STAFF REPORTS

NO. 1122
SEPTEMBER 2024

REVISED
OCTOBER 2024

Financial System Architecture and Technological Vulnerability

Selman Erol | Michael Junho Lee

FEDERAL RESERVE BANK of NEW YORK

Financial System Architecture and Technological Vulnerability
Selman Erol and Michael Junho Lee
Federal Reserve Bank of New York Staff Reports, no. 1122
September 2024; revised October 2024
https://doi.org/10.59576/sr.1122

Abstract

This paper presents a framework to study the technological resiliency of financial system architecture. Financial market infrastructures, or platforms, compete with services that play critical functions along various stages in the lifecycle of a trade, and make investments in technological resiliency to guard against attackers seeking to exploit system weaknesses. Platforms' financial network effects attenuate competition between platforms on security. Exposure to vulnerabilities is magnified in the presence of strategic adversaries. Private provision of technological resiliency is generally sub-optimal, with over- and under-investment in security depending on market structure. Vulnerabilities evolve over the maturity of a financial system, but there generically exists a tipping point at which technological resiliency diverges from optimal and creates technological drag on the financial system. We find supportive evidence in tri-party repo settlement: the exit of duopolist resulted in a significant drop in IT-related investment by the sole provider, even as peer firms ramp up investment.

JEL classification: D82, D86, D47, G29
Key words: financial market architecture, technological vulnerability, cyber risk, financial stability

---

Lee: Federal Reserve Bank of New York (email: michael.j.lee@ny.frb.org). Erol: Carnegie Mellon University, David A. Tepper School of Business (email: erol@cmu.edu). The authors thank Sakshee Singh for exceptional research assistance.

1 Introduction

The architecture of the financial system is significantly concentrated. The pace of financial markets, along with financial regulation, have increased the importance of economizing on liquidity. Financial architecture has co-evolved with financial activity to develop arrangements to economize on liquidity and digest the higher velocity and volume of trade. Concurrently, the underlying infrastructure that facilitates trade, clearing, and settlement has undergone dramatic consolidation (Figure 1). At heart are financial market infrastructures (FMIs), including exchanges, clearinghouses, and settlement providers, banks, and other non-bank financial institutions, which offer services that generate value through network effects, and thus tend to be natural monopolies (Pirrong 2011). Appropriately, the systemic importance of these key financial institutions have garnered significant attention, particularly with respect to the concentration of financial risk (Duffie 2014).

Disruptions in the non-redundant operations of key financial institutions can lead directly to dysfunction and broad failures across the entire financial system. This paper studies technological vulnerabilities in the financial system, an under-explored determinant of financial stability, which arise from dependence and concentration of the underlying architecture. Notably, risks arising from cyberattacks have steadily risen to the forefront of financial stability considerations. Cyberattacks on financial institutions are more disruptive and monetarily more profitable, making them prime targets for cyber adversaries (Aldasoro, Gambacorta, Giudici and Leach 2020). Significant increases in the cyber criminal enterprises, nation-state actors, and geopolitical tensions have contributed to growing concern and actual losses in the financial sector. Despite the broad acknowledgment of policy makers and industry about the systemic risks posed by cyber risk, there is little work on the mechanisms and to assess whether and when the technological resiliency of the financial system is adequate.

We propose a theory of financial system architecture, in which the market structure and its technological vulnerabilities of the underlying infrastructure are endogenously determined by competition between these financial intermediaries, or *platforms*. Platforms facilitate financial transactions between their network of users and safeguard their system from technological risks, including those arising from cyber adversaries seeking to infiltrate its systems and data. Platforms offer enhanced trading and liquidity-saving mechanisms that scale with the size of their networks.

We draw three main insights. First, we establish an inverse relation between security and key financial market characteristics, and show that the private provision of technological resiliency is generally sub-optimal, with the possibility of both over- and under-investment.

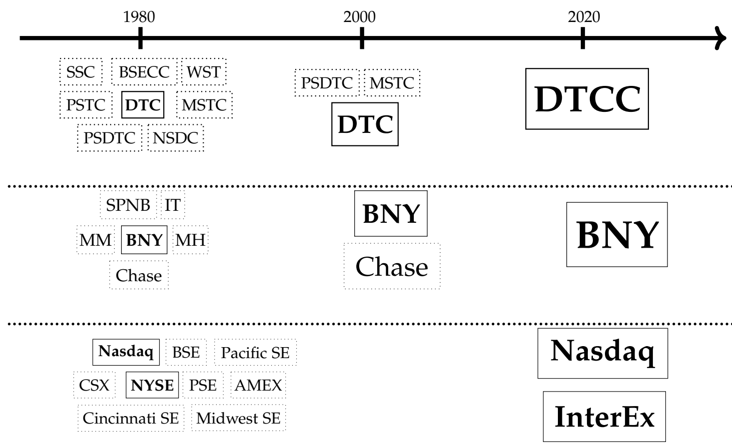

Figure 1: **Consolidation of financial system architecture.**

Second, in a dynamic environment where market structure and vulnerability of a financial system evolves over time, we show that the financial system gravitates towards consolidation, and identify a ”tipping point” at which point long-term divergence occurs between the private and socially optimal level of technological resiliency. Third, these vulnerabilities translate into a *technological drag* on the financial system, characterized by market-wide losses resulting from susceptibility to cyber attacks, and materially impacts aggregate trading patterns and capacity. We find empirical evidence in support of this link between market structure and technological investment in the tri-party repo settlement.

In our model, traders seek each other to enter financial transactions that generate gains from trade when successfully settled. Settlement requires traders to pledge collateral, making liquidity a scarce resource. Traders individually choose a financial platform to join. On the financial side, platforms increase financial efficiency in two ways: first, they improve matching efficiency by increasing the set of latent trading opportunities that are realized; second, they offer liquidity-saving through multilateral netting between traders on their platforms. Both enhancements create natural economies of scale.

On the technological side, platforms make investments to strengthen technological resiliency and reduce the possibility of disruptions to their operations. In particular, platforms must defend against cyber adversaries, who attempt to compromise their systems and data, i.e. orchestrate a cyberattack. Attackers choose the intensity with which they attempt to infiltrate each platform. Our model is sufficiently flexible to capture a comprehensive set of cyber attack types that compromise the availability, integrity, and confidentiality of platforms' systems or data (Curti, Gerlach, Kazinnik, Lee and Mihov 2023). Our framework allows us to map attacks to disruptions in trade and settlement, and even lock up liquidity. Platforms choose security investments that enhance technological resiliency, particularly against risks posed by cyber adversaries.

We first show that attack intensity is proportional to a platform's technological vulnerability and the size of the platform. Consequently, weaknesses are amplified through the strategic behavior of cyber adversaries. Thus, in the model, cyber risk is distinguished by the strategic interaction between platforms and attackers. In contrast to other forms of operational risk, cyber risk is affected by the intensity of the attack.

Traders choose platforms based on two dimensions: financial value and technological value. The relative financial value of platforms consist of trade surplus arising from platform membership, which depends on match efficiency, netting benefits, and network size. The relative technological value of platforms is based on expected losses arising from technological vulnerability, which depend on trade-related losses and collateral-related losses, which are realized in the event of a successful attack.

Platforms compete to broaden their networks, and the resulting competition for traders determines equilibrium investment for both platforms. In equilibrium, cyberattacks introduce "cyber diseconomies," i.e. larger platforms are exposed to greater adversarial attack intensity due to being higher value targets. Still, in equilibrium, larger platforms garner larger shares by making sufficient security investments to ensure that they broaden their share of the market. We map key primitive features of the financial market directly to technological vulnerability. In general, we find that technological vulnerability rises with the financial value of platforms and relaxes as markets become more concentrated.

We find that a public-private wedge in technological resilience generically arises. The socially-optimal level of investment explicitly takes into account the expected losses from vulnerabilities, which incorporates both the size of the platform and losses conditional on a successful attack. In contrast, private investments are driven by competition for users, which hinges on the *marginal* trader's choice. As a result, the public-private wedge can go both ways – when competition is relatively even, over-investment occurs, as platforms shore up technological resilience to extend their shares. When one platform is sufficiently dominant, however, the public-private wedge is negative, resulting in excessively high probabilities of attacks.

The interplay between market structure and technological resiliency motivates a dynamic extension. We consider a dynamic environment, in which platform concentration and technological vulnerability co-evolve over time. Initially, competition for member-traders intensifies, as traders become more pivotal. However, once sufficient concentration is established, traders' platform choice is predominantly determined by network benefits over concerns over technological vulnerability, and the dominant platform eases investments, resulting in under-investment. This under-investment results in *technological drag* on efficiency, characterized by excessive cyber disruptions that significantly affect normal functioning of financial systems.

Altogether, our analysis shows that as a financial system matures, its underlying architecture can become concentrated and become technologically vulnerable. Over time, all traders in the market are left with a dominant infrastructure with no other viable option. These circumstances could fundamentally influence both the nature and volume of trading itself. To examine this, we extend the model to endogenize trading. Traders internalize technological risks by trading defensively: they shift away from lucrative but liquidity-intensive trading strategies, thereby lowering expected losses associated with liquidity and collateral lock-up in the event of a cyberattack. Moreover, as financial system architecture becomes more concentrated and vulnerable, the aggregate trading capacity diminishes. Both changes in trading activity serve to magnify technological drag.

A resounding implication is that in the absence of regulation, technological vulnerabilities rise when markets become consolidated. Broad consolidation throughout the financial system points to the possibility for under-provision of technological resiliency. We test this empirical implication by analyzing technological expenditure of settlement banks in the tri-party repo market. Specifically, we exploit the exit of a duopolist in tri-party repo settlement in 2019, which resulted in a single bank solely responsible for the entire market. We find a significant drop in IT expenditure over assets in the years following the exit. Importantly, IT expenditure increases secularly for other large banks throughout the entire sample period, as documented by Modi, Pierri, Timmer, Peria and Peria (2022).

Relative to the rapid consensus between private and public sectors regarding the grave risks posed by a well-orchestrated cyber campaign against the financial system, progress on policies and regulatory guardrails to safeguard the financial system from systemic cyber risk is relatively slow. These issues are especially salient for FMIs, who are crucial, and often times provide non-redundant services, to normal financial market functioning. Our analysis highlights tradeoffs that arise between financially optimal design and technological resilience. Greater concentration in financial market infrastructure can allow for more efficient financial arrangements, including those involving collateral and liquidity. However, in the absence of regulation, private provision of technological resilience can be lacking when platforms become too concentrated. Furthermore, from a practical standpoint, our results suggest that historical performance on technological resiliency may not be a sufficient indicator for current technological resiliency, especially with industry consolidation.

Our paper relates to a growing literature that studies the impact of cyber risk on financial stability (Kashyap and Wetherilt (2019), Brando, Kotidis, Kovner, Lee and Schreft (2022)). Studies have the potential for cyber attacks to spark runs at financial institutions (Duffie and Younger 2019). Anand, Duley and Gai (2022) studies tradeoffs arising between ex-ante and ex-post investments when banks defending against cyber attacks. Eisenbach, Kovner and Lee (2022) shows that an attack on key financial institutions and FMUs can result in significant liquidity dislocations and have systemic consequences. Eisenbach, Kovner and Lee (2024) finds systemic cyber risk rises during stressed financial conditions, shedding further light on the tail-risk nature of cyber risk. Kotidis and Schreft (2023) finds supportive evidence in an actual cyber attack that hit a financial service provider and whose disruptions propagated throughout the network. In a corporate context, Crosignani, Macchiavelli and Silva (2020) studies and finds evidence of network propagation, indicating that potential cyberattacks have aggregate implications. Our paper studies how technological vulnerabilities of financial market infrastructure translate into systemic exposure to cyber attacks.

These concerns are ever more evident in the case of financial market infrastructures, whose criticality has long been recognized and shown to be prime targets (Aldasoro, Gambacorta, Giudici and Leach 2022). A focal theme has been considerations regarding risk concentration and amplifications (Duffie 2014). Our paper also rationalizes the observed concentration of the underlying financial system architecture, as described Awrey and Macey (2021), which shows that technological investments are a key driver of consolidation. We bridge the gap by exploring a less-explored dimension regarding technological vulnerabilities, and show how market structure and technological resiliency go hand-in-hand.

Our paper contributes to the literature on information and system security (e.g. Anderson (2001), Anderson and Moore (2006)). Fainmesser, Galeotti and Momot (2019) studies the collection and protection of user data and shows that the security of user data depends on the underlying business model of the firm. Brolley, Cimon and Riordan (2020) considers the provision of security when security investments are unobservable and finds under-provision to occur. Our paper dynamically links security investment to the underlying market structure and studies the relation between cyber vulnerability and the lifecycle of the financial industry.

2 Baseline Model

We start by outlining the static environment. We consider a model of a financial system in which traders enter financial transactions that generate gains from trade when successfully settled. Settlement requires traders to pledge collateral, making liquidity a scarce resource. *Platforms* are FMUs that offer trading, clearing, and settlement services. These services enhance financial efficiency in two ways: first, they improve matching efficiency by increasing the set of latent trading opportunities that are realized; second, they offer liquidity-saving, for example, through multilateral netting. Importantly, both create natural economies of scale. Traders select a platform to process transactions. A key complication is the presence of a malicious actor, or *attacker*, who attempts to compromise the system and data of platforms, e.g. orchestrate a cyber attack. Platforms choose security investments that enhance their operational resiliency, including those against cyber attacks.

