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Abstract

A limit theory is developed for the least squares estimator for mildly and purely explosive autoregressions under drifting sequences of parameters with autoregressive roots ρ_n satisfying

$$\rho_n \to \rho \in (-\infty, -1] \cup [1, \infty) \quad \text{and} \quad n\left(|\rho_n| - 1\right) \to \infty.$$

Drifting sequences of innovations and initial conditions are also considered. A standard specification of a short memory linear process for the autoregressive innovations is extended to a triangular array formulation both for the deterministic weights and for the primitive innovations of the linear process, which are allowed to be heteroskedastic L_1-mixingales. The paper provides conditions that guarantee the validity of Cauchy limit distribution for the OLS estimator and standard Gaussian limit distribution for the t-statistic under this extended explosive and mildly explosive framework.

JEL classification: C12, C18, C22

Key words: triangular array, explosive autoregression, linear process, conditional heteroskedasticity, mixingale, Cauchy distribution

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1 Introduction

First order autoregressive processes with an explosive root, i.i.d. Gaussian innovations and zero initial condition were first analysed by White (1958) who, using a moment generating function approach, derived a Cauchy limit theory for the maximum likelihood estimator. Using martingale methods, Anderson (1959) arrived to the same conclusion and showed that the Cauchy limit theory is not invariant to deviations from Gaussianity and that, in general, the limit distribution of the OLS estimator depends on the distribution of the innovation sequence. Invariance of the Cauchy least squares regression limit theory to the distribution of the innovations can be recovered when the explosive root approaches unity as the sample size tends to infinity at sufficiently slow rate. This invariance, first established by Phillips and Magdalinos (2007a, hereafter PM2007a), allows for semiparametric inference within the class of mildly explosive autoregressions: a property that has been employed to construct inferential procedures for the detection and dating of financial bubbles by Phillips, Wu and Yu (2011) and Phillips and Yu (2011) among others.

The invariance result of PM2007 was extended in various directions to include innovation sequences that are weakly dependent (Phillips and Magdalinos (2007b)), strongly dependent (Magdalinos (2012)), conditionally heteroskedastic (Arvanitis and Magdalinos (2019)) and to a class of stationary processes that includes long memory and antipersistence (Wang (2023)). Aue and Horvath (2007) relaxed the moment conditions on the innovations by considering an i.i.d. innovation sequence that belongs to the domain of attraction of a α-stable law and showed that, in general, the normalised and centred OLS estimator converges to a ratio of two independent α-stable random variables which is Cauchy distributed only when the innovation sequence belongs to the Gaussian domain of attraction with $\alpha = 2$.

All works listed above consider drifting sequences of autoregressive parameters that converge to unity from above at the mildly explosive rate. Drifting sequences of autoregressive parameters have been employed as early as Phillips (1987b) in the analysis of the discontinuity of inference in different regions of the parameter space. More recent work considers certain distributional aspects of the innovation sequence of an autoregression as an infinite dimensional nuisance parameter: from this viewpoint, an analysis of drifting sequences of innovation processes provides information on the sensitivity of autoregressive inference to the innovations' distributional characteristics.

Andrews and Guggenberger (2012, 2014) consider the distribution function of a stationary innovation sequence as part of the parameter space and derive an OLS and GLS limit theory along drifting sequences of autoregressive parameters on $[-1+\delta, 1]$ for some $\delta > 0$ and along drifting sequences of innovation processes belonging to a class of (possibly conditionally heteroskedastic) martingale differences. Recent work by Magdalinos and Petrova (2023) proposes an endogenously generated instrumental variable procedure for autoregression and predictive regression with uniform asymptotic size properties over an autoregressive parameter space of the form $[-M, M]$ for some $M > 0$ which, in addition to (near) stationary and unit roots, includes explosive and mildly explosive autoregressive roots. Any attempt to introduce aspects of the innovation sequence as a nuisance parameter in a parameter space containing autoregressive roots in $(-\infty, -1] \cup [1, \infty)$ would require the development of limit distribution theory along drifting sequences of both (mildly) explosive roots and innovation processes. Such limit theory is not available, even in the simplest case of the OLS estimator and the current paper aims to fill this gap in the literature.

Given a sample $t \in \{1, ..., n\}$, we consider a sequence of linear process innovations $u_{n,t} = \sum_{j=0}^{\infty} c_{n,j} e_{n,t-j}$ where $(e_{n,t})$ is a (possibly conditionally heteroskedastic) martingale difference array and $(c_{n,j})$ is an array of numbers satisfying a short memory array condition. A law of large numbers is derived for sample variance and covariance of $(e_{n,t})$ (Lemma 1). In the mildly explosive case, $|\rho_n| \to 1$, the paper employs martingale approximation in the spirit of PM2007a (Lemma 2) and provides a direct extension of the Cauchy limit distribution result of that paper. In the purely explosive case, $\rho_n \to \rho \in (-\infty, -1) \cup (1, \infty)$, the array structure of $(e_{n,t})$ invalidates the martingale convergence theorem and raises significant challenges in showing that the denominator of the ratio that arises as an approximation of the normalised and centred OLS estimator is *a.s.* non-zero. This issue is dealt with by showing that $(e_{n,t}, \mathcal{F}_{n,t})$ may be approximated by a martingale difference sequence $(e_t, \mathcal{F}_t)$ by taking $e_t := \liminf_{n\to\infty} e_{n,t}$ and $\mathcal{F}_t := \sigma\left(\liminf_{n\to\infty} \mathcal{F}_{n,t}\right)$. This approximation may be employed to show that the denominator of the normalised and centred OLS estimator is non-zero with probability tending to one (Lemma 3). Lemma 4 derives new limit theory for (mildly) explosive processes with a negative root $\rho_n \to \rho \in (-\infty, -1]$. Theorems 1 and 2 provide the limit distribution theory for the OLS estimator and the t-statistic respectively. All proofs are included in the Appendix.

2 Main Results

Consider a first order autoregressive process of the form

$$x_{n,t} = \mu + X_{n,t}, \quad X_{n,t} = \rho_n X_{n,t-1} + u_{n,t}, \quad t \in \{1, ..., n\} \tag{1}$$

where $X_{n,t}$ is initialised at $X_{n,0}$. The intercept is introduced in the model in a way that it may contribute but does not dominate asymptotically in the form of a deterministic trend when ρ_n is in a vicinity of unity; this specification goes back to Andrews (1993) and has been employed by numerous papers that wish to introduce an intercept to nonstationary models while maintaining their stochastic nature. It is easy to see that upon recursive substitution, (1) can be written as

$$x_{n,t} = \mu (1 - \rho_n) + \rho_n x_{n,t-1} + u_{n,t}, \quad x_{n,0} = \mu + X_{n,0}. \tag{2}$$

As mentioned in the introduction, we consider a drifting sequence of innovation processes $(u_{n,t})$ in order to provide the possibility of including some of the distributional properties of $(u_{n,t})$ in a parameter space of an autoregression as a nuisance parameter. We do not consider drifting sequences of intercepts since any critical region based on the OLS estimator

$$\hat{\rho}_n = \frac{\sum_{t=1}^{n} x_{n,t} x_{n,t-1} - n \bar{x}_{n,n} \bar{x}_{n,n-1}}{\sum_{t=1}^{n} x_{n,t-1}^2 - n \bar{x}_{n,n-1}^2} \tag{3}$$

is exactly invariant to μ.

We present a formal set of assumptions on the drifting sequences of parameters ρ_n, $u_{n,t}$ and $X_{n,0}$ in (1)-(2).

Assumption 1 (AR root). *The sequence $(\rho_n)_{n \in \mathbb{N}}$ satisfies $\rho_n \to \rho \in (-\infty, -1] \cup [1, \infty)$ and $n (|\rho_n| - 1) \to \infty$*

Assumption 2 (innovation sequence). *For each $n \in \mathbb{N}$, the sequence $(u_{n,t})_{t \in \mathbb{N}}$ in (2) is a stationary linear process of the form*

$$u_{n,t} = \sum_{j=0}^{\infty} c_{n,j} e_{n,t-j} \quad \textit{with} \quad \sup_{n \geq 1} \sum_{j=0}^{\infty} |c_{n,j}| < \infty \tag{4}$$

where $(c_{n,j})$ *is an array of numbers satisfying* $c_{n,0} = 1$, $\lim_{n\to\infty} \sum_{j=0}^{\infty} c_{n,j}^2 > 0$ *and*

$$C(\rho) := \lim_{n\to\infty} \sum_{j=0}^{\infty} \rho^{-j} c_{n,j} \neq 0 \;\; \textit{for } \rho \in (-\infty, -1] \cup [1, \infty). \tag{5}$$

Given a sequence of filtrations $(\mathcal{F}_{n,t})_{t\in\mathbb{Z}}$*, the sequence* $(e_{n,t}, \mathcal{F}_{n,t})_{t\in\mathbb{Z}}$ *in (2) is a martingale difference array such that*

$$\liminf_{n\to\infty} \liminf_{t\to\infty} \mathbb{E}_{\mathcal{F}_{n,t-1}} |e_{n,t}| > 0 \;\; a.s., \tag{6}$$

$\max_{1\le t\le n} \mathbb{E}\left(e_{n,t}^2 \mathbf{1}\left\{e_{n,t}^2 > \lambda_n\right\}\right) \to 0$ *when* $\lambda_n \to \infty$, $\sigma_n^2 := \mathbb{E}\left(e_{n,t}^2\right) \to \sigma^2 > 0$ *and* $\sigma_{n,t}^2 := \mathbb{E}_{\mathcal{F}_{n,t-1}}\left(e_{n,t}^2\right)$ *satisfies the following:* $\left(\sigma_{n,t}^2\right)_{t\in\mathbb{Z}}$ *is strictly stationary for each* n *with* $\sigma_{n,t}^2 > 0$ *a.s*

$$\limsup_{n\to\infty} \sup_{t\in\mathbb{N}} \sigma_{n,t}^2 < \infty \;\; a.s., \tag{7}$$

and there exist $b > 0$ *and sequences of positive numbers* $(\psi_m)_{m\in\mathbb{N}}$ *and* $(\varphi_n)_{n\in\mathbb{N}}$ *satisfying* $\psi_m \to 0$ *and* $\varphi_n \to 0$ *such that*

$$\sup_{t\ge 1} \left\|\mathbb{E}_{\mathcal{F}_{n,t-1-m}}\left(\sigma_{n,t}^2 - \sigma_n^2\right)\right\|_{L_1} \le b\left(\psi_m + \varphi_n\right) \quad \textit{for all } m, n \ge 1. \tag{8}$$

In the special case of $e_{n,t}$ being conditionally homoskedastic, $\sigma_{n,t}^2 = \sigma_n^2$ for all t, so strict stationarity of $\left(\sigma_{n,t}^2\right)_{t\in\mathbb{Z}}$ is immediate and (7) and (8) hold trivially, the former by convergence of $(\sigma_n^2)_{n\in\mathbb{N}}$ and the latter since the left side of (8) is equal to 0 by the tower property of conditional expectations.

Assumption 3 (initial condition).

(i) *When* $|\rho_n| \to 1$, $X_{n,0} = o_p\left[\left(\rho_n^2 - 1\right)^{-1/2}\right]$.

(ii) *When* $|\rho_n| \to |\rho| > 1$, $X_{n,0} \to_d X_0$ *where* X_0 *is an* $\mathcal{F}_0$*-measurable random variable, where* $\mathcal{F}_0 := \sigma\left(\cup_{n\in\mathbb{N}} \mathcal{F}_{n,0}\right)$ *with* $(\mathcal{F}_{n,t})_{t\in\mathbb{Z}}$ *defined in Assumption 2.*

We provide some discussion of Assumptions 1-3. Assumption 1 includes explosive roots $\rho_n \to \rho \in (-\infty, -1) \cup (1, \infty)$ and mildly explosive roots $\rho_n \to 1$ with $n(\rho_n - 1) \to \infty$ and $\rho_n \to -1$ with $n(\rho_n + 1) \to \infty$; when

$|\rho_n| \to 1$, the convergence to 1 or -1 takes place at rate strictly dominated by the n^{-1}: the local to unity rate of near nonstationary processes (see Phillips (1987b)). Unlike much of the existing literature on mildly explosive processes (Phillips and Magdalinos (2007b), Magdalinos (2012), Arvanitis and Magdalinos (2019)), the mildly explosive rate is not restricted by a parametrisation and is allowed to be arbitrary only required to satisfy $n(|\rho_n| - 1) \to \infty$.

Assumption 2 is an array generalisation of a stationary short memory linear process with respect to both the non-stochastic weights $c_{n,j}$ and the innovation sequence $e_{n,t}$ which is now an array of martingale differences. Since

$$\sum_{h=0}^{\infty} \left|\gamma_{u_n}(h)\right| \leq \sigma_n^2 \left(\sum_{i=0}^{\infty} |c_{n,i}|\right)^2, \tag{9}$$

(4) implies that $\sup_{n\geq 1} \sum_{h=0}^{\infty} \left|\gamma_{u_n}(h)\right| < \infty$. The existence of the limits of $\sum_{j=0}^{\infty} c_{n,j}^2$ and $\sum_{j=0}^{\infty} \rho^{-j} c_{n,j}$ is ensured on a subsequence $(m_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ by (4) and the Bolzano-Weierstrass (BW) theorem; this is usually enough to establish the uniformity of asymptotic size of critical regions and confidence intervals, the proof of which typically relies on subsequential arguments, see Andrews, Cheng and Guggenberger (2020). The existence of such limits along $\mathbb{N}$ when conducting asymptotics of estimators along drifting sequences is typically assumed for notational economy with a proper BW analysis conducted when computing the asymptotic size of critical regions. The same holds for the existence of the limit of σ_n^2, guaranteed subsequentially by stationarity and boundedness of the sequence $\mathbb{E}\left(\sigma_{n,0}^2\right)$.

Condition (5) assumes away antipersistence: the usual requirement $C(1) \neq 0$ needs to be extended over the entire range $\rho \in (-\infty, -1] \cup [1, \infty)$ in order to avoid degeneracy in the long run variance. When $\rho = -1$, (5) requires that

$$C(-1) = \lim_{n\to\infty} \sum_{j=0}^{\infty} (-1)^{-j} c_{n,j} = \lim_{n\to\infty} \left(\sum_{j=0}^{\infty} c_{n,2j} - \sum_{j=0}^{\infty} c_{n,2j+1}\right) \neq 0.$$

Conditions (6) and (7) are required for the proof of the local Marcinkiewicz-Zygmund conditions for a martingale difference e_t that approximates $e_{n,t}$ (see Lemma 3 and its proof). A martingale difference sequence $(\eta_t, \mathcal{H}_t)_{t\in\mathbb{N}}$ is said to satisfy the local Marcinkiewicz-Zygmund (MZ) conditions if

$$\liminf_{t\to\infty} \mathbb{E}\left(|\eta_t| \,|\, \mathcal{H}_{t-1}\right) \ a.s. \ \text{ and } \ \sup_{t\in\mathbb{N}} \mathbb{E}\left(\eta_t^2 \middle| \mathcal{H}_{t-1}\right) < \infty \ a.s. \tag{10}$$

see Lai and Wei (1983). The local MZ conditions are used in conjunction with Corollary 2 of Lai and Wei (1983) to show that the denominator of the ratio that approximates the centred and normalised OLS estimator in the explosive case is *a.s.* non-zero; Lemma 3 extends this approach to arrays of martingale differences to which the martingale convergence theorem and the Lai and Wei (1983) result do not directly apply.

Assumption 2 accommodates a large class of stationary conditional heteroskedastic processes. Condition (8) is a slight generalisation of the L_1-mixingale array assumption of Andrews (1988). By using similar methods to Example 1 of Arvanitis and Magdalinos (2019) and the results of Giraitis *et.al.* (2000), we may show that (8) is satisfied by an array of stationary ARCH(∞) processes. The L_1-mixingale array condition (8) is useful for the validity of a law of large numbers for $(\rho_n^2 - 1)\sum_{t=1}^n \rho_n^{-2t}\sigma_{n,t}^2$ derived by Arvanitis and Magdalinos (2019) and for deriving a law of large numbers for $n^{-1}\sum_{t=1}^n u_{n,t}^2$ in Lemma 1 below.

Under mild explosivity, Assumption 3(i) imposes the usual order of magnitude on the initial condition $X_{n,0}$ that guarantees its asymptotic negligibility from OLS asymptotics. For explosive processes with $|\rho| > 1$, $X_{n,0}$ contributes to the limit distribution of the OLS estimator; we denote by X_0 its limit in distribution, which, in line with the discussion for the existence of the limits of $\sum_{j=0}^\infty c_{n,j}^2$ and $\sum_{j=0}^\infty \rho^{-j} c_{n,j}$, is ensured subsequentially by tightness of the sequence $(X_{n,0})_{n\in\mathbb{N}}$, in other words by the condition $X_{n,0} = O_p(1)$.

Under Assumption 2, we may prove the following law of large numbers, which is useful for deriving the asymptotic distribution of the t-statistic in Theorem 2. Denoting by $\gamma_{u_n}(h) = \mathbb{E}(u_{n,t}u_{n,t-h})$ and $\omega_n^2 := \left(\sum_{j=0}^\infty c_{n,j}\right)^2 \sigma_n^2$ the autocovariance function and long run variance of $(u_{n,t})$, (9), the identity

$$\omega_n^2 = \sigma_n^2 \sum_{j=0}^\infty c_{n,j}^2 + 2\sum_{h=1}^\infty \gamma_{u_n}(h) \tag{11}$$

and the convergence of the sequences $\sum_{j=0}^\infty c_{n,j}$, $\sum_{j=0}^\infty c_{n,j}^2$ and σ_n^2 (the former by (5)) imply that $\lim_{n\to\infty}\sum_{h=1}^\infty \gamma_{u_n}(h)$ exists in $\mathbb{R}$.

Lemma 1. *Let Assumption 2 hold and K be a non-negative bounded function on $[0,1]$ satisfying $K(0) = 1$. Then:*

(i) $\left\| \frac{1}{n}\sum_{t=1}^n u_{n,t}^2 - \sigma^2 \lim_{n\to\infty}\sum_{j=0}^\infty c_{n,j}^2 \right\|_{L_1} \to 0.$

(ii) *If, in addition,*

$$\sup_{n\geq 1}\sup_{h\geq 1}\left|cov\left(e_{n,t}^{2},e_{n,t+h}^{2}\right)\right|<\infty \tag{12}$$

holds, then

$$\left\|\frac{1}{n}\sum_{h=1}^{M}K\left(\frac{h}{M}\right)\sum_{t=h+1}^{n}u_{n,t}u_{n,t-h}-\lim_{n\to\infty}\sum_{h=1}^{\infty}\gamma_{u_{n}}\left(h\right)\right\|_{L_{1}}\to 0$$

whenever $M\to\infty$ *and* $M/n^{1/2}\to 0$.

