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Abstract

Economic growth depends on combining existing inputs into new inventions. However, these combinations are not always physically feasible. One important example is dark silicon, where power and cooling limits prevent all the transistors on a chip from operating at once. I develop a growth model motivated by these limitations and establish conditions for a unique balanced growth path where electronic miniaturization and power efficiency grow at the same rate. When power-efficiency research becomes harder, both total research employment and productivity growth fall on the new balanced growth path. With gradual adjustment of research employment, I construct a local equilibrium transition in which more researchers temporarily coexist with slower productivity growth. This Eroom's law result holds for small shocks in a two-dimensional design space with sufficiently similar industry constraints.

JEL classification: O31, O33, O41

Key words: miniaturization, power efficiency, directed innovation, economic growth

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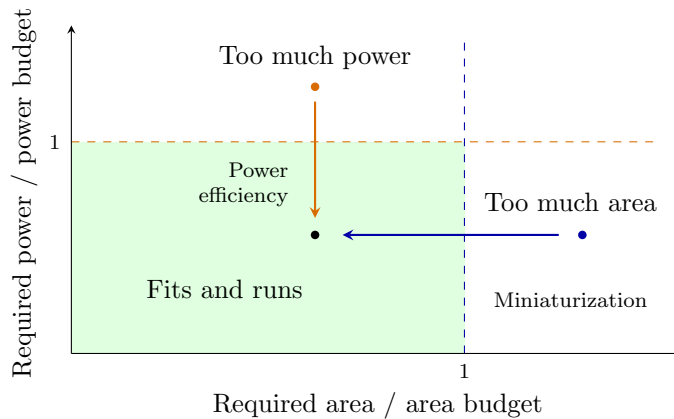
To view the authors' disclosure statements, visit
https://www.newyorkfed.org/research/staff_reports/sr1209.html.

1 Introduction

Moore’s law describes the increase in the number of components on a chip (Moore, 1965). Advances in information technology contributed to the U.S. productivity resurgence from the mid-1990s through the early 2000s (Jorgenson et al., 2008), before productivity growth slowed in the mid-2000s (Fernald, 2014). During the earlier period, Dennard scaling allowed smaller transistors to operate at lower voltage and higher speed while keeping power per unit of chip area constant (Dennard et al., 1974). As voltage reductions became harder, power and cooling limits increasingly constrained the use of additional transistors. This creates dark silicon: transistors may fit on a chip but cannot all operate at once within its power budget (Esmaeilzadeh et al., 2011). These developments connect to Eroom’s law: sustaining technological progress requires increasing research effort. Bloom et al. (2020) document declining research productivity across industries, including semiconductors, where maintaining Moore’s law has required a substantial increase in researchers.

I model a new chip design as a combination of electronic components that satisfies certain size and power efficiency constraints. Firms can produce any design that satisfies both constraints. As researchers relax these constraints, more designs become feasible and can be produced as specialized varieties. The increase in variety raises productivity, following Romer (1990). Figure 1 gives a simple example of the constraints in the model. A manufacturer has a fixed amount of space and a fixed power budget. The shaded rectangle contains the designs that it can use. A design outside the rectangle becomes usable when reductions in its size or power needs bring it inside.

Figure 1: A Design Must Both Fit and Operate



Note: The horizontal and vertical coordinates give a design’s area and power requirements, each divided by the corresponding industry budget. Only designs in the shaded rectangle can be used. The arrows illustrate how miniaturization or lower power requirements can bring a design into that rectangle.

In this model, I show that there exists a balanced growth path (BGP), and—under certain regularity conditions—the BGP is unique. In the BGP, area and power requirements for specified electronic functions decline at the same rate. Their ratio is constant because research responds to whichever constraint is tighter. This is an equilibrium result of the model; the engineering relationship underlying Dennard scaling motivates the two constraints.

One of the main consequences of the model is a mechanism through which the end of Dennard scaling and the dark silicon problem can contribute to the productivity slowdown. Proposition 1 shows that—when it is harder to find new ideas for power efficiency—a larger share of researchers works on power efficiency, while total research employment and productivity growth fall on the new balanced growth path. The harder problem takes a larger share of a smaller research workforce; the number of researchers in that activity need not rise. Continued miniaturization does not fully make up for the loss, because power constraints prevent the use of many newly miniaturized designs.

I also construct a local equilibrium transition that links the end of Dennard scaling to Eroom’s law. Following a sufficiently small shock, total research employment temporarily exceeds its pre-shock level while productivity growth is below its previous rate. The result holds with a two-dimensional design space and one industry, and persists when several industries have sufficiently similar constraints. New miniaturization researchers enter while existing power-research projects wind down. The equilibrium eventually converges to the new BGP with lower research employment and productivity growth.

1.1 Related Work

The model builds on work in which new combinations drive economic growth. [Weitzman \(1998\)](#) studies the production of ideas by combining existing knowledge, while [Acemoglu and Azar \(2020\)](#) study production using different sets of input suppliers. The number of feasible combinations then determines the range of specialized inputs available for production. [Azar \(2022\)](#) imposes size constraints on combinations and estimates miniaturization’s contribution to productivity growth over 1960–2019, with the largest effects in the 1990s and early 2000s. In this paper, I microfound [Romer \(1990\)](#)’s expanding varieties model using both size and power constraints to study how power limits can reduce the growth gains from continued miniaturization. While the model takes inspiration from previous combinatorial growth papers, it fits within [Romer \(1990\)](#)’s framework and allows us to derive a BGP.

[Jones \(2011\)](#) studies how complementarity lets a weak link in a production chain depress output. In my model, complementarity arises because of two interlocking constraints: chips must be small enough and power efficient enough to be used. The same researchers can

relax either of the constraints that prevent a design from being used. Progress on one dimension changes the return to improving the other. Complementary research tasks can already generate an aggregate slowdown when one task becomes harder.

The allocation of research between area and power is a directed technical change problem in the sense of [Acemoglu \(2002\)](#). In his CES benchmark, fixed substitution parameters govern how technical change affects relative factor rewards. Here, the effective elasticity of substitution between technological capacities is endogenous: component requirements determine its dependence on relative capacity, and research allocation determines where the economy operates on that relationship. [Appendix C](#) derives the implied substitution elasticity. This concerns substitution between area and power capacity; the CES elasticity across produced varieties remains a primitive. [Lemma 1](#) derives the underlying design-volume elasticities and their switching thresholds, yielding uniqueness when industries’ switching intervals form a connected set. The geometry also gives a finite area-capacity threshold beyond which miniaturization adds no designs at fixed power capacity.

Semiconductors are also a general-purpose technology, as studied by [Helpman and Trajtenberg \(1998\)](#); [Jovanovic and Rousseau \(2005\)](#). In the model, several industries use the same components. Whether an improvement makes a new design possible in a given industry depends on that industry’s space and power budgets.