Agents. There are three types of agents: platforms $p \in \{1, 2\}$, an attacker (or adversary), and a set of traders U, where the total mass of traders is $\mu \equiv |U|$. For each platform p, there is a mass of initial traders, denoted η_p, and $\eta = \eta_1 + \eta_2$. The remaining mass of traders, denoted $\lambda = \mu - \eta$, choose at most one platform to join. Let λ_p be the mass of traders that join platform p, and let the total mass of traders be denoted μ_p.

Trading. A set of bilateral trading opportunities are stochastically realized, each of size 1. For any given trade, the gross position of the trader is 1 or -1. A trade generates a surplus of β for both parties. The set of opportunities can depend on whether both traders are on the same platform, or*on-platform*, or not. We refer to trades that occur between two traders on different platforms as *off-platform*. We use $\theta \in \{p, 0\}$ to represent on- and off-platform, respectively. At the trading stage, the main benefit from on-platform trades arises from more trading opportunities being generated between traders of the same platform.

Formally, we represent the set of trading opportunities as an endogenous Erdos-Renyi graphon. Each trader has an independent trading type $t \sim U[0, 1]$. For two traders of type t and t', the probability of getting matched and the velocity of trades conditional on getting matched are encoded in the velocity function $\tau(t, t') \in [0, 1]$. Conditional on being matched, the nature of the trade is reflected by the position function $\pi(t, t') \in \{-1, 0, 1\}$, where $\pi(t, t') = 1$ if a trader of type t holds a long position, $\pi(t, t') = -1$ if she holds a short position in matches with trader type t', and $\pi(t', t) = -\pi(t, t')$. When there is no trading opportunity between the two types, $\pi(t, t') = \pi(t', t) = 0$. Together, the expected position of a trader of type t with a trader of type t' is $\tau(t, t') \cdot \pi(t, t')$.

The velocity function $\tau(t, t')$ shapes the *intensity* of trading activity. The velocity function is given by $\tau(t, t') = \tau e_\theta$, where $\theta \in \{0, p\}$, and comprised of two parts. The first component, τ, represents a market-wide trading intensity common across all traders. In the baseline model, we take the velocity to be constant τ, which is endogenized in Section 6. The second component, e_θ, represents pairwise trading intensity, which is determined by which platforms the traders are on. The probability of getting matched is e_p if the trade is on-platform, and e_0 if the trade is off-platform, where $e_p \geq e_0$. If, for example, platforms provide improved matching technologies, then $e_p > e_0$. Consequently, the velocity function $\tau(t, t') = e_\theta \tau$ is endogenously determined through individual trader's platform choices.

The position function determines the *pattern* of trading activity. We specify a simple ring structure for the position function that allows for a rich possibility of aggregate outcomes. Specifically, there are $2m$ groups of trader types: $g_1^1 = [0, \frac{1}{2m}), g_2^1 = [\frac{1}{2m}, \frac{2}{2m}), ..g_m^1 = [\frac{m-1}{2m}, \frac{1}{2}]$ and $g_1^2 = [\frac{1}{2}, \frac{m+1}{2m}), g_2^2 = [\frac{m+1}{2m}, \frac{m+2}{2m}), ..g_m^2 = [\frac{2m-1}{m}, 1]$. With some abuse of notation, let $\pi(g, g') = \pi(t, t')$ for all $t \in g$ and $t' \in g'$, where:

- For all $i = 1, 2, ..., m$ and both $k = 1, 2$

$$\pi(g_i^k, g_{i+1}^k) = \pi(g_i^k, g_{i+2}^k) = ... = \pi(g_i^k, g_{i+n_1 \texttt{ mod m}}^k) = 1;$$

- For all $i = 1, 2, ..., m$,

$$\pi(g_i^1, g_{i+1}^2) = \pi(g_i^1, g_{i+2}^2) = ... = \pi(g_i^1, g_{i+n_2 \texttt{ mod m}}^2) = 1;$$

- and $\pi(i, j) = 0$ for all remaining pairs;

for some n_1, n_2 where $n_1 < \frac{m}{2}$ and $n_2 \leq m$. Holding fixed the intensity of trade, a trader in a group realizes trading opportunities with traders belonging to $(2n_1 + n_2)$ other groups. Among these matches, $2n_1$ matches belong to a cyclic pattern and multilaterally net to zero. n_2 matches are non-cyclic with no netting.

Collateral and Liquidity. In the model, liquidity is scarce, which makes relevant the settlement stage of a trade. In particular, traders are required to post collateral in order to ensure that settlement occurs. The amount of collateral required to settle trades depends on whether trades are settled on- or off-platform. If a trade is settled bilaterally, 1 unit of collateral must be posted from both parties per trade. If both traders belong to the same platform, then traders have the option to execute and settle trades on the shared platform. Platforms provide liquidity-saving benefits to their members by multilateral netting all trades executed on the platform. Let κ be the outside return of collateral utilization, where $\kappa \in (0, \beta)$.

We provide an illustration of trade and settlement patterns in Figure 2. Every group g_i^k has n_1 long and short positions from "close" groups g_j^k, and n_2 positions with other groups $g_j^{k'}$ for $k' \neq k$. For an individual trader, the liquidity cost associated with settling these positions off-

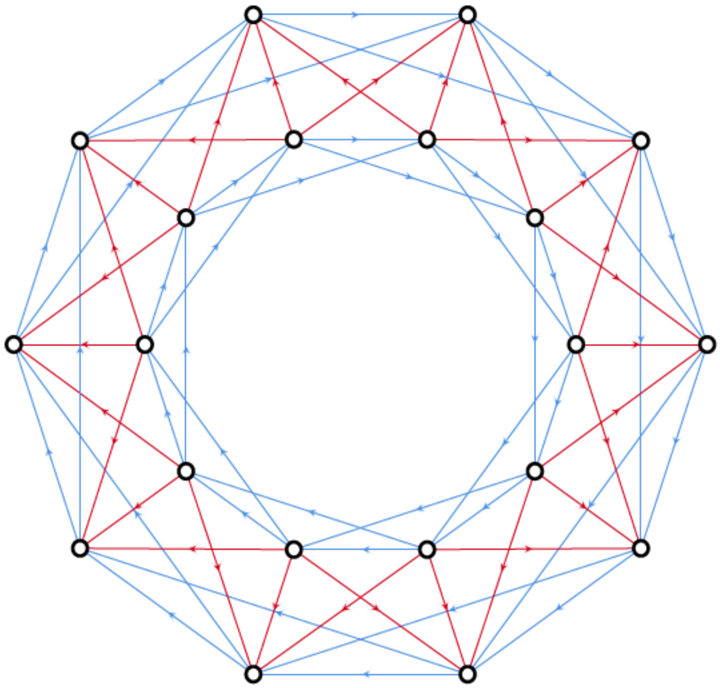

Figure 2: **Example of trading pattern by groups of traders with $m = 10$, $n_1 = 2$, and $n_2 = 3$.** Blue positions are cyclic and provide netting benefits. Red positions are bipartite and acyclic, and do not net.

platform is $(2n_1 + n_2)$. The liquidity cost associated with settling these positions on-platform is n_2.

We refer to the ratio of on-platform liquidity cost to off-platform liquidity cost, $\frac{n_2}{2n_1+n_2}$, as the *liquidity-cost ratio*. This captures the liquidity cost achieved from trading with those on-platform relative to those off-platform, and relates to the extent to which network effects arise from coordinating on a shared platform for settlement purposes. The possible values of the liquidity-cost ratio span $[0, 1]$, with smaller values indicating higher liquidity savings from on-platform trading. For simplicity, we take n_1, n_2 to be such that the liquidity-saving ratio is given by $\frac{1}{m}$.[1]

Security and Attacks. Platforms compete for users by offering a network of users (which enhances the value of being a member) bundled with its operational security (which lowers the probability of a disruption). Platform p's choose a security investment level $s_p \in [0, 1]$ at

[1]This holds for odd m and where $n_1 = \frac{m-1}{2}$ and $n_2 = 1$. These assumptions are purely expositional.

cost $c_s s_p$ to maximize their payoff, where the unit payoff from a trader is normalized to 1:[2]

$$\mu_p - c_s s_p. \tag{1}$$

Security investment increases the resiliency of the platform to malicious actors intent on disrupting and extracting value from the platform. There is a representative attacker who chooses the intensity at which it attempts to infiltrate and compromise the data and systems of the platform. We denote this attack intensity on platform p as $a_p \in [0, 1]$, which comes at a cost $c_a a_p$.

The probability of a successful attack on platform p is determined by both the level of the platform's security investment and the attacker's attack intensity. The probability of a successful attack is $\min\{\chi_p, 1\}$ where

$$\chi_p = \sqrt{\phi(s_p) a_p}, \text{ where } \phi(s_p) \equiv g\mathrm{e}^{-Gs_p} \tag{2}$$

for some parameters $g, G > 0$, where $\phi(s_p)$ represents the *technological vulnerability* of the platform. If the attack on platform p is successful, the attacker obtains a payoff that increases in the size of the platform μ_p:

$$H(\mu_p) \equiv 1 - \frac{h}{\mu_p}$$

We take $h < \eta_1, \eta_2$ so that $H > 0$. Altogether, the attacker chooses attack intensity a_p to maximize

$$\chi_p H(\mu_p) - c_a a_p. \tag{3}$$

The losses accrued to a trader of platform p resulting from a successful attack depend on the nature of the attack. We allow for three channels through which losses are accrued to traders. First, traders may incur non-trade losses, associated with leakage of private information, including data revealing credentials, personally-identifiable information (PII), and proprietary information regarding trading activity. Second, collateral held by the platform could be inaccessible, in which case traders incur additional losses, not only from failing to net trades but also from having collateral encumbered but not accessible. Third, trades themselves could be disrupted and may be difficult to reconcile or replicate.

To account for an array of potential disruptions, we consider a general loss function. Conditional on a successful attack, trader t at platform p incurs three potential losses:

[2] Our main results are robust to alternative payoff structure, such as those proportional to trading activity.

- *non-trade loss* y;
- *trade loss* $(1-r)$;
- and *collateral loss* $(1-f)$;

where r and f are the fraction of trades and collateral recovered following an attack, respectively. We allow for f and r to be negative, in which case the attack affects the firm's trading and collateral position beyond the scope of current outstanding trades. This could happen, for example, if an outage locks up collateral for extended periods of time or if regulatory reporting or trading operations are dependent on the platform's services.

Trader Preferences. After observing each platform's existing traders and security choice, each trader chooses at most one platform to join. In addition to the expected common value of membership on a platform, trader t has idiosyncratic preference for platform 1 over 2 $x_t\tau$, where $x_t \sim U[-\sigma, \sigma]$, and such that preferences scale with trade intensity τ.
Timeline The events of the model are as follows:

$t=1$ Platforms make security investment.

$t=2$ Traders choose the platform to join.

$t=3$ Attacker chooses attack intensity.

$t=4$ Trading opportunities are realized. Trades are executed and collateral is posted.

$t=5$ Attack is lodged. Payoffs are realized.

3 Equilibrium Analysis

We characterize the equilibrium by solving backwards, starting with the attacker's attack intensity toward each platform.

Attacker's Problem. At $t=3$, the attacker chooses attack intensities, taking as given platforms' security choices and traders' platform choices. Specifically, the attacker chooses a_p to maximize $\sqrt{\phi(s_p)a_p}H(\mu_p) - c_a a_p$. The (interior) solution is given by:

$$\sqrt{a_p} = \frac{1}{2c_a}\sqrt{\phi(s_p)}H(\mu_p),$$

and implies a probability of a successful attack on platform p of

$$\chi_p = \sqrt{\phi(s_p)a_p} = \sqrt{\phi(s_p)}\frac{1}{2c_a}\sqrt{\phi(s_p)}H(\mu_p) \tag{4}$$

$$= \frac{1}{2c_a}\phi(s_p)H(\mu_p), \tag{5}$$

where $H(\mu_p) = 1 - \frac{h}{\mu_p}$. We summarize the attacker's strategy and the resulting probability of an attack, as a function of s_p, below:

Proposition 1. *Suppose that $F < 2c_a$. The attacker chooses*

$$a_p = \phi(s_p)\left(\frac{H(\mu_p)}{2c_a}\right)^2 \tag{6}$$

The probability of a successful attack on p is

$$\chi_p = \frac{1}{2c_a}\phi(s_p)H(\mu_p). \tag{7}$$

The attacker's optimal attack intensity is proportional to the technological vulnerability $\phi(s_p)$ of platform p. As such, any variation in security investment by a platform is matched by the attacker, who adjusts its intensity accordingly. Second, the attacker gains more from a successful attack on a larger platform and thus devotes more resources to attacking larger platforms.

We will hold throughout the paper that the attacker's cost c_a is sufficiently large such that attacker's problem has an interior solution.

Assumption 1. $c_a > \frac{g}{2}$.

A distinguishing feature of cyber risk, relative to other forms of operational risk, is that it is brought forth by a strategic adversary (Eisenbach et al. 2022). In our model, we see this actualize through the attacker's equilibrium attack strategy $\frac{1}{c_a}\sqrt{\phi(s_p)}H(\mu_p)$, which targets platforms with greater vulnerability and more members. In contrast, operational risk also loads onto technological vulnerabilities of a platform, but external risk factors are less likely to be endogenously affected by the vulnerability itself. For example, operational risk may be determined by some factor $\bar{a}$ such that a major operational event, such as system outages arising from a natural disaster, arises with probability $\sqrt{\phi(s_p) \cdot \bar{a}}$. In comparison, a cyber attacker *amplifies* vulnerabilities through its attack intensity, resulting in a probability of a successful attack that is significantly more exposed to potential shortfalls in security investments made by the platform.