Note that the covariance condition (12) in part (ii) does not impose finite fourth moment on $e_{n,t}$ (since $h>0$) and that is it is automatically satisfied under conditional homoskedasticity by the law of iterated expectations. The above law of large numbers together with the law of large numbers established by Lemma 1 of Arvanitis and Magdalinos (2019) will be sufficient for the asymptotic development of the paper. We proceed by extending the asymptotic approximations that lead to the Cauchy-distributed ratio to mildly explosive arrays. In doing so, we will require a strengthening of the summability condition (4) to

$$\sup_{n\geq 1}\sum_{j=0}^{\infty}j^{\delta}\left|c_{n,j}\right|<\infty \quad \text{for some } \delta>0. \tag{13}$$

The reason for this is that, unlike the standard non-array case $c_{n,j}=c_{j}$, (4) does not guarantee

$$\sum_{j=m_{n}}^{\infty}\left|c_{n,j}\right|\to 0 \;\text{ when } m_{n}\to\infty \tag{14}$$

which is very useful in establishing approximations with mildly explosive processes. A counterexample to (14) is easy to construct: $c_{n,j}^{2}:=\left(\phi_{n}^{2}-1\right)\phi_{n}^{2j}$ where $\phi_{n}\to 1$ with $n\left(\phi_{n}-1\right)\to\infty$ satisfies (4) since $\sum_{j=0}^{\infty}c_{n,j}^{2}\to 1$; taking (m_{n}) to be any sequence satisfying $m_{n}\to\infty$ and $m_{n}\left(\phi_{n}-1\right)\to 0$ we obtain $\sum_{j=0}^{m_{n}-1}c_{n,j}^{2}\to 0$ and $\sum_{j=m_{n}}^{\infty}c_{n,j}^{2}\to 1$. On the other hand, since $\sum_{j=m_{n}}^{\infty}\left|c_{n,j}\right|\leq m_{n}^{-\delta}\sum_{j=m_{n}}^{\infty}j^{\delta}\left|c_{n,j}\right|=O\left(m_{n}^{-\delta}\right)$, (14) is satisfied under (13). In order not to impose the summability condition (13) unnecessarily in the non-array case $c_{n,j} = c_j$, we prove Lemma 2 under (13) or under the following dominance condition: there exists $(c_j)_{j\geq 0}$ satisfying $\sum_{j=0}^{\infty} |c_j| < \infty$ and $\sum_{j=0}^{\infty} c_j \neq 0$ such that

$$|c_{n,j}| \leq b\,|c_j| \quad \text{for some } b > 0 \tag{15}$$

where b is independent of n and j.

Consider the stochastic sequences

$$\xi_n = \left(\rho_n^2 - 1\right)^{1/2} \sum_{t=1}^{n} \rho_n^{-t} u_{n,t} \tag{16}$$

and

$$\mathbf{X}_n = \left(\rho_n^2 - 1\right)^{1/2} \rho_n^{-n} x_{n,n} = \xi_n + \left(\rho_n^2 - 1\right)^{1/2} X_{n,0} + \rho_n^{-n} \mu \tag{17}$$

$$\mathbf{Y}_n = \left(\rho_n^2 - 1\right)^{1/2} \sum_{t=1}^{n} \rho_n^{-(n-t+1)} u_{n,t}. \tag{18}$$

When $\rho_n \to \rho \geq 1$, denote $C(\rho) = \lim_{n\to\infty} \sum_{j=0}^{\infty} \rho^{-j} c_{n,j}$ (the existence of the limit is ensured by Assumption 2) and

$$\tilde{X}_n(\rho) = C(\rho) \left(\rho_n^2 - 1\right)^{1/2} \sum_{t=1}^{n} \rho_n^{-t} e_{n,t} \tag{19}$$

$$\tilde{Y}_n(\rho) = C(\rho) \left(\rho_n^2 - 1\right)^{1/2} \sum_{t=1}^{n} \rho_n^{-(n-t+1)} e_{n,t}. \tag{20}$$

Denote by $C(1) = \lim_{n\to\infty} \sum_{j=0}^{\infty} c_{n,j}$ and by $\tilde{X}_n(1)$ and $\tilde{Y}_n(1)$ the sequences in (19) and (20) with $C(\rho)$ replaced by $C(1)$.

Lemma 2. *Let Assumption 1 with $\rho_n \to 1$ and Assumption 2 with either (13) or (15) hold. Then $\left\|\xi_n - \tilde{X}_n(1)\right\|_{L_2} \to 0$, $\left\|\mathbf{Y}_n - \tilde{Y}_n(1)\right\|_{L_2} \to 0$ and*

$$[\mathbf{X}_n, \mathbf{Y}_n] \to_d \left[\tilde{X}(1), \tilde{Y}(1)\right] \tag{21}$$

as $n \to \infty$, where $\tilde{X}(1)$ and $\tilde{Y}(1)$ are independent $N(0, \omega^2)$ random variables with $\omega^2 = \sigma^2 C(1)^2$.

It is worth noting that the martingale approximation of $\mathbf{Y}_n$ by $\tilde{Y}_n(1)$ is new and relaxes the summability condition $\sum_{j=1}^{\infty} j\,|c_j| < \infty$ of Phillips and Magdalinos (2007b) to (13) or the classical short memory condition $\sum_{j=1}^{\infty} |c_j| < \infty$ in the non-array case.

The next result deals with martingale approximation of sample moments of (purely) explosive arrays. This is more challenging than the mildly explosive case since the development of the theory of explosive autoregressions is based on the martingale convergence theorem which, unlike the weak convergence arguments employed in the mildly explosive case, does not admit a triangular array generalisation. In what follows, we approximate the martingale difference array $(e_{n,t}, \mathcal{F}_{n,t})$ by a martingale difference sequence $(e_t, \mathcal{F}_t)$ along a subsequence using Levy's upward lemma for conditional expectations (see 14.2 in Williams (1991)). The martingale difference sequence $(e_t, \mathcal{F}_t)$ is shown to satisfy the local MZ conditions (10), thus ensuring that the denominator of the ratio that approximates the centred and normalised OLS estimator is small with probability tending to 0; see Lemma 3(ii) below. The theoretical development of Lemma 3 is new and is necessitated by the lack of available martingale convergence theory for an array of martingale differences $e_{n,t}$ resulting from considering drifting sequences of autoregressive innovations $(u_{n,t})$.

Lemma 3. *Let Assumption 1 with $\rho_n \to \rho \in (1, \infty)$, and Assumption 2 with (13) or (15) hold. For each $t \in \mathbb{Z}$ let $e_t := \liminf_{n\to\infty} e_{n,t}$ and*

$$\mathcal{G}_{n,t} := \cap_{j=n}^{\infty} \mathcal{F}_{j,t} \;\; \text{and} \;\; \mathcal{F}_t := \sigma\left(\cup_{n=1}^{\infty} \mathcal{G}_{n,t}\right) = \sigma\left(\liminf_{n\to\infty} \mathcal{F}_{n,t}\right). \tag{22}$$

(i) *The sequence $(e_t, \mathcal{F}_t)_{t\in\mathbb{Z}}$ is a martingale difference satisfying the local MZ conditions (10).*

(ii) *For any subsequence of $(\mathbf{X}_n)_{n\in\mathbb{N}}$ there exists a further subsequence that converges in distribution. If $\mathbf{X}_{k_n} \to_d \mathbf{X}_\infty$ for some subsequence $(\mathbf{X}_{k_n})_{n\in\mathbb{N}}$ of $(\mathbf{X}_n)_{n\in\mathbb{N}}$, then $\mathbb{P}(\mathbf{X}_\infty = 0) = 0$.*

(iii) $\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}(|\mathbf{X}_n| \le \delta) = 0$ *and* $|\mathbf{Y}_n| / |\mathbf{X}_n| = O_p(1)$.

The next result extends the approximation results for the numerator and denominator, (23) and (24) respectively, of the centred and normalised OLS estimator. While (24) is a straightforward extension, the proof of (23) contains new theory even for the non-array case $c_{n,j} = c_j$ when $\rho_n^{-n} n \nrightarrow 0$: this may only occur for roots ρ_n that lie logarithmically close to the local to unity region: $\rho_n - 1 = O(\log n/n)$ see (74) in the Appendix. Such rates are assumed away by the polynomial parametrisation $\rho_n = 1 + c/n^\alpha$ for $c > 0$ and $\alpha \in (0,1)$ employed by Magdalinos (2012) and Arvanitis and Magdalinos (2019) but are allowed by Assumption 1 which postulates mildly explosive and explosive roots in full generality.

Lemma 4. *Let Assumption 1 with $\rho_n \to \rho \geq 1$, and Assumption 2 with (13) or (15) hold. Then, as $n \to \infty$,*

$$\left(\rho_n^2 - 1\right) \rho_n^{-n} \sum_{t=1}^n x_{n,t-1} u_{n,t} = \mathbf{X}_n \mathbf{Y}_n + o_p(1) \tag{23}$$

$$\left(\rho_n^2 - 1\right)^2 \rho_n^{-2n} \sum_{t=1}^n x_{n,t-1}^2 = \mathbf{X}_n^2 + o_p(1) \tag{24}$$

where $\mathbf{X}_n$ and $\mathbf{Y}_n$ are given in (17) and (18).

Lemmata 2, 3 and 4 provide the essential elements for the approximation of $\left(\rho_n^2 - 1\right)^{-1} \rho_n^n \left(\hat{\rho}_n - \rho_n\right)$ when $\rho_n \to \rho \in [1, \infty)$. The limit theory for $\rho_n \to \rho \in (-\infty, -1]$ may be derived as a mirror image of the $\rho \in [1, \infty)$ case by employing the transformation $x_t \mapsto (-1)^{-t} x_t$. Denoting

$$\check{x}_{n,t} = (-1)^{-t} x_{n,t},\ \check{X}_{n,t} = (-1)^{-t} X_{n,t},\ \check{u}_{n,t} = (-1)^{-t} u_{n,t}, \tag{25}$$

it is easy to see that $\check{X}_{n,t}$ satisfies the recursion

$$\check{X}_{n,t} = |\rho_n| \check{X}_{n,t-1} + \check{u}_{n,t}. \tag{26}$$

As long as we establish that the innovation sequence $(\check{u}_{n,t})$ satisfies Assumption 2, Lemma 4 will imply that

$$\left(\rho_n^2 - 1\right) |\rho_n|^{-n} \sum_{t=1}^n \check{x}_{n,t-1} \check{u}_{n,t} = \check{\mathbf{X}}_n \check{\mathbf{Y}}_n + o_p(1) \tag{27}$$

$$\left(\rho_n^2 - 1\right)^2 \rho_n^{-2n} \sum_{t=1}^n \check{x}_{n,t-1}^2 = \check{\mathbf{X}}_n^2 + o_p(1) \tag{28}$$

where $\check{\xi}_n = \left(\rho_n^2 - 1\right)^{1/2} \sum_{t=1}^n |\rho_n|^{-t} \check{u}_{n,t}$ and

$$\left[\check{\mathbf{X}}_n, \check{\mathbf{Y}}_n\right] = \left[\check{\xi}_n + \left(\rho_n^2 - 1\right)^{1/2} X_{n,0}, \left(\rho_n^2 - 1\right)^{1/2} \sum_{t=1}^n |\rho_n|^{-(n-t+1)} \check{u}_{n,t}\right]. \tag{29}$$

The fact that

$$\check{u}_{n,t} = \sum_{j=0}^\infty \check{c}_{n,t} \check{e}_{n,t-j}, \quad \text{with } \check{e}_{n,t} = (-1)^{-t} e_{n,t},\ \check{c}_{n,t} = (-1)^{-t} c_{n,t} \tag{30}$$

satisfies Assumption 2 is established in the proof of Theorem 1. Hence, Lemma 2 and Lemma 3 continue to apply with $\left[\mathbf{X}_n, \tilde{X}_n(\rho)\right]$ replaced by $\left[\check{\mathbf{X}}_n, \check{X}_n(\rho)\right]$ and $\left[\mathbf{Y}_n, \tilde{Y}_n(\rho)\right]$ replaced by $\left[\check{\mathbf{Y}}_n, \check{Y}_n(\rho)\right]$ where

$$\left[\check{X}_n(\rho), \check{Y}_n(\rho)\right] = C(|\rho|)\left(\rho_n^2 - 1\right)^{1/2}\left[\sum_{t=1}^{n} |\rho_n|^{-t} \check{e}_{n,t}, \sum_{t=1}^{n} |\rho_n|^{-(n-t+1)} \check{e}_{n,t}\right]. \tag{31}$$

Combining Lemmata 2-4 and using (27)-(31), we arrive to the following result for the OLS estimator in (3). Denote $C_{nj} = \sum_{t=1}^{\infty} \rho^{-t} c_{n,j+t}$.

Theorem 1. *Consider the process $x_{n,t}$ in (1)-(2) under Assumption 1, Assumption 2 with either (13) or (15) and Assumption 3. The following limit theory applies to the OLS estimator in (3) as $n \to \infty$:*

(i) When $|\rho_n| \to 1$

$$\left(\rho_n^2 - 1\right)^{-1} |\rho_n|^n (\hat{\rho}_n - \rho_n) \to_d \mathcal{C}$$

where $\mathcal{C}$ denotes a standard Cauchy random variable.

(ii) *When $|\rho_n| \to |\rho| > 1$,*

$$\left(\rho_n^2 - 1\right)^{-1} |\rho_n|^n (\hat{\rho}_n - \rho_n) = \frac{\mathbf{Y}_n}{\mathbf{X}_n} \mathbf{1}\{\rho > 1\} - \frac{\check{\mathbf{Y}}_n}{\check{\mathbf{X}}_n} \mathbf{1}\{\rho < -1\} + o_p(1) \tag{32}$$

where $\mathbf{X}_n = \xi_n + \left(\rho_n^2 - 1\right)^{1/2} X_{n,0}$, $\check{\mathbf{X}}_n = \check{\xi}_n + \left(\rho_n^2 - 1\right)^{1/2} X_{n,0}$, the elements of $\left\{\xi_n, \check{\xi}_n, \mathbf{Y}_n, \check{\mathbf{Y}}_n\right\}$ have the same variance for each n and satisfy $\lim_{n\to\infty} \mathbb{E}(\xi_n \mathbf{Y}_n) = 0$ and $\lim_{n\to\infty} \mathbb{E}\left(\check{\xi}_n \check{\mathbf{Y}}_n\right) = 0$. In particular,

$$\hat{\rho}_n - \rho_n = O_p\left(|\rho_n|^{-n}\right) \tag{33}$$

If $(e_{n,t})_{t\in\mathbb{Z}}$ is Gaussian, $\upsilon(\rho) = \lim_{n\to\infty} \sum_{j=0}^{\infty} C_{nj}^2$ exists and $X_{n,0}$ converges in distribution jointly with $\sum_{j=0}^{\infty} C_{nj} e_{n,-j}$, then

$$\left(\rho_n^2 - 1\right)^{-1} |\rho_n|^n (\hat{\rho}_n - \rho_n) \to_d \frac{\zeta}{\xi + X_0 \left\{C(\rho)^2 \left(\rho^2 - 1\right)^{-1} + \upsilon(\rho)\right\}^{-1/2}} \tag{34}$$

where ζ and ξ are independent $N(0, \sigma^2)$ random variables; if $X_0 = 0$ a.s. the right side of (34) follows a Cauchy distribution.

Having characterised the limit distribution of the OLS estimator, we proceed to discussing the limit distribution of the resulting t-statistic

$$T_n(\rho_n) = \frac{\left(\sum_{t=1}^{n} \left(x_{n,t-1} - \bar{x}_{n,n-1}\right)^2\right)^{1/2}}{\hat{\omega}_n} \left(\hat{\rho}_n - \rho_n\right) \tag{35}$$

where $\hat{\omega}_n^2 = \hat{\sigma}_{n,u}^2 + 2n^{-1}\sum_{h=1}^{M} K\left(\frac{h}{M}\right)\sum_{t=h+1}^{n} \hat{u}_{n,t}\hat{u}_{n,t-h}$ is an estimator of the long run variance ω^2, with K a kernel function satisfying the usual conditions, $\hat{\sigma}_{n,u}^2 = n^{-1}\sum_{t=1}^{n} \hat{u}_{n,t}^2$ is an estimator of $\mathbb{E}u_{n,t}^2 = \sigma_n^2 \sum_{j=0}^{\infty} c_{n,j}^2$ and $\hat{u}_{n,t}$ denote the OLS residuals from (2).

Theorem 2. *Consider the process $x_{n,t}$ in (1)-(2) under Assumption 1, Assumption 2 with either (13) or (15) and Assumption 3. If K satisfies the assumptions of Lemma 1, $M \to \infty$, $M/n^{1/2} \to 0$ and (12) holds, the t-statistic in (35) satisfies $T_n(\rho_n) \to_d N(0,1)$ under each of the following conditions:*

(i) $|\rho_n| \to 1$

(ii) $|\rho_n| \to |\rho| > 1$, *$(e_{n,t})_{t\in\mathbb{Z}}$ is Gaussian and $c_{n,j} = 0$ for all $j \geq 1$.*

Remarks. Theorems 1 and 2 extend the scope of available limit theory on the right side of unity to general drifting sequences of autoregressive parameters, innovation sequences and initial conditions. A summary of the different directions of this extension follows.

1. The mildly explosive specification of Assumption 1 includes neighbourhoods of unity that may approach the boundary with local to unity processes: for such neighbourhoods ρ_n^n is no longer guaranteed to have an exponential rate as in the case with a polynomial root parametrisation of the form $\rho_n = 1 + c/n^\alpha$ with $c > 0$ and $\alpha \in (0,1)$ frequently assumed in the literature. Assumption 1 also includes drifting sequences of explosive autoregressions with roots on $(-\infty,-1) \cup (1,\infty)$ as well as drifting sequences of mildly explosive roots converging to -1. As far as we are aware, this work is the first to provide a full development of OLS limit theory for mildly explosive processes at -1, even in the standard case of non-array autoregressive innovations.

2. Assumption 2 extends the standard specification of a short memory linear process for the autoregressive innovations to a triangular array

formulation both for the deterministic weights and for the primitive innovations of the linear process. The triangular arrays of primitive innovations are assumed to be possibly conditionally heteroskedastic martingale differences satisfying an L_1-mixingale condition. To our knowledge, our work is the first to introduce a triangular array formulation of (short memory) linear correlation, with existing work by Andrews and Guggenberger (2012) introducing an array framework to a conditionally heteroskedastic (but uncorrelated) martingale difference sequence.

3. We show that the OLS estimator generated by mildly explosive autoregression continues to conform to central limit theory under drifting sequences of autoregressive roots to $\{1, -1\}$ and drifting sequences of short memory autoregressive innovations. The Cauchy limit distribution for the OLS estimator and the standard normal distribution for the corresponding t-statistic continue to hold under drifting sequences of autoregressive roots and innovations. A direct extension is possible since mildly explosive limit theory employs weak convergence methods (essentially the martingale central limit theorem) which are well-known to accommodate easily sample moments of triangular arrays of random variables. On the other hand, the asymptotic analysis of explosive autoregressions with root in $(-\infty, -1) \cup (1, \infty)$ depends on the martingale convergence theorem which does not extend to sums of martingale difference arrays. For this reason, we are only able to obtain the exact rate of convergence of the OLS estimator and the approximation in (32) under the full generality of Assumption 2. As in the standard non-array case, asymptotic normality of the t-statistic is only achieved under independent, Gaussian innovation errors $u_{n,t}$.