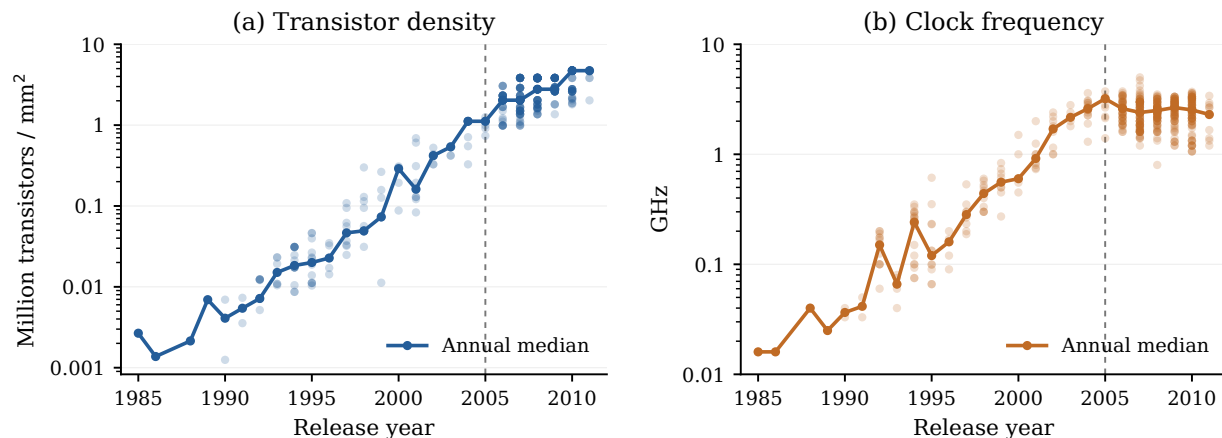
The paper proceeds as follows. [Section 2](#) presents the motivating facts, [Section 3](#) introduces the model, [Section 4](#) solves for the BGP, [Section 5](#) analyzes the equilibrium outside the BGP, and shows how Eroom’s law can appear in the transition dynamics. [Section 6](#) concludes.

2 Engineering Constraints and Motivating Facts

I begin with the engineering changes behind the model and some evidence on research and productivity. These observations motivate the model. Moore’s law describes the increase in the number of components on a chip ([Moore, 1965](#)). For a time, making transistors smaller also allowed their voltage to fall. Under Dennard scaling, these reductions increased switching speed while keeping power per unit of chip area constant ([Dennard et al., 1974](#)). A chip could hold more transistors without producing more heat per unit of area. In [Figure 2](#), transistor density keeps rising, but processor clock speeds level off in the mid-2000s.

Once Dennard scaling ended, voltage no longer fell as it had before, even as more transistors fit on a chip. Power limits could then prevent all the transistors that fit on a chip from operating together, the problem known as dark silicon ([Esmaeilzadeh et al., 2011](#)). In [Figure 3](#), the cooling requirements measured by TDP increased proportionally with chip

Figure 2: Transistor Density and Processor Clock Speed



Note: CPUDB processor records released in 1985–2011 with positive transistor count, die area, thermal design power, and clock frequency (560 records). Dots are records and lines are annual medians; both vertical axes are logarithmic. The dashed line marks 2005 as a reference year. *Source:* [Danowitz et al. \(2012\)](#).

area between 1985–2004. But after 2005, the cooling requirements for even moderately sized chips jumped to the top of the range, regardless of size.

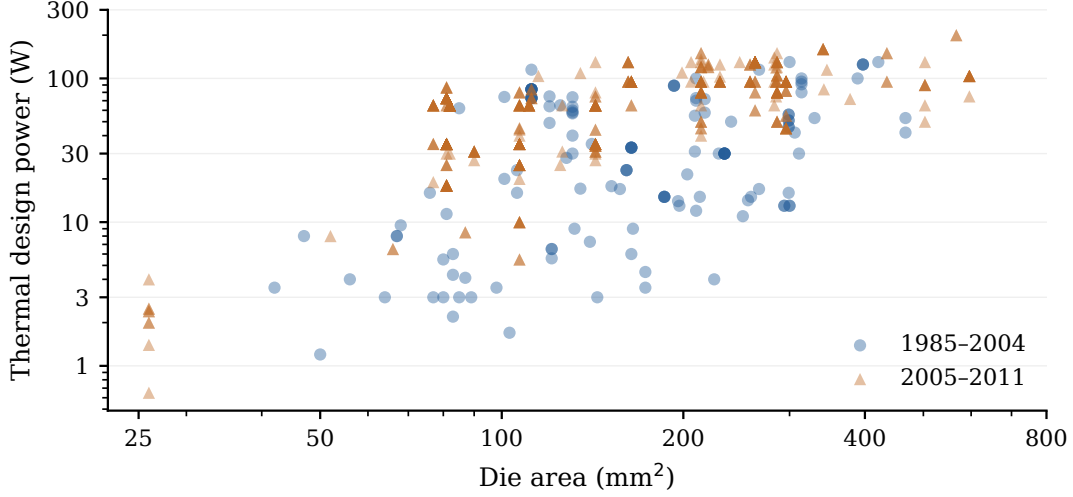
Research effort and patenting. The broad and narrow semiconductor research-input series of [Bloom et al. \(2020\)](#), measured as R&D divided by a wage deflator, rise by factors of 1.62 and 1.15 between their 1995–2004 and 2005–2014 averages. Power-saving patent tags also become more frequent. Among granted patents with semiconductor classifications, their share rises from 0.060 percent in filing years 1995–2004 to 0.128 percent in 2005–2014 (78 of 129,479 patents and 245 of 191,874). Adding computing and logic circuits to the universe gives an increase from 1.195 to 2.019 percent.¹

Productivity growth. [Fernald \(2014\)](#) documents slower TFP growth in IT-producing industries, IT-intensive users, and private business. Duration-weighted averages of the published intervals fall from 14.44 to 6.98 percent, 2.08 to 0.90 percent, and 1.85 to 0.78 percent, respectively, between 1995–2004 and 2004–2011.²

¹PatentsView April 2026 current CPC and application tables, covering grants through December 2025 and downloaded September 8, 2026 ([PatentsView, 2026](#)). I count a patent as a semiconductor patent if it has an H01L or H10 code. For the broader count I also include G06F and H03K. I identify power-saving inventions by G06F 1/32 or H03K 19/0016, including their coded subgroups. The counts use filing years, with each granted patent counted once in each universe.

²I use Table 1 of [Fernald \(2014\)](#), rows 7, 9, and 1. The periods 1995–2000 and 2000–2004 last five and four years, respectively; I use these durations as weights in the earlier average. The corresponding weights for 2004–2007 and 2007–2011 are three and four years. IT-intensive users exclude IT producers, natural resources, construction, finance, insurance, and real estate.

Figure 3: Processor Area and Power Requirements



Note: Die area and thermal design power (TDP) for the same 560 CPUDB records as Figure 2, on logarithmic axes. TDP is a design power rating used to size a cooling system. The points compare heterogeneous processors, including differences in performance and intended use. Between 1985 and 2004, TDP grew with the size of the chip, while after 2005, TDP jumped to the top of the range even for moderately sized chips. *Source:* Danowitz et al. (2012).