Traders' Problem. We turn to the platform choice problem of traders. At a high level, traders value platforms based on two dimensions. The first dimension is financial value, determined by the expected gains that accrue from executing and settling traders on a platform. Platforms offer improved matching between member traders and liquidity savings through multilateral netting, which makes activities on a platform advantageous relative to private activity. The economies of scale present on a platform are endogenously determined by the size of its user base. The second dimension is technological value, determined by the expected losses arising from cyber attacks that exploit vulnerabilities in the platform's systems. This second dimension is endogenously determined by platforms' security investments and attackers' attack intensity. As represented in the attacker's payoff function, gains from a successful attack are assumed to grow with a platform's user base. Hence, all else equal, attacks are intensified as a platform grows, and stands as a source of dis-economies of scale.

We first show that an individual trader's preference for platform 1 *relative* to platform 2 can be represented as the following:

Lemma 1 (FT Decomposition)**.** *An individual trader's valuation of the membership of platform* 1 *(relative to platform* 2*) can be represented as:*

$$\tau x_t - \underbrace{y(\chi_1 - \chi_2)}_{\text{relative non-trade losses}} + \underbrace{\tau F(\mu_1 - \mu_2)}_{\text{relative financial value}} - \underbrace{\tau T(\mu_1\chi_1 - \mu_2\chi_2)}_{\text{relative technological value}}, \tag{8}$$

where F and T take the general formulation:

$$F = \underbrace{e_p\left(\beta - \frac{1}{m}\kappa\right)}_{\text{trade surplus from platform}} - \underbrace{e_0(\beta - \kappa)}_{\text{trade surplus from private}}, \tag{9}$$

$$T = \underbrace{\beta e_p(1-r)}_{\text{trade-related loss}} - \underbrace{\kappa e_p\left(\frac{1}{m}f - r\right)}_{\text{collateral-related loss}}. \tag{10}$$

In addition to an individual trader's idiosyncratic preference x_t, the relative value of platforms is comprised of two components: $F(\mu_1 - \mu_2)$, which captures relative financial value, or the unconditional relative gains that accrue from the relative size of platforms; and $T(\mu_1\chi_1 - \mu_2\chi_2)$, which represents relative technological value, or the difference in expected losses from operational vulnerabilities. This characterization offers an interpretable representation of individual traders that will allow us to re-map equilibrium outcomes to the primitives of the economic and operational dimensions. To focus exposition, we set $y = 0$ from here forth.

With this characterization, consider the relative payoff of choosing 1 over 2 for trader t shown in Lemma 1. Substituting in the probability of an attack χ_p, we obtain:

$$x_t + F(\mu_1 - \mu_2) - T(\mu_1\chi_1 - \mu_2\chi_2) = x_t + F(\mu_1 - \mu_2) - T\left(\mu_1 \frac{1}{2c_a}\phi(s_1)H(\mu_1) - \mu_2 \frac{1}{2c_a}\phi(s_2)H(\mu_2)\right) \tag{11}$$

Recall, the attack intensity is comprised of $H(\mu_p) = 1 - \frac{h}{\mu_p}$. Further simplifying the expression, we obtain:

$$x_t + \underbrace{F(\mu_1 - \mu_2)}_{\text{Platform economies}} \underbrace{- \frac{T}{2c_a}((\phi(s_1)\mu_1 - \phi(s_2)\mu_2)}_{\text{Relative losses}} + \underbrace{\frac{T \cdot h}{2c_a}(\phi(s_1) - \phi(s_2))}_{\text{Cyber diseconomies}} \tag{12}$$

The above equation provides further decomposition of the relative technological value. In addition to the relative financial value, which is determined by platforms' economies of scale (i.e. $F(\mu_1 - \mu_2)$), the relative technological value is comprised of two parts. The first part, *relative losses*, represents the absolute difference in the security stances of each platform. The second part represents a *cyber diseconomy*, which arises due to the greater intensity of attacks expected on platforms of larger size. Users rationally anticipate that platforms with more users are also more attractive targets and incorporate this into their platform choice.

In an interior equilibrium, a trader of type t^* is indifferent between the two platforms. In such an equilibrium, by the uniformity of x_t, a $\frac{\sigma - x^*}{2\sigma}$ fraction of traders with $x_t > x^*$ prefer 1 over 2, and the remaining $\frac{x^* + \sigma}{2\sigma}$ of traders with $x_t < x^*$ prefer 2 over 1. This implies $\mu_1 = \eta_1 + \lambda\frac{\sigma - x^*}{2\sigma}$. Substituting in the expression for x^* yields:

$$\frac{2\sigma}{\lambda}(\mu_1 - \eta_1) - \sigma = \mu_1 \underbrace{\left(2F - \frac{T}{2c_a}(\phi(s_1) + \phi(s_2))\right)}_{\text{marginal network gain of platform 1 over 2}} - \mu \underbrace{\left(F - \frac{T}{2c_a}\phi(s_2)\right)}_{\text{network gain of platform 2}} + \frac{Th}{2c_a}(\phi(s_1) - \phi(s_2)) \tag{13}$$

As long as idiosyncratic preferences are sufficiently heterogeneous (i.e., $\sigma > \lambda F$), an interior equilibrium arises in which a trader with preferences $x_t = x^*$ is indifferent between the two platforms, summarized below[3]:

Proposition 2. *Let $\sigma > \lambda F$. The equilibrium mass of traders in platform* 1 *is given uniquely by*

$$\mu_1 = \max\left\{\eta_1, \min\left\{\eta_1 + \lambda, \tilde{\mu}_1\right\}\right\} \tag{14}$$

where

$$\tilde{\mu}_1 = \frac{\sigma + \frac{2\sigma}{\lambda}\eta_1 - \mu\left(F - \frac{T}{2c_a}\phi(s_2)\right) + \frac{Th}{2c_a}\left(\phi(s_1) - \phi(s_2)\right)}{\frac{2\sigma}{\lambda} - \left(2F - \frac{T}{2c_a}\left(\phi(s_1) + \phi(s_2)\right)\right)}.$$

Platform security. The above proposition pins down the equilibrium mass of traders at each platform, taking as given security investments of the platforms. At $t = 0$, platforms simultaneously choose security investments. To simplify the expressions, we focus on *interior equilibria*, defined as $s_p^* \geq 0, \tilde{\mu}_p^* \in [\eta_p, \eta_p + \lambda]$ for both platforms.

Platform p chooses the level of security investments to maximize:

$$\tilde{\mu}_p - c_s s_p. \tag{15}$$

Because successful attacks primarily inflict costs directly on users, platforms do not explicitly factor this into their decision to lower technological vulnerability. However, the platform's profits are proportional to the mass of traders on its platform, and hence, it internalizes these risks through its impact on its ability to attract members, $\tilde{\mu}_p$. The mass of traders who choose the platform, in turn, depends on the competitiveness of their offering relative to that offered by their competitor.

We solve for platforms' equilibrium security investment strategies, summarized in the following proposition.

Proposition 3. *In any interior equilibrium, the investment of platform p is given by*

$$s_p^* = \frac{1}{G}\ln\left(1 - N_p - \frac{1}{N_p}\frac{c_s}{G\left(\mu - 2h\right)}\right) + \frac{1}{G}\ln\frac{TgG\left(\mu - 2h\right)\lambda}{4c_a c_s\left(\sigma - \lambda F\right)}$$

where

$$N_p \equiv \frac{1}{2} - \frac{\sigma}{2\left(\sigma - \lambda F\right)\left(\mu - 2h\right)}\left(\eta_p - \eta_{p'}\right)$$

[3] The set of equilibria in the general case can be found in the proof of Proposition 2.

and yields

$$\phi(s_p^*) = \frac{2c_a c_s \left(\frac{2\sigma}{\lambda} - 2F\right)}{TG\left(\mu - 2h\right)} \frac{1}{1 - N_p - \frac{1}{N_p}\frac{c_s}{G(\mu-2h)}}. \tag{16}$$

We are now able to fully characterize the interior equilibrium, its associated market structure, and fully outline the probability of a successful attack:

Theorem 2. *In any interior equilibrium,*

$$\mu_p^* = \eta_p + \frac{\lambda}{2}\left(1 + \left(\eta_p - \eta_{p'}\right)\frac{F}{\sigma - \lambda F}\right)$$

Assuming that η_1, η_2 are bounded away from h and $\frac{c_s}{G}$ is small enough, the unique equilibrium is interior if and only if $\frac{\sigma}{\lambda} - F \geq |\eta_1 - \eta_2|$.[4] *The probability of a successful attack on platform p is given by:*

$$\chi_p^* = \frac{c_s}{TG\mu_p^*} \cdot \left(\frac{2\sigma}{\lambda} - \left(2F - \frac{T}{2c_a}\left(\phi(s_1^*) + \phi(s_2^*)\right)\right)\right) \tag{17}$$

$$= \frac{c_s\left(\frac{2\sigma}{\lambda} - 2F\right)}{TG\left(\mu - 2h\right)} \frac{1}{1 - N_p - \frac{1}{N_p}\frac{c_s}{G(\mu-2h)}} \left[1 - \frac{h}{\eta_p + \frac{\lambda}{2}\left(1 + \left(\eta_p - \eta_{p'}\right)\frac{F}{\sigma - \lambda F}\right)}\right] \tag{18}$$

The probability of a cyberattack represents the equilibrium level of technological risk posed to users as a consequence of the security choice of platforms and the attack intensity chosen by the adversary. In equilibrium, a platform faces a lower likelihood of a cyberattack when it has more traders (i.e. larger μ_p) and when losses associated with technological failure are greater (i.e. larger T). It is worthwhile noting that a positive probability of a cyberattack is not necessarily inefficient if security costs become prohibitively high. We examine the conditions for inefficiency in the next section.

4 Public-Private Wedge in Technological Resiliency

In the absence of regulation, security levels are determined competitively. Platforms offer outside options to non-users, and hence, users' platform choice acts as a disciplining device. Whether equilibrium security levels at each respective platform are inefficient largely depends on the extent to which competitive forces align with optimal levels.

[4] A sharp characterization of when the equilibrium is interior can be found in the appendix in Proposition (9).

To evaluate divergence from the social optimum, we analyze the optimal security levels, taking the distribution of users between platforms as given. Before proceeding, it is useful to overview the rationale for this benchmark choice. This ex-post benchmark is chosen for both theoretical and practical reasons. From a theoretical standpoint, network effects arise from both the financial and technological value of a platform. Trade surplus, and associated liquidity benefits are maximized by connecting all traders on a single platform, which naturally biases the optimum toward concentration. Economies also arise from concentrating security investments into a single platform instead of dispersed investment into multiple platforms. In the context of our model, both forces push the social optimum to a solution with a single platform with significant levels of security investment.[5] Unsurprisingly, foundational financial infrastructures, such as wholesale payment systems or Treasury security settlement systems, are often run directly by central banks, who are positioned to fully internalize the value of technological resiliency.[6]

From an institutional standpoint, the benchmark solution is consistent with common regulatory frameworks, which are mandated to establish supervisory standards for firms, *taking as given* the underlying market structure. Market structure and risks associated with concentration (or lack thereof) are implicitly considered in regulatory choices through their implications on vulnerabilities, but not policies are not geared toward affecting the relative competitiveness of a firm, for various reasons, including issues of political economy and fairness. Hence, a benchmark on ex-post security choice offers implications that are more readily applicable to policy.

Consider a planner's objective to maximize utilitarian welfare, taking as given μ_p.

$$\tau\mu_p^2\left(e_p\left(\beta - \frac{1}{m}\kappa\right) - T\chi_p\right) - c_s s_p \tag{19}$$

Intuitively, the planner chooses security levels, taking into account the volume of trade between traders of a given platform, $e_p\tau\mu_p^2$. Taking the first order condition yields:

$$\tau\mu_p^2\frac{T}{2c_a}\phi(s_p)H(\mu_p) - c_s = 0 \tag{20}$$

[5]The argument is straightforward. Setting aside x_t for a moment, the first-best allocation of traders taking as given optimal security levels is $\tau\sum_p e_p\bar{\beta}\mu_p^2 - \frac{c_s}{\tau G}$, which increases in $\mu_{p'}$, where p' is the dominant platform. This implies that with x_t, optimal allocation balances between private preferences and network efficiency, whereby the marginal value of adding a trader to the dominant platform is equal to x_t (which is negative if $p' = 1$).

[6]A notable exception is on the question of redundancies in critical infrastructure. Although our model does not directly capture redundancies in the midst of an attack, users' platform choice takes into account the likelihood of larger platforms to be favorable targets.

Rearranging the equation, we pin down the optimal security investment for each platform:

Proposition 4. *At the optimal security investment level s_p^{soc}, technological vulnerability of platform p is given by:*

$$\phi(s_p^{soc}) = \frac{1}{\tau\mu_p}\frac{2c_a c_s}{(\mu_p - h)TG}. \tag{21}$$

The socially optimal investment level directly takes into account expected losses associated with an attack, which include platform size μ_p, trade activity τ, and conditional losses T. In addition, it internalizes the attacker's cost c_a, and increases with the platform investment cost c_s.