4. Theorems 1 and 2 provide limit distribution theory along drifting sequences of parameters that may be used for interval estimation. Phillips, Wu and Yu (2011) and Phillips and Yu (2011) apply the construction of Cauchy confidence intervals for the detection of financial bubbles. The results of Theorems 1 and 2 could be used in order to assess the uniformity properties of the asymptotic coverage of these confidence intervals, with the autoregressive innovation sequence viewed as an infinite dimensional nuisance parameter.

3 Conclusion

The paper provides generic limit theory for the OLS estimator and the associated t-statistic for a first order autoregression on the explosive side of unity under drifting sequences of parameters. A general (mildly) explosive autoregressive root is considered that may approach the boundary with processes that are local to 1 and -1 at arbitrary rate. Drifting sequences for the innovation processes in the autoregression are also considered that take the form of triangular arrays of short memory linear processes with primitive errors that are (possibly conditionally heteroskedastic) martingale difference arrays. The asymptotic development of the paper provides the necessary apparatus for considering autoregressive innovation processes as part of the statistical model (in the form of an infinite dimensional nuisance parameter) and for assessing their effect on the asymptotic size of OLS-based procedures in the explosive and mildly explosive region.

4 Proofs

This section contains the proofs of mathematical statements in the paper. We begin by proving an extension of Proposition A1(b) of Phillips and Magdalinos (2007).

Lemma A1. *Under Assumption 1, for any $p \geq 0$ the following hold:*

(i) $[n(|\rho_n| - 1)]^p |\rho_n|^{-n} \to 0$

(ii) *When* $|\rho_n| \to 1$, $(|\rho_n| - 1)^{1+p} \sum_{t=1}^n t^p |\rho_n|^{-t} \to \Gamma(p+1)$.

Proof. For part (i), write

$$|\rho_n|^{-n} = \exp\{-n \log |\rho_n|\} = \exp\{-n \log(1 + |\rho_n| - 1)\}.$$

When $|\rho_n| \to |\rho| > 1$, $n|\rho_n|^{-n} = n \exp\{-n \log |\rho| (1 + o(1))\} = o(1)$ since $\log |\rho| > 0$; when $|\rho_n| \to 1$, $\log(1+x) = x + O(x)$ as $x \to 0$ implies that

$$|\rho_n|^{-n} = \exp\{-n(|\rho_n| - 1)(1 + o(1))\} = o\left([n(|\rho_n| - 1)]^{-p}\right)$$

for any $p \geq 0$ as required, since $n(|\rho_n| - 1) \to \infty$ and $\lim_{M\to\infty} M^p e^{-\gamma M} = 0$ for any $\gamma > 0$. For part (ii), the case $p = 0$ is just a geometric progression. For

$p > 0$, employing an Euler summation argument and the change of variables $s = (|\rho_n| - 1)\, t$

$$\begin{aligned} (|\rho_n| - 1)^{1+p} \sum_{t=1}^{n} t^p \, |\rho_n|^{-t} &= (|\rho_n| - 1)^{1+p} \int_1^{n+1} \lfloor t \rfloor^p \, |\rho_n|^{-\lfloor t \rfloor} \, dt \\ &= \int_{|\rho_n|-1}^{(n+1)(|\rho_n|-1)} \left(\frac{\lfloor (|\rho_n|-1)^{-1} s \rfloor}{(|\rho_n|-1)^{-1}} \right)^p |\rho_n|^{\lfloor (|\rho_n|-1)^{-1} s \rfloor} \, ds \quad (36) \end{aligned}$$

Since $|\rho_n| - 1 \to 0$, $n\,(1 - |\rho_n|) \to \infty$ and

$$\begin{aligned} |\rho_n|^{\lfloor (|\rho_n|-1)^{-1} s \rfloor} &= (1 + (|\rho_n| - 1))^{\lfloor (|\rho_n|-1)^{-1} s \rfloor} \\ &= \exp\left\{ \left\lfloor (|\rho_n| - 1)^{-1} s \right\rfloor \log\left(1 + |\rho_n| - 1\right) \right\} \\ &= \exp\left\{ \left\lfloor (|\rho_n| - 1)^{-1} s \right\rfloor (|\rho_n| - 1) \left[1 + O\left(|\rho_n| - 1\right)\right] \right\} \\ &\to e^{-s} \end{aligned}$$

the dominated convergence theorem implies that the integral on the right side of (36) converges to $\int_0^\infty s^p e^{-s} ds = \Gamma(p+1)$, completing the proof.

Proof of Lemma 1. Writing

$$\begin{aligned} n^{-1} \sum_{t=1}^{n} u_{n,t}^2 &= n^{-1} \sum_{j=0}^{\infty} c_{n,j}^2 \sum_{t=1}^{n} e_{n,t-j}^2 + 2n^{-1} \sum_{j=0}^{\infty} \sum_{i=0}^{j-1} c_{n,j} c_{n,i} \sum_{t=1}^{n} e_{n,t-j} e_{n,t-i} \\ &= A_n + 2B_n \end{aligned}$$

in order of appearance, we obtain

$$\begin{aligned} \|B_n\|_{L_1} &\le \sum_{j=0}^{\infty} \sum_{i=0}^{j-1} |c_{n,j}|\,|c_{n,i}| \left\| \frac{1}{n} \sum_{t=1}^{n} e_{n,t-j} e_{n,t-i} \right\|_{L_1} \\ &= \sum_{j=0}^{\infty} \sum_{i=0}^{j-1} |c_{n,j}|\,|c_{n,i}| \left\| \frac{1}{n} \sum_{t=1}^{n} e_{n,t-j} \left(\mathbf{1}\left\{ e_{n,t-j}^2 \le l_n \right\} + \mathbf{1}\left\{ e_{n,t-j}^2 > l_n \right\} \right) e_{n,t-i} \right\|_{L_1} \\ &\le \sum_{j=0}^{\infty} \sum_{i=0}^{j-1} |c_{n,j}|\,|c_{n,i}| \frac{1}{n} \sum_{t=1}^{n} \left\| e_{n,t-j} \mathbf{1}\left\{ e_{n,t-j}^2 > l_n \right\} e_{n,t-i} \right\|_{L_1} \\ &\quad + \sum_{j=0}^{\infty} \sum_{i=0}^{j-1} |c_{n,j}|\,|c_{n,i}| \left\| \frac{1}{n} \sum_{t=1}^{n} e_{n,t-j} \mathbf{1}\left\{ e_{n,t-j}^2 \le l_n \right\} e_{n,t-i} \right\|_{L_2} \quad (37) \end{aligned}$$

where the last line follows by the Lyapounov inequality and $(l_n)_{n\in\mathbb{N}}$ is a sequence satisfying $l_n \to \infty$ and $l_n/n \to 0$. The summand inside the above L_2 norm is an $\mathcal{F}_{n,t-i}$-martingale difference array (since $i < j$) so

$$\begin{aligned}\left\|\frac{1}{n}\sum_{t=1}^{n} e_{n,t-j}\mathbf{1}\left\{|e_{n,t-j}| \le l_n\right\} e_{n,t-i}\right\|_{L_2} &= \left(\frac{1}{n^2}\sum_{t=1}^{n}\mathbb{E}\left(e_{n,t-j}^2\mathbf{1}\left\{e_{n,t-j}^2 \le l_n\right\} e_{n,t-i}^2\right)\right)^{1/2} \\ &\le \left(\frac{l_n}{n}\sigma_n^2\right)^{1/2}\end{aligned}$$

so the second term on the right of (37) is bounded by

$$\left(\sup_{n\ge 1}\sum_{j=0}^{\infty}|c_{n,j}|\right)^2 \sigma_n\sqrt{l_n/n} \to 0$$

since $l_n/n \to 0$. For the first term on the right of (37), the CS inequality gives

$$\left\|e_{n,t-j}\mathbf{1}\left\{e_{n,t-j}^2 > l_n\right\} e_{n,t-i}\right\|_{L_1} \le \left(\mathbb{E}e_{n,t-j}^2\mathbf{1}\left\{e_{n,t-j}^2 > l_n\right\}\right)^{1/2}\left(\mathbb{E}e_{n,t-i}^2\right)^{1/2}$$

so the first term on the right of (37) is bounded by

$$\left(\sup_{n\ge 1}\sum_{j=0}^{\infty}|c_{n,j}|\right)^2 \sigma_n\left(\sup_{j\in\mathbb{Z}}\mathbb{E}e_{n,j}^2\mathbf{1}\left\{e_{n,j}^2 > l_n\right\}\right)^{1/2} \to 0$$

since $\left(e_{n,j}^2\right)$ is a UI family. We conclude that

$$\left\|\frac{1}{n}\sum_{t=1}^{n}u_{n,t}^2 - \sum_{j=0}^{\infty}c_{n,j}^2\frac{1}{n}\sum_{t=1}^{n}e_{n,t-j}^2\right\|_{L_1} \to 0. \tag{38}$$

Let $(\lambda_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ be integer-valued sequences satisfying

$$\lambda_n \to \infty,\ k_n \to \infty,\ \lambda_n k_n/\sqrt{n} \to 0. \tag{39}$$

Using the identity

$$\sum_{t=1}^{n}e_{n,t-j}^2\mathbf{1}\left\{e_{n,t-j}^2 \le \lambda_n\right\} = \sum_{l=0}^{k_n-1}M_{n,j,l} + \sum_{t=1}^{n}\mathbb{E}_{\mathcal{F}_{n,t-j-k_n}}\left(e_{n,t-j}^2\mathbf{1}\left\{e_{n,t-j}^2 \le \lambda_n\right\}\right)$$

where

$$M_{n,j,l}=\sum_{t=1}^{n}\left(\mathbb{E}_{\mathcal{F}_{n,t-j-l}}\left(e_{n,t-j}^{2}\mathbf{1}\left\{e_{n,t-j}^{2}\leq\lambda_{n}\right\}\right)-\mathbb{E}_{\mathcal{F}_{n,t-j-l-1}}\left(e_{n,t-j}^{2}\mathbf{1}\left\{e_{n,t-j}^{2}\leq\lambda_{n}\right\}\right)\right)$$

is a martingale array, we may write

$$\begin{aligned}
\frac{1}{n}\sum_{t=1}^{n}e_{n,t-j}^{2} &= \frac{1}{n}\sum_{t=1}^{n}e_{n,t-j}^{2}\mathbf{1}\left\{e_{n,t-j}^{2}\leq\lambda_{n}\right\}+\frac{1}{n}\sum_{t=1}^{n}e_{n,t-j}^{2}\mathbf{1}\left\{e_{n,t-j}^{2}>\lambda_{n}\right\} \\
&= \frac{1}{n}\sum_{l=0}^{k_n-1}M_{n,j,l}+\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}_{\mathcal{F}_{t-j-k_n}}\left(e_{n,t-j}^{2}\mathbf{1}\left\{e_{n,t-j}^{2}\leq\lambda_{n}\right\}\right) \\
&\quad +\frac{1}{n}\sum_{t=1}^{n}e_{n,t-j}^{2}\mathbf{1}\left\{e_{n,t-j}^{2}>\lambda_{n}\right\} \\
&= \frac{1}{n}\sum_{l=0}^{k_n-1}M_{n,j,l}+\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}_{\mathcal{F}_{t-j-k_n}}\left(e_{n,t-j}^{2}\right)+\frac{1}{n}N_{n,j} \qquad (40)
\end{aligned}$$

where $N_{n,j}=\sum_{t=1}^{n}\left[e_{n,t-j}^{2}\mathbf{1}\left\{e_{n,t-j}^{2}>\lambda_{n}\right\}-\mathbb{E}_{\mathcal{F}_{t-k_n}}\left(e_{n,t-j}^{2}\mathbf{1}\left\{e_{n,t-j}^{2}>\lambda_{n}\right\}\right)\right]$. By the triangle inequality and the Jensen inequality for conditional expectations,

$$\left\|\frac{1}{n}N_{n,j}\right\|_{L_1}\leq 2\sup_{j\in\mathbb{Z}}\mathbb{E}\left(e_{n,j}^{2}\mathbf{1}\left\{e_{n,j}^{2}>\lambda_{n}\right\}\right). \qquad (41)$$

By the Lyapounov inequality and the martingale property of $M_{n,j}$

$$\begin{aligned}
&\left\|\frac{1}{n}\sum_{l=0}^{k_n-1}M_{n,j,l}\right\|_{L_1} \\
\leq\ & \frac{1}{n}\sum_{l=0}^{k_n-1}\left\|M_{n,j,l}\right\|_{L_2} \\
=\ & \frac{1}{n}\sum_{l=0}^{k_n-1}\left\{\sum_{t=1}^{n}\left(\mathbb{E}_{\mathcal{F}_{n,t-j-l}}\left(e_{n,t-j}^{2}\mathbf{1}\left\{e_{n,t-j}^{2}\leq\lambda_{n}\right\}\right)-\mathbb{E}_{\mathcal{F}_{n,t-j-l-1}}\left(e_{n,t-j}^{2}\mathbf{1}\left\{e_{n,t-j}^{2}\leq\lambda_{n}\right\}\right)\right)^{2}\right\}^{1/2} \\
\leq\ & \frac{1}{n}k_n\left(\lambda_n^{2}n\right)^{1/2}=k_n\lambda_n/\sqrt{n}. \qquad (42)
\end{aligned}$$

Since the bounds in (41) and (42) are independent of j,

$$\begin{aligned}\left\|\sum_{j=0}^{\infty}c_{n,j}^{2}\left(\frac{1}{n}\sum_{l=0}^{k_n-1}M_{n,j,l}+\frac{1}{n}N_{n,j}\right)\right\|_{L_1} &\leq \sum_{j=0}^{\infty}c_{n,j}^{2}\left(\left\|\frac{1}{n}\sum_{l=0}^{k_n-1}M_{n,j,l}\right\|_{L_1}+\left\|\frac{1}{n}N_{n,j}\right\|_{L_1}\right)\\ &\leq \left(2\sup_{j\in\mathbb{Z}}\mathbb{E}\left(e_{n,j}^{2}\mathbf{1}\left\{e_{n,j}^{2}>\lambda_n\right\}\right)+\frac{k_n\lambda_n}{\sqrt{n}}\right)\sup_{n\geq 1}\sum_{j=0}^{\infty}c_{n,j}^{2}\\ &= o(1)\end{aligned}$$

by (39) and UI of $\left(e_{n,j}^{2}\right)_{j\in\mathbb{Z}}$. Hence, (38) and (40) imply that

$$\left\|\frac{1}{n}\sum_{t=1}^{n}u_{n,t}^{2}-\sum_{j=0}^{\infty}c_{n,j}^{2}\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}_{\mathcal{F}_{t-j-k_n}}\left(e_{n,t-j}^{2}\right)\right\|_{L_1}\to 0. \tag{43}$$

Note that $\mathbb{E}_{\mathcal{F}_{t-j-k_n}}\left(e_{n,t-j}^{2}\right)=\mathbb{E}_{\mathcal{F}_{t-j-k_n}}\mathbb{E}_{\mathcal{F}_{t-j-1}}\left(e_{n,t-j}^{2}\right)=\mathbb{E}_{\mathcal{F}_{t-j-k_n}}\sigma_{n,t-j}^{2}$ by the tower property. (43) implies that the lemma will follow from

$$\left\|\sum_{j=0}^{\infty}c_{n,j}^{2}\frac{1}{n}\sum_{t=1}^{n}\left(\mathbb{E}_{\mathcal{F}_{t-j-k_n}}\sigma_{n,t-j}^{2}-\sigma_{n}^{2}\right)\right\|_{L_1}\to 0. \tag{44}$$

Using the mixingale property (8),

$$\begin{aligned}\left\|\sum_{j=0}^{\infty}c_{n,j}^{2}\frac{1}{n}\sum_{t=1}^{n}\left(\mathbb{E}_{\mathcal{F}_{t-j-k_n}}\sigma_{n,t-j}^{2}-\sigma_{n}^{2}\right)\right\|_{L_1} &\leq \sum_{j=0}^{\infty}c_{n,j}^{2}\frac{1}{n}\sum_{t=1}^{n}\left\|\left(\mathbb{E}_{\mathcal{F}_{t-j-k_n}}\sigma_{n,t-j}^{2}-\sigma_{n}^{2}\right)\right\|_{L_1}\\ &\leq b\left(\psi_{k_n}+\varphi_n\right)\sup_{n\geq 1}\sum_{j=0}^{\infty}c_{n,j}^{2}=o(1)\end{aligned}$$

as required, since $k_n\to\infty$ by (39).

For part (ii), write

$$\begin{aligned}\frac{1}{n}\sum_{t=h+1}^{n}u_{n,t}u_{n,t-h} &= \sum_{j,i=0}^{\infty}c_{n,j}c_{n,i}\frac{1}{n}\sum_{t=h+1}^{n}e_{n,t-j}e_{n,t-(i+h)}\\ &= \sum_{i=0}^{\infty}c_{n,i}\sum_{j\neq i+h}c_{n,j}\frac{1}{n}\sum_{t=h+1}^{n}e_{n,t-j}e_{n,t-(i+h)}\\ &\quad+\sum_{i=0}^{\infty}c_{n,i}c_{n,i+h}\frac{1}{n}\sum_{t=1}^{n-h}e_{n,t-i}^{2}.\end{aligned}$$

When $j \neq i+h$, the martingale difference property of $(e_{n,t})$ gives

$$\begin{aligned}
\left\| \frac{1}{n} \sum_{t=h+1}^{n} e_{n,t-j} e_{n,t-(i+h)} \right\|_{L_2}^2 &= \frac{1}{n^2} \sum_{t=h+1}^{n} \mathbb{E} e_{n,t-j}^2 e_{n,t-(i+h)}^2 \\
&= \frac{1}{n^2} \sum_{t=h+1}^{n} \mathbb{E} e_{n,t-j}^2 \left(e_{n,t-(i+h)}^2 - \sigma_n^2 \right) + \frac{n-h}{n^2} \sigma_n^4 \\
&= \frac{1}{n^2} \sum_{t=h+1}^{n} \gamma_{e_{n,t}^2} (i+h-j) + \frac{n-h}{n^2} \sigma_n^4 \\
&\leq \frac{1}{n} \sup_{h \geq 1} \left| \gamma_{e_{n,t}^2} (h) \right| + \frac{1}{n} \sigma_n^4 = O\left(\frac{1}{n} \right).
\end{aligned}$$

Since $\sup_{r \in [0,1]} K(r) < \infty$, the above implies that

$$\sum_{h=1}^{M} K(h/M) \sum_{i=0}^{\infty} |c_{n,i}| \sum_{j \neq i+h} |c_{n,j}| \left\| \frac{1}{n} \sum_{t=h+1}^{n} e_{n,t-j} e_{n,t-(i+h)} \right\|_{L_1} = O\left(\frac{M}{n^{1/2}} \right) = o(1)$$

by the choice of M. Hence

$$\left\| \frac{1}{n} \sum_{h=1}^{M} K(h/M) \sum_{t=h+1}^{n} u_{n,t} u_{n,t-h} - \sum_{h=1}^{M} K(h/M) \sum_{i=0}^{\infty} c_{n,i} c_{n,i+h} \frac{1}{n} \sum_{t=1}^{n-h} e_{n,t-i}^2 \right\|_{L_1} = O\left(\frac{M}{n^{1/2}} \right).$$