3 The Model

3.1 Environment

Time $t \in \mathbb{R}_{++}$ is continuous. There is one representative household and a competitive producer of the final good. Intermediate industries are indexed by $i \in \mathcal{I}$, a finite set. The household has one unit of labor, which it supplies inelastically. It consumes the final good and has preferences

$$U_0 = \int_0^\infty e^{-\rho t} \log C_t dt,$$

where $\rho > 0$ is the discount rate. All firms belong to the household, which receives their profits.

Initial technology stocks and the household's ownership of their patents are given. Write B_t for the market value of the household's patent claims. Their initial value B_0 is determined by equilibrium patent prices; it is not specified independently. At those prices, the household chooses consumption and asset holdings subject to

$$\dot{B}_t = r_t B_t + w_t - C_t.$$

Labor income is w_t , since the household supplies one unit. B_t is subject to the no-Ponzi

condition

$$\liminf_{T \rightarrow \infty} \exp \left\{ - \int_0^T r_u du \right\} B_T \geq 0.$$

The household's optimal plan also satisfies the transversality condition

$$\lim_{T \rightarrow \infty} e^{-\rho T} \frac{B_T}{C_T} = 0.$$

The first-order conditions for the household's problem imply $\frac{\dot{C}_t}{C_t} = r_t - \rho$.

The final good is produced competitively from the outputs of the intermediate industries:

$$Y_t = \prod_{i \in \mathcal{I}} Y_{it}^{\omega_i}, \quad \omega_i > 0, \quad \sum_{i \in \mathcal{I}} \omega_i = 1.$$

We take the final good as the numeraire. Research uses labor but no final goods, so final-good market clearing requires

$$C_t = Y_t.$$

Each intermediate industry produces a different type of good, such as medical devices, communications units, or vehicle controllers. The industry's blueprints specify combinations of electronic functions. Some combinations take too much space or need too much power. Reductions in these requirements allow firms to produce designs they could not previously use.

Let $R_{e,t}$ denote labor devoted to power-efficiency research and $R_{s,t}$ labor devoted to miniaturization. All other workers produce intermediate goods:

$$\sum_{i \in \mathcal{I}} L_{it} + R_{e,t} + R_{s,t} = 1.$$

An improvement is available to all industries. The same improvement may admit new designs in one industry and none in another.

3.2 Area and Power Constraints

I describe a blueprint by a vector of components, $x = (x_1, \dots, x_d) \in \mathbb{R}_+^d$, where x_j gives the amount of component j . Each industry i has different requirements for the specific components. For industry i , each component j occupies a_{ij} much space and requires b_{ij} units of power. The vectors $a_i, b_i \in \mathbb{R}_{++}^d$ give area and power per function, each normalized by the total amount of area and power that can be provided by the electronics packaging in

industry i . At time $t = 0$, the size and power constraints in industry i are therefore given by

$$\underbrace{s_0 a_i \cdot x \leq 1}_{\text{area budget}}, \quad \underbrace{e_0 b_i \cdot x \leq 1}_{\text{power budget}}.$$

At time t , area and power requirements are scaled by s_t and e_t , respectively. Summing the requirements of the functions in a blueprint gives a time- t constraint

$$\underbrace{s_t a_i \cdot x \leq 1}_{\text{area budget}}, \quad \underbrace{e_t b_i \cdot x \leq 1}_{\text{power budget}}.$$

Each constraint defines a simplex in design space. The feasible region is their intersection. Define

$$\mathcal{S}_{it} = \{x \in \mathbb{R}_+^d : s_t a_i \cdot x \leq 1\}, \quad (1)$$

$$\mathcal{E}_{it} = \{x \in \mathbb{R}_+^d : e_t b_i \cdot x \leq 1\}. \quad (2)$$

The feasible design region is

$$\mathcal{F}_{it} = \mathcal{S}_{it} \cap \mathcal{E}_{it}. \quad (3)$$

Figure 4 illustrates this construction in two design dimensions.

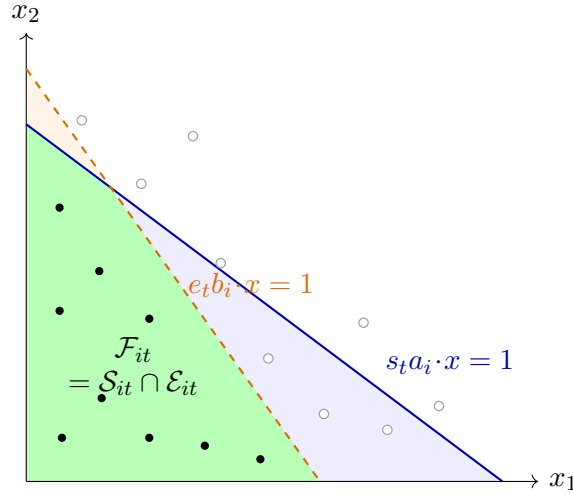


Figure 4: The feasible design region

Note: Each point is a potential blueprint. The solid blue line is the area constraint and the dashed orange line is the power constraint. The green region is their intersection. Filled points are feasible; hollow points violate at least one constraint.

3.3 Production and the Static Equilibrium

The equilibrium analysis uses a continuum of potential blueprints in each industry, with measure $\mu_i dx$ on $\mathbb{R}_+^d$. To motivate this specification, first consider a Poisson sample Π_i with intensity μ_i , evaluated along a fixed capacity path. The corresponding continuum production technology is stated below. At each date t , every feasible blueprint in industry i can be operated as a distinct specialized variety. Let $\mathcal{J}_{it} = \Pi_i \cap \mathcal{F}_{it}$ denote the set of feasible blueprints. The number of available varieties is $N_{it} = |\mathcal{J}_{it}|$. In this sampling illustration, sufficiently large initial expected counts $\mu_i |\mathcal{F}_{i0}|$ make the probability that any industry has no feasible blueprint arbitrarily small.³ A variety $\omega \in \mathcal{J}_{it}$ is produced using labor according to $y_{i\omega t} = \ell_{i\omega t}$.

Industry i combines the outputs of its feasible varieties using the constant-elasticity-of-substitution technology

$$Y_{it} = \left(\sum_{\omega \in \mathcal{J}_{it}} y_{i\omega t}^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}}, \quad \varepsilon > 1.⁴$$

The different blueprints are imperfect substitutes. Consequently, an industry with access to more blueprints can produce more output from a given amount of labor. This is the familiar gain from specialization in an expanding-variety model.

When $N_{it} > 0$, feasible varieties enter symmetrically and have the same marginal cost of production, so the static equilibrium allocates industry i 's labor equally across them. If $L_{it} = \sum_{\omega \in \mathcal{J}_{it}} \ell_{i\omega t}$, then $\ell_{i\omega t} = \frac{L_{it}}{N_{it}}$. Substituting this allocation into the production function gives

$$Y_{it} = \left[N_{it} \left(\frac{L_{it}}{N_{it}} \right)^{(\varepsilon-1)/\varepsilon} \right]^{\varepsilon/(\varepsilon-1)} = N_{it}^{1/(\varepsilon-1)} L_{it}.$$

Industry productivity is therefore $A_{it} \equiv N_{it}^{\frac{1}{\varepsilon-1}}$. The productivity formula extends to $N_{it} = 0$ by setting $A_{it} = 0$, without dividing labor by the number of varieties.