We revisit the equilibrium strategies and outcomes in the competitive environment. Equilibrium security investment strategies and technological vulnerabilities are complicated expressions. We exploit an equilibrium characteristic that assists with analysis:

Corollary 1. *In equilibrium, technological vulnerability of platform p satisfies:*

$$\phi(s_p^*) = \frac{2c_a c_s \Omega}{TG(\mu_p^* - h)}, \text{ where } \Omega = \frac{2\sigma}{\lambda} - \left(2F - \frac{T}{2c_a}(\phi(s_1^*) + \phi(s_2^*))\right). \tag{22}$$

Now, consider the difference in technological vulnerability of the platforms implied by the public and private investment choices. We use the above corollary to re-formulate $\phi(s_p^*)$ in terms of $\phi(s_p^{soc})$:

$$\phi(s_p^*) = \underbrace{\tau\mu_p\left(\frac{2\sigma}{\lambda} - \left(2F - \frac{T}{2c_a}(\phi(s_1^*) + \phi(s_2^*))\right)\right)}_{=\tau\mu_p\Omega\equiv\Omega_p}\phi(s_p^{soc}) \tag{23}$$

Here, Ω_p represents a *public-private wedge* in technological vulnerability for platform p. Private investment is lower (greater) than what is publicly optimal if $\Omega_p > 1$ ($\Omega_p \leq 1$).

Lemma 2. *The public-private wedge in technological vulnerability is given by:*

$$\Omega_p^* = \underbrace{\tau\mu_p}_{\text{platform-specific vulnerability}} \cdot \underbrace{\Omega}_{\text{common vulnerability}} \tag{24}$$

The public-private wedge Ω_p of platform p is comprised of two components. The first component $\tau\mu_p$ is platform-specific vulnerability which represents the level of financial market activity on a platform. This arises primarily because the platforms focus on user base, rather than activity, in determining their security choice. As market-level activity, whether through user base or trade volume, increases, the wedge increases. The second component Ω represents a common vulnerability shared across platforms and results from potential inefficiencies arising from discipline brought forth through competition. To see this, observe that Ω can be expressed in terms of market share:

$$\Omega = \frac{2\sigma}{\lambda} - 2F + \frac{T}{2c_a}\tau\Omega\sum_p \mu_p \phi(s_p^{soc}) \tag{25}$$

$$\Omega = \frac{\frac{2\sigma}{\lambda} - 2F}{1 - \sum_p \frac{c_s}{G(\mu_p^* - h)}}, \tag{26}$$

which implies that Ω increases as platforms' market shares become more asymmetric. We make the following observation.

Proposition 5. *The public-private wedge increases in market size $\tau\mu$, and Ω increases in imbalances in market share $|\mu_p - \mu_{p'}|$.*

In other words, competition, as a disciplining mechanism to incentivize security investments by platforms, is most effective when platforms are even in market shares, and deteriorates as they become more asymmetric.

More generally, we attain the following result:

Theorem 3. *Private security investment is generally inefficient, that is, $\Omega \neq 1$. Moreover, Let $\frac{\sigma}{\lambda} - F \geq \frac{1}{2} - \frac{2c_s}{(\mu - 2h)G}$. Then, the nature of inefficiency is:*

- *Inefficiently high investment for low $|\eta_p - \eta_{p'}|$;*
- *and inefficiently low investment for large $|\eta_p - \eta_{p'}|$.*

Intuitively, one aspect to think about is that private incentives to invest are driven by the value accruing to the "marginal" user, and not the overall cost. This results in overinvestment when the marginal user is more sensitive to security strength, and less so once there is sufficient difference in size between the two platforms.

5 Dynamics of Technological Vulnerability

In the static equilibrium, inefficiencies in investment can go either way and are determined by the underlying market structure. This points to the importance of considering the dynamic evolution of market structure in concurrence with technological vulnerabilities of the financial system. In this section, we explore this issue by considering a dynamic extension of the model.

Dynamic Extension. We now extend our model to a dynamic environment. There are infinitely many periods $t = 1, 2, 3..., \infty$. We use superscripts to denote the period. At the beginning of subsequent periods $t \geq 1$, each platform chooses its level of security investment. Investments into security is assumed to fully depreciate every period.[7]

As before, there is a μ mass of traders. Each platform has a mass $\bar{\eta}$ are long-term "anchor" traders, who remain a user at the respective platform indefinitely. Anchor traders may be those who prefer a platform for strong idiosyncratic preferences or unmodeled feasibility constraints arising from geographical or regulatory factors. The remaining $\mu_* \equiv \mu - 2\bar{\eta}$ mass of traders are distributed across platforms, with a total of μ_p^0 on platform p, where $\sum_p \mu_p^0 = \mu$. In each period, traders (excluding anchor traders) independently receive an opportunity to switch platforms, with probability α. All other traders remain on their platform from the previous period $t-1$.

After traders choose their platforms, an adversary arrives with probability γ. As before, the adversary chooses the attack intensity for each platform. We assume that all agents behave myopically and make decisions to maximize current period payoffs.[8]

Dynamic Equilibrium. In effect, the baseline game is played in each period. The masses of incumbent traders (η_1^t, η_2^t) are determined by the previous period's platform adoption $(\mu_1^{t-1}, \mu_2^{t-1})$.

Law of motion. At the end of period $t-1$, the mass of traders on platform p is μ_p^{t-1}. The mass of incumbent traders at platform p at the beginning of period t is

$$\eta_p^t = (1-\alpha)\left(\mu_p^t - \bar{\eta}\right) + \bar{\eta}. \tag{27}$$

We assume that $\alpha < \frac{\sigma}{(\mu - 2\bar{\eta})F}$, which roughly corresponds to the dynamic analog of our baseline assumption $\sigma > F\lambda$. Then Theorem 2 defines the law of motion of the mass of traders in each platform as long as the solution is interior.

[7]In addition to tractability, this captures the notion that investments must be made to adapt platform security to the quickly evolving nature of cyber risk.

[8]Our object of interest is the evolution of the mass of traders across platforms. The myopia across periods is an innocuous assumption in the sense that it simply attenuates the convergence rates and the laws of motion, but it buys simplicity and tractability. The assumption that investment in security depreciates ensures that qualitative results are robust.

Denote $R = 1 - \alpha \frac{\sigma - (\mu - 2\bar{\eta})F}{\sigma - \alpha(\mu - 2\bar{\eta})F}$. Note $R \in (0,1)$ if $\sigma > (\mu - 2\bar{\eta})\,F$ and $R > 1$ if $\sigma < (\mu - 2\bar{\eta})\,F$.

We first characterize the set of possible equilibrium paths of market dynamics:

Proposition 6. *For small enough $\frac{c_s}{G}$ the following hold.*[9] *If $\mu_1^0 = \mu_2^0$, then $\mu_2^t = \frac{\mu}{2}$ at every t. Otherwise, let $\mu_1^0 > \mu_2^0$ without loss of generality.*

- *If $\frac{\sigma}{F} - \lambda \geq (1-\alpha)\left(\mu_1^0 - \mu_2^0\right)$ and $\sigma > (\mu - 2\bar{\eta})\,F$, then equilibrium is interior in every period and $\mu_p^t = \frac{\mu}{2} + R^t\left(\mu_p^0 - \frac{\mu}{2}\right)$. Both μ_p^t converge monotonically to $\frac{\mu}{2}$;*
- *If $\frac{\sigma}{F} - \lambda \geq (1-\alpha)\left(\mu_1^0 - \mu_2^0\right)$ and $(\mu - 2\bar{\eta})\,F > \sigma$, then equilibrium is interior during $t \leq \bar{t} \equiv \left\lfloor \frac{\ln\left(\frac{\sigma}{F} - \lambda\right) - \ln\left(2(1-\alpha)\left|\mu_p^0 - \frac{\mu}{2}\right|\right)}{\ln R} \right\rfloor$ and boundary for $t > \bar{t}$. During $t \leq \bar{t}$, $\mu_p^t = \frac{\mu}{2} + R^t\left(\mu_p^0 - \frac{\mu}{2}\right)$. During $t > \bar{t}$, $\mu_2^t = \bar{\eta} + (1-\alpha)^{t-\bar{t}}\left(\frac{\mu}{2} - R^{\bar{t}}\left(\frac{\mu}{2} - \mu_2^0\right)\right)$ and $\mu_1^t = \mu - \mu_2^t$;*
- *If $(1-\alpha)\left(\mu_1^0 - \mu_2^0\right) > \frac{\sigma}{F} - \lambda$, then equilibrium is corner every period, $\mu_2^t = \bar{\eta} + (1-\alpha)^t\left(\mu_2^0 - \bar{\eta}\right)$ and $\mu_1^t = \mu - \mu_2^t$. Platform 2 does not invest.*

Intuitively, if the platforms are sufficiently unequal from the beginning (i.e. $(1-\alpha)\left(\mu_1^0 - \mu_2^0\right) > \frac{\sigma}{F} - \lambda$), network effects dominate and all "free" traders move to the larger platform at every period. Consequently, the difference in platform size, $\mu_1^t - \mu_2^t$, widens, and the smaller platform shrinks to size f at the limit.

In the case where platforms begin with roughly equal (but not identical) size (i.e. $0 < (1-\alpha)\left(\mu_1^0 - \mu_2^0\right) < \frac{\sigma}{F} - \lambda$), the ratio between variation in idiosyncratic preferences and network effects determines the long-term market structure. In particular, if the variation in preference σ is sufficiently large relative to complementarities $(\mu - 2\bar{\eta})\,F$, neither platform is "sticky" and the market is split in the long-run, resulting in a shrinking gap in size $\mu_1^t - \mu_2^t$ over time. However, with large complementarities $(\mu - 2\bar{\eta})\,F$, the larger platform eventually captures the majority of the market over time, and the gap $\mu_1^t - \mu_2^t$ grows at a rate R that is proportional to the relative complementarity $\frac{(\mu - 2\bar{\eta})F}{\sigma}$ up to the point $\bar{t}$. After this point, the gap grows faster as all "free" traders choose the dominant platform regardless of their idiosyncratic preferences. It is also worthwhile to mention that both platforms invest in interior equilibria. It is easy to see that the smaller platform does not invest in corner equilibria as there is no benefit to investing in technology. Consequently, Proposition 6 also shows that the smaller platform eventually stops investing unless the heterogeneity is large relative to complementarities and the initial gap is not too wide.

[9] $\frac{c_s}{G} < \left(\frac{\mu - 2h}{2\bar{\eta} - 2h} + \left(\frac{\sigma}{\lambda} - A\right)\frac{4c_a}{TF}\right)^{-1}(\bar{\eta} - h)$

A dynamic implication is that along any path of increased consolidation, within a discrete number of periods, security investments drop below the social optimum:

Theorem 4 (Tipping point)**.** *For any equilibrium path of consolidation, there exists a tipping point at which investment by both platforms drop below the socially optimal.*

We call this the tipping point: when private competition between platforms is no longer sufficient to ensure adequate security levels and technological vulnerability is above what is desirable from a social standpoint. Past this tipping point, there is loss in efficiency directly as a result of inefficiently high exposure to operational disruptions, and in particular, cyber adversaries.

We refer to efficiency loss as *technological drag*. Descriptively, as markets consolidate around a dominant platform, the platform lowers in security investment, and this directly translates into an increased barrage of attacks from cyber adversaries. This increases the likelihood of a cyber attack and also increases the impact of an attack conditional on being successful, as more traders are dependent on the platform's technological resiliency. Given the potential for material impact on traders and their activities, it is natural to think about how changes in technological resiliency affect trading itself and how it might amplify inefficiencies. We explore this in the next section.

6 Trading, Amplification, and Technological Drag

So far, platforms affected trading activity only through agglomeration in activity through improved on-platform matching. In this section, we extend the benchmark model to endogenize trading along two dimensions, trading strategy and intensity, and analyze how cyber vulnerability is affected and eveen amplified by endogenous financial activity.

Endogenous Trading. Consider the following modifications. At the beginning of the period, and before idiosyncratic preferences are realized, traders collectively commit to an investment in τ at some cost $c_\tau(\tau)$, where $c'_\tau(\cdot), c''_\tau(\cdot) > 0$.[10] This investment represents costs associated with the acquisition of resources to maintain a trading capacity of τ. Afterwards, as before, platforms make security choices. Idiosyncratic preferences are realized and traders choose platforms. After choosing platforms, traders select their trading strategies z, which is either H or L, which differ in terms of the trading return β^z and netting benefit m^z.

[10] To isolate the effects arising from the interplay between traders and platforms, we assume investment choice is collectively determined.

Specifically, each trader chooses whether to shift its on-platform trading strategy from a baseline strategy of L, which involves high return and low netting benefit, to H, which involves low return and high netting benefit. In order to isolate the impact of technological risk on trading patterns, the net financial return of strategies is assumed to be identical, i.e.

$$\beta^H - \frac{1}{m^H}\kappa = \beta^L - \frac{1}{m^L}\kappa. \tag{28}$$

This implies that in the absence of technological risk, traders would be indifferent between the two, and intuitively, both lie on the "liquidity-return" frontier. We use F^z and T^z to denote the financial and technological values given trading strategy z. By shifting to H, the trader is assumed to implement an H-trading-strategy with other traders who choose H, and maintain an L-trading-strategy with others who keep to L.

We start by considering the trading strategy of trader t on platform p. A trader t on platform p's payoff is given by:

$$x_t + e_p\left(\beta^z - \frac{1}{m^z}\kappa\right)\mu_p + e_0(\beta - \kappa)\mu_{-p} - T^z\mu_p\chi_p \tag{29}$$

Since net return $\left(\beta^z - \frac{1}{m^z}\kappa\right)$ is identical across trading patterns, the trader maximizes payoff by minimizing T. Note that:

$$T^H - T^L = \left(\beta^H - \beta^L\right)e_p(1-r) - \kappa e_p\left(\left(\frac{1}{m^H} - \frac{1}{m^L}\right)f\right) \tag{30}$$

$$= e_p\left[\left(\beta^H - \beta^L\right)(1 - r - f)\right]. \tag{31}$$

Thus, traders strictly prefer a trading strategy with lower β and higher m if $f < 1 - r$:

Lemma 3. *Suppose that $f \leq 1 - r$. Traders' optimal trading strategy is $z = H$.*

Whenever two trading strategies deliver identical net financial return, traders prefer the strategy that results in lower exposure to technological risk. A cyberattack results in trade losses that scale with β^z, and collateral losses, which scale with the level of pledged collateral. To minimize losses, traders choose the strategy that effectively lowers T. As long as collateral recovery is not too high $f < 1 - r$, strategy H delivers both lower trade losses and requires less collateral per trade, making it strictly preferable to strategy L.