Using the same argument employed to show (43) and (44),

$$\left\| \sum_{h=1}^{M} K(h/M) \sum_{i=0}^{\infty} c_{n,i} c_{n,i+h} \frac{1}{n} \sum_{t=1}^{n-h} \left(e_{n,t-i}^2 - \sigma_n^2 \right) \right\|_{L_1} \to 0. \qquad (45)$$

It remains to determine the limit of the sequence

$$w_n := \sigma_n^2 \sum_{h=1}^{M} K\left(\frac{h}{M} \right) \sum_{i=0}^{\infty} c_{n,i} c_{n,i+h} \left(1 - \frac{h}{n} \right).$$

Recalling that $\gamma_{u_n}(h) = \sigma_n^2 \sum_{h=1}^{M} K\left(\frac{h}{M}\right) \sum_{i=0}^{\infty} c_{n,i} c_{n,i+h}$ for $h \geq 0$ and that K is bounded and $K(0) = 1$, write

$$\begin{aligned}
w_n &= \sum_{h=1}^{M} K\left(\frac{h}{M} \right) \gamma_{u_n}(h) + O\left(\frac{M}{n} \right) \\
&= \sum_{h=1}^{\infty} \gamma_{u_n}(h) + \sum_{h>M} \gamma_{u_n}(h) + \sum_{h=1}^{M} (K(\frac{h}{M}) - 1) \gamma_{u_n}(h) + O(\frac{M}{n}). \qquad (46)
\end{aligned}$$

Since $\sum_{h=M+1}^{\infty} \left|\gamma_{u_n}(h)\right| \leq \sigma_n^2 \sum_{i=0}^{\infty} |c_{n,i}| \sum_{h=M+1}^{\infty} |c_{n,h}| = O\left(M^{-\delta}\right)$ by (13), the second term of (46) tends to 0 as $M \to \infty$; for the third term, since $\sup_{r\in[0,1]} |K(r)| < \infty$, we obtain

$$\sum_{h=1}^{M} \left|K\left(\frac{h}{M}\right) - 1\right| \left|\gamma_{u_n}(h)\right| \leq \left(\sup_{r\in[0,1]} |K(r)| + 1\right) \sup_{n\geq 1} \sum_{h=1}^{\infty} \left|\gamma_{u_n}(h)\right| < \infty$$

so the fact that $K(0) = 1$ and the dominated convergence theorem imply that

$$\lim_{n\to\infty} \sum_{h=1}^{M_n} \left(K\left(\frac{h}{M_n}\right) - 1\right) \gamma_{u_n}(h) = 0.$$

Finally, from the discussion preceding Lemma 1, the convergence of the sequences $\sum_{j=0}^{\infty} c_{n,j}$, $\sum_{j=0}^{\infty} c_{n,j}^2$ and σ_n^2 by Assumption 2 imply that $\lim_{n\to\infty} \sum_{h=1}^{\infty} \gamma_{u_n}(h)$ exists in $\mathbb{R}$; hence $w_n \to \lim_{n\to\infty} \sum_{h=1}^{\infty} \gamma_{u_n}(h)$ and completes the proof.

Proof of Lemma 2. Write

$$\begin{aligned}
\xi_n &= \left(\rho_n^2 - 1\right)^{1/2} \sum_{t=1}^{n} \rho_n^{-t} \sum_{j=0}^{t-1} c_{n,j} e_{n,t-j} + \left(\rho_n^2 - 1\right)^{1/2} \sum_{t=1}^{n} \rho_n^{-t} \sum_{j=t}^{\infty} c_{n,j} e_{n,t-j} \\
&= \left(\rho_n^2 - 1\right)^{1/2} \sum_{j=0}^{n-1} \rho_n^{-j} c_{n,j} \sum_{t=1}^{n-j} \rho_n^{-t} e_{n,t} + \sum_{j=0}^{\infty} C_{n,j} e_{n,-j} \qquad (47)
\end{aligned}$$

where $C_{n,j} := \left(\rho_n^2 - 1\right)^{1/2} \sum_{t=1}^{n} \rho_n^{-t} c_{n,j+t}$. For the second term on the right of (47),

$$\begin{aligned}
\left\| \sum_{j=0}^{\infty} C_{n,j} e_{n,-j} \right\|_{L_2}^2 &= \sigma_n^2 \sum_{j=0}^{\infty} C_{n,j}^2 = \sigma_n^2 \left(\rho_n^2 - 1\right) \sum_{j=0}^{\infty} \left(\sum_{t=1}^{n} \rho_n^{-t} c_{n,j+t} \right)^2 \\
&\leq \sigma_n^2 \left(\rho_n^2 - 1\right) \sum_{j=0}^{\infty} \sum_{t=1}^{n} |\rho_n|^{-t} |c_{n,j+t}| \sum_{s=1}^{n} |\rho_n|^{-s} |c_{n,j+s}| \\
&\leq \sigma_n^2 \sum_{s=1}^{\infty} |c_{n,s}| \left(\rho_n^2 - 1\right) \sum_{t=1}^{n} |\rho_n|^{-t} \sum_{j=t}^{\infty} |c_{n,j}| \\
&= \sigma_n^2 \sum_{s=1}^{\infty} |c_{n,s}| \left(\rho_n^2 - 1\right) \left(\sum_{t=1}^{m_n-1} |\rho_n|^{-t} \sum_{j=t}^{\infty} |c_{n,j}| + \sum_{t=m_n}^{n} |\rho_n|^{-t} \sum_{j=t}^{\infty} |c_{n,j}| \right) \\
&= b_{1n} + b_{2n}
\end{aligned}$$

in order of appearance, where (m_n) is a sequence chosen to satisfy $m_n \to \infty$ and $m_n(\rho_n - 1) \to 0$. Now

$$b_{1n} \leq \sigma_n^2 \left(\sup_{n\geq 1} \sum_{s=1}^{\infty} |c_{n,s}| \right)^2 m_n \left(\rho_n^2 - 1\right) \to 0$$

and

$$\begin{aligned} b_{2n} &\leq \sigma_n^2 \sum_{s=1}^{\infty} |c_{n,s}| \sum_{j=m_n}^{\infty} \left(\frac{j}{m_n}\right)^{\delta} |c_{n,j}| \left(\rho_n^2 - 1\right) \sum_{t=m_n}^{n} |\rho_n|^{-t} \\ &\leq \frac{1}{m_n^{\delta}} \sigma_n^2 \left(\sup_{n\geq 1} \sum_{j=1}^{\infty} j^{\delta} |c_{n,j}| \right) \left(\sup_{n\geq 1} \sum_{s=1}^{\infty} |c_{n,s}| \right) \left(\rho_n^2 - 1\right) \sum_{t=m_n}^{n} |\rho_n|^{-t} \\ &= O\left(\frac{1}{m_n^{\delta}}\right). \end{aligned}$$

We conclude that the second term on the right of (47) tends to 0 in L_2. For the first term

$$\begin{aligned} \left\| \left(\rho_n^2 - 1\right)^{1/2} \sum_{j=0}^{n-1} \rho_n^{-j} c_{n,j} \sum_{t=n-j+1}^{n} \rho_n^{-t} e_{n,t} \right\|_{L_2} &\leq \left(\rho_n^2 - 1\right)^{1/2} \sum_{j=0}^{n-1} \rho_n^{-j} |c_{n,j}| \left\| \sum_{t=n-j+1}^{n} \rho_n^{-t} e_{n,t} \right\|_{L_2} \\ &= \left(\rho_n^2 - 1\right)^{1/2} \sigma_n \sum_{j=0}^{n-1} \rho_n^{-j} |c_{n,j}| \left(\sum_{t=n-j+1}^{n} \rho_n^{-2t} \right)^{1/2} \\ &= \sigma_n \rho_n^{-n} \sum_{j=0}^{n-1} |c_{n,j}| \left(\left(\rho_n^2 - 1\right) \sum_{t=1}^{j} \rho_n^{-2t} \right)^{1/2} \\ &= O\left(\rho_n^{-n}\right) \end{aligned}$$

so (47) implies that

$$\left\| \xi_n - \left(\sum_{j=0}^{n-1} \rho_n^{-j} c_{n,j} \right) \left(\rho_n^2 - 1\right)^{1/2} \sum_{t=1}^{n} \rho_n^{-t} e_{n,t} \right\|_{L_2} \to 0. \qquad (48)$$

Since $\sup_{n\geq 1} \left\| \left(\rho_n^2 - 1\right)^{1/2} \sum_{t=1}^{n} \rho_n^{-t} e_{n,t} \right\|_{L_2}^2 < \infty$, the approximation $\left\| \xi_n - \tilde{X}_n(1) \right\|_{L_2} \to 0$ will follow from (48) if

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \rho_n^{-j} c_{n,j} = \lim_{n\to\infty} \sum_{j=0}^{\infty} c_{n,j}.$$

Choosing again $m_n \to \infty$ and $m_n (\rho_n - 1) \to 0$, write

$$\begin{aligned}
\left| \sum_{j=0}^{n-1} \rho_n^{-j} c_{n,j} - \sum_{j=0}^{\infty} c_{n,j} \right| &\leq \sum_{j=0}^{n-1} \left| \rho_n^{-j} - 1 \right| |c_{n,j}| \\
&\leq \sum_{j=0}^{m_n - 1} \left| \rho_n^{-j} - 1 \right| |c_{n,j}| + 2 \sum_{j=m_n}^{n-1} |c_{n,j}| \\
&\leq \sup_{0 \leq k < m_n} \left(1 - \rho_n^{-k}\right) \sum_{j=0}^{m_n - 1} |c_{n,j}| + \frac{2}{m_n^{\delta}} \sum_{j=m_n}^{n-1} j^{\delta} |c_{n,j}| \\
&\leq \left(\sup_{0 \leq k < m_n} \left(1 - \rho_n^{-k}\right) + \frac{2}{m_n^{\delta}} \right) \sup_{n \geq 1} \sum_{j=0}^{\infty} j^{\delta} |c_{n,j}| .
\end{aligned}$$

Since $m_n^{\delta} \to \infty$, it is enough to show that $\sup_{0 \leq k < m_n} \left(1 - \rho_n^{-k}\right) \to 0$. Apply the mean value theorem to the function $x \mapsto \rho_n^{-x}$ around $(0, k)$: there exists $k_0 \in (0, k)$ such that $1 - \rho_n^{-k} = k \rho_n^{-k_0} \log \rho_n$, i.e.

$$\sup_{0 \leq k < m_n} \left(1 - \rho_n^{-k}\right) \leq m_n \log \rho_n = m_n \log (1 + \rho_n - 1) = m_n O (\rho_n - 1) = o(1)$$

from the choice of (m_n), where we used the fact that $\log(1 + x) = O(x)$ as $x \to 0$. This completes the proof of $\left\| \xi_n - \tilde{X}_n (1) \right\|_{L_2} \to 0$, which, by Assumption 3(i) and (17), implies that $\mathbf{X}_n = \tilde{X}_n (1) + o_p (1)$.

For the approximation for $\mathbf{Y}_n$, write

$$\begin{aligned}
\mathbf{Y}_n &= \left(\rho_n^2 - 1\right)^{1/2} \sum_{t=1}^{n} \rho_n^{-t} u_{n,n-t+1} = \left(\rho_n^2 - 1\right)^{1/2} \sum_{j=0}^{\infty} c_{n,j} \sum_{t=1}^{n} \rho_n^{-t} e_{n,n-t-j+1} \\
&= \left(\rho_n^2 - 1\right)^{1/2} \sum_{j=0}^{\infty} \rho_n^{j} c_{n,j} \sum_{t=j+1}^{n+j} \rho_n^{-t} e_{n,n-t+1} \\
&= \left(\rho_n^2 - 1\right)^{1/2} \left(\sum_{t=1}^{n-1} \rho_n^{-t} e_{n,n-t+1} \sum_{j=0}^{t-1} \rho_n^{j} c_{n,j} + \sum_{t=n}^{\infty} \rho_n^{-t} e_{n,n-t+1} \sum_{j=t-n}^{t-1} \rho_n^{j} c_{n,j} \right) \\
&= \left(\rho_n^2 - 1\right)^{1/2} \left(\sum_{t=1}^{n-1} \rho_n^{-t} e_{n,n-t+1} \left[\sum_{j=0}^{t-1} c_{n,j} + \sum_{j=0}^{t-1} \left(\rho_n^{j} - 1\right) c_{n,j} \right] + \sum_{t=0}^{\infty} \bar{C}_{nj} e_{n,-(t-1)} \right) \\
&= Y_{1n} + Y_{2n} + Y_{3n}
\end{aligned}$$

in order of appearance, where $\bar{C}_{nj} = (\rho_n^2 - 1)^{1/2} \sum_{j=0}^{n-1} \rho_n^{-(n-j)} c_{n,j+t}$. Since $\sum_{j=0}^{n-1} \rho_n^{-(n-j)} = \sum_{j=1}^{n} \rho_n^{-j}$, it is easy to see $\sum_{j=0}^{\infty} \bar{C}_{nj}^2$ has the same upper bound as $\sum_{j=0}^{\infty} C_{nj}^2$ so that that $\|Y_{3n}\|_{L_2}^2 \to 0$. For Y_{2n}, note that $\rho_n \to 1$ implies that $\log \rho_n \to 0$; choosing a sequence $m_n \to \infty$ and $m_n \log \varphi_{2n} \to 0$ and using the inequality

$$\rho_n^j - 1 \leq j \rho_n^j \log \rho_n \tag{49}$$

obtained by applying the mean value theorem to the increasing function $x \mapsto \rho_n^x$ around $(0, j)$, we can write

$$\begin{aligned}
\|Y_{2n}\|_{L_2}^2 &= \sigma_n^2 \left(\rho_n^2 - 1\right) \sum_{t=1}^{n-1} \rho_n^{-2t} \left(\sum_{j=0}^{t-1} \left(\rho_n^j - 1\right) c_{n,j} \right)^2 \\
&\leq \sigma_n^2 \left(\rho_n^2 - 1\right) (\log \rho_n)^2 \sum_{t=1}^{n-1} \rho_n^{-2t} \left(\sum_{j=0}^{t-1} j \rho_n^j |c_{n,j}| \right)^2 \\
&= \sigma_n^2 \left(\rho_n^2 - 1\right) (\log \rho_n)^2 \sum_{t=1}^{n-1} \rho_n^{-2t} \sum_{j=0}^{t-1} j \rho_n^j |c_{n,j}| \sum_{i=0}^{t-1} i \rho_n^i |c_{n,i}| \\
&= \sigma_n^2 (\log \rho_n)^2 \sum_{j=0}^{n-2} j \rho_n^j |c_{n,j}| \sum_{i=0}^{n-2} i \rho_n^i |c_{n,i}| \left(\rho_n^2 - 1\right) \sum_{t=(j \vee i)+1}^{n-1} \rho_n^{-2t} \\
&= \sigma_n^2 (\log \rho_n)^2 \sum_{j=0}^{n-2} j \rho_n^j |c_{n,j}| \sum_{i=0}^{n-2} i \rho_n^i |c_{n,i}| \rho_n^{-2(j \vee i)} \left(\rho_n^2 - 1\right) \sum_{t=1}^{n-(j \vee i)-1} \rho_n^{-2t} \\
&\leq \left(\log \rho_n \sum_{j=0}^{n-2} j \rho_n^{-j} |c_{n,j}| \right)^2 O(1) \\
&\leq \left(m_n \log \rho_n \sum_{j=0}^{m_n} |c_{n,j}| + \sum_{j=m_n+1}^{n-2} \frac{\log \rho_n^j}{\rho_n^j} |c_{n,j}| \right)^2 O(1) \\
&\leq \left(O(m_n \log \rho_n) + \sum_{j=m_n+1}^{n-2} |c_{n,j}| \right)^2 O(1) = o(1)
\end{aligned}$$

from the inequality $\log x \leq x$ for $x \geq 1$. Finally,

$$\begin{aligned}
\left\| Y_{1n} - \left(\rho_n^2 - 1\right)^{1/2} \sum_{j=0}^{n-1} c_{n,j} \sum_{t=1}^{n-1} \rho_n^{-t} e_{n,n-t+1} \right\|_{L_2}^2 &= \sigma_n^2 \left(\rho_n^2 - 1\right) \sum_{t=1}^{n-1} \rho_n^{-2t} \left(\sum_{j=t}^{n-1} c_{n,j} \right)^2 \\
&\leq \left(\sum_{j=k_n}^{n-1} |c_{n,j}| \right)^2 O(1) + O\left(k_n \left(\rho_n^2 - 1\right)\right) \\
&= o(1)
\end{aligned}$$

by writing $\sum_{t=1}^{n-1} \rho_n^{-2t} \left(\sum_{j=t}^{n-1} c_{n,j} \right)^2 = \left(\sum_{t=k_n}^{n-1} + \sum_{t=1}^{k_n - 1} \right) \rho_n^{-2t} \left(\sum_{j=t}^{n-1} |c_{n,j}| \right)^2$ with $k_n \to \infty$ and $k_n (\rho_n^2 - 1) \to 0$. We conclude that

$$\left\| \mathbf{Y}_n - \tilde{Y}_n(1) \right\|_{L_2} \leq \left| C(1) - \sum_{j=0}^{n-1} c_{n,j} \right| \left\| \left(\rho_n^2 - 1\right)^{1/2} \sum_{t=1}^{n-1} \rho_n^{-t} e_{n,n-t+1} \right\|_{L_2} + o(1) = o(1)$$

since $\sum_{j=0}^{n-1} c_{n,j} \to C(1)$ by Assumption 2.