The Poisson structure links the number of varieties available to industry i to the volume of its feasible design region. Since the Poisson process has constant intensity $\mu_i > 0$, we have $N_{it} \sim \text{Poisson}(\mu_i |\mathcal{F}_{it}|)$. Define the deterministic-equivalent number of varieties by $\bar{N}_{it} \equiv \mu_i |\mathcal{F}_{it}|$. In particular, $\Pr(N_{it} = 0) = e^{-\bar{N}_{it}}$. As the expected number of feasible blueprints becomes large, $\frac{N_{it}}{\bar{N}_{it}} \xrightarrow{p} 1$. Define deterministic-equivalent industry productivity by

³For any tolerance $\delta \in (0, 1)$, it suffices that $\mu_i |\mathcal{F}_{i0}| \geq \log(|\mathcal{I}|/\delta)$ for every industry. The Poisson formula and the union bound then give $\Pr(\exists i \in \mathcal{I} : N_{i0} = 0) \leq \sum_{i \in \mathcal{I}} e^{-\mu_i |\mathcal{F}_{i0}|} \leq \delta$. Since research expands feasible regions, an industry with a feasible blueprint initially retains at least one at all subsequent dates.

⁴The sum over an empty set is zero. Thus, if no blueprint is feasible in industry i , then $N_{it} = 0$ and $Y_{it} = 0$: the industry cannot produce until a blueprint becomes feasible. This case is part of the Poisson specification, which assigns positive probability to an empty feasible set.

$\bar{A}_{it} \equiv \bar{N}_{it}^{\frac{1}{\varepsilon-1}}$. The realized productivity process remains a jump process, but $\frac{A_{it}}{\bar{A}_{it}} \xrightarrow{P} 1$ as the expected number of feasible blueprints becomes large. These are concentration statements at fixed dates along the specified capacity path.

Combining industry production with the final-good technology determines the allocation of labor across industries and aggregate productivity at each date. Let $L_t = 1 - R_{e,t} - R_{s,t}$ denote labor available for production. When every industry has at least one feasible blueprint, the competitive allocation satisfies $L_{it} = \omega_i L_t$. Aggregate output is therefore $Y_t = L_t \prod_{i \in \mathcal{I}} (\omega_i A_{it})^{\omega_i}$. Apart from the constant term generated by the expenditure shares, aggregate productivity is $A_t = \prod_{i \in \mathcal{I}} A_{it}^{\omega_i}$. Define its deterministic equivalent as $\bar{A}_t \equiv \prod_{i \in \mathcal{I}} \bar{A}_{it}^{\omega_i}$. If any industry has no feasible blueprint, its zero output implies $Y_t = 0$ under the Cobb–Douglas final-good technology. The aggregate output formula therefore extends to this case, although the labor allocation need not be unique.

For the equilibrium analysis, feasible blueprints instead have the exact continuum mass $\bar{N}_{it} = \int_{\mathcal{F}_{it}} \mu_i dx = \mu_i |\mathcal{F}_{it}|$. Replacing the sums above by integrals gives the production technology

$$Y_{it} = \left(\int_{\mathcal{F}_{it}} y_{ixt}^{(\varepsilon-1)/\varepsilon} \mu_i dx \right)^{\varepsilon/(\varepsilon-1)}, \quad L_{it} = \int_{\mathcal{F}_{it}} \ell_{ixt} \mu_i dx,$$

where $y_{ixt} = \ell_{ixt}$. Symmetry gives $Y_{it} = \bar{N}_{it}^{1/(\varepsilon-1)} L_{it}$ and the same labor-allocation formulas derived above. Patent profits and values below are measured per unit of blueprint mass. I retain the bars and the term “deterministic equivalent,” but these are exact objects of this continuum economy. Positive capacities and $\mu_i > 0$ give positive feasible mass in every industry. All equilibrium, transition, and welfare results concern this economy; no convergence of finite Poisson equilibria is assumed.

3.4 Research and the Evolution of Technological Constraints

Researchers can work on making electronic functions smaller or on reducing the power they need to operate. All industries use these functions, so an improvement can make designs feasible in more than one industry.

Let $R_{e,t}$ and $R_{s,t}$ denote labor devoted to power efficiency and miniaturization, respectively. Their effects on the two technological constraints are

$$-\frac{\dot{e}_t}{e_t} = \chi_e R_{e,t}^\alpha \quad \text{and} \quad -\frac{\dot{s}_t}{s_t} = \chi_s R_{s,t}^\alpha,$$

where $\chi_e, \chi_s > 0$ measure research productivity and $\alpha \in (0, 1)$. Here e_t scales the power needed for a specified electronic function at a fixed performance level. It is not total chip

power or TDP, which also depend on the functions and performance chosen. The parameter χ_e measures how effectively research reduces that requirement. A lower χ_e represents costlier power-efficiency improvements after voltage scaling becomes harder. This experiment abstracts from the engineering coupling between miniaturization and power: the two research activities remain separate, so their common growth rate can be restored at a lower level.

A reduction in e_t or s_t expands the feasible region. In the continuum economy, the added region $\mathcal{F}_{i,t+dt} \setminus \mathcal{F}_{it}$ contains additional blueprint mass. Hence

$$d\bar{N}_{it} = \mu_i d|\mathcal{F}_{it}|. \quad (4)$$

Research brings these existing blueprints into production. Its return depends on the size of the added region. Define the power and area elasticities of feasible volume as

$$\eta_{e,it} = -\frac{\partial \log |\mathcal{F}_{it}|}{\partial \log e_t}, \quad \eta_{s,it} = -\frac{\partial \log |\mathcal{F}_{it}|}{\partial \log s_t}. \quad (5)$$

To derive aggregate growth, take logarithms of aggregate productivity and substitute the industry productivity formula:

$$\begin{aligned} \log \bar{A}_t &= \sum_{i \in \mathcal{I}} \omega_i \log \bar{A}_{it} \\ &= \frac{1}{\varepsilon - 1} \sum_{i \in \mathcal{I}} \omega_i (\log \mu_i + \log |\mathcal{F}_{it}|). \end{aligned}$$

Since μ_i , ω_i , and ε are constant, the total time derivative is

$$\frac{d}{dt} \log \bar{A}_t = \frac{\dot{\bar{A}}_t}{\bar{A}_t} = \frac{1}{\varepsilon - 1} \sum_{i \in \mathcal{I}} \omega_i \frac{d}{dt} \log |\mathcal{F}_{it}|.$$