A shift in traders' trading strategy to lower technological risk exposure directly impacts platform's security choices. Specifically, let us revisit platform p's technological vulnerability

$\phi(s_p^*)$ given T^z,

$$\phi(s_p^*) = \frac{2c_a c_s \Omega}{T^z G(\mu_p^* - h)}, \tag{32}$$

which strictly decreases in T^z. This implies the following:

Proposition 7. *Choosing lower T increases vulnerability $\phi(s_p^*)$ and increases probability of attack χ_p.*

The intuition for this result is the following. Technological considerations lead to a clear tie-breaking between any two strategies with similar net financial returns but differential exposure to technological risk. By choosing trading strategies that lower technological exposure, financial markets have greater tolerance to cyberattacks more broadly. This tolerance, resulting from the endogenous reaction to lower exposure, leads to a decrease in security investment by platforms, and correspondingly, to an increase in vulnerability and probability of a cyberattack.

Now, consider the trading capacity τ chosen by traders, who anticipate platforms' market shares and security investments.

$$\tau E\left[x_t + e_p\left(\beta^H - \frac{1}{m^H}\kappa\right)\mu_p + e_0\left(\beta-\kappa\right)\mu_{-p} - T^H\mu_p\chi_p\right] - c_\tau(\tau) \tag{33}$$

$$= \tau\left[\begin{array}{c} Pr(x_t > x^*)\left(E[x_t|x_t > x^*] + F(\mu_1-\mu_2) - T^H(\mu_1\chi_1 - \mu_2\chi_2)\right) \\ +\left(e_p(\beta^H - \frac{\kappa}{m^H})\mu_2 + e_0\left(\beta-\kappa\right)\mu_1 - T^H\mu_2\chi_2\right)\end{array}\right] - c_\tau(\tau) \tag{34}$$

$$= \tau\left[\underbrace{\frac{\sigma - x^*}{2\sigma}\left(E[x_t|x_t > x^*] - x^*\right) + \left(e_p\left(\beta^H - \frac{\kappa}{m^H}\right)\mu_2 + e_0\left(\beta-\kappa\right)\mu_1 - T^H\mu_2\chi_2\right)}_{\star(s_1,s_2)}\right] - c_\tau(\tau) \tag{35}$$

Trading capacity τ increases in $\star$, the expression in the bracket, which we can further simplify:

$$\frac{\sigma - x^*}{2\sigma}\left(E[x_t|x_t > x^*] - x^*\right) + \left(e_p\left(\beta^H - \frac{\kappa}{m^H}\right)\mu_2 + e_0\left(\beta-\kappa\right)\mu_1 - T^H\mu_2\chi_2\right) \tag{36}$$

$$= \frac{\left(\frac{2\sigma}{\lambda}(\mu_1 - \eta_1)\right)^2}{4\sigma} + e_p\left(\beta^H - \frac{\kappa}{m^H}\right)\mu - F\mu_1 - T^H\mu_2\chi_2. \tag{37}$$

Since equilibrium market shares μ_1 an μ_2 are invariant with respect to technological vulnerability, the main factor to consider is $T^H \mu_2 \chi_2$:

$$T^H \mu_2 \chi_2 = T^H \mu_2 \phi(s_2) \frac{1}{2c_a} H(\mu_2) \tag{38}$$

$$= \frac{c_s}{G} \Omega \tag{39}$$

Recall that by Proposition 5, Ω increases in the asymmetry of market shares between platforms. Hence, as the common vulnerability component Ω increases, investment into τ decreases.

Proposition 8. *As common vulnerability Ω increases, τ^* decreases.*

Technological drag arises from both an ex-post and an ex-ante standpoint. From an ex-ante standpoint, as the common vulnerability component Ω increases, traders lower trading capacity τ. To see this, let $c_\tau(\tau) = \frac{c_\tau}{2}\tau^2$, for which we attain $\tau^* = \frac{\star(s_1,s_2)}{\tau_c}$. From an ex-ante standpoint, traders lower investment into their trading capacity in direct response to technological vulnerabilities, and this corresponds to a decrease in trading intensity by $\frac{c_s}{Gc_\tau}$ as Ω increases.

From an ex-post standpoint, disruptions in platform operations result system-wide economic costs arising from trade and collateral losses, with inefficiently heightened probability of cyberattacks resulting in excess cost per unit trade of:

$$\tau T^H \sum_p \mu_p^2 \left(\chi_p(s_p^*) - \chi_p(s_p^{soc})\right) = \tau T^H \sum_p \mu_p^2 (\Omega_p - 1) \phi(s_p^{soc}) \frac{H(\mu_p)}{2c_a} \tag{40}$$

$$= \frac{c_s(\tau\mu\Omega - 2)}{G}, \tag{41}$$

which is positive if $\tau\mu\Omega > 2$. Since this condition relaxes as market shares become more asymmetric, we have:

Theorem 5 (Technological drag)**.** *Technological drag increases as the system becomes more concentrated.*

7 Evidence from the Tri-Party Repo Market

The theory indicates that in the absence of regulation, technological vulnerabilities rise when markets become consolidated. Given system-wide consolidation in the past half-century, this points to the possibility for under-provision of technological resiliency across various parts of the financial system. There are severe limitations to testing implications of the theory. First, most firms generally do not report information on investments into technology, let alone cybersecurity. Second, the intensity at which attackers target any individual institution is not directly observable. Third, even if investments are observable, identifying the link between market structure and technological investment can be challenging since they are co-determined.

To tackle this, we exploit the exit of a duopolist in tri-party repo settlement, which resulted in a single bank solely responsible for the entire market. As background, the tri-party repo market consists of lenders, borrowers, and clearing banks. Clearing banks act as platforms by facilitating clearing and settlement for repo transactions, acting as the "third-party." By 2005, only two clearing banks existed in the market: Bank of New York Mellon (BNY) and JPMorgan Chase (JPMC). In 2016, JPMC announced its intent to wind down its tri-party operations and fully exited in 2019. Consequently, as of 2019, BNY represents the sole operator of tri-party repo, which is vital to treasury and money markets, and is key to monetary policy operations as well, including the repo and reverse-repo facilities.

Data and empirical strategy. We use the full exit of JPMC as a shock to market structure, and examine whether there are material changes in BNY's investments in technology. In this setting, we are able to alleviate some aforementioned limitations. First, although JPMC's market share declined over time, its full exit and choice not to sell its business to a different party signified full consolidation of tri-party settlement and represented fundamental change to the market structure. This enables us to identify changes in investment triggered by full consolidation. Second, as a regulated institution, BNY reports detailed information about its operations, including its IT-related expenses. We use the IT investment series provided by Modi et al. (2022), which identifies technological expenditures on regulatory filings and disclosures. IT investments include various expenditures, including cyber-related investments, which by industry reports, represent 6-14 percent of IT expenditure Insights (2019). Although many banks do not report consistently over the entire sample, BNY consistently reported expenses throughout the entire sample period from 2001 to 2021.[11] Finally, the intensity of cyberattacks over our sample period from 2001 to 2021 follow a general upward trend based on industry reports, which would suggest absolute drop in investment would indicate under-investment.[12]

We augment the data with bank-level characteristics from Call Reports. We obtain repo market share data by combining NY Fed Tri-Party Repo data and quarterly and annual reports

[11] See Modi et al. (2022) for detailed description of the data.

[12] For example, see the Chapter 3 of the IMF Global Financial Stability Report.

from BNY and JPMC, which allows us to verify the timing of full exit. A notable event during our sample period is the tri-party repo market reform, which was initiated in response to weaknesses revealed during the financial crisis.[13] One of the actionable steps required settlement banks, namely BNY and JPMC, to make technological investments to improve resiliency and compliance. The process for technological improvements began in 2010, and necessary investments were reportedly met by 2014.

We take a difference-in-difference approach to test whether technological investments at BNY changes following the exit of JPMorgan. Our most saturated specification is:

$$IT_{it} = Reform_t \times BNY + Exit_t \times BNY + \delta_i + \gamma_t + \varepsilon_{it} \tag{42}$$

where IT_{it} is the IT investment over assets of firm i in year t, $Reform_t$ is 1 for observations starting 2010 to 2014, during which tri-party market infrastructure reform took place, and $Exit_t$ is 1 for the period following JPMC's exit. We also include bank and year fixed effects. In the univariate specification, we include the average IT expenditure over assets of the top 30 banks, excluding BNY and JPMC, which takes into account IT investment trends of other large banks.

The results for the univariate and panel regressions, reported in Tables 1 and 2, offer a consistent picture. We find broad evidence of a significant decline in IT investment at BNY following JPMC exit from tri-party repo settlement. JPMC's exit is estimated to have resulted in a drop of 0.148 IT expenditure per asset dollar at BNY. After accounting for excess IT expenses to have occurred as a result of the market reform, the estimate remains roughly the same, at about 0.131 IT expenditure per asset dollar. Altogether, we find a strong association between IT investment and market consolidation.

The drop in technological investment at BNY is evident in Figure 3 in absolute scale and relative to other large banks. Consistent with large investments necessary to meet objectives of the tri-party reforms, BNY investments jump in 2010 and stay elevated relative to trend until 2014, at which point reform goals were met. It continues to rise until 2019, at which investments significantly decline relative to trend. During our entire sample period, technological investment at other large banks grows, as documented in Modi et al. (2022).

We do not see any material change in BNY's investment trend in response to the announcement of JPMC's exit in 2016. As explained earlier, in principle, JPMC could have chosen to spin off or sell its tri-party repo settlement business instead of winding down its operations,

[13] See here for more detail.

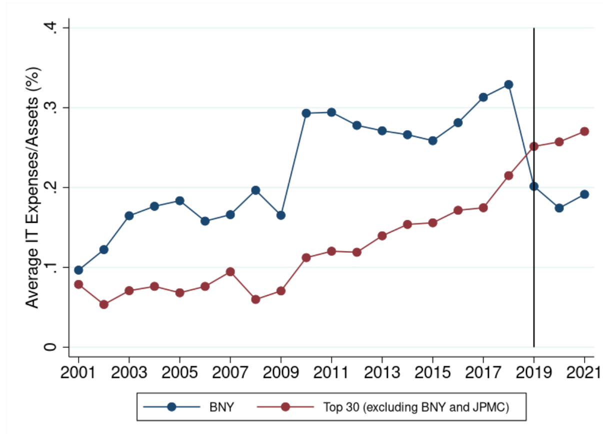

Figure 3: **IT expenditure of BNY and other large banks.** The blue line represents the IT expense over assets for BNY, and the red line represents the average IT expense over assets for the top 30 banks excluding BNY and JPMC.

which would have preserved the duopoly market structure. Doing so could have maintained competitive dynamics that would incentivize technological investments to retain and extend market shares. Continuing investment would be consistent with our model predictions, in which the dominant platform maintains investments to extend its market share. In this respect, the point of full market consolidation (and clean exit of JPMC) may have represented the actual point at which certain resolutions were reached in the competitive landscape of tri-party repo settlement.

8 Conclusion

This paper proposes a theory of financial system architecture that incorporates technological vulnerabilities that are determined endogenously through platforms' investment decisions. In our model, platforms provide enhancements along both trade and post-trade stages, both which generate network effects. At the same time, platforms must also invest into security to strengthen technological resiliency of the financial system against attackers seeking to exploit system weaknesses. Natural complementarities in platforms' services attenuate technological resiliency. Our framework also links primitive market features to technological vulnerability, including the velocity of financial markets.

A key result is that a wedge generically arises between the socially optimal and privately optimal level of security. This points to the importance of corrective measures, such as regulation, to ensure that adequate levels of investment are taking place that fully internalize the system-level costs associated with a cyber impairment. We find these vulnerabilities evolve non-monotonically over the maturity of a financial system, and in particular, exhibit a tipping point at which technological resiliency diverges from optimal. This also happens to be the circumstance under which the underlying architecture becomes sufficiently consolidated, a pattern that is observed empirically. Under such circumstances, in the short-run, technological vulnerabilities create nontrivial losses, as traders have no viable outside option and incur costs associated with trade and collateral impairments. In the long run, the heightened frequency of successful attacks creates drag on the financial system.

References

Aldasoro, Inaki, Leonardo Gambacorta, Paolo Giudici, and Thomas Leach, "The drivers of cyber risk," 2020.

Aldasoro, Iñaki, Leonardo Gambacorta, Paolo Giudici, and Thomas Leach, "The drivers of cyber risk," *Journal of Financial Stability*, 2022, *60*, 100989.

Anand, Kartik, Chanelle Duley, and Prasanna Gai, "Cybersecurity and financial stability," 2022.

Anderson, Ross, "Why information security is hard-an economic perspective," in "Seventeenth Annual Computer Security Applications Conference" IEEE 2001, pp. 358–365.

—— **and Tyler Moore**, "The economics of information security," *science*, 2006, *314* (5799), 610–613.

Awrey, Dan and Joshua Macey, "Open Access, Interoperability, and the DTCC's Unexpected Path to Monopoly," 2021.