To show (21), write

$$\left[\tilde{X}_n(1), \tilde{Y}_n(1) \right]' = \sum_{t=1}^{n} \zeta_{n,t}$$

where $\zeta_{n,t} := C(1) (\rho_n^2 - 1)^{1/2} \left[\rho_n^{-t}, \rho_n^{-(n-t+1)} \right]' e_{n,t}$ is a $\mathcal{F}_{n,t}$-martingale difference array. We apply a standard martingale central limit theorem (e.g. Corollary 3.1 of Hall and Heyde (1980)):

$$\begin{aligned}
\sum_{t=1}^{n} \mathbb{E}_{\mathcal{F}_{n,t-1}} \left(\zeta_{n,t} \zeta_{n,t}' \right) &= C(1)^2 \begin{bmatrix} (\rho_n^2 - 1) \sum_{t=1}^n \rho_n^{-2t} \sigma_{n,t}^2 & (\rho_n^2 - 1) \rho_n^{-n-1} \sum_{t=1}^n \sigma_{n,t}^2 \\ (\rho_n^2 - 1) \rho_n^{-n-1} \sum_{t=1}^n \sigma_{n,t}^2 & (\rho_n^2 - 1) \sum_{t=1}^n \rho_n^{-2(n-t+1)} \sigma_{n,t}^2 \end{bmatrix} \\
&= C(1)^2 \sigma^2 I_2 + o_p(I_2) \qquad (50)
\end{aligned}$$

since $\left\| (\rho_n^2 - 1) \rho_n^{-n-1} \sum_{t=1}^n \sigma_{n,t}^2 \right\|_{L_1} \leq (\rho_n^2 - 1) \rho_n^{-n-1} n \max_{1 \leq t \leq n} \mathbb{E} e_{n,t}^2 \to 0$ and the law of large numbers in Lemma 1(ii) of Arvanitis and Magdalinos with $a_{n,t} = (\rho_n^2 - 1) \rho_n^{-2t}$ and $y_t = \sigma_{n,t}^2 - \mathbb{E} \sigma_{n,t}^2$ gives

$$\left(\rho_n^2 - 1\right) \sum_{t=1}^{n} \rho_n^{-2t} \sigma_{n,t}^2 = \left(\rho_n^2 - 1\right) \sum_{t=1}^{n} \rho_n^{-2t} \mathbb{E} \sigma_{n,t}^2 + o_p(1) = \sigma^2 + o_p(1)$$

and the same result with $a_{n,t} = (\rho_n^2 - 1)\,\rho_n^{-2(n-t+1)}$ implies that

$$\left(\rho_n^2 - 1\right)\sum_{t=1}^{n}\rho_n^{-2(n-t+1)}\sigma_{n,t}^2 = \left(\rho_n^2 - 1\right)\sum_{t=1}^{n}\rho_n^{-2(n-t+1)}\mathbb{E}\sigma_{n,t}^2 + o_p(1) = \sigma^2 + o_p(1).$$

For the Lindeberg condition,

$$\begin{aligned}\left\|\zeta_{n,t}\right\|^2 &= C(1)^2\left(\rho_n^2 - 1\right)\left(\rho_n^{-2t} + \rho_n^{-2(n-t+1)}\right)e_{n,t}^2 \\ &\le C(1)^2\left(\rho_n^2 - 1\right)\left(\rho_n^{-2t} + \rho_n^{-2(n-t+1)}\right)\left(\lambda_n + \mathbf{1}\left\{e_{n,t}^2 > \lambda_n\right\}\right)\end{aligned}$$

and we may write

$$\begin{aligned}\mathcal{L}_n(\delta) &= \sum_{t=1}^{n}\mathbb{E}\left(\left\|\zeta_{n,t}\right\|^2 1\left\{\left\|\zeta_{n,t}\right\| > \delta\right\}\right) \\ &\le \max_{1\le t\le n}\left\{\lambda_n\mathbb{P}\left(\left\|\zeta_{n,t}\right\| > \delta\right) + \mathbb{E}\left(e_{n,t}^2\mathbf{1}\left\{e_{n,t}^2 > \lambda_n\right\}\right)\right\}2C(1)^2\left(\rho_n^2 - 1\right)\sum_{t=1}^{n}\rho_n^{-2t} \\ &\le 2C(1)^2\left\{\frac{1}{\delta^2}\lambda_n\max_{1\le t\le n}\mathbb{E}\left\|\zeta_{n,t}\right\|^2 + \max_{1\le t\le n}\mathbb{E}\left(e_{n,t}^2\mathbf{1}\left\{e_{n,t}^2 > \lambda_n\right\}\right)\right\} \\ &\le 2C(1)^2\left\{\frac{1}{\delta^2}C(1)^2\left(\rho_n^2 - 1\right)\lambda_n\max_{1\le t\le n}\mathbb{E}e_{n,t}^2 + \max_{1\le t\le n}\mathbb{E}\left(e_{n,t}^2\mathbf{1}\left\{e_{n,t}^2 > \lambda_n\right\}\right)\right\}.\end{aligned}$$

Choosing $\lambda_n \to \infty$ and $(\rho_n^2 - 1)\lambda_n \to 0$, the first term tends to 0 by the choice of λ_n and the second term tends to 0 by the uniform integrability of the sequence $\left\{e_{n,t}^2 : t \in \mathbb{Z}, n \in \mathbb{N}\right\}$. Having proved the Lindeberg condition $\mathcal{L}_n(\delta) \to 0$ for all $\delta > 0$, (21) follows by (50) and Corollary 3.1 of Hall and Heyde (1980) (since the limit in (50) is non-random, the martingale CLT holds without the requirement that $\mathcal{F}_{n,t}$ is increasing in n).

Proof of Lemma 3. We show that

$$e_t := \liminf_{n\to\infty} e_{n,t} = \lim_{n\to\infty} e_{k_n,t} \text{ exists } a.s. \text{ in } \mathbb{R} \text{ and in } L_2 \text{ for each } t \in \mathbb{Z} \quad (51)$$

for some subsequence $(k_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$. Almost sure convergence of $(e_{k_n,t})_{n\in\mathbb{N}}$ in $[-\infty, \infty]$ for each t along a subsequence $(k_n)_{n\in\mathbb{N}}$ follows since the limit inferior of $(e_{n,t})_{n\in\mathbb{N}}$ is an accumulation point. By the Fatou lemma,

$$\mathbb{E}\left|e_t\right| \le \liminf_{n\to\infty}\mathbb{E}\left|e_{n,t}\right| \le \sup_{n\in\mathbb{N}}\mathbb{E}\left|e_{n,t}\right| < \infty$$

which implies that $\mathbb{P}(|e_t| < \infty) = 1$, showing that e_t in (51) exists $a.s.$ in $\mathbb{R}$. For L_2 convergence,

$$\begin{aligned}\mathbb{E}\left(e_{k_n,t} - e_{k_m,t}\right)^2 &\leq \mathbb{E}\left[\left(e_{k_n,t} - e_{k_m,t}\right)^2 \mathbf{1}\left\{\left(e_{k_n,t} - e_{k_m,t}\right)^2 \leq \lambda\right\}\right] \\ &\quad + \mathbb{E}\left[\left(e_{k_n,t} - e_{k_m,t}\right)^2 \mathbf{1}\left\{\left(e_{k_n,t} - e_{k_m,t}\right)^2 > \lambda\right\}\right] \\ &\leq \mathbb{E}\left[\left(e_{k_n,t} - e_{k_m,t}\right)^2 \mathbf{1}\left\{\left(e_{k_n,t} - e_{k_m,t}\right)^2 \leq \lambda\right\}\right] + 2\sup_{n,t\in\mathbb{N}} \mathbb{E}\left[e_{n,t}^2 \mathbf{1}\left\{e_{n,t}^2 > \lambda/2\right\}\right].\end{aligned}$$

Since $\lim_{n\to\infty} \mathbb{E}\left[\left(e_{k_n,t} - e_{k_m,t}\right)^2 \mathbf{1}\left\{\left(e_{k_n,t} - e_{k_m,t}\right)^2 \leq \lambda\right\}\right] = 0$ for all $\lambda > 0$ and $t \in \mathbb{Z}$ by the bounded convergence theorem and

$$\limsup_{n\to\infty} \mathbb{E}\left(e_{k_n,t} - e_{k_m,t}\right)^2 \leq 2 \limsup_{\lambda\to\infty} \sup_{n\in\mathbb{N}} \sup_{t\in\mathbb{Z}} \mathbb{E}\left[e_{n,t}^2 \mathbf{1}\left\{e_{n,t}^2 > \lambda/2\right\}\right] = 0$$

by the uniform integrability of the $\left\{e_{n,t}^2 : n \in \mathbb{N}, t \in \mathbb{Z}\right\}$ in Assumption 2. Convergence in L_2 of $\left(e_{k_n,t}\right)_{n\in\mathbb{N}}$ follows and implies that $\mathbb{E}e_t^2 < \infty$, completing the proof of (51). Recall the definitions in (22) and the fact that $A \in \liminf_{n\to\infty} \mathcal{F}_{n,t}$ if and only if $A \in \mathcal{F}_{n,t}$ for all but finitely many n for any $A \subseteq \Omega$. Since $e_{k_n,t}$ is $\mathcal{F}_{k_n,t}$-measurable for each t and there exists $n_0 \in \mathbb{N}$ such that $\mathcal{F}_{k_n,t} \subseteq \mathcal{F}_t$ for all $n \geq n_0$, $e_{k_n,t}$ is $\mathcal{F}_t$-measurable for for all $n \geq n_0$ which implies that $e_t = \lim_{n\to\infty} e_{k_n,t}$ is $\mathcal{F}_t$-measurable for each t. Since $\sup_{t\in\mathbb{Z}} \mathbb{E}e_t^2 < \infty$, $\left(e_t, \mathcal{F}_t\right)_{t\in\mathbb{Z}}$ will be a martingale difference sequence if $\mathbb{E}\left(e_t | \mathcal{F}_{t-1}\right) = 0$ $a.s.$ for each t. To prove this, we use the fact that $\mathcal{G}_{n,t}$ is an increasing sequence of σ-algebras in n so that the sequence $M_n = \mathbb{E}\left(e_t | \mathcal{G}_{n,t-1}\right)$ is a uniformly integrable martingale with respect to $\mathcal{G}_{n,t-1}$ for each t and employ Levy's upward theorem (see 14.2 in Williams (1991)) which, in our context, states that: for each $t \in \mathbb{Z}$

$$\mathbb{E}\left(e_t | \mathcal{G}_{n,t-1}\right) \to \mathbb{E}\left(e_t | \mathcal{F}_{t-1}\right) \ (n \to \infty) \ a.s. \text{ and in } L_2 \tag{52}$$

with $\mathcal{G}_{n,t}$ and $\mathcal{F}_t$ defined in (22). While $e_{n,t}$ is not a $\mathcal{G}_{n,t}$-martingale difference (it is not $\mathcal{G}_{n,t}$-adapted), $\mathcal{G}_{n,t-1} \subseteq \mathcal{F}_{n,t-1}$ so the tower property of conditional expectations implies that

$$\mathbb{E}\left(e_{n,t} | \mathcal{G}_{n,t-1}\right) = \mathbb{E}\left(\left[\mathbb{E}\left(e_{n,t} | \mathcal{F}_{n,t-1}\right)\right] | \mathcal{G}_{n,t-1}\right) = 0. \tag{53}$$

We conclude that for each $t \in \mathbb{Z}$

$$\begin{aligned}\left\|\mathbb{E}\left(e_t | \mathcal{F}_{t-1}\right)\right\|_{L_1} &\leq \left\|\mathbb{E}\left(e_t | \mathcal{G}_{k_n,t-1}\right) - \mathbb{E}\left(e_t | \mathcal{F}_{t-1}\right)\right\|_{L_1} + \left\|\mathbb{E}\left(e_t | \mathcal{G}_{k_n,t-1}\right)\right\|_{L_1} \\ &= \left\|\mathbb{E}\left(e_t | \mathcal{G}_{k_n,t-1}\right) - \mathbb{E}\left(e_t | \mathcal{F}_{t-1}\right)\right\|_{L_1} + \left\|\mathbb{E}\left(e_t - e_{k_n,t} | \mathcal{G}_{k_n,t-1}\right)\right\|_{L_1} \\ &\leq \left\|\mathbb{E}\left(e_t | \mathcal{G}_{k_n,t-1}\right) - \mathbb{E}\left(e_t | \mathcal{F}_{t-1}\right)\right\|_{L_1} + \left\|e_{k_n,t} - e_t\right\|_{L_1} \end{aligned} \tag{54}$$

where the equality follows from $\mathbb{E}\left(e_{k_n,t}\middle|\mathcal{G}_{k_n,t-1}\right) = 0$ in (53) and the last inequality by the Jensen inequality and the law of iterated expectations. Taking limits as $n \to \infty$, the first term on the right of (54) tends to 0 by (52) ($(k_n)_{n\in\mathbb{N}}$ is a subsequence of $\mathbb{N}$) and the second term tends to 0 by (51); hence, the right side of (54) tends to 0 as $n \to \infty$ pointwise for each $t \in \mathbb{Z}$ which implies that $\left\|\mathbb{E}\left(e_t\middle|\mathcal{F}_{t-1}\right)\right\|_{L_1} = 0$ for each t. Hence, $\mathbb{E}\left(e_t\middle|\mathcal{F}_{t-1}\right) = 0$ $a.s.$ for each t as required.

We now show that the martingale difference sequence $(e_t, \mathcal{F}_t)$ satisfies the local MZ conditions (10). To this end, we first prove that $\left\{\sigma^2_{n,t} : n \in \mathbb{N}, t \in \mathbb{Z}\right\}$ is a uniformly integrable sequence. Denote

$$\upsilon(x) = \sup_{n\in\mathbb{N}} \sup_{t\in\mathbb{Z}} \mathbb{E}\left(e^2_{n,t}\mathbf{1}\left\{e^2_{n,t} > x\right\}\right).$$

The $\mathcal{F}_{n,t-1}$-measurability of $\sigma^2_{n,t}$ implies that

$$\begin{aligned}
\mathbb{E}\left(\sigma^2_{n,t}\mathbf{1}\left\{\sigma^2_{n,t} > \lambda\right\}\right) &= \mathbb{E}\left(\mathbb{E}\left(e^2_{n,t}\mathbf{1}\left\{\sigma^2_{n,t} > \lambda\right\}\middle|\mathcal{F}_{n,t-1}\right)\right) = \mathbb{E}\left(e^2_{n,t}\mathbf{1}\left\{\sigma^2_{n,t} > \lambda\right\}\right) \\
&\leq \mathbb{E}\left(e^2_{n,t}\mathbf{1}\left\{e^2_{n,t} \leq \lambda^{1/2}\right\}\mathbf{1}\left\{\sigma^2_{n,t} > \lambda\right\}\right) + \mathbb{E}\left(e^2_{n,t}\mathbf{1}\left\{e^2_{n,t} > \lambda^{1/2}\right\}\right) \\
&\leq \lambda^{1/2}\mathbb{P}\left(\sigma^2_{n,t} > \lambda\right) + \upsilon\left(\lambda^{1/2}\right) \\
&\leq \lambda^{-1/2}\sup_{n\in\mathbb{N}}\mathbb{E}\left(\sigma^2_{n,1}\right) + \upsilon\left(\lambda^{1/2}\right)
\end{aligned}$$

by the Markov inequality. Since the right side is independent of n and t and $\lim_{\lambda\to\infty}\upsilon\left(\lambda^{1/2}\right) = 0$ by the uniform integrability of $\left\{\sigma^2_{n,t} : n \in \mathbb{N}, t \in \mathbb{Z}\right\}$ taking $\lambda \to \infty$ shows that $\left\{\sigma^2_{n,t} : n \in \mathbb{N}, t \in \mathbb{Z}\right\}$ is a UI sequence. Next we show that

$$\left\|\limsup_{n\to\infty}\sigma^2_{n,t}\right\|_{L_1} < \infty \text{ for each } t \in \mathbb{N}. \tag{55}$$

Since $\limsup_{n\to\infty}\sigma^2_{n,t}$ is an accumulation point of $\left\{\sigma^2_{n,t} : n \in \mathbb{N}\right\}$, there exists a subsequence $(m_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\limsup_{n\to\infty}\sigma^2_{n,t} = \lim_{n\to\infty}\sigma^2_{m_n,t}$; using the uniform integrability of the sequence $\left\{\sigma^2_{n,t} : n \in \mathbb{N}\right\}$ to interchange limit and expectation, we may write

$$\left\|\limsup_{n\to\infty}\sigma^2_{n,t}\right\|_{L_1} = \left\|\lim_{n\to\infty}\sigma^2_{m_n,t}\right\|_{L_1} = \lim_{n\to\infty}\left\|\sigma^2_{m_n,t}\right\|_{L_1} = \lim_{n\to\infty}\sigma^2_{m_n} = \sigma^2$$

showing (55).

We now show the second part of (10) for $(e_t, \mathcal{F}_t)$. Since e_t^2 is integrable and $\mathcal{G}_{m,t-1} \uparrow \mathcal{F}_{t-1}$ as $m \to \infty$, Levy's upward theorem and the Fatou lemma give

$$\begin{aligned}
\mathbb{E}\left(e_t^2 \middle| \mathcal{F}_{t-1}\right) &= \lim_{m\to\infty} \mathbb{E}\left(e_t^2 \middle| \mathcal{G}_{m,t-1}\right) = \lim_{m\to\infty} \mathbb{E}\left(\liminf_{n\to\infty} e_{n,t}^2 \middle| \mathcal{G}_{m,t-1}\right) \\
&\leq \lim_{m\to\infty} \liminf_{n\to\infty} \mathbb{E}\left(e_{n,t}^2 \middle| \mathcal{G}_{m,t-1}\right) \\
&= \lim_{m\to\infty} \liminf_{n\to\infty} \mathbb{E}\left(\mathbb{E}\left(e_{n,t}^2 \middle| \mathcal{F}_{n,t-1}\right) \middle| \mathcal{G}_{m,t-1}\right) \qquad (56)
\end{aligned}$$

for all $n \geq m$, by the tower property of conditional expectations since $\mathcal{G}_{m,t-1} \subseteq \mathcal{G}_{n,t-1} \subseteq \mathcal{F}_{n,t-1}$ when $n \geq m$. The uniform integrability of $\left\{\sigma_{n,t}^2 : n \in \mathbb{N}\right\}$ and the reverse Fatou lemma ensure that

$$\liminf_{n\to\infty} \mathbb{E}\left(\mathbb{E}\left(e_{n,t}^2 \middle| \mathcal{F}_{n,t-1}\right) \middle| \mathcal{G}_{m,t-1}\right) \leq \mathbb{E}\left(\limsup_{n\to\infty} \mathbb{E}\left(e_{n,t}^2 \middle| \mathcal{F}_{n,t-1}\right) \middle| \mathcal{G}_{m,t-1}\right)$$

for each m, so (56) gives

$$\mathbb{E}\left(e_t^2 \middle| \mathcal{F}_{t-1}\right) \leq \lim_{m\to\infty} \mathbb{E}\left(\limsup_{n\to\infty} \mathbb{E}\left(e_{n,t}^2 \middle| \mathcal{F}_{n,t-1}\right) \middle| \mathcal{G}_{m,t-1}\right). \qquad (57)$$

The integrability in (55) implies that Levy's upward theorem applies to the right side of (57) and gives

$$\begin{aligned}
\lim_{m\to\infty} \mathbb{E}\left(\limsup_{n\to\infty} \mathbb{E}\left(e_{n,t}^2 \middle| \mathcal{F}_{n,t-1}\right) \middle| \mathcal{G}_{m,t-1}\right) &= \mathbb{E}\left(\limsup_{n\to\infty} \mathbb{E}\left(e_{n,t}^2 \middle| \mathcal{F}_{n,t-1}\right) \middle| \mathcal{F}_{t-1}\right) \\
&= \limsup_{n\to\infty} \mathbb{E}\left(e_{n,t}^2 \middle| \mathcal{F}_{n,t-1}\right) \quad a.s. \qquad (58)
\end{aligned}$$

because $\mathcal{F}_{n,t-1} \subseteq \mathcal{F}_{t-1}$ for all but finitely many n. Combining (57) and (58), we obtain

$$\mathbb{E}\left(e_t^2 \middle| \mathcal{F}_{t-1}\right) \leq \limsup_{n\to\infty} \mathbb{E}\left(e_{n,t}^2 \middle| \mathcal{F}_{n,t-1}\right) \ a.s.$$

for each $t \in \mathbb{N}$. We conclude that

$$\sup_{t\in\mathbb{N}} \mathbb{E}\left(e_t^2 \middle| \mathcal{F}_{t-1}\right) \leq \limsup_{n\to\infty} \sup_{t\in\mathbb{N}} \mathbb{E}\left(e_{n,t}^2 \middle| \mathcal{F}_{n,t-1}\right) \ a.s.$$

and (7) implies that $\sup_{t\in\mathbb{N}} \mathbb{E}\left(e_t^2 \middle| \mathcal{F}_{t-1}\right) < \infty$ *a.s.*, showing the second part of (10).