With industry requirements fixed, feasible volume changes through e_t and s_t . Applying the chain rule in logarithms gives

$$\begin{aligned} \frac{d}{dt} \log |\mathcal{F}_{it}| &= \frac{\partial \log |\mathcal{F}_{it}|}{\partial \log e_t} \frac{\dot{e}_t}{e_t} + \frac{\partial \log |\mathcal{F}_{it}|}{\partial \log s_t} \frac{\dot{s}_t}{s_t} \\ &= -\eta_{e,it} \frac{\dot{e}_t}{e_t} - \eta_{s,it} \frac{\dot{s}_t}{s_t} \\ &= \eta_{e,it} \chi_e R_{e,t}^\alpha + \eta_{s,it} \chi_s R_{s,t}^\alpha. \end{aligned}$$

In the second line I substitute the elasticity definitions; in the third I substitute the research

equations. Substitution into the expression for aggregate growth gives

$$\frac{\dot{\bar{A}}_t}{\bar{A}_t} = \frac{1}{\varepsilon - 1} \sum_{i \in \mathcal{I}} \omega_i (\eta_{e,it} \chi_e R_{e,t}^\alpha + \eta_{s,it} \chi_s R_{s,t}^\alpha). \quad (6)$$

This is the exact productivity growth rate in the continuum economy. Thus, the return to each direction of research depends both on the productivity of research and on the amount of economically relevant design space exposed by moving the corresponding constraint.

4 Solving for Equilibrium

I now solve for the number of researchers and their allocation between the two research activities. An improvement is worth the profits from the designs it makes feasible, so the research problem depends on the production decisions derived in the previous section. The equilibrium conditions must account for these profits as well as the labor required to obtain them.

For this purpose, I assume that a laboratory receives a perpetual exclusive license to each blueprint that its improvement first makes feasible. It may produce the variety itself, using labor, or sell the license to a producer. Either way it receives the value of the monopoly profits. Production must still satisfy the area and power constraints at the required performance. The assignment is unambiguous to first order: with continuous improvements, blueprints first exposed by both boundaries simultaneously have zero first-order measure. This rule is an assumption about ownership in the model, not a characterization of semiconductor patent law. I derive the resulting research rewards in Section 4.4. The two theorems below state the existence and uniqueness results.

For compactness, define $H \equiv \chi_e^{-1/\alpha} + \chi_s^{-1/\alpha}$.

Theorem 1 (Existence of an interior BGP) *Suppose $\rho > 0$, $\varepsilon > 1$, $\alpha \in (0, 1)$, and $\chi_e, \chi_s > 0$. Suppose also that $d \geq 2$ and that a_i and b_i are strictly positive and nonproportional for every industry i . Then the continuum benchmark admits at least one stationary interior balanced-growth configuration with relative capacity $z^* > 0$. It generates an exact balanced-growth equilibrium from any initial stocks satisfying $E_0/S_0 = z^*$. From other initial ratios, reaching this configuration requires a transition. Every such equilibrium has the same total research employment, research allocation, and growth rates.*

Total research employment is the unique $R^ \in (0, 1/\varepsilon)$ satisfying*

$$(\varepsilon - 1)\rho R^* = d \left(\frac{R^*}{H} \right)^\alpha (1 - \varepsilon R^*).$$

The implied allocation and common rate of technological progress are

$$\begin{aligned} R_e^* &= \frac{\chi_e^{-1/\alpha}}{H} R^*, & R_s^* &= \frac{\chi_s^{-1/\alpha}}{H} R^*, \\ L^* &= 1 - R^*, & g^* &= \left(\frac{R^*}{H} \right)^\alpha. \end{aligned}$$

Along every balanced-growth equilibrium,

$$\begin{aligned} -\frac{\dot{e}_t}{e_t} &= -\frac{\dot{s}_t}{s_t} = g^* > 0, \\ g_A^* &= \frac{d}{\varepsilon - 1} g^*. \end{aligned}$$

The existence argument has two parts. First, at a sufficiently low level of research employment, expected patent income makes entry worthwhile. Further entry reduces that income through congestion and the loss of production labor. The free-entry condition is satisfied at a positive employment level where researchers earn the production wage.

Second, the returns must support an allocation that keeps the capacity ratio constant. If power is sufficiently scarce, power research is more profitable than miniaturization; if area is sufficiently scarce, the comparison is reversed. By continuity, there is an intermediate ratio at which both research activities pay the production wage and can deliver equal rates of progress. The less productive activity needs more researchers to achieve that rate. Equal progress then preserves the ratio and the relative returns to research.

The units used to measure progress matter for interpreting this result. Under ideal constant-field scaling, reducing transistor dimensions and voltage by a factor k reduces both transistor area and switching power by k^2 , when switching frequency rises by k (Dennard et al., 1974). In that comparison, performance changes along with transistor dimensions. In the model, performance is specified, and equal progress follows from the allocation of researchers. I do not assume an engineering relationship between the two research technologies. This allows me to consider a decline in the productivity of power research in Section 5.2.

Uniqueness theorem The remaining question is whether only one balance between power and area capacity supports steady growth. As power requirements fall relative to area requirements, industries shift from being limited by power to being limited by area. If the ranges over which they shift overlap or meet, leaving no gaps, the relative incentive to improve power efficiency falls strictly across the combined range. It can then support the required research allocation at only one relative capacity, z^* . The following theorem states this condition in terms of industries' switching intervals.

Theorem 2 (Uniqueness of the BGP) *Maintain the conditions of Theorem 1, and let $z \equiv E/S = s/e$ denote power capacity relative to area capacity. Let z^* denote its constant value along a balanced-growth equilibrium. For each industry i , define its switching interval by*

$$I_i \equiv [\underline{z}_i, \bar{z}_i], \quad \underline{z}_i \equiv \min_{1 \leq j \leq d} \frac{b_{ij}}{a_{ij}}, \quad \bar{z}_i \equiv \max_{1 \leq j \leq d} \frac{b_{ij}}{a_{ij}}. \quad (7)$$

This is the range of relative capacity over which industry i changes from being power constrained to being area constrained. If $\bigcup_{i \in \mathcal{I}} I_i$ is connected, then z^ , the equilibrium relative capacity of the two constraints, is unique. Consequently, the interior balanced-growth equilibrium is unique up to the initial common level of power and area capacity.*

Industries can have different constraints and still satisfy the theorem's condition. Their switching intervals must overlap or meet, so that an increase in relative power capacity always reduces power sensitivity in at least one industry with positive expenditure. Aggregate payoffs then fall strictly across the combined range, giving a single crossing with relative research costs. If there is a gap, payoffs can be constant over it. At the particular research costs consistent with those payoffs, a whole interval of capacity ratios supports balanced growth. The existence result still holds in this case.

The remainder of this section constructs the equilibrium and proves Theorems 1 and 2.

4.1 The Geometry of Technological Progress

Define power and area capacity as $E_t \equiv \frac{1}{e_t}$, $S_t \equiv \frac{1}{s_t}$. The feasible design region can then be written as

$$\mathcal{F}_i(E_t, S_t) = \{x \in \mathbb{R}_+^d : a_i \cdot x \leq S_t, b_i \cdot x \leq E_t\}.$$

Let $\Phi_i(E_t, S_t) \equiv |\mathcal{F}_i(E_t, S_t)|$ denote its volume. The continuum mass of feasible blueprints is $\bar{N}_{it} = \mu_i \Phi_i(E_t, S_t)$.

