_{i0}(\rho - 1) + \eta_{i0}^2(\rho - 1)^2\rho + \eta_{i0}(\rho - 1)^2 v_{i1} + \eta_{i0}(\rho - 1)v_{i2} \\
&+ v_{i1} + v_{i1}\epsilon_{i2} - v_{i1}\epsilon_{i1} + v_{i1}\eta_{i0}(\rho - 1)\rho + (\rho - 1)v_{i1}^2 + v_{i1}v_{i2})
\end{aligned}
$$

G.2 Insights on the effects of the model's features on the aggregates

Consider the following linear AR(1) model for the TFP of a firm i at time t, with $i = 1, ..., N$: $o_{i,t+1} = \rho o_{i,t} + \nu_{i,t+1}$, where innovations ν are normally distributed with variance σ_ν^2. Define the aggregated TFP $\tilde{G}_t = \sum_i o_{i,t} - \sum_i o_{i,t-1} = \sum_i (o_{i,t} - o_{i,t-1})$. We are interested in assessing the importance of ρ and σ_ν^2 on the variance and autocorrelation of $\tilde{G}_t$. Let us assume that $o_{i,t}$ are i.i.d., which is an unappealing assumption that we make here only to illustrate the mechanisms. Let us also assume that $\rho < 1$. For the variance of $\tilde{G}_t$, we get:

$$
\begin{aligned}
Var(\tilde{G}_t) &= Var\left(\sum_i (o_{i,t} - o_{i,t-1})\right) = \sum_i Var(o_{i,t} - o_{i,t-1}) \\
&= \sum_i Var\left(-(1-\rho)o_{i,t-1} + \nu_{i,t}\right) = N\left((1-\rho)^2 \frac{\sigma_\nu^2}{1-\rho^2} + \sigma_\nu^2\right) \\
&= 2N\frac{\sigma_\nu^2}{1+\rho}
\end{aligned}
$$

For the first order autocovariance of $\tilde{G}_t$, we get:

$$
\begin{aligned}
Cov(\tilde{G}_t, \tilde{G}_{t-1}) &= Cov\left(\left(\sum_i (o_{i,t} - o_{i,t-1})\right), \left(\sum_i (o_{i,t-1} - o_{i,t-2})\right)\right) \\
&= NCov\left((-(1-\rho)\rho o_{i,t-2} - (1-\rho)\nu_{i,t-1} + \nu_{i,t})\right), \left((-(1-\rho)o_{i,t-2} + \nu_{i,t-1})\right) \\
&= N\left(\frac{-(1-\rho)\sigma_\nu^2}{1+\rho}\right)
\end{aligned}
$$

Second, we consider a model that also features a Gaussian transitory component as follows: $o_{i,t+1} = \eta_{i,t+1} + \varepsilon_{i,t+1}$, where $\eta_{i,t+1} = \rho\eta_{i,t} + \nu_{i,t+1}$, and innovations ν and ε are normally distributed with respective variances σ_ν^2 and σ_ε^2. Under the same assumptions considered above, it is possible to show that now:

$$
\begin{aligned}
Var(\tilde{G}_t) &= Var\left(\sum_i (o_{i,t} - o_{i,t-1})\right) = \sum_i Var(o_{i,t} - o_{i,t-1}) \\
&= \sum_i Var\left(-(1-\rho)\eta_{i,t-1} + \nu_{i,t} + \varepsilon_{i,t} - \varepsilon_{i,t-1}\right) \\
&= 2N\left(\frac{\sigma_\nu^2}{1+\rho} + \sigma_\varepsilon^2\right)
\end{aligned}
$$

Thus, as expected ε increases the volatility of $\tilde{G}_t$. As for the first order autocovariance of $\tilde{G}_t$, we now get:

$$
\begin{aligned}
Cov(\tilde{G}_t, \tilde{G}_{t-1}) &= Cov\left(\left(\sum_i (o_{i,t} - o_{i,t-1})\right), \left(\sum_i (o_{i,t-1} - o_{i,t-2})\right)\right) \\
&= N\left((1-\rho)^2 \rho \frac{\sigma_\nu^2}{1-\rho^2} - (1-\rho)\sigma_\nu^2 - \sigma_\varepsilon^2\right) \\
&= N\left(\frac{-(1-\rho)\sigma_\nu^2}{1+\rho} - \sigma_\varepsilon^2\right)
\end{aligned}
$$

Thus, as expected ε decreases the autocorrelation of $\tilde{G}_t$.