Next, we show the first part of (10) for $(e_t, \mathcal{F}_t)$. Levy's upward theorem, the fact that $\liminf_{n\to\infty} |e_{n,t}|$ is the limit of a subsequence $(e_{r_n,t})_{n\in\mathbb{N}}$ and the dominated convergence theorem imply that for some subsequence $(r_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$,

$$\begin{aligned}\mathbb{E}\left(|e_t| \middle| \mathcal{F}_{t-1}\right) &= \lim_{m\to\infty} \mathbb{E}\left(|e_t| \middle| \mathcal{G}_{m,t-1}\right) = \lim_{m\to\infty} \mathbb{E}\left(\liminf_{n\to\infty} |e_{n,t}| \middle| \mathcal{G}_{m,t-1}\right) \\ &= \lim_{m\to\infty} \mathbb{E}\left(\lim_{n\to\infty} |e_{r_n,t}| \middle| \mathcal{G}_{m,t-1}\right) = \lim_{m\to\infty} \lim_{n\to\infty} \mathbb{E}\left(|e_{r_n,t}| \middle| \mathcal{G}_{m,t-1}\right) \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \mathbb{E}\left(\mathbb{E}\left(|e_{r_n,t}| \middle| \mathcal{F}_{r_n,t-1}\right) \middle| \mathcal{G}_{m,t-1}\right) \qquad (59)\end{aligned}$$

for all $r_n \geq m$, by the tower property. Now for each m, the monotone convergence theorem implies that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left(\mathbb{E}\left(|e_{r_n,t}| \middle| \mathcal{F}_{r_n,t-1}\right) \middle| \mathcal{G}_{m,t-1}\right) &\geq \lim_{n\to\infty} \mathbb{E}\left(\inf_{j\geq r_n} \mathbb{E}\left(|e_{j,t}| \middle| \mathcal{F}_{j,t-1}\right) \middle| \mathcal{G}_{m,t-1}\right) \\ &= \mathbb{E}\left(\lim_{n\to\infty} \inf_{j\geq r_n} \mathbb{E}\left(|e_{j,t}| \middle| \mathcal{F}_{j,t-1}\right) \middle| \mathcal{G}_{m,t-1}\right) \\ &= \mathbb{E}\left(\liminf_{n\to\infty} \mathbb{E}\left(|e_{r_n,t}| \middle| \mathcal{F}_{r_n,t-1}\right) \middle| \mathcal{G}_{m,t-1}\right) \\ &\geq \mathbb{E}\left(\liminf_{n\to\infty} \mathbb{E}\left(|e_{n,t}| \middle| \mathcal{F}_{n,t-1}\right) \middle| \mathcal{G}_{m,t-1}\right)\end{aligned}$$

since $(r_n)_{n\in\mathbb{N}}$ is a subsequence of $\mathbb{N}$. Substituting into (59) and using the integrability of $\liminf_{n\to\infty} \mathbb{E}\left(|e_{n,t}| \middle| \mathcal{F}_{n,t-1}\right)$ (guaranteed by (55)), Levy's upward theorem gives

$$\begin{aligned}\mathbb{E}\left(|e_t| \middle| \mathcal{F}_{t-1}\right) &\geq \lim_{m\to\infty} \mathbb{E}\left(\liminf_{n\to\infty} \mathbb{E}\left(|e_{n,t}| \middle| \mathcal{F}_{n,t-1}\right) \middle| \mathcal{G}_{m,t-1}\right) \\ &= \mathbb{E}\left(\liminf_{n\to\infty} \mathbb{E}\left(|e_{n,t}| \middle| \mathcal{F}_{n,t-1}\right) \middle| \mathcal{F}_{t-1}\right) \\ &= \liminf_{n\to\infty} \mathbb{E}\left(|e_{n,t}| \middle| \mathcal{F}_{n,t-1}\right) \quad a.s.\end{aligned}$$

for each t, which implies that

$$\liminf_{t\to\infty} \mathbb{E}\left(|e_t| \middle| \mathcal{F}_{t-1}\right) \geq \liminf_{n\to\infty} \liminf_{t\to\infty} \mathbb{E}\left(|e_{n,t}| \middle| \mathcal{F}_{n,t-1}\right) > 0 \ a.s.$$

by (6). This shows that $(e_t, \mathcal{F}_t)_{t\in\mathbb{Z}}$ is a martingale difference sequence that satisfies the local MZ condition (10) and completes the proof of part (i).

For part (ii), we begin by showing that for each $t \in \mathbb{N}$

$$\left| \sum_{t=1}^{k_n} \rho_{k_n}^{-t} e_{k_n,t} - \sum_{t=1}^{k_n} \rho^{-t} e_t \right| \rightarrow_{a.s.} 0 \tag{60}$$

as $n \rightarrow \infty$ along the same subsequence (k_n) for which convergence in (51) obtains. We first show that

$$R_n := \sum_{j=1}^{\infty} j^2 \left| \rho_n^{-j} - \rho^{-j} \right| \rightarrow 0. \tag{61}$$

We apply the mean value theorem to the function $x \mapsto x^{-j}$: $\rho_n^{-j} - \rho^{-j} = -(\rho_n - \rho)\, j \phi_n^{-j-1}$ for some $\phi_n \rightarrow \rho$; hence, we may choose $\eta \in (0, \rho - 1)$ and $n_0(\eta) \in \mathbb{N}$ such that for all $n \geq n_0(\eta)$: $\phi_n > \rho - \eta$ which implies that

$$\sum_{j=1}^{\infty} j^2 \left| \rho_n^{-j} - \rho^{-j} \right| = |\rho_n - \rho| \sum_{j=1}^{\infty} j^3 \phi_n^{-j-1} \leq |\rho_n - \rho| \sum_{j=1}^{\infty} j^3 (\rho - \eta)^{-j-1} \rightarrow 0$$

since $\rho - \eta > 1$ from the choice $\eta \in (0, \rho - 1)$ implies that $\sum_{j=1}^{\infty} j^3 (\rho - \eta)^{-j-1} < \infty$. Having established (61), write

$$\begin{aligned} \left| \sum_{t=1}^{k_n} \rho_{k_n}^{-t} e_{k_n,t} - \sum_{t=1}^{k_n} \rho^{-t} e_t \right| &\leq \left| \sum_{t=1}^{k_n} \rho_{k_n}^{-t} e_{k_n,t} - \sum_{t=1}^{k_n} \rho^{-t} e_{k_n,t} \right| + \left| \sum_{t=1}^{k_n} \rho^{-t} e_{k_n,t} - \sum_{t=1}^{k_n} \rho^{-t} e_t \right| \\ &\leq \sum_{t=1}^{\infty} t \left| \rho_{k_n}^{-t} - \rho^{-t} \right| \frac{|e_{k_n,t}|}{t} + \sum_{t=1}^{\infty} \rho^{-t} |e_{k_n,t} - e_t| \\ &\leq R_n^{1/2} \left(\sum_{t=1}^{\infty} \frac{e_{k_n,t}^2}{t^2} \right)^{1/2} + \sum_{t=1}^{\infty} \rho^{-t} |e_{k_n,t} - e_t| . \end{aligned} \tag{62}$$

Since $\sum_{t=1}^{\infty} t^{-2} < \infty$, dominated convergence implies that $\sum_{t=1}^{\infty} t^{-2} e_{k_n,t}^2 \rightarrow \sum_{t=1}^{\infty} t^{-2} e_t^2 < \infty$ $a.s.$ since $\sum_{t=1}^{\infty} t^{-2} \mathbb{E} e_t^2 < \infty$, so the first term of (62) tends to 0 $a.s.$ by (61). The second term of (62) tends to 0 $a.s.$ since $\sum_{t=1}^{\infty} \rho^{-t} < \infty$ by (51) and dominated convergence. This completes the proof of (60).

Using (61) we have $\left\| \xi_n - (\rho^2 - 1)^{1/2} \sum_{t=1}^{n} \rho^{-t} u_{n,t} \right\|_{L_1} \rightarrow 0$ when $\rho > 1$;

also,

$$\begin{aligned}\sum_{t=1}^{n}\rho^{-t}u_{n,t} &= \sum_{t=1}^{n}\rho^{-t}\sum_{j=0}^{t-1}c_{n,j}e_{n,t-j}+\sum_{t=1}^{n}\rho^{-t}\sum_{j=t}^{\infty}c_{n,j}e_{n,t-j}\\ &= \sum_{j=0}^{n-1}c_{n,j}\sum_{t=j+1}^{n}\rho^{-t}e_{n,t-j}+\sum_{t=1}^{n}\rho^{-t}\sum_{j=t}^{\infty}c_{n,j}e_{n,t-j}\\ &= \sum_{j=0}^{n-1}\rho^{-j}c_{n,j}\sum_{t=1}^{n-j}\rho^{-t}e_{n,t}+\sum_{t=1}^{n}\rho^{-t}\sum_{j=t}^{\infty}c_{n,j}e_{n,t-j}\\ &= \sum_{j=0}^{n-1}\rho^{-j}c_{n,j}\sum_{t=1}^{n}\rho^{-t}e_{n,t}-\rho^{-n}\sum_{j=0}^{n-1}c_{n,j}\sum_{t=1}^{j}\rho^{-t}e_{n,t+n-j}+\sum_{t=1}^{n}\rho^{-t}\sum_{j=0}^{\infty}c_{n,j+t}e_{n,-j}\end{aligned}$$

and $\left\|\rho^{-n}\sum_{j=0}^{n-1}c_{n,j}\sum_{t=1}^{j}\rho^{-t}e_{n,t+n-j}\right\|_{L_1}=O\left(\rho^{-n}\right)$ imply that

$$\mathbf{X}_n=\tilde{X}_n\left(\rho\right)+G_{n,0}\left(\rho\right)+o_p\left(1\right) \tag{63}$$

where

$$G_{n,0}=\left(\rho^2-1\right)^{1/2}\left(\sum_{t=1}^{n}\rho^{-t}\sum_{j=0}^{\infty}c_{n,j+t}e_{n,-j}+X_{n,0}\right). \tag{64}$$

is an $\mathcal{F}_{n,0}$-measurable random variable. Consider an arbitrary subsequence $\left(\mathbf{X}_{m_n}\right)_{n\in\mathbb{N}}\subseteq\left(\mathbf{X}_n\right)_{n\in\mathbb{N}}$. Let $\tilde{e}_t:=\liminf_{n\to\infty}e_{m_n,t}$; by part (i) applied to the martingale difference array $\left(e_{m_n,t},\mathcal{F}_{m_n,t}\right)$ and $\tilde{\mathcal{F}}_t:=\sigma\left(\liminf_{n\to\infty}\mathcal{F}_{m_n,t}\right)$, $\left(\tilde{e}_t,\tilde{\mathcal{F}}_t\right)_{t\in\mathbb{N}}$ is a martingale difference sequence satisfying the local MZ conditions (10). Also, there exists a subsequence $\left(k_n\right)_{n\in\mathbb{N}}\subseteq\left(m_n\right)_{n\in\mathbb{N}}$ such that (60) applies with e_t replaced by $\tilde{e}_t$, so that

$$\tilde{X}_{k_n}\left(\rho\right)\to_{a.s.}\sum_{t=1}^{\infty}\pi_t\tilde{e}_t,\quad \pi_t:=C\left(\rho\right)\left(\rho^2-1\right)^{1/2}\rho^{-t} \tag{65}$$

by the martingale convergence theorem. Since $\left(X_{n,0}\right)_{n\in\mathbb{N}}$ converges in distribution and

$$\sup_{n\in\mathbb{N}}\left\|\sum_{t=1}^{n}\rho^{-t}\sum_{j=0}^{\infty}c_{n,j+t}e_{n,-j}\right\|_{L_1}\leq\frac{1}{\rho-1}\sup_{n\in\mathbb{N}}\left\|e_{n,0}\right\|_{L_1}\sup_{n\in\mathbb{N}}\sum_{j=0}^{\infty}\left|c_{n,j}\right|<\infty,$$

$\{G_{n,0}(\rho) : n \in \mathbb{N}\}$ is a tight sequence; since tightness extends to the subsequence $\{G_{k_n,0}(\rho) : n \in \mathbb{N}\}$ with $(k_n)_{n\in\mathbb{N}}$ satisfying (65), there exists a subsequence $(r_n)_{n\in\mathbb{N}} \subseteq (k_n)_{n\in\mathbb{N}}$ such that $G_{r_n,0}(\rho) \to_d G_0(\rho)$ where $G_0(\rho)$ is $\mathcal{F}_0$-measurable, since each $G_{r_n,0}(\rho)$ is $\mathcal{F}_{r_n,0}$-measurable and $\mathcal{F}_{r_n,0} \subseteq \mathcal{F}_0$ for all but finitely many n. Since (65) obtains along the subsequence $(r_n)_{n\in\mathbb{N}}$, we conclude from (63) that: for any subsequence $(m_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ there exists a further subsequence $(r_n)_{n\in\mathbb{N}} \subseteq (m_n)_{n\in\mathbb{N}}$ such that

$$\mathbf{X}_{r_n} \to_d \mathbf{X}_\infty := \sum_{t=1}^{\infty} \pi_t \tilde{e}_t + G_0(\rho) \text{ and } \mathbb{P}(\mathbf{X}_\infty = 0) = 0 \qquad (66)$$

where the distribution of $\mathbf{X}_\infty$ may depend on the subsequence $(r_n)_{n\in\mathbb{N}}$. The fact that $\mathbf{X}_\infty \neq 0$ *a.s.* follows by by Corollary 2 of Lai and Wei (1983) since $\left(\tilde{e}_t, \tilde{\mathcal{F}}_t\right)_{t\in\mathbb{N}}$ is a martingale difference sequence satisfying the local MZ conditions (10), $\pi_t \neq 0$ for all t and $G_0(\rho)$ is $\mathcal{F}_0$-measurable. Since $(\mathbf{X}_{m_n})_{n\in\mathbb{N}}$ is an arbitrary subsequence of $(\mathbf{X}_n)_{n\in\mathbb{N}}$, part (ii) follows.

Since the limit superior is an accumulation point of the real sequence $\{\mathbb{P}(|\mathbf{X}_n| \leq \delta) : n \in \mathbb{N}\}$, there exists a subsequence $(m_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that

$$\limsup_{n\to\infty} \mathbb{P}(|\mathbf{X}_n| \leq \delta) = \lim_{n\to\infty} \mathbb{P}(|\mathbf{X}_{m_n}| \leq \delta). \qquad (67)$$

By (66), there exists a subsequence $(r_n)_{n\in\mathbb{N}} \subseteq (m_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \mathbb{P}(|\mathbf{X}_{r_n}| \leq x) = \mathbb{P}(|\mathbf{X}_\infty| \leq x) =: F_\infty(x)$$

at all continuity points x of the distribution function $F_\infty(\cdot)$ of $|\mathbf{X}_\infty|$. Since $\mathbb{P}(\mathbf{X}_\infty = 0) = 0$, $F_\infty(x) = 0$ for all $x \leq 0$ so F_∞ is left-continuous (and hence continuous) at $x = 0$; since 0 is a continuity point of F_∞ and F_∞ has countably many points of discontinuity, there exists $\delta_0 > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}(|\mathbf{X}_{r_n}| \leq \delta) = \mathbb{P}(|\mathbf{X}_\infty| \leq \delta) \quad \text{for all } \delta < \delta_0. \qquad (68)$$

Since $\{\mathbb{P}(|\mathbf{X}_{r_n}| \leq \delta) : n \in \mathbb{N}\}$ is a subsequence of the convergent sequence $\{\mathbb{P}(|\mathbf{X}_{m_n}| \leq \delta) : n \in \mathbb{N}\}$, (67) and (68) imply that

$$\limsup_{n\to\infty} \mathbb{P}(|\mathbf{X}_n| \leq \delta) = \mathbb{P}(|\mathbf{X}_\infty| \leq \delta) \quad \text{for all } \delta < \delta_0$$

so the continuity of F_∞ at 0 implies that

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}(|\mathbf{X}_n| \leq \delta) = \mathbb{P}(|\mathbf{X}_\infty| \leq 0) = \mathbb{P}(|\mathbf{X}_\infty| = 0) = 0$$

as required.

For the last part, $|\mathbf{Y}_n| / |\mathbf{X}_n| = O_p(1)$ is equivalent to

$$\lim_{\lambda\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(|\mathbf{Y}_n| / |\mathbf{X}_n| \geq \lambda\right) = 0. \tag{69}$$

To prove this fix $\alpha \in (0,1)$ and write

$$\begin{aligned}
\mathbb{P}\left(|\mathbf{Y}_n/\mathbf{X}_n| \geq \lambda\right) &= \mathbb{P}\left(|\mathbf{Y}_n/\mathbf{X}_n| \geq \lambda, |\mathbf{X}_n| > \lambda^{-\alpha}\right) + \mathbb{P}\left(|\mathbf{Y}_n/\mathbf{X}_n| \geq \lambda, |\mathbf{X}_n| \leq \lambda^{-\alpha}\right) \\
&\leq \mathbb{P}\left(\frac{|\mathbf{Y}_n|}{\lambda^{-\alpha}} \geq \lambda\right) + \mathbb{P}\left(|\mathbf{X}_n| \leq \lambda^{-\alpha}\right) \\
&= \mathbb{P}\left(|\mathbf{Y}_n| \geq \lambda^{1-\alpha}\right) + \mathbb{P}\left(|\mathbf{X}_n| \leq \lambda^{-\alpha}\right) \\
&\leq \frac{1}{\lambda^{1-\alpha}} \sup_{n\in\mathbb{N}} \mathbb{E}\,|\mathbf{Y}_n| + \mathbb{P}\left(|\mathbf{X}_n| \leq \lambda^{-\alpha}\right).
\end{aligned}$$

Since $\lim_{\lambda\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(|\mathbf{X}_n| \leq \lambda^{-\alpha}\right) = 0$, $\alpha \in (0,1)$ and $\sup_{n\in\mathbb{N}} \mathbb{E}\,|\mathbf{Y}_n| < \infty$, (69) follows.