Recall that the elasticities of feasible design volume with respect to power and area capacity are $\eta_{e,it} = \frac{\partial \log \Phi_i(E_t, S_t)}{\partial \log E_t}$ and $\eta_{s,it} = \frac{\partial \log \Phi_i(E_t, S_t)}{\partial \log S_t}$. Each elasticity gives the proportional response of feasible blueprint mass to a proportional increase in the corresponding capacity.

Multiplying both capacities by $\lambda > 0$ scales every feasible design by the same factor: $\mathcal{F}_i(\lambda E, \lambda S) = \lambda \mathcal{F}_i(E, S)$. Because the design space has dimension d , $\Phi_i(\lambda E, \lambda S) = \lambda^d \Phi_i(E, S)$. Thus, Φ_i is homogeneous of degree d . Euler's theorem implies $\eta_{e,it} + \eta_{s,it} = d$.

Define the economically weighted elasticities $\mathcal{E}_{e,t} = \sum_{i \in \mathcal{I}} \omega_i \eta_{e,it}$ and $\mathcal{E}_{s,t} = \sum_{i \in \mathcal{I}} \omega_i \eta_{s,it}$. Since the final-good expenditure shares sum to one, $\mathcal{E}_{e,t} + \mathcal{E}_{s,t} = d$.

4.2 A Candidate Balanced-Growth Path

We look for an interior balanced-growth equilibrium in which research employment is constant and the relative position of the power and area frontiers is stationary. Let $z_t \equiv \frac{E_t}{S_t} = \frac{s_t}{e_t}$. A larger z_t means that power capacity is abundant relative to area capacity. Stationarity of z_t requires $\frac{\dot{E}_t}{E_t} = \frac{\dot{S}_t}{S_t} \equiv g$. The research technologies therefore imply $\chi_e R_e^\alpha = \chi_s R_s^\alpha = g$. Equal growth is necessary for a constant ratio, but it is not sufficient for equilibrium. One still needs to find a z^* and total research employment that satisfy both free-entry conditions and clear the labor market.

To see how the common capacity growth rate translates into growth in feasible volume, apply the chain rule to $\Phi_{it} = \Phi_i(E_t, S_t)$:

$$\begin{aligned} \frac{\dot{\Phi}_{it}}{\Phi_{it}} &= \frac{E_t}{\Phi_{it}} \frac{\partial \Phi_i}{\partial E}(E_t, S_t) \frac{\dot{E}_t}{E_t} + \frac{S_t}{\Phi_{it}} \frac{\partial \Phi_i}{\partial S}(E_t, S_t) \frac{\dot{S}_t}{S_t} \\ &= \eta_{e,it} g + \eta_{s,it} g = g(\eta_{e,it} + \eta_{s,it}). \end{aligned}$$

The first equality separates the effects of expanding power and area capacity; the second uses their common growth rate. Because Φ_i is homogeneous of degree d , Euler's theorem gives

$$E_t \frac{\partial \Phi_i}{\partial E}(E_t, S_t) + S_t \frac{\partial \Phi_i}{\partial S}(E_t, S_t) = d \Phi_{it}.$$

Dividing by Φ_{it} shows that $\eta_{e,it} + \eta_{s,it} = d$, so feasible volume grows at rate dg . Finally, $\bar{N}_{it} = \mu_i \Phi_{it}$ with constant μ_i , so the deterministic-equivalent number of blueprints grows at the same rate:

$$\frac{\dot{\bar{N}}_{it}}{\bar{N}_{it}} = \frac{\mu_i \dot{\Phi}_{it}}{\mu_i \Phi_{it}} = \frac{\dot{\Phi}_{it}}{\Phi_{it}} = dg.$$

Deterministic-equivalent productivity in every industry, and hence aggregate productivity, grows at rate $g_A \equiv \frac{\dot{\bar{A}}_{it}}{\bar{A}_{it}} = \frac{\dot{A}_t}{A_t} = \frac{d}{\varepsilon-1}g$. Production labor is constant along the balanced-growth path, so final output and consumption also grow at rate g_A . The household Euler equation then gives the constant interest rate $r = \rho + g_A$.

4.3 Profits and the Patent HJB

The monopolistic producer of a blueprint has marginal cost w_t . Its markup under CES demand with elasticity $\varepsilon > 1$ is $\frac{\varepsilon}{\varepsilon-1}$. For industry i , the expenditure share ω_i is divided equally among the symmetric varieties. Revenue from each blueprint is therefore $\frac{\omega_i Y_t}{\bar{N}_{it}}$. Multiplying this revenue by the profit margin $1/\varepsilon$ gives $\pi_{it} = \frac{\omega_i Y_t}{\varepsilon \bar{N}_{it}}$.

Let v_{it} denote the market value of a blueprint in industry i . We are already restricting

attention to a balanced-growth path, so the interest rate $r = \rho + g_A$ is constant. The patent holder's Hamilton–Jacobi–Bellman equation is

$$rv_{it} = \pi_{it} + \dot{v}_{it}. \quad (8)$$

To solve this equation, first use $\pi_{it} = (\omega_i/\varepsilon)Y_t/\bar{N}_{it}$. Taking growth rates gives $\frac{\dot{\pi}_{it}}{\pi_{it}} = \frac{\dot{Y}_t}{Y_t} - \frac{\dot{\bar{N}_{it}}}{\bar{N}_{it}} = g_A - dg \equiv g_\pi$. The first term reflects growth in industry expenditure; the second reflects the dilution of that expenditure across a growing number of blueprints.

Because r and g_π are constant, the no-bubble solution to the linear HJB is proportional to current profits. Thus $\dot{v}_{it} = g_\pi v_{it}$, and substituting into Equation (8) gives

$$\begin{aligned} v_{it} &= \frac{\pi_{it}}{r - g_\pi} \\ &= \frac{\pi_{it}}{r - (g_A - dg)} \\ &= \frac{\pi_{it}}{\rho + dg} \\ &= \frac{\omega_i Y_t}{\varepsilon \bar{N}_{it} (\rho + dg)}. \end{aligned} \quad (9)$$

The other solution to the HJB adds a term proportional to e^{rt} ; the patent no-bubble condition rules out that term. Equation (9) is therefore the unique patent value used in the research-entry conditions below.

4.4 Innovation Values and Free Entry

A marginal increase in power capacity makes $\bar{N}_{it}\eta_{e,it}$ additional blueprints feasible in industry i ; that is, $\partial \bar{N}_{it}/\partial \log E_t = \bar{N}_{it}\eta_{e,it}$. Similarly, $\partial \bar{N}_{it}/\partial \log S_t = \bar{N}_{it}\eta_{s,it}$.