Brando, Danny, Antonis Kotidis, Anna Kovner, Michael Lee, and Stacey L Schreft, "Implications of cyber risk for financial stability," 2022.

Brolley, Michael, David Cimon, and Ryan Riordan, "Efficient Cyber Risk: Security and Competition in Financial Markets," *Available at SSRN*, 2020.

Crosignani, Matteo, Marco Macchiavelli, and André F Silva, "Pirates without Borders: The Propagation of Cyberattacks through Firms' Supply Chains," *FRB of New York Staff Report*, 2020, (937).

Curti, Filippo, Jeffrey Gerlach, Sophia Kazinnik, Michael Lee, and Atanas Mihov, "Cyber risk definition and classification for financial risk management," *Journal of Operational Risk*, 2023, *18* (2).

Duffie, Darrell, "Financial market infrastructure," *Across the Great Divide: New Perspectives on the Financial Crisis*, 2014, *652*, 251.

—— **and Joshua Younger**, *Cyber runs*, Brookings, 2019.

Eisenbach, Thomas M, Anna Kovner, and Michael Junho Lee, "Cyber risk and the US financial system: A pre-mortem analysis," *Journal of Financial Economics*, 2022, *145* (3), 802–826.

——, ——, and ——, "When It Rains, It Pours: Cyber Vulnerability and Financial Conditions," 2024.

Fainmesser, Itay P, Andrea Galeotti, and Ruslan Momot, "Digital privacy," *Available at SSRN*, 2019.

Insights, Deloitte, "Pursuing Cybersecurity Maturity at Financial Institutions," *Survey Spotlights Key Traits among More Advanced Risk Managers. Results and Analysis from the Second Annual FS-ISAC/Deloitte Cyber Risk Services CISO Survey. Deloitte. March*, 2019, *2.*

Kashyap, Anil K and Anne Wetherilt, "Some principles for regulating cyber risk," in "AEA Papers and Proceedings," Vol. 109 American Economic Association 2014 Broadway, Suite 305, Nashville, TN 37203 2019, pp. 482–487.

Kotidis, Antonis and Stacey Schreft, "The Propagation of Cyberattacks through the Financial System: Evidence from an Actual Event," 2023.

Modi, Kosha, Mr Nicola Pierri, Mr Yannick Timmer, Maria Martinez Peria, and Mr Maria Soledad Martinez Peria, "The anatomy of banks' IT investments: Drivers and implications," 2022.

Pirrong, Craig, "The economics of central clearing: theory and practice," 2011.

A Tables

Table 1: Impact of JPMC Exit on BNY's IT Expenditure

	BNY IT Expenditure$_{it}$ Assets$_{it}$ (1)	(2)
Exit$_t$	-0.212***	-0.181***
	(-6.026)	(-6.025)
Big IT$_t$	1.205***	1.094***
	(6.714)	(7.329)
Reform$_t$		0.052***
		(3.234)
Constant	0.088***	0.086***
	(3.991)	(4.808)
R^2	0.72	0.83
Observations	21	21

t statistics in parentheses

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

This table provides regression estimates of the change in BNY's IT expenditure, normalized by assets, in the period following JPMC's exit. BIG IT$_t$ is the average IT expenditure over assets of the top 30 banks by size excluding BNY and JPMC.Significance at the 10% level is denoted by *; 5%, by **; and 1%, by ***

Table 2: Impact of JPMC Exit on BNY's IT Expenditure

	IT Expenditure_{it} Assets_{it}			
	(1)	(2)	(3)	(4)
$\text{Exit}_t \times$ BNY	-0.144***	-0.128***	-0.148***	-0.131***
	(-3.078)	(-2.663)	(-3.274)	(-2.833)
$\text{Reform}_t \times$ BNY		0.061		0.059
		(1.554)		(1.570)
Firm FE	Y	Y	Y	Y
Time FE	N	N	Y	Y
R^2	0.80	0.80	0.82	0.82
Observations	629	629	629	629

t statistics in parentheses

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

This table provides panel regression estimates of the change in IT expenditure of the top 30 banks (excluding JPMC), normalized by assets, in the period following JPMC's exit. Significance at the 10% level is denoted by *; 5%, by **; and 1%, by ***

B Proofs

Proof of Proposition 1. The attacker chooses a_p to maximize $\sqrt{\phi(s_p)a_p}H(\mu_p) - c_a a_p$. The solution is given by the FOC:

$$\sqrt{a_p} = \frac{1}{2c_a}\sqrt{\phi(s_p)}H(\mu_p).$$

Consequently, the successful attack probability is

$$\chi_p = \sqrt{\phi(s_p)a_p} = \sqrt{\phi(s_p)}\frac{1}{2c_a}\sqrt{\phi(s_p)}H(\mu_p) = \frac{1}{2c_a}\phi(s_p)H(\mu_p).$$

□

Proof of Proposition 2. Recall that the relative payoff of choosing 1 over 2 for trader t is

$$\begin{aligned}
&x_t + F(\mu_1 - \mu_2) - T(\mu_1\chi_1 - \mu_2\chi_2) \\
=&x_t + F(\mu_1 - \mu_2) - T\left(\mu_1\frac{1}{2c_a}\phi(s_1)H(\mu_1) - \mu_2\frac{1}{2c_a}\phi(s_2)H(\mu_2)\right) \\
=&x_t + \mu_1\left(2F - t\frac{1}{2c_a}(\phi(s_1)+\phi(s_2))\right) + \left(-F\mu + T\frac{1}{2c_a}(\mu\phi(s_2) + h(\phi(s_1)-\phi(s_2)))\right)
\end{aligned}$$

Suppose that all traders join the same platform, say 1. This is an equilibrium if and only if trader t with $x_t = -\sigma$ prefers platform 1 when $\mu_1 = \eta_1 + \lambda$. Call this corner-1 PCE (platform choice equilibrium) of the subgame. The condition is given by

$$0 < -\sigma + (\lambda+\eta_1)\left(2F - T\frac{1}{2c_a}(\phi(s_1)+\phi(s_2))\right) + \left(-F\mu + T\frac{1}{2c_a}(\phi(s_2)\mu + h(\phi(s_1)-\phi(s_2)))\right). \tag{43}$$

Similar argument holds for platform 2. If trader with $x_t = \sigma$ prefers platform 2 when $\mu_1 = \eta_1$, then all traders join platform 2 is an equilibrium. Call this corner-2 PCE. The condition is

$$0 > \sigma + \eta_1\left(2F - T\frac{1}{2c_a}(\phi(s_1)+\phi(s_2))\right) + \left(-F\mu + t\frac{1}{2c_a}(\phi(s_2)\mu + h(\phi(s_1)-\phi(s_2)))\right). \tag{44}$$

Next suppose there is a marginal trader who is indifferent between the two platforms, t^*. Denote $x^* = x_{t^*}$. Call the corresponding equilibrium the interior PCE.

By the uniformity of x_t, $\frac{\sigma - x^*}{2\sigma}$ fraction of traders have $x_t > x^*$ and prefer 1 over 2, whereas $\frac{x^*+\sigma}{2\sigma}$ fraction of traders have $x_t < x^*$ and prefer 2 over 1, implying $\mu_1 = \eta_1 + \lambda\frac{\sigma - x^*}{2\sigma}$. Equivalently, $x^* = \sigma - \frac{2\sigma}{\lambda}(\mu_1 - \eta_1)$. Then for an interior solution to $\mu_1 \in [\eta_1, \eta_1 + \lambda]$, we have

$$\begin{aligned}
0 &= x^* + \mu_1 \left(2F - T\frac{1}{2c_a}(\phi(s_1) + \phi(s_2))\right) + \left(-F\mu + t\frac{1}{2c_a}(\mu\phi(s_2) + h(\phi(s_1) - \phi(s_2)))\right) \\
&= \sigma - \frac{2\sigma}{\lambda}(\mu_1 - \eta_1) + \mu_1 \left(2F - T\frac{1}{2c_a}(\phi(s_1) + \phi(s_2))\right) + \left(-F\mu + t\frac{1}{2c_a}(\mu\phi(s_2) + h(\phi(s_1) - \phi(s_2)))\right) \\
&= \mu_1 \left(-\frac{2\sigma}{\lambda} + 2F - T\frac{1}{2c_a}(\phi(s_1) + \phi(s_2))\right) + \left(\sigma + \frac{2\sigma}{\lambda}\eta_1 - F\mu + T\frac{1}{2c_a}(\phi(s_2)\mu + h(\phi(s_1) - \phi(s_2)))\right)
\end{aligned} \tag{45}$$

Here (45) is linear in μ_1. So (45)$= 0$ has a solution μ_1 between $[\eta_1, \eta_1 + \lambda]$ if and only if 0 lies on the interval between the two extreme values of (45) obtained by plugging in η_1 and $\eta_1 + \lambda$ into μ_1. Notice that if we substitute in $\mu_1 = \lambda + \eta_1$, (45) becomes exactly (43). If we substitute in $\mu_1 = \eta_1$, (45) becomes exactly (44). Together this implies the corner value of (45) at η_1 is (44) and at $\eta_1 + \lambda$ is (45). Summarizing,

- If (43)$> 0 >$(44), both corners are PCEs and there is also an interior PCE given by (45)$= 0$. There are three PCEs.
- If (43)$< 0 <$(44), neither corners are PCEs and the unique PCE given by (45)$= 0$.
- If (43),(44)> 0, only corner-1 is a PCE.
- If (43),(44)< 0, only corner-2 is a PCE.

Define $\tilde{\mu}_1$ as the solution to (45)$= 0$. Under $F < \frac{\sigma}{\lambda}$, (45) is decreasing in μ_1. Then the possible cases are

- If (43)$< 0 <$(44), the unique PCE is (45)$= 0$, and $\mu_1 = \min\{\eta_1 + \lambda, \max\{\eta_1, \tilde{\mu}_1\}\} = \tilde{\mu}_1$.
- If (43),(44)> 0, corner-1 is the unique PCE and $\mu_1 = \eta_1 + \lambda$. In this case, $\tilde{\mu}_1 > \eta_1 + \lambda$ since (45) is decreasing and (43)> 0. So $\mu_1 = \min\{\eta_1 + \lambda, \max\{\eta_1, \tilde{\mu}_1\}\} = \eta_1 + \lambda$.
- If (43),(44)< 0, corner-2 is the unique PCE and $\mu_1 = \eta_1$. In this case, $\tilde{\mu}_1 < \eta_1$ since (45) is decreasing and (44)< 0. So $\mu_1 = \min\{\eta_1 + \lambda, \max\{\eta_1, \tilde{\mu}_1\}\} = \eta_1$.

Altogether, the general solution is $\mu_1 = \min\{\eta_1 + \lambda, \max\{\eta_1, \tilde{\mu}_1\}\}$. Finally, it follows that

$$\begin{aligned}\tilde{\mu}_1 &= \frac{\sigma + \frac{2\sigma}{\lambda}\eta_1 - F\mu + T\frac{1}{2c_a}\left(\phi(s_2)\mu + h\left(\phi(s_1) - \phi(s_2)\right)\right)}{\frac{2\sigma}{\lambda} - 2F + T\frac{1}{2c_a}\left(\phi(s_1) + \phi(s_2)\right)} \\ &= \frac{\sigma + \frac{2\sigma}{\lambda}\left(\eta_1 - h\right) + \left(\mu - 2h\right)\left(T\frac{1}{2c_a}\phi(s_2) - F\right)}{\frac{2\sigma}{\lambda} - 2F + T\frac{1}{2c_a}\left(\phi(s_1) + \phi(s_2)\right)} + h.\end{aligned}$$

□

Proof of Proposition 3. Conditioning on an interior equilibrium, given s_2, platform 1 maximizes $\tilde{\mu}_1 - c_s s_1$. First we show that $\tilde{\mu}_1 - h$ is log-concave in s_1. After some normalization, there are positive constants C and C' such that

$$\tilde{\mu}_1 - h = \frac{C}{C' + \mathrm{e}^{-Gs_1}}$$

In particular,

$$\begin{aligned}CgT\frac{1}{2c_a} &= \sigma + \frac{2\sigma}{\lambda}\left(\eta_1 - h\right) + \left(\mu - 2h\right)\left(T\frac{1}{2c_a}\phi(s_2) - F\right) \\ C'gT\frac{1}{2c_a} &= \frac{2\sigma}{\lambda} - 2F + T\frac{1}{2c_a}\phi(s_2).\end{aligned}$$

Note $C' > 0$ by $\sigma > \lambda A$. In the interior equilibrium, $\tilde{\mu}_1 \geq \eta_1 > h$, so $C > 0$. Then $\ln\left(\tilde{\mu}_1 - h\right) = \ln C - \ln\left(C' + \mathrm{e}^{-Gs_1}\right)$. The derivative of $\ln\left(\tilde{\mu}_1 - h\right)$ w.r.t. s_1 is then

$$\frac{G\mathrm{e}^{-Gs_1}}{C' + \mathrm{e}^{-Gs_1}} = G\left(1 - \frac{C'}{C' + \mathrm{e}^{-Gs_1}}\right).$$

This is clearly decreasing in s_1. So $\tilde{\mu}_1 - h$ is log-concave, thus quasi-concave. Then so is $\tilde{\mu}_1 - c_s s_1$. Hence, the FOC gives the solution to the unconstrained problem of maximizing $\tilde{\mu}_1 - c_s s_1$ under parametric restrictions that yield an interior equilibrium.