Proof of Lemma 4. Recursing (2) we obtain

$$x_{n,t} = \mu + X_{n,0}\rho_n^t + \xi_{n,t}, \quad \xi_{n,t} := \textstyle\sum_{j=1}^{t} \rho_n^{t-j} u_{n,j} \tag{70}$$

where $\xi_{n,t}$ is the restriction of $x_{n,t}$ when $\mu = X_{n,0} = 0$ and satisfies

$$\left(\rho_n^2 - 1\right)^{1/2} \rho_n^{-n} \xi_{n,n} = \xi_n$$

in (16). We first prove (23). Using (70), we obtain

$$\left(\rho_n^2 - 1\right) \rho_n^{-n} \sum_{t=1}^{n} x_{n,t-1} u_{n,t} = \left(\rho_n^2 - 1\right) \rho_n^{-n} \sum_{t=1}^{n} \xi_{n,t-1} u_{n,t} + X_{n,0} \left(\rho_n^2 - 1\right)^{1/2} \mathbf{Y}_n + o_p(1) \tag{71}$$

since $\left(\rho_n^2 - 1\right) \rho_n^{-n} \sum_{t=1}^{n} u_{n,t} = O_p\left(n^{1/2} \left(\rho_n^2 - 1\right) \rho_n^{-n}\right) = o_p\left(\left(\rho_n^2 - 1\right)^{1/2}\right)$ by Lemma A1. By Assumption 3, the second term of (71) will be $o_p(1)$ when

$\rho_n \to 1$ but will contribute asymptotically when $\rho_n \to \rho > 1$. Writing

$$\begin{aligned}
\sum_{t=1}^{n} \xi_{n,t-1} u_{n,t} &= \sum_{t=1}^{n} \left(\sum_{i=1}^{t-1} \rho_n^{t-1-i} u_{n,i} \right) u_{n,t} \\
&= \sum_{t=1}^{n} \rho_n^{t-1} \left(\sum_{i=1}^{t-1} \rho_n^{-i} u_{n,i} \right) u_{n,t} \\
&= \sum_{i=1}^{n-1} \rho_n^{-i} u_{n,i} \sum_{t=i+1}^{n} \rho_n^{t-1} u_{n,t} \\
&= \sum_{i=1}^{n-1} \rho_n^{-i} u_{n,i} \sum_{t=1}^{n} \rho_n^{t-1} u_{n,t} - \sum_{i=1}^{n-1} \rho_n^{-i} u_{n,i} \sum_{t=1}^{i} \rho_n^{t-1} u_{n,t},
\end{aligned}$$

showing that

$$N_n = \left(\rho_n^2 - 1\right) \rho_n^{-n} \sum_{i=1}^{n-1} \rho_n^{-i} u_{n,i} \sum_{t=1}^{i} \rho_n^{t-1} u_{n,t} = o_p(1) \tag{72}$$

will imply that

$$\left(\rho_n^2 - 1\right) \rho_n^{-n} \sum_{t=1}^{n} \xi_{n,t-1} u_{n,t} = \xi_n \mathbf{Y}_n + o_p(1) \tag{73}$$

and (23) will follow by combining (71) and (73) and using the definition of $\mathbf{X}_n$ in (17). It remains to prove (72). When $\rho_n^{-n} n \to 0$, the proof is easy:

$$\begin{aligned}
\|N_n\|_{L_1} &\leq \left(\rho_n^2 - 1\right) \rho_n^{-n} \sum_{i=1}^{n-1} \rho_n^{-i} \sum_{t=1}^{i} \rho_n^{t-1} \mathbb{E}\left|u_{n,i} u_{n,t}\right| \\
&\leq \left(\rho_n^2 - 1\right) \rho_n^{-n} \sum_{i=1}^{n-1} \rho_n^{-i} \sum_{t=1}^{i} \rho_n^{t-1} \left(\mathbb{E} u_{n,i}^2\right)^{1/2} \left(\mathbb{E} u_{n,t}^2\right)^{1/2} \\
&\leq \max_{1 \leq j \leq n} \mathbb{E} u_{n,j}^2 \left(\rho_n^2 - 1\right) \rho_n^{-n} \sum_{i=1}^{n-1} \rho_n^{-i} \sum_{t=1}^{i} \rho_n^{t-1} \\
&= \max_{1 \leq j \leq n} \mathbb{E} u_{n,j}^2 \frac{\rho_n^2 - 1}{\rho_n - 1} \rho_n^{-n} \left(n - \sum_{i=1}^{n-1} \rho_n^{-i} \right) = O\left(n \rho_n^{-n}\right)
\end{aligned}$$

showing (72) when $\rho_n^{-n} n \to 0$. When $\rho_n^{-n} n \not\to 0$ we show that, for all but finitely many n

$$\rho_n - 1 \leq c \frac{\log n}{n} \text{ for all } c > 1. \tag{74}$$

To see this, write

$$n\rho_n^{-n} = n\left\{(1 + \rho_n - 1)^{(\rho_n - 1)^{-1}}\right\}^{-n(\rho_n - 1)} \sim ne^{-n(\rho_n - 1)}$$

as $n \to \infty$ so that $n\rho_n^{-n}$ is a decreasing function of $\rho_n - 1$ for all but finitely many n. When $\rho_n - 1 = c \log n/n$ for some $c > 0$, $n\rho_n^{-n} \sim ne^{-c \log n} = n^{1-c} \to 0$ for all $c > 1$; since $n\rho_n^{-n}$ eventually decreases as $\rho_n - 1$ increases, $n\rho_n^{-n} \to 0$ whenever $\rho_n - 1 \geq c \log n/n$ eventually for some $c > 1$, which proves (74). Hence, when $\rho_n^{-n} n \not\to 0$,

$$\begin{aligned} \left(\rho_n^2 - 1\right)\rho_n^{-n} &\leq 2\rho_n^{-n}\frac{\log n}{n} = \frac{\log n}{n} o\left[\left(n\left(\rho_n - 1\right)\right)^{-2}\right] \\ &= \frac{\log n}{n} o\left[(\log n)^{-2}\right] = o\left(\frac{1}{n \log n}\right) \end{aligned}$$

since $\rho_n^{-n} = o\left[\left(n\left(\rho_n - 1\right)\right)^{-k}\right]$ for all $k \in \mathbb{N}$ by Lemma A1. Therefore in order to prove (72) when $\rho_n^{-n} n \not\to 0$, it is sufficient to show that

$$N_n' = \frac{1}{n \log n} \sum_{i=1}^{n-1} \rho_n^{-i} u_{n,i} \sum_{t=1}^{i} \rho_n^{t-1} u_{n,t} = o_p(1). \tag{75}$$

Letting $S_{n,t} := \sum_{j=1}^{t} u_{n,j}$ and using the summation by parts formula in the spirit of Phillips (1987b), we can write

$$\begin{aligned} N_n' &= \frac{1}{n \log n} \sum_{i=1}^{n-1} \rho_n^{-i} u_{n,i} \sum_{t=1}^{i} \rho_n^{t-1} \Delta S_{n,t} \\ &= \frac{1}{n \log n} \sum_{i=1}^{n-1} u_{n,i} S_{n,i} - \frac{1}{n \log n}\left(\rho_n - 1\right) \sum_{i=1}^{n-1} \rho_n^{-i} u_{n,i} \sum_{t=1}^{i} S_{n,t} \rho_n^{t-1}. \end{aligned} \tag{76}$$

The first term on the right is $O_p\left((\log n)^{-1}\right)$ since

$$\frac{1}{n} \sum_{i=1}^{n-1} u_{n,i} S_{n,i} = \frac{1}{n} \sum_{i=1}^{n-1} u_{n,i}^2 + \frac{1}{n} \sum_{i=1}^{n-1} u_{n,i} S_{n,i-1} = O_p(1)$$

by Lemma 1 and the standard result $n^{-1}\sum_{i=1}^{n-1} S_{n,i-1}u_{n,i} \to_d \int_0^1 B(t)\,dB(t)$ (Phillips (1987a)) where is a Brownian motion with variance $C(1)^2\sigma^2$ arising from the FCLT

$$n^{-1/2}\sum_{i=1}^{\lfloor nt\rfloor} u_{n,i} \Rightarrow B(t) \quad \text{on } D[0,1] \tag{77}$$

(see e.g. Jacod and Shiryaev (2003) for the triangular array formulation). For the second term of (76), (74) implies that it is enough to show that

$$N_n'' = \frac{1}{n^2}\sum_{i=1}^{n-1}\rho_n^{-i}u_{n,i}\sum_{t=1}^{i} S_{n,t}\rho_n^{t-1} = o_p(1). \tag{78}$$

Denoting $\tau_n := (\rho_n - 1)^{-1}$ for brevity, write

$$\begin{aligned}
N_n'' &= \frac{1}{n^2}\left(\sum_{i=1}^{n-1}\rho_n^{-i}u_{n,i}\sum_{t=1}^{i} S_{n,t-1}\rho_n^{t-1} + O(\rho_n - 1)\sum_{i=1}^{n-1}u_{n,i}^2\right)\\
&= \frac{1}{n^2}\sum_{i=2}^{n}\rho_n^{-(i-1)}u_{n,i-1}\int_0^{i-1} S_{n,\lfloor t\rfloor}\rho_n^{\lfloor t\rfloor}dt + O_p\left(n^{-1}\right)\\
&= \frac{\tau_n}{n^2}\sum_{i=2}^{n}\rho_n^{-i-1}\left(\int_0^{(i-1)\tau_n^{-1}} S_{n,\lfloor \tau_n t\rfloor}\rho_n^{\lfloor \tau_n t\rfloor}dt\right)\Delta S_{n,i-1} + O_p\left(n^{-1}\right)\\
&= \frac{\tau_n}{n^2}\int_1^{n}\rho_n^{-\lfloor r\rfloor}\int_0^{\lfloor r\rfloor\tau_n^{-1}} S_{n,\lfloor \tau_n t\rfloor}\rho_n^{\lfloor \tau_n t\rfloor}dt\,dS_{n,\lfloor r\rfloor} + O_p\left(n^{-1}\right)\\
&= \frac{\tau_n^2}{n^2}I_n + O_p\left(n^{-1}\right)
\end{aligned}$$

where the weak convergence theory of Kurtz and Protter (1991) gives

$$\begin{aligned}
I_n &= \int_{\tau_n^{-1}}^{n\tau_n^{-1}}\rho_n^{-\lfloor \tau_n r\rfloor}\left(\int_0^{\lfloor \tau_n r\rfloor\tau_n^{-1}}\frac{S_{n,\lfloor \tau_n t\rfloor}}{\tau_n^{1/2}}\rho_n^{\lfloor \tau_n t\rfloor}dt\right)d\left(\frac{S_{n,\lfloor \tau_n r\rfloor}}{\tau_n^{1/2}}\right)\\
&\to_d \int_0^{\infty} e^{-r}\left(\int_0^{r} e^{t}B(t)\,dt\right)dB(r).
\end{aligned}$$

It is easy to see that the limit stochastic integral has zero mean and finite variance, so $I_n = O_p(1)$ and $N_n'' = O_p\left(n^{-2}(\rho_n - 1)^{-2}\right) = o_p(1)$ as required. This completes the proof of (72) and (23).

The argument for (24) is standard: the recursion in (1) gives the identity

$$\begin{aligned}\left(\rho_n^2-1\right)\sum_{t=1}^{n}\xi_{n,t-1}^2 &= \xi_{n,n}^2-2\rho_n\sum_{t=1}^{n}\xi_{n,t-1}u_{n,t}-\sum_{t=1}^{n}u_{n,t}^2\\ &= \rho_n^{2n}\left(\rho_n^2-1\right)^{-1}\xi_n^2+O_p\left[\left(\rho_n^2-1\right)^{-1}\rho_n^n\right]+O_p(n)\end{aligned}$$

by (23) and Lemma 1. Since $\xi_n^2=O_p(1)$ exactly and $n\left(\rho_n^2-1\right)\rho_n^{-n}\to 0$ by Lemma A1, we conclude that $\left(\rho_n^2-1\right)^2\rho_n^{-2n}\sum_{t=1}^{n}\xi_{n,t-1}^2=\xi_n^2+o_p(1)$. Now (70) gives

$$\left(\rho_n^2-1\right)^2\rho_n^{-2n}\sum_{t=1}^{n}x_{n,t-1}^2=\xi_n^2+\left(\rho_n^2-1\right)^2\rho_n^{-2n}\left\{2X_{n,0}\sum_{t=1}^{n}\xi_{n,t-1}\rho_n^{t-1}+X_{n,0}^2\frac{\rho_n^{2n}}{\rho_n^2-1}\right\}+O_p\left(\rho_n^{-n}\right)$$

since $\sum_{t=1}^{n}\xi_{n,t-1}=(1+o_p(1))\left(\rho_n-1\right)^{-1}\xi_{n,n}=O_p\left(\left(\rho_n^2-1\right)^{-2}\rho_n^n\right)$. Now

$$\sum_{t=1}^{n}\xi_{n,t-1}\rho_n^{t-1}=\sum_{t=1}^{n}\sum_{i=1}^{t-1}\rho_n^{2(t-1)-i}u_{n,i}=\sum_{t=0}^{n-1}\rho_n^{2t}\sum_{i=1}^{n-1}\rho_n^{-i}u_{n,i}+\sum_{t=1}^{n}\sum_{i=t}^{n-1}\rho_n^{2(t-1)-i}u_{n,i}$$

and $\left(\rho_n^2-1\right)^2\rho_n^{-2n}\left\|\sum_{t=1}^{n}\sum_{i=t}^{n-1}\rho_n^{2(t-1)-i}u_{n,i}\right\|_{L_1}=O\left(\rho_n^{-n}\right)$ implies that

$$\begin{aligned}\left(\rho_n^2-1\right)^2\rho_n^{-2n}\sum_{t=1}^{n}x_{n,t-1}^2 &= \xi_n^2+2\left(\rho_n^2-1\right)^{1/2}X_{n,0}\xi_n+\left(\rho_n^2-1\right)X_{n,0}^2+O_p\left(\rho_n^{-n}\right)\\ &= \left[\xi_n+\left(\rho_n^2-1\right)^{1/2}X_{n,0}\right]^2+o_p(1)\end{aligned}$$

and (24) follows by (17).

Proof of Theorem 1.

(a) Let $\rho_n\to\rho\in[1,\infty)$.

By (17),

$$\left(\rho_n^2-1\right)^{1/2}\rho_n^{-n}x_{n,n}=\mathbf{X}_n=\xi_n+\left(\rho_n^2-1\right)^{1/2}X_{n,0}+o_p(1). \qquad (79)$$

We begin by proving

$$\left(\rho_n-1\right)^{3/2}\rho_n^{-n}\sum_{t=1}^{n}x_{n,t-1}=\mathbf{X}_n+o_p(1),\quad \bar{x}_{n,n-1}=O_p\left(n^{-1}\left(\rho_n-1\right)^{-3/2}\rho_n^n\right). \qquad (80)$$

Summing (2) over $t \in \{1, ..., n\}$ gives

$$\begin{aligned}
(\rho_n - 1)^{3/2} \rho_n^{-n} \sum_{t=1}^{n} x_{n,t-1} &= \left(\rho_n^2 - 1\right)^{1/2} \rho_n^{-n} \left\{ x_{n,n} - x_{n,0} - n\mu\left(1 - \rho_n\right) - \sum_{t=1}^{n} u_{n,t} \right\} \\
&= \left(\rho_n^2 - 1\right)^{1/2} \rho_n^{-n} x_{n,n} + O_p\left[\rho_n^{-n} n \left(\rho_n - 1\right)\right] \\
&= \mathbf{X}_n + o_p(1)
\end{aligned}$$

showing the first part; the second part follows from the first since $\mathbf{X}_n = O_p(1)$. Next, we show that

$$\left(\rho_n^2 - 1\right)^2 \rho_n^{-2n} n \bar{x}_{n,n-1}^2 = o_p(1) \quad \text{and} \quad \left(\rho_n^2 - 1\right) \rho_n^{-n} n \bar{x}_{n,n-1} \bar{u}_n = o_p(1). \quad (81)$$

For the first part, (80) implies that $\left(\rho_n^2 - 1\right)^2 \rho_n^{-2n} n \bar{x}_{n,n-1}^2 = O_p\left(n^{-1}\left(\rho_n - 1\right)^{-1}\right)$; for the second part, $\bar{u}_n = O_p\left(n^{-1/2}\right)$ by the CLT, so (**??**) implies that

$$\left(\rho_n^2 - 1\right) \rho_n^{-n} n \bar{x}_{n,n-1} \bar{u}_n = O_p\left(\left(\rho_n - 1\right)^{-1/2}\right) \bar{u}_n = O_p\left(n^{-1/2}\left(\rho_n - 1\right)^{-1/2}\right).$$

For the OLS estimator, we may write

$$\begin{aligned}
\left(\rho_n^2 - 1\right)^{-1} \rho_n^n \left(\hat{\rho}_n - \rho_n\right) &= \frac{\left(\rho_n^2 - 1\right) \rho_n^{-n} \left(\sum_{t=1}^{n} x_{n,t-1} u_{n,t} - n \bar{x}_{n,n-1} \bar{u}_n\right)}{\left(\rho_n^2 - 1\right)^2 \rho_n^{-2n} \left(\sum_{t=1}^{n} x_{n,t-1}^2 - n \bar{x}_{n,n-1}^2\right)} \\
&= \frac{\left(\rho_n^2 - 1\right) \rho_n^{-n} \sum_{t=1}^{n} x_{n,t-1} u_{n,t}}{\left(\rho_n^2 - 1\right)^2 \rho_n^{-2n} \sum_{t=1}^{n} x_{n,t-1}^2} + o_p(1) \\
&= \frac{\mathbf{X}_n \mathbf{Y}_n}{\mathbf{X}_n^2} + o_p(1) = \frac{\mathbf{Y}_n}{\mathbf{X}_n} + o_p(1) \qquad (82)
\end{aligned}$$

where the second line follows by (81) and the third line by Lemma 4.

When $\rho_n \to 1$, $\left(\rho_n^2 - 1\right)^{1/2} X_{n,0} \to_p 0$, so Lemma 2 implies that

$$\left(\rho_n^2 - 1\right)^{-1} \rho_n^n \left(\hat{\rho}_n - \rho_n\right) = \frac{\mathbf{Y}_n}{\mathbf{X}_n} + o_p(1) = \frac{\tilde{Y}_n(1)}{\tilde{X}_n(1)} + o_p(1) \to_d \mathcal{C}$$

where convergence in distribution follows by (21). This proves part (i) when $\rho \geq 1$.