The patent values of marginal power- and area-capacity improvements are

$$\begin{aligned} V_{e,t} &= \sum_{i \in \mathcal{I}} v_{it} \bar{N}_{it} \eta_{e,it}, \\ V_{s,t} &= \sum_{i \in \mathcal{I}} v_{it} \bar{N}_{it} \eta_{s,it}. \end{aligned}$$

Substituting the solved patent value from Equation (9) yields $V_{e,t} = \frac{Y_t}{\varepsilon(\rho+dg)} \mathcal{E}_e$ and $V_{s,t} = \frac{Y_t}{\varepsilon(\rho+dg)} \mathcal{E}_s$. This yields $\frac{V_{e,t}}{V_{s,t}} = \frac{\mathcal{E}_e}{\mathcal{E}_s}$.

Research congestion generates decreasing returns within each direction. Taking aggregate research employment as given, an individual researcher in direction $k \in \{e, s\}$ produces innovations at rate $\chi_k R_{k,t}^{\alpha-1}$. This is the private arrival rate: an atomistic laboratory does

not internalize the congestion its entry imposes on other researchers.

Free entry into each active research direction requires the expected value of innovation to equal the wage:

$$w_t = \chi_e R_{e,t}^{\alpha-1} V_{e,t},$$

$$w_t = \chi_s R_{s,t}^{\alpha-1} V_{s,t}.$$

4.5 Equilibrium Research Employment

To close the labor market, let $L_t = 1 - R_{e,t} - R_{s,t}$ denote employment in intermediate production. The constant CES markup implies that the production wage bill is the fraction $(\varepsilon - 1)/\varepsilon$ of final output: $w_t L_t = \frac{\varepsilon-1}{\varepsilon} Y_t$. The equilibrium production wage is therefore $w_t = \frac{\varepsilon-1}{\varepsilon} \frac{Y_t}{L_t}$.

Combining the solved innovation values with the production wage and the two free-entry conditions gives, for each $k \in \{e, s\}$, $\frac{\varepsilon-1}{L} = \frac{\chi_k \mathcal{E}_k R_k^{\alpha-1}}{\rho + dg}$. Using $\chi_k R_k^\alpha = g$, this becomes $R_k = \frac{\mathcal{E}_k g L}{(\varepsilon-1)(\rho + dg)}$.

Let $R \equiv R_e + R_s$ denote total research employment. Summing the two free-entry conditions and using $\mathcal{E}_e + \mathcal{E}_s = d$ and $L = 1 - R$ gives $R = \frac{dg(1-R)}{(\varepsilon-1)(\rho + dg)}$. Equivalently,

$$(\varepsilon - 1)\rho R = dg(1 - \varepsilon R). \quad (10)$$

The common-growth condition determines the research allocation required to sustain a given g : $R_e = \left(\frac{g}{\chi_e}\right)^{1/\alpha}$ and $R_s = \left(\frac{g}{\chi_s}\right)^{1/\alpha}$. Define $H \equiv \chi_e^{-1/\alpha} + \chi_s^{-1/\alpha}$. It follows that $R = Hg^{1/\alpha}$ and hence $g = \left(\frac{R}{H}\right)^\alpha$.

Substituting this expression into Equation (10) produces a single equation for equilibrium research employment:

$$(\varepsilon - 1)\rho R = d \left(\frac{R}{H}\right)^\alpha (1 - \varepsilon R). \quad (11)$$

Rearrange this as

$$\frac{(\varepsilon - 1)\rho R H^\alpha}{d R^\alpha} + \varepsilon R = 1. \quad (12)$$

Because $\alpha < 1$, both terms on the left hand side strictly increase with R . Note the left hand side is 0 at $R = 0$ and greater than 1 at $R = \frac{1}{\varepsilon}$. Therefore, there is a unique non-zero solution $R^* \in (0, \frac{1}{\varepsilon})$.

The equilibrium rate of semiconductor progress is $g^* = \left(\frac{R^*}{H}\right)^\alpha$, and the allocation of research labor is $R_e^* = \frac{\chi_e^{-1/\alpha}}{H} R^*$ and $R_s^* = \frac{\chi_s^{-1/\alpha}}{H} R^*$.

4.6 The Equilibrium Position of the Two Constraints

Along a candidate balanced-growth path, homogeneity implies that the aggregate power-capacity elasticity depends on E and S only through their ratio $z = E/S$.⁵ Write

$$\mathcal{E}_e(z) \equiv \sum_{i \in \mathcal{I}} \omega_i \left. \frac{\partial \log \Phi_i(E, S)}{\partial \log E} \right|_{E/S=z}.$$

Recall that $R_k = \frac{\mathcal{E}_k g L}{(\varepsilon-1)(\rho+dg)}$ and $R = \frac{dg L}{(\varepsilon-1)(\rho+dg)}$. Dividing R_k by R yields $\frac{\mathcal{E}_e}{d} = \frac{R_e^*}{R^*}$ and $\frac{\mathcal{E}_s}{d} = \frac{R_s^*}{R^*}$. Consequently, the stationary relative capacity z^* satisfies

$$\frac{\mathcal{E}_e(z^*)}{d} = \frac{\chi_e^{-1/\alpha}}{\chi_e^{-1/\alpha} + \chi_s^{-1/\alpha}}. \quad (13)$$

To complete the characterization of the BGP, we need to show that this equation has at least one solution under the conditions of Theorem 1, and exactly one under the conditions of theorem 2.

The required behavior of $\mathcal{E}_e(z)$ follows from the geometry of the two simplices rather than from a separate assumption. The endpoints of the switching interval in Equation (7) are the relative capacities at which the identity of industry i 's binding constraint begins and finishes changing.

Lemma 1 (Single crossing for two simplices) *Suppose $d \geq 2$ and a_i and b_i are not proportional. Then $\eta_{e,i}(z)$ is continuous and nonincreasing in z , with $\eta_{e,i}(z) = d$ for $z \leq \underline{z}_i$, $\eta_{e,i}(z) = 0$ for $z \geq \bar{z}_i$. Moreover, it is strictly decreasing on $(\underline{z}_i, \bar{z}_i)$.*

To interpret the lemma, fix area capacity and start with power capacity low enough that the area constraint does not bind. Increasing power capacity then enlarges the whole feasible simplex. A one-percent increase expands it in all d design dimensions, raising its volume by d percent, so $\eta_{e,i} = d$.

For larger z , part of the feasible set reaches the area limit. In a direction where this has happened, further power improvements have no effect: the area constraint is fixed. Every direction can therefore switch from power constrained to area constrained, but not back again. The elasticity declines with z , although the number of additional blueprints need not decline. Once the area limit has been reached in all directions, $\eta_{e,i} = 0$. Nonproportionality of a_i and b_i means that the switching ratios differ across directions. Appendix A proves these claims for proportional changes in feasible volume.

⁵See Appendix for details.

Figure 5 illustrates this sequence in two design dimensions. The area simplex is fixed while the power simplex expands outward as $z = E/S$ rises.