The FOC for 1's problem is

$$-\frac{\sigma + \frac{2\sigma}{\lambda}\left(\eta_1 - h\right) + \left(\mu - 2h\right)\left(T\frac{1}{2c_a}\phi(s_2) - F\right)}{\left(\frac{2\sigma}{\lambda} - 2F + T\frac{1}{2c_a}\left(\phi(s_1) + \phi(s_2)\right)\right)^2}T\frac{1}{2c_a}\phi'(s_1) - c_s = 0.$$

Notice $\phi'(s_1) = -FGe^{-Gs_1} = -G\phi(s_1)$. So

$$\frac{\sigma + \frac{2\sigma}{\lambda}(\eta_1 - h) + (\mu - 2h)\left(T\frac{1}{2c_a}\phi(s_2) - F\right)}{\left(\frac{2\sigma}{\lambda} - 2F + T\frac{1}{2c_a}(\phi(s_1) + \phi(s_2))\right)^2} T\frac{1}{2c_a}G\phi(s_1) - c_s = 0. \tag{46}$$

This similarly holds for platform 2. Denote

$$X_p = \sigma + \frac{2\sigma}{\lambda}(\eta_p - h) - (\mu - 2h)F.$$

So we have

$$\frac{X_1 + (\mu - 2h)T\frac{1}{2c_a}\phi(s_2)}{\left(\frac{2\sigma}{\lambda} - 2F + T\frac{1}{2c_a}(\phi(s_1) + \phi(s_2))\right)^2}\frac{TG}{2c_a}\phi(s_1) = c_s = \frac{X_2 + (\mu - 2h)\left(T\frac{1}{2c_a}\phi(s_1)\right)}{\left(\frac{2\sigma}{\lambda} - 2F + T\frac{1}{2c_a}(\phi(s_1) + \phi(s_2))\right)^2}\frac{TG}{2c_a}\phi(s_2). \tag{47}$$

Thus, we have

$$\left(X_1 + (\mu - 2h)\left(T\frac{1}{2c_a}\phi(s_2)\right)\right)\phi(s_1) = \left(X_2 + (\mu - 2h)\left(T\frac{1}{2c_a}\phi(s_1)\right)\right)\phi(s_2) \implies X_1\phi(s_1) = X_2\phi(s_2).$$

This also implies $\phi(s_1) = \frac{X_2}{X_1}\phi(s_2)$. Plugging this back into the FOC for 1 in (47), we get

$$\begin{aligned} c_s &= \frac{X_1 + (\mu - 2h)T\frac{1}{2c_a}\phi(s_2)}{\left(\frac{2\sigma}{\lambda} - 2F + T\frac{1}{2c_a}(\phi(s_1) + \phi(s_2))\right)^2}\frac{TG}{2c_a}\phi(s_1) \\ \implies \frac{2c_a c_s}{TG} &= \frac{X_1 + (\mu - 2h)T\frac{1}{2c_a}\phi(s_2)}{\left(\frac{2\sigma}{\lambda} - 2F + T\frac{1}{2c_a}\phi(s_2)\left(\frac{X_1+X_2}{X_1}\right)\right)^2}\frac{X_2}{X_1}\phi(s_2). \end{aligned} \tag{48}$$

Note that $X_1 + X_2 = (\mu - 2h)\left(\frac{2\sigma}{\lambda} - 2F\right)$. Then

$$\frac{2\sigma}{\lambda} - 2F + T\frac{1}{2c_a}\phi(s_2)\left(\frac{X_1 + X_2}{X_1}\right) = \frac{\frac{2\sigma}{\lambda} - 2F}{X_1}\left(X_1 + (\mu - 2h)T\frac{1}{2c_a}\phi(s_2)\right).$$

Therefore, (48) becomes

$$
\begin{aligned}
\frac{2c_a c_s}{TG} &= \frac{X_1 + (\mu - 2h)\, t \frac{1}{2c_a} \phi(s_2)}{\left(\frac{\frac{2\sigma}{\lambda} - 2F}{X_1} \left(X_1 + (\mu - 2h)\, T \frac{1}{2c_a} \phi(s_2) \right) \right)^2} \frac{X_2}{X_1} \phi(s_2) \\
&= \frac{X_1 X_2}{\left(\frac{2\sigma}{\lambda} - 2F \right)^2 \left(X_1 + (\mu - 2h)\, T \frac{1}{2c_a} \phi(s_2) \right)} \phi(s_2) \\
\implies X_1 X_2 \phi(s_2) &= \frac{2c_a c_s}{TG} \left(\frac{2\sigma}{\lambda} - 2F \right)^2 \left(X_1 + (\mu - 2h)\, T \frac{1}{2c_a} \phi(s_2) \right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\phi(s_2) &= \frac{\frac{2c_a c_s}{TG} \left(\frac{2\sigma}{\lambda} - 2F \right)^2 X_1}{X_1 X_2 - \frac{2c_a c_s}{TG} \left(\frac{2\sigma}{\lambda} - 2F \right)^2 (\mu - 2h)\, T \frac{1}{2c_a}} \\
&= \frac{2c_a c_s \left(\frac{2\sigma}{\lambda} - 2F \right)}{TG\, (\mu - 2h)} \times \frac{1}{1 - \frac{1}{\left(\frac{2\sigma}{\lambda} - 2F \right) (\mu - 2h)} X_1 - \frac{1}{X_1} \left(\frac{2\sigma}{\lambda} - 2F \right) \frac{c_s}{G}}.
\end{aligned}
$$

This means,

$$
s_2 = \frac{1}{G} \ln \left(1 - \frac{\lambda X_1}{2\, (\sigma - \lambda F)} \frac{1}{\mu - 2h} - \frac{2\, (\sigma - \lambda F)}{\lambda X_1} \frac{c_s}{G} \right) + \frac{1}{G} \ln \frac{TgG\, (\mu - 2h)\, \lambda}{4 c_a c_s\, (\sigma - \lambda)}.
$$

Note that we have

$$
\begin{aligned}
\frac{\lambda X_1}{2\, (\sigma - \lambda A)\, (\mu - 2h)} &= \frac{\lambda \left(\sigma + \frac{2\sigma}{\lambda} (\eta_1 - h) - (\mu - 2h)\, A \right)}{2\, (\sigma - \lambda A)\, (\mu - 2h)} \\
&= \frac{\lambda\sigma + 2\sigma\, (\eta_1 - h) - \sigma\, (\eta_1 + \eta_2 + \lambda - 2h)}{2\, (\sigma - \lambda A)\, (\mu - 2h)} + \frac{1}{2} \\
&= \frac{\sigma\, (\eta_1 - \eta_2)}{2\, (\sigma - \lambda A)\, (\mu - 2h)} + \frac{1}{2} = N_2
\end{aligned}
$$

So

$$
\begin{aligned}
\phi(s_2) &= \frac{2c_a c_s \left(\frac{2\sigma}{\lambda} - 2F \right)}{TG\, (\mu - 2h)} \frac{1}{1 - \frac{1}{\left(\frac{2\sigma}{\lambda} - 2F \right) (\mu - 2h)} X_1 - \frac{1}{X_1} \left(\frac{2\sigma}{\lambda} - 2F \right) \frac{c_s}{G}} \\
&= \frac{2c_a c_s \left(\frac{2\sigma}{\lambda} - 2F \right)}{TG\, (\mu - 2h)} \frac{1}{1 - N_2 - \frac{1}{N_2} \frac{c_s}{G(\mu - 2h)}}. \qquad (49)
\end{aligned}
$$

and

$$s_2 = \frac{1}{G} \ln \left(1 - N_2 - \frac{1}{N_2} \frac{c_s}{G(\mu - 2h)} \right) + \frac{1}{G} \ln \frac{TgG(\mu - 2h)\lambda}{4c_a c_s (\sigma - \lambda F)}. \tag{50}$$

This is the joint solution to the unconstrained optimization problems for platforms 1 and 2, with a necessary condition for an interior equilibrium ($s_p \geq 0$ and $\tilde{\mu}_p \in [\eta_p, \eta_p + \lambda]$ for both platforms) if it exists. □

Proof of Theorem 2. We continue with the proof of Proposition 3. Recall the FOC of platform 1:

$$\frac{X_1 + (\mu - 2h) T \frac{1}{2c_a} \phi(s_2)}{\left(\frac{2\sigma}{\lambda} - 2F + T \frac{1}{2c_a} (\phi(s_1) + \phi(s_2)) \right)^2} \frac{TG}{2c_a} \phi(s_1) = c_s$$

Notice that

$$\begin{aligned}
\tilde{\mu}_1 - h &= \frac{\sigma + \frac{2\sigma}{\lambda}(\eta_1 - h) + (\mu - 2h)\left(T \frac{1}{2c_a} \phi(s_2) - A \right)}{\frac{2\sigma}{\lambda} - 2F + T \frac{1}{2c_a} (\phi(s_1) + \phi(s_2))} \\
&= \frac{X_1 + (\mu - 2h) T \frac{1}{2c_a} \phi(s_2)}{\frac{2\sigma}{\lambda} - 2F + T \frac{1}{2c_a} (\phi(s_1) + \phi(s_2))}
\end{aligned}$$

so

$$\frac{X_1 + (\mu - 2h) T \frac{1}{2c_a} \phi(s_2)}{\left(\frac{2\sigma}{\lambda} - 2F + T \frac{1}{2c_a} (\phi(s_1) + \phi(s_2)) \right)^2} = \frac{(\tilde{\mu}_1 - h)^2}{X_1 + (\mu - 2h) B \frac{1}{2c_a} \phi(s_2)}$$

So we have

$$\begin{aligned}
(\tilde{\mu}_1 - h)^2 &= \frac{2c_s c_a}{TG} \frac{X_1 + (\mu - 2h) T \frac{1}{2c_a} \phi(s_2)}{\phi(s_1)} \\
&= \frac{2c_s c_a}{TG} \frac{X_1 + (\mu - 2h) T \frac{1}{2c_a} \frac{X_1}{X_2} \phi(s_1)}{\phi(s_1)} \\
&= \frac{2c_s c_a}{TG} X_1 \left(\frac{1}{\phi(s_1)} + (\mu - 2h) B \frac{1}{2c_a} \frac{1}{X_2} \right)
\end{aligned}$$

Plug in $\phi(s_1)$ and cancel out terms to find

$$
\begin{aligned}
&(\tilde{\mu}_1 - h)^2 = \frac{2c_s c_a}{TG} X_1 \left(\frac{1}{\phi(s_1)} + (\mu - 2h)\, T \frac{1}{2c_a} \frac{1}{X_2} \right) \\
&= \frac{2c_s c_a}{TG} X_1 \left(\frac{TG\,(\mu - 2h)}{2c_a c_s \left(\frac{2\sigma}{\lambda} - 2F\right)} \left(1 - \frac{1}{\left(\frac{2\sigma}{\lambda} - 2F\right)(\mu - 2h)} X_1 - \frac{1}{X_1} \left(\frac{2\sigma}{\lambda} - 2F \right) \frac{c_s}{G} \right) + (\mu - 2h)\, T \frac{1}{2c_a} \frac{1}{X_2} \right) \\
&= X_1 (\mu - 2h) \left(\frac{1}{\frac{2\sigma}{\lambda} - 2F} \left(\frac{X_1}{\left(\frac{2\sigma}{\lambda} - 2F\right)(\mu - 2h)} \right) \right) = \left(\frac{X_1}{\frac{2\sigma}{\lambda} - 2F} \right)^2 .
\end{aligned}
$$

Then $\left(\tilde{\mu}_p - h\right)^2 = \left(\frac{X_p}{\frac{2\sigma}{\lambda} - 2A}\right)^2$ for both p. Interior equilibrium also necessitates $\tilde{\mu}_p \geq \eta_p > h$. So $\tilde{\mu}_p - h > 0$. Interior equilibrium also necessitates $\phi(s_p) > 0$ for both p. We also have $X_1\phi(s_1) = X_2\phi(s_2)$. So X_1 and X_2 have the same sign. If both are negative, then $\tilde{\mu}_p - h = \frac{-X_p}{\frac{2\sigma}{\lambda} - 2F}$. Adding these up over p gives $\mu - 2h = -\frac{X_1 + X_2}{\frac{2\sigma}{\lambda} - 2F} = 2h - \mu$, which is a contradiction. So $X_1, X_2 > 0$. Then in any interior equilibrium

$$
\begin{aligned}
\tilde{\mu}_1 &= \frac{X_1}{\frac{2\sigma}{\lambda} - 2F} + h = \frac{\sigma + \frac{2\sigma}{\lambda}(\eta_1 - h) - (\mu - 2h) F}{\frac{2\sigma}{\lambda} - 2F} + h \\
&= \frac{\sigma + \frac{2\sigma}{\lambda}\eta_1 - \mu F}{\frac{2\sigma}{\lambda} - 2F} = \frac{\sigma - \lambda F + F(\eta_1 - \eta_2)}{\frac{2\sigma}{\lambda} - 2F} + \eta_1 \\
&= \frac{1}{2} \frac{F}{\frac{\sigma}{\lambda} - F} (\eta_1 - \eta_2) + \frac{\lambda}{2} + \eta_1
\end{aligned}
$$

Consequently, we have

$$
\tilde{\mu}_1 = \frac{X_1}{\frac{2\sigma}{\lambda} - 2F} + h = \eta_1 + \frac{\lambda}{2} \left(1 + (\eta_1 - \eta_2) \frac{F}{\sigma - \lambda F} \right), \tag{51}
$$

and for which we provide the conditions under which the equilibrium is interior in Proposition (9). □

Proposition 9. *The quantities (50) and (51) yield the equilibrium if and only if* $1 > N_p + \frac{1}{N_p} \frac{c_s}{G(\mu - 2h)} + \left(\frac{\sigma}{\lambda} - F\right) \frac{c_s}{G(\mu - 2h)} \frac{4c_a}{Tg}$ *for both* p*, and* $\frac{\sigma}{F} - \lambda \geq |\eta_1 - \eta_2|$.