In the explosive case $\rho_n \to \rho > 1$, (82) and Lemma 3(iii) imply that $\left|\hat{\rho}_n - \rho_n\right| = O_p\left(\rho_n^{-n}\right)$. To prove that ξ_n and $\mathbf{Y}_n$ are asymptotically uncorrelated, letting

$$\bar{G}_{n,0} = G_{n,0} - \left(\rho^2 - 1\right)^{1/2} X_{n,0} = \left(\rho^2 - 1\right)^{1/2} \sum_{t=1}^{n} \rho^{-t} \sum_{j=0}^{\infty} c_{n,j+t} e_{n,-j},$$

(63) and (17) imply that

$$\begin{aligned}\xi_n &= \tilde{X}_n'(\rho) + \bar{G}_{n,0} + o_p(1) \\ \tilde{X}_n'(\rho) &= \left(\rho^2 - 1\right)^{1/2} \sum_{j=0}^{n-1} \rho^{-j} c_{n,j} \sum_{t=1}^{\lfloor n/2 \rfloor} \rho^{-t} e_{n,t} \qquad (83)\end{aligned}$$

because $\left\| \sum_{j=0}^{n-1} \rho^{-j} c_{n,j} \sum_{t>\lfloor n/2 \rfloor} \rho^{-t} e_{n,t} \right\|_{L_1} = O\left(\rho^{-\lfloor n/2 \rfloor}\right)$; also

$$\begin{aligned}\mathbf{Y}_n &= \left(\rho^2 - 1\right)^{1/2} \sum_{j=0}^{\infty} c_{n,j} \sum_{t=1}^{n} \rho^{-t} e_{n,n-t-j+1} \\ &= \left(\rho^2 - 1\right)^{1/2} \sum_{j=0}^{\lfloor n/4 \rfloor} c_{n,j} \sum_{t=1}^{\lfloor n/4 \rfloor} \rho^{-t} e_{n,n-t-j+1} + o_p(1) \qquad (84)\end{aligned}$$

because $\left\| \sum_{j=0}^{\infty} c_{n,j} \sum_{t>\lfloor n/4 \rfloor} \rho^{-t} e_{n,n-t-j+1} \right\|_{L_1} = O\left(\rho^{-\lfloor n/4 \rfloor}\right)$ and

$$\left\| \sum_{j>\lfloor n/4 \rfloor} c_{n,j} \sum_{t=1}^{\lfloor n/4 \rfloor} \rho^{-t} e_{n,n-t-j+1} \right\|_{L_1} \leq \|e_{n,1}\|_{L_2} \frac{1}{\rho - 1} \sum_{j>\lfloor n/4 \rfloor} |c_{n,j}| = O\left(n^{-\delta}\right).$$

Comparing (83) and (84), we conclude that $\xi_n = \xi_n' + o_p(1)$ and $\mathbf{Y}_n = \mathbf{Y}_n' + o_p(1)$, where $\mathbb{E}\left(\xi_n' \mathbf{Y}_n'\right) = 0$ so ξ_n and $\mathbf{Y}_n$ are asymptotically uncorrelated. Finally, denoting by $\gamma_{u_n}(h) = \mathbb{E}\left(u_{n,t} u_{n,t-h}\right)$ the autocovariance function of $(u_{n,t})$, it is easy to see that

$$\begin{aligned}var\left(\mathbf{Y}_n\right) &= \left(\rho_n^2 - 1\right) \sum_{t=1}^{n} \sum_{s=1}^{n} \rho_n^{-(n-t+1)-(n-s+1)} \gamma_{u_n}(t-s) \\ &= \left(\rho_n^2 - 1\right) \sum_{t=1}^{n} \sum_{s=1}^{n} \rho_n^{-t-s} \gamma_{u_n}(s-t) \\ &= var\left(\xi_n\right)\end{aligned}$$

so that the zero mean random variables $\mathbf{Y}_n$ and ξ_n have the same variance for all n.

(b) Let $\rho_n \to \rho \in (-\infty, -1]$.

We first prove that the sequence $(\check{u}_{n,t})$ in (30) satisfies Assumption 2. Note that $(\check{e}_{n,t}, \mathcal{F}_{n,t})_{t\in\mathbb{Z}}$ is a martingale difference sequence with the same conditional variance $\mathbb{E}_{\mathcal{F}_{n,t-1}}\left(\check{e}_{n,t}^2\right) = \sigma_{n,t}^2$ as $e_{n,t}$; in particular, (8) holds for $\check{e}_{n,t}$. Since the requirements on uniform integrability of the squared process, (6) and (4) are the same for $\check{e}_{n,t}$ and $e_{n,t}$, Assumption 2 is satisfied with $e_{n,t}$ replaced by $\check{e}_{n,t}$. Since $|\check{c}_{n,j}| = |c_{n,j}|$ and $\check{c}_{n,j}^2 = c_{n,j}^2$, the only thing left to prove is (5), namely that the limit of the sequence $\left\{\sum_{j=0}^{\infty} |\rho|^{-j}\, \check{c}_{n,j} : n \in \mathbb{N}\right\}$ exists and is non-zero. Since $|\rho| = -\rho$,

$$\sum_{j=0}^{\infty} |\rho|^{-j}\, \check{c}_{n,j} = \sum_{j=0}^{\infty} (-\rho)^{-j} (-1)^{-j} c_{n,j} = \sum_{j=0}^{\infty} \rho^{-j} c_{n,j} \to C(\rho) \neq 0$$

by the original condition (5) of Assumption 2. This completes the proof that $\check{u}_{n,t} = (-1)^{-t} u_{n,t}$ satisfies Assumption 2. Hence, (27) and (28) hold by (26) and Lemma 4.

Since

$$\check{x}_{n,t} = \check{X}_{n,t} + |\rho_n|^t X_{n,0} + (-1)^{-t} \mu \tag{85}$$

the sample mean $n^{-1}\sum_{t=1}^{n} \check{x}_{n,t-1}$ has the same order of magnitude as $\bar{x}_{n,n-1}$ in (80) with ρ_n replaced by $|\rho_n|$ and

$$\begin{aligned}\left(\rho_n^2 - 1\right)^{1/2} |\rho_n|^{-n} \check{x}_{n,n} &= \left(\rho_n^2 - 1\right)^{1/2} |\rho_n|^{-n} \check{X}_{n,n} + \left(\rho_n^2 - 1\right)^{1/2} X_{n,0} + o_p(1) \\ &= \check{\xi}_n + \left(\rho_n^2 - 1\right)^{1/2} X_{n,0} = \check{\mathbf{X}}_n + o_p(1).\end{aligned}$$

We conclude that

$$\begin{aligned}\left(\rho_n^2 - 1\right)^{-1} |\rho_n|^n (\hat{\rho}_n - \rho_n) &= \frac{(\rho_n^2 - 1)\,|\rho_n|^{-n} \left(\sum_{t=1}^{n} x_{n,t-1}u_{n,t} - n\bar{x}_{n,n-1}\bar{u}_n\right)}{(\rho_n^2 - 1)^2 \rho_n^{-2n} \left(\sum_{t=1}^{n} x_{n,t-1}^2 - n\bar{x}_{n,n-1}^2\right)} \\ &= \frac{-(\rho_n^2 - 1)\,|\rho_n|^{-n} \left(\sum_{t=1}^{n} \check{x}_{n,t-1}\check{u}_{n,t} - n^{-1}\sum_{t=1}^{n} \check{x}_{n,t-1}\sum_{j=1}^{n} \check{u}_{n,j}\right)}{(\rho_n^2 - 1)^2 \rho_n^{-2n} \left(\sum_{t=1}^{n} \check{x}_{n,t-1}^2 - n^{-1}\left(\sum_{t=1}^{n} \check{x}_{n,t-1}\right)^2\right)} \\ &= -\frac{(\rho_n^2 - 1)\,|\rho_n|^{-n} \sum_{t=1}^{n} \check{x}_{n,t-1}\check{u}_{n,t}}{(\rho_n^2 - 1)^2 \rho_n^{-2n} \sum_{t=1}^{n} \check{x}_{n,t-1}^2} + o_p(1) \\ &= -\frac{\check{\mathbf{Y}}_n}{\check{\mathbf{X}}_n} + o_p(1).\end{aligned} \tag{86}$$

When $\rho_n \to -1$, $(\rho_n^2 - 1)^{1/2} X_{n,0} = o_p(1)$ and Lemma 2 and (21) imply that

$$\left[\check{\mathbf{Y}}_n, \check{\mathbf{X}}_n\right] = \left[\check{X}_n(-1), \check{Y}_n(-1)\right] + o_p(1) \to_d \left[\check{X}(-1), \check{Y}(-1)\right] \tag{87}$$

where are defined in (31) with $C(\rho) = C(-1)$ and $\check{X}(-1), \check{Y}(-1)$ are independent $N\left(0, \sigma^2 C(-1)^2\right)$ random variables. Combining (86) and (87) implies that

$$\left(\rho_n^2 - 1\right)^{-1} |\rho_n|^n (\hat{\rho}_n - \rho_n) \to_d \mathcal{C} \text{ for } \rho \leq -1$$

(using the symmetry of the Cauchy distribution) and completes the proof of part (i).

In the explosive case $\rho_n \to \rho < -1$, (86) and Lemma 3(ii) applied to $\check{\mathbf{X}}_n$ and $\check{\mathbf{Y}}_n$ (this is permissible since $|\rho_n| \to |\rho| > 1$ and $(\check{u}_{n,t})$ in (30) satisfies Assumption 2) imply that $|\hat{\rho}_n - \rho_n| = O_p\left(|\rho_n|^{-n}\right)$. The argument leading to (83) and (84) also applies to $\check{\xi}_n$ and $\check{\mathbf{Y}}_n$ and shows that $\check{\xi}_n = \check{\xi}_n' + o_p(1)$ and $\check{\mathbf{Y}}_n = \check{\mathbf{Y}}_n' + o_p(1)$ where $\mathbb{E}\check{\xi}_n' \check{\mathbf{Y}}_n' = 0$. Finally, it is easy to verify that $var\left(\check{\xi}_n\right) = var\left(\check{\mathbf{Y}}_n\right)$ in the same way as in (a).

Under Gaussianity of $(e_{n,t})$, ξ_n and $\mathbf{Y}_n$ are zero mean asymptotically independent Gaussian processes, so $(\xi_n, \mathbf{Y}_n)$ will converge in distribution if and only if the common variance of ξ_n and $\mathbf{Y}_n$

$$\begin{aligned} \mathbb{E}\xi_n^2 &= \mathbb{E}\tilde{X}_n(\rho)^2 + \left(\rho^2 - 1\right)\mathbb{E}e_{n,0}^2 \sum_{j=0}^{\infty}\left(\sum_{t=1}^{n} \rho^{-t} c_{n,j+t}\right)^2 \\ &= \left(\rho^2 - 1\right)\sigma_n^2 \left\{\frac{C(\rho)^2}{\rho^2 - 1} + \sum_{j=0}^{\infty} C_{nj}^2\right\} + o(1) \end{aligned} \quad (88)$$

converges as $n \to \infty$, i.e. if and only if the sequence $\left\{\sum_{j=0}^{\infty} C_{nj}^2 : n \in \mathbb{N}\right\}$ converges. By (83), $\mathbf{X}_n = \tilde{X}_n'(\rho) + \bar{G}_{n,0} + (\rho_n^2 - 1)^{1/2} X_{n,0} + o_p(1)$ with $\tilde{X}_n'(\rho)$ independent of $\left(\bar{G}_{n,0}, X_{n,0}\right)$, and $\tilde{X}_n'(\rho)$ and $\bar{G}_{n,0}$ converging in distribution by Gaussianity and existence of the limit $\upsilon(\rho)$ and $X_{n,0} \to_d X_0$ by Assumption 3, where X_0 is $\mathcal{F}_0$-measurable, so $X_{n,0}$ is independent of $\tilde{X}_n'(\rho)$; hence, joint convergence in distribution of $\left(\bar{G}_{n,0}, X_{n,0}\right)$ guarantees convergence in distribution of $(\mathbf{X}_n)$ and asymptotic independence of $(\mathbf{X}_n)$ and $(\mathbf{Y}_n)$ implies convergence in distribution of the ratio in (34).

Proof of Theorem 2. The OLS residuals can be written as

$$\hat{u}_{n,t} = u_{n,t} - \bar{u}_n - (\hat{\rho}_n - \rho_n)(x_{n,t-1} - \bar{x}_{n-1}), \quad (89)$$

so, since $\hat{\rho}_n - \rho_n = O_p\left(\rho_n^{-n}\left(\rho_n^2-1\right)\right)$, $\sum_{t=1}^{n}\left(x_{n,t-1}-\bar{x}_{n-1}\right)^2 = O_p\left(\rho_n^{2n}\left(\rho_n^2-1\right)^{-2}\right)$, $\sum_{t=1}^{n}\left(x_{n,t-1}-\bar{x}_{n,n-1}\right)\left(u_t-\bar{u}_n\right) = O_p\left(\rho_n^{n}\left(\rho_n^2-1\right)^{-1}\right)$ and $\bar{u}_{n,n} = O_p\left(n^{-1/2}\right)$, we obtain

$$\begin{aligned}\hat{\sigma}_n^2 &= \frac{1}{n}\sum_{t=1}^{n}\hat{u}_{n,t}^2 = \frac{1}{n}\sum_{t=1}^{n}u_{n,t}^2 - \bar{u}_n^2 + \left(\hat{\rho}_n-\rho_n\right)^2\frac{1}{n}\sum_{t=1}^{n}\left(x_{n,t-1}-\bar{x}_{n-1}\right)^2 \\ &\quad -2\left(\hat{\rho}_n-\rho_n\right)\frac{1}{n}\sum_{t=1}^{n}\left(x_{n,t-1}-\bar{x}_{n-1}\right)\left(u_{n,t}-\bar{u}_n\right) \\ &= \frac{1}{n}\sum_{t=1}^{n}u_{n,t}^2 + O_p\left(\frac{1}{n}\right)\end{aligned}$$

which, combined with Lemma 1(i) show that $\hat{\sigma}_n^2 \to_p \sigma^2 \lim_{n\to\infty}\sum_{j=0}^{\infty}c_{n,j}^2$. We now show that

$$\hat{\omega}_n^2 \to_p \omega^2 = \sigma^2 C\left(1\right)^2. \tag{90}$$

Using (89) we may write

$$\begin{aligned}\frac{1}{n}\sum_{t=h+1}^{n}\hat{u}_{n,t}\hat{u}_{n,t-h} &= \frac{1}{n}\sum_{t=h+1}^{n}u_{n,t}u_{n,t-h} - \bar{u}_n\bar{u}_{n-h} \\ &\quad -\frac{1}{n}\left(\hat{\rho}_n-\rho_n\right)\left(\sum_{t=h+1}^{n}u_{n,t}x_{n,t-h-1} + \sum_{t=h+1}^{n}x_{n,t-1}u_{n,t-h}\right) \\ &\quad +\left(\hat{\rho}_n-\rho_n\right)\left(\bar{x}_{n-1}\bar{u}_{n-h}+\bar{u}_n\bar{x}_{n-h-1}\right) \\ &\quad +\left(\hat{\rho}_n-\rho_n\right)^2\left(\frac{1}{n}\sum_{t=h+1}^{n}x_{n,t-1}x_{n,t-h-1} - \bar{x}_{n-1}\bar{x}_{n-h-1}\right).\end{aligned} \tag{91}$$

Since $\bar{u}_{n,n}\bar{u}_{n,n-h} = O_p\left(n^{-1}\right)$ and (80) implies that

$$\left(\hat{\rho}_n-\rho_n\right)\left(\bar{x}_{n-1}\bar{u}_{n-h}+\bar{u}_n\bar{x}_{n-h-1}\right) = O_p\left(n^{-3/2}\left(\rho_n-1\right)^{-1/2}\right) = O_p\left(n^{-1}\right)$$

and $\left(\hat{\rho}_n-\rho_n\right)^2\bar{x}_{n-1}\bar{x}_{n-h-1} = O_p\left(n^{-2}\left(\rho_n-1\right)^{-1}\right) = O_p\left(n^{-1}\right)$, all terms involving $\bar{u}_{n,n}$ and $\bar{x}_{n,n-1}$ in (91) are $O_p\left(n^{-1}\right)$ uniformly over h. Also,

$$\begin{aligned}\left(\hat{\rho}_n-\rho_n\right)^2\frac{1}{n}\left|\sum_{t=h+1}^{n}x_{n,t-1}x_{n,t-h-1}\right| &\leq \left(\hat{\rho}_n-\rho_n\right)^2\frac{1}{n}\left(\sum_{t=h+1}^{n}x_{n,t-1}^2\right)^{1/2}\left(\sum_{t=1}^{n-h}x_{n,t-1}^2\right)^{1/2} \\ &\leq \left(\hat{\rho}_n-\rho_n\right)^2\frac{1}{n}\sum_{t=1}^{n}x_{n,t-1}^2 = O_p\left(\frac{1}{n}\right)\end{aligned}$$

uniformly over h,

$$\begin{aligned}\frac{1}{n}\left|\hat{\rho}_n-\rho_n\right|\left|\sum_{t=h+1}^{n}u_{n,t}x_{n,t-h-1}+\sum_{t=h+1}^{n}x_{n,t-1}u_{n,t-h}\right| &\leq \frac{1}{n}\left|\hat{\rho}_n-\rho_n\right|\left(\sum_{t=1}^{n}x_{n,t-1}^2\right)^{1/2}\left(\sum_{t=1}^{n}u_{n,t}^2\right)^{1/2}\\ &= \frac{1}{n}O_p\left(n^{1/2}\right)=O_p\left(n^{-1/2}\right)\end{aligned}$$

uniformly over h. We conclude that

$$\max_{1\leq h\leq M}\left|\frac{1}{n}\sum_{t=h+1}^{n}\hat{u}_{n,t}\hat{u}_{n,t-h}-\frac{1}{n}\sum_{t=h+1}^{n}u_{n,t}u_{n,t-h}\right|=O_p\left(n^{-1/2}\right) \tag{92}$$

which implies that

$$\begin{aligned}\frac{1}{n}\sum_{h=1}^{M}K\left(\frac{h}{M}\right)\left|\sum_{t=h+1}^{n}\hat{u}_{n,t}\hat{u}_{n,t-h}-\sum_{t=h+1}^{n}u_{n,t}u_{n,t-h}\right| &\leq O_p\left(n^{-1/2}\right)\sum_{h=1}^{M}K\left(\frac{h}{M}\right)\\ &= O_p\left(Mn^{-1/2}\right).\end{aligned}$$

Since $Mn^{-1/2}\to 0$ by assumption, the law of large number of Lemma 1(ii) implies that

$$\begin{aligned}\frac{1}{n}\sum_{h=1}^{M}K\left(\frac{h}{M}\right)\sum_{t=h+1}^{n}\hat{u}_{n,t}\hat{u}_{n,t-h} &= \frac{1}{n}\sum_{h=1}^{M}K\left(\frac{h}{M}\right)\sum_{t=h+1}^{n}u_{n,t}u_{n,t-h}+O_p\left(Mn^{-1/2}\right)\\ &= \lim_{n\to\infty}\sum_{h=1}^{\infty}\gamma_{u_n}(h)+o_p(1)\end{aligned}$$

completing the proof of (90).

The approximations of (82) and (86) and (90) yield the following for the t-statistic in (35):

$$T_n\left(\rho_n\right)=\left(1+o_p(1)\right)\frac{1}{\omega}\left(\mathbf{Y}_n\mathbf{1}\left\{\rho\geq 1\right\}-\check{\mathbf{Y}}_n\mathbf{1}\left\{\rho\leq -1\right\}\right). \tag{93}$$

In the mildly explosive case of part (i), $|\rho|=1$, Lemma 2 implies that both $\mathbf{Y}_n$ and $\check{\mathbf{Y}}_n$ are asymptotically normally distributed with mean 0 and common asymptotic variance equal to $\omega^2=\sigma^2C(1)^2$ by (31). Part (i) of the theorem follows by (93). For part (ii), (93) still holds but the asymptotic variance of $\mathbf{Y}_n$ (and of $\check{\mathbf{Y}}_n$) is given by (88) and is no longer to ω^2; we may recover the correct asymptotic variance only when the (Gaussian) sequence $(u_{n,t})$ is independent, i.e. when $c_{n,j} = 0$ for all $j \geq 1$ and $\omega^2 = \sigma^2$. In this case, $T_n(\rho_n) \to_d N(0,1)$ follows immediately from (93) with $\mathbf{Y}_n$ and $\check{\mathbf{Y}}_n$ being identically distributed zero mean Gaussian sequences for each n with (88) giving

$$var(\mathbf{Y}_n) = \sigma_n^2 C(\rho)^2 = \sigma_n^2 \lim_{n\to\infty} \left(c_{n,0} + \sum_{j=1}^{\infty} \rho^{-j} c_{n,j} \right)^2 = \sigma_n^2 \lim_{n\to\infty} c_{n,0}^2 \to \sigma^2$$

completing the proof of the theorem.

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