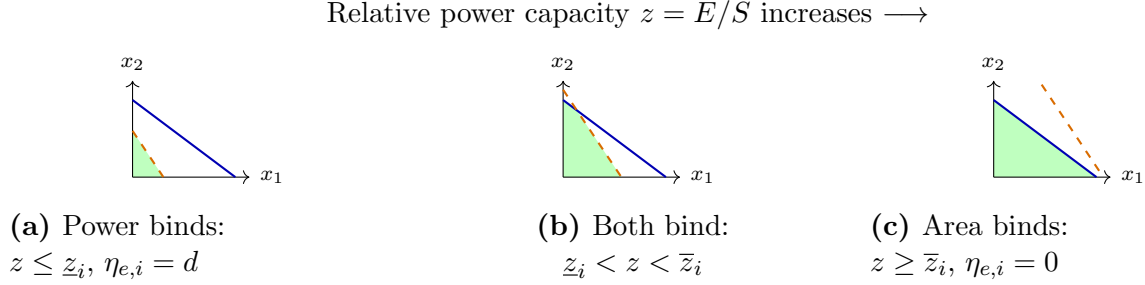


Figure 5: The falling marginal value of power capacity

Note: The solid blue line is the fixed area boundary $a_i \cdot x = 1$; the dashed orange line is the power boundary $b_i \cdot x = z$; and the shaded region is their intersection. In the middle panel, area limits some directions and power limits the others. Expanding the power budget leaves a smaller set of power-constrained directions and reduces $\eta_{e,i}(z)$.

Let $\theta_e \equiv \frac{\chi_e^{-1/\alpha}}{\chi_e^{-1/\alpha} + \chi_s^{-1/\alpha}} \in (0, 1)$. If the nonproportionality condition in Lemma 1 holds for every industry, the lemma implies that $\mathcal{E}_e(z)/d$ is continuous and nonincreasing, equals one for sufficiently small z , and equals zero for sufficiently large z . The intermediate value theorem therefore proves that Equation (13) has at least one solution. Moreover, if the closed intervals $I_i = [\underline{z}_i, \bar{z}_i]$, $i \in \mathcal{I}$, have a connected union, every nontrivial subinterval of the transition range intersects the interior of at least one industry's switching range. In that case $\mathcal{E}_e(z)$ is strictly decreasing over the relevant range and the solution z^* is unique. The vectors a_i and b_i suffice to check the overlap condition. If it fails, a stationary ratio still exists. A gap between switching ranges can, however, support a whole interval of stationary ratios at a particular value of θ_e .

Endogenous substitution between capacities. When both constraints matter, more area capacity can compensate for less power capacity while keeping productivity unchanged: different designs become feasible. The amount of extra room needed depends on the relative abundance of the two capacities. The elasticity of substitution describes how this tradeoff changes. In a CES technology, this elasticity is constant. Here, component requirements and industry weights determine how it varies with relative capacity. Research changes that ratio, so it can also change how readily the two capacities substitute for one another. If every industry is constrained only by power, extra room adds no designs. Appendix C gives the formula and its derivation.

Asset-market consistency. The preceding conditions determine the real allocation without treating household wealth as an additional state variable. Once patent values have been

solved, asset-market clearing simply requires the household to hold the outstanding patents:

$$B_t = \sum_{i \in \mathcal{I}} \bar{N}_{it} v_{it} = \frac{Y_t}{\varepsilon(\rho + dg)}.$$

In the full-appropriation benchmark this identity is consistent with the household budget constraint. To see this, first separate the change in household patent wealth into the value of newly created patents and the capital gain on existing patents:

$$\dot{B}_t = \sum_{i \in \mathcal{I}} v_{it} \dot{\bar{N}}_{it} + \sum_{i \in \mathcal{I}} \bar{N}_{it} \dot{v}_{it}. \quad (14)$$

The patent HJB implies $\dot{v}_{it} = r_t v_{it} - \pi_{it}$. Hence, the second term in Equation (14) equals the required return on the household's existing portfolio less the profits paid out by that portfolio: $\sum_{i \in \mathcal{I}} \bar{N}_{it} \dot{v}_{it} = r_t B_t - \sum_{i \in \mathcal{I}} \bar{N}_{it} \pi_{it} = r_t B_t - \frac{Y_t}{\varepsilon}$. The first term in Equation (14) is the market value of the patents created by current research. By the definitions of $V_{e,t}$ and $V_{s,t}$, the laws of motion for technological capacity, and the two free-entry conditions multiplied by their respective research employment levels,

$$\begin{aligned} \sum_{i \in \mathcal{I}} v_{it} \dot{\bar{N}}_{it} &= V_{e,t} \frac{\dot{E}_t}{E_t} + V_{s,t} \frac{\dot{S}_t}{S_t} \\ &= V_{e,t} \chi_e R_{e,t}^\alpha + V_{s,t} \chi_s R_{s,t}^\alpha \\ &= w_t R_{e,t} + w_t R_{s,t} = w_t R_t. \end{aligned} \quad (15)$$

Combining Equations (14)–(15) gives

$$\dot{B}_t = r_t B_t - \frac{Y_t}{\varepsilon} + w_t R_t. \quad (16)$$

Now use $R_t = 1 - L_t$ and the production-labor wage bill $w_t L_t = (\varepsilon - 1)Y_t/\varepsilon$. The two non-asset terms in Equation (16) become $-\frac{Y_t}{\varepsilon} + w_t R_t = -\frac{Y_t}{\varepsilon} + w_t - w_t L_t = w_t - Y_t$. Because final-good market clearing gives $C_t = Y_t$, the law of motion for patent wealth is therefore $\dot{B}_t = r_t B_t + w_t - C_t$. The resulting expression is the household budget constraint. Thus the household can hold all outstanding patents at the values already derived. There is no additional equation for R^* , g^* , or z^* from the ownership of these assets.

5 Unbalanced Growth and Eroom's Law

In this section, I analyze the equilibrium outside the BGP, and show how Eroom's law can appear in the transition dynamics.

5.1 Unbalanced Technological Progress

Along any differentiable capacity path, power and area capacity need not improve at the same rate. Let $g_{e,t} \equiv \frac{\dot{E}_t}{E_t}$, $g_{s,t} \equiv \frac{\dot{S}_t}{S_t}$. Industry i 's feasible design volume then grows at $\frac{\dot{\Phi}_{it}}{\Phi_{it}} = \eta_{e,it}g_{e,t} + \eta_{s,it}g_{s,t}$. Its deterministic-equivalent productivity growth rate is therefore

$$g_{\bar{A}_{i,t}} = \frac{1}{\varepsilon - 1} (\eta_{e,it}g_{e,t} + \eta_{s,it}g_{s,t}). \quad (17)$$

Using $\eta_{e,it} + \eta_{s,it} = d$, Equation (17) can be written as $g_{\bar{A}_{i,t}}$ equals $[dg_{s,t} + \eta_{e,it}(g_{e,t} - g_{s,t})]/(\varepsilon - 1)$. Consequently, the difference in productivity growth between industries i and j is $g_{\bar{A}_{i,t}} - g_{\bar{A}_{j,t}}$, which equals $(g_{e,t} - g_{s,t})(\eta_{e,it} - \eta_{