Assume that there is some $h' > h$ *such that* $\eta_1, \eta_2 \geq h'$*. Then under*

$$
\frac{c_s}{G} < \left(\frac{\mu - 2h}{2h' - 2h} + \left(\frac{\sigma}{\lambda} - F \right) \frac{4c_a}{Tg} \right)^{-1} (h' - h)
$$

the equilibrium is interior if and only if $\frac{\sigma}{F} - \lambda \geq |\eta_1 - \eta_2|$.

Proof of Proposition 9. We have $h < \eta_1, \eta_2$ so $2h < \mu$. If $1 - N_p - \frac{1}{N_p}\frac{c_s}{G(\mu-2h)} \leq 0$ for some p, then (49) can not hold so there is no interior equilibrium. Suppose $1 - N_p - \frac{1}{N_p}\frac{c_s}{G(\mu-2h)} > 0$ for both p. So (49) is positive. In this case, keeping $\tilde{\mu}_1, \tilde{\mu}_2$ relaxed, an interior s_1, s_2 is characterized by (49)$\leq F$ for both p. Notice $1 - N_p - \frac{1}{N_p}\frac{c_s}{G(\mu-2h)} > 0$ and (49)$\leq F$ can be jointly written as $\frac{2c_a c_s\left(\frac{2\sigma}{\lambda}-2F\right)}{TG(\mu-2h)g} < 1 - N_p - \frac{1}{N_p}\frac{c_s}{G(\mu-2h)}$.

Given (51), the condition $\tilde{\mu}_p \in [\eta_p, \eta_p + \lambda]$ for both p is simply equivalent to $\frac{\sigma}{F} - \lambda \geq |\eta_1 - \eta_2|$. So the necessary and sufficient conditions for the equilibrium to be interior are (a) s_p and $\tilde{\mu}_p$ given by (50) and (51), (b) $1 > N_p + \frac{1}{N_p}\frac{c_s}{G(\mu-2h)} + \left(\frac{\sigma}{\lambda} - F\right)\frac{c_s}{G(\mu-2h)}\frac{4c_a}{Tg}$ for both p, and (c) $\frac{\sigma}{F} - \lambda \geq |\eta_1 - \eta_2|$.

Take any h' such that $\eta_1, \eta_2 \geq h' > h$. Let $\varepsilon = \frac{h'-h}{\mu-2h}$. Suppose that (c) holds. Wlog let $\eta_1 \geq \eta_2$. Then (c) is $\frac{\sigma}{F} - \lambda \geq \eta_1 - \eta_2$. Then we have

$$
\begin{aligned}
N_1 &= \frac{1}{2} - \frac{\sigma}{2(\sigma - \lambda F)(\mu - 2h)}(\eta_1 - \eta_2) \\
&= \frac{1}{2}\left(1 - \frac{\eta_1 - \eta_2}{\mu - 2h}\left(1 + \frac{1}{\frac{\sigma}{F\lambda} - 1}\right)\right) \\
&> \frac{1}{2}\left(1 - \frac{\eta_1 - \eta_2}{\mu - 2h}\left(1 + \frac{1}{\frac{\eta_1 - \eta_2}{\lambda}}\right)\right) \\
&= \frac{1}{2}\left(1 - \frac{\eta_1 - \eta_2 + \lambda}{\mu - 2h}\right) \\
&\geq \frac{1}{2}\left(1 - \frac{\eta - 2h' + \lambda}{\mu - 2h}\right) = \varepsilon
\end{aligned}
$$

and

$$
N_2 = \frac{1}{2} - \frac{\sigma}{2(\sigma - \lambda F)(\mu - 2h)}(\eta_2 - \eta_1) > \frac{1}{2} > \varepsilon
$$

This is, (c) implies $[N_p > 2\varepsilon$ for both $p]$.

Note $N_1 + N_2 = 1$. If [$N_p > 2\varepsilon$ for both p], then [$N_p < 1 - 2\varepsilon$ for both p]. Then

$$N_p + \frac{1}{N_p} \frac{c_s}{G(\mu - 2h)} + \left(\frac{\sigma}{\lambda} FA\right) \frac{c_s}{G(\mu - 2h)} \frac{4c_a}{Tg} < 1 - 2\varepsilon + \frac{1}{2\varepsilon} \frac{c_s}{G(\mu - 2h)} + \left(\frac{\sigma}{\lambda} - F\right) \frac{c_s}{G(\mu - 2h)} \frac{4c_a}{Tg}$$
$$= 1 - 2\varepsilon + \frac{c_s}{G(\mu - 2h)} \left(\frac{1}{2\varepsilon} + \left(\frac{\sigma}{\lambda} - F\right) \frac{4c_a}{Tg}\right)$$
$$< 1 - \varepsilon \iff$$
$$\frac{c_s}{G} < \overline{C} \equiv \left(\frac{1}{2\varepsilon} + \left(\frac{\sigma}{\lambda} - F\right) \frac{4c_a}{Tg}\right)^{-1} \varepsilon(\mu - 2h).$$

Thus under $\frac{c_s}{G} < \overline{C}$, [$N_p > 2\varepsilon$ for both p] implies (b).

Then, under $\frac{c_s}{G} < \overline{C}$, (c) implies (b). Therefore, under $\frac{c_s}{G} < \overline{C}$, the equilibrium is interior if and only if (c) holds, and the equilibrium quantities are given by (a). □

Proof of Proposition 6. Set $h' = f$ in Proposition 9 and assume

$$\frac{c_s}{G} < \overline{C} = \left(\frac{1}{2\varepsilon} + \left(\frac{\sigma}{\lambda} - A\right) \frac{4c_a}{Tg}\right)^{-1} \varepsilon(\mu - 2h)$$
$$= \left(\frac{\mu - 2h}{2(f - h)} + \left(\frac{\sigma}{\lambda} - A\right) \frac{4c_a}{Tg}\right)^{-1} (f - h)$$

Then by Proposition 9, the equilibrium is interior if and only if $\frac{\sigma}{A} - \lambda \geq |\eta_1 - \eta_2|$. Note $\frac{\sigma}{A} - \lambda \geq |\eta_1 - \eta_2| \iff \tilde{\mu}_p \in [\eta_p, \eta_p + \lambda]$.

WLOG, let $\eta_1^t > \eta_2^t$. Note $\frac{\sigma}{F} - \lambda > \eta_1^{t+1} - \eta_2^{t+1} \implies \mu_1^{t+1} = \tilde{\mu}_1^{t+1} < \eta_1^{t+1} + \lambda^{t+1}$ and $\frac{\sigma}{F} - \lambda < \eta_1^{t+1} - \eta_2^{t+1} \implies \tilde{\mu}_1^{t+1} > \mu_1^{t+1} = \eta_p^{t+1} + \lambda^{t+1}$.

Denote $\eta_{*p}^t = \eta_p^t - f$, $\mu_{*p}^t = \mu_p^t - f$, $\mu_*^t = \mu^t - 2f$. Note $\eta_1^{t+1} - \eta_2^{t+1} = \eta_{*1}^{t+1} - \eta_{*2}^{t+1} = (1 - \alpha)\left(\mu_{*1}^t - \mu_{*2}^t\right) = (1 - \alpha)\left(2\mu_{*1}^t - \mu_*\right) = 2(1 - \alpha)\left(\mu_{*1}^t - \frac{\mu_*}{2}\right)$.

Lemma 4. *If $\frac{\sigma}{F} - \lambda \geq 2(1 - \alpha)\left|\mu_{*p}^t - \frac{\mu_*}{2}\right|$, which is equivalent to $\frac{\sigma}{F} - \lambda \geq \eta_1^{t+1} - \eta_2^{t+1}$, which is* *equivalent to $\mu_1^{t+1} < \eta_1^{t+1} + \lambda^{t+1}$, we have*

$$
\begin{aligned}
\mu_p^{t+1} &= \eta_p^{t+1} + \frac{\lambda^{t+1}}{2}\left(1 + \left(\eta_p^{t+1} - \eta_{p'}^{t+1}\right)\frac{F}{\sigma - \lambda^{t+1}F}\right) \\
&= (1-\alpha)\,\mu_{*p}^t + f + \frac{\alpha\mu_*}{2}\left(1 + (1-\alpha)\left(\mu_{*p}^t - \mu_{*p'}^t\right)\frac{F}{\sigma - \alpha\mu_* F}\right) \\
\mu_{*p}^{t+1} &= (1-\alpha)\,\mu_{*p}^t + \frac{\alpha\mu_*}{2}\left(1 + (1-\alpha)\left(\mu_{*p}^t - \mu_{*p'}^t\right)\frac{F}{\sigma - \alpha\mu_* F}\right) \\
&= (1-\alpha)\,\mu_{*p}^t + \frac{\alpha\mu_*}{2}\left(1 + (1-\alpha)\left(2\mu_{*p}^t - \mu_*\right)\frac{F}{\sigma - \alpha\mu_* F}\right) \\
&= (1-\alpha)\,\mu_{*p}^t\left(1 + \frac{\alpha\mu_*}{2}2\frac{F}{\sigma - \alpha\mu_* F}\right) + \frac{\alpha\mu_*}{2}\left(1 - (1-\alpha)\,\mu_*\frac{F}{\sigma - \alpha\mu_* F}\right) \\
&= \mu_{*p}^t\frac{(1-\alpha)\,\sigma}{\sigma - \alpha\mu_* F} + \frac{\alpha\mu_*}{2}\left(\frac{\sigma - \mu_* F}{\sigma - \alpha\mu_* F}\right) \\
\mu_{*p}^{t+1} - \frac{\mu_*}{2} &= \mu_{*p}^t\frac{(1-\alpha)\,\sigma}{\sigma - \alpha\mu_* F} + \frac{\alpha\mu_*}{2}\left(\frac{\sigma - \mu_* F}{\sigma - \alpha\mu_* F}\right) - \frac{\mu_*}{2} \\
&= \frac{(1-\alpha)\,\sigma}{\sigma - \alpha\mu_* F}\left(\mu_{*p}^t - \frac{\mu_*}{2}\right) = R\left(\mu_{*p}^t - \frac{\mu_*}{2}\right)
\end{aligned}
$$

Lemma 5. *If $\frac{\sigma}{F} - \lambda < 2(1-\alpha)\left|\mu_{*p}^t - \frac{\mu_*}{2}\right|$, which is equivalent to $\frac{\sigma}{F} - \lambda > \eta_1^{t+1} - \eta_2^{t+1}$, which is equivalent to $\mu_1^{t+1} = \eta_1^{t+1} + \lambda^{t+1}$, we have*

$$
\begin{aligned}
\mu_2^{t+1} &= \eta_2^{t+1} = (1-\alpha)\,\mu_{*2}^t + f \\
\mu_{*2}^{t+1} &= (1-\alpha)\,\mu_{*2}^t
\end{aligned}
$$

Using these lemmas, we analyze four cases.

Case 1: $\frac{\sigma}{F} - \lambda \geq 2(1-\alpha)\left|\mu_{*p}^0 - \frac{\mu_*}{2}\right|$ and $\sigma > \mu_* F$. In this case, by Lemma 4, equilibrium starts and remains interior

$$
\mu_{*p}^t = \frac{\mu_*}{2} + R^t\left(\mu_{*p}^0 - \frac{\mu_*}{2}\right) \implies \mu_p^t = \frac{\mu}{2} + R^t\left(\mu_p^0 - \frac{\mu}{2}\right)
$$

Case 2: $\frac{\sigma}{F} - \lambda \geq 2(1-\alpha)\left|\mu_{*p}^0 - \frac{\mu_*}{2}\right|$ and $\sigma < \mu_* F$. In this case, by Lemma 4 equilibrium starts and remains interior until

$$
\left|\mu_{*p}^t - \frac{\mu_*}{2}\right| = \left(1 - \alpha\frac{\sigma - \mu_* F}{\sigma - \alpha\mu_* F}\right)^t\left|\mu_p^0 - \frac{\mu_*}{2}\right| > \frac{\frac{\sigma}{F} - \lambda}{2(1-\alpha)}
$$

for the first time. This happens at $\bar{t}$ by definition of $\bar{t}$. After this point $\bar{t}$, equilibrium turns to and remains corner with $\mu_{*2}^{t'+\bar{t}} = (1-\alpha)^{t'} \mu_{*2}^{\bar{t}}$. Thus

$$\mu_2^{t'+\bar{t}} = f + (1-\alpha)^{t'} \left(\mu_2^{\bar{t}} - f\right) = f + (1-\alpha)^{t'} \left(\frac{\mu_*}{2} + R^{\bar{t}} \left(\mu_{*2}^0 - \frac{\mu_*}{2}\right) - f\right)$$
$$= f + (1-\alpha)^{t'} \left(\frac{\mu}{2} - R^{\bar{t}} \left(\frac{\mu}{2} - \mu_2^0\right)\right)$$

Case 3: $\frac{\sigma}{F} - \lambda < 2(1-\alpha)\left|\mu_{*p}^0 - \frac{\mu_*}{2}\right|$. In this case, the equilibrium starts and remains corner and $\mu_{*2}^t = (1-\alpha)^t \mu_{*2}^0$. So

$$\mu_2^t = f + (1-\alpha)^t \left(\mu_2^0 - f\right)$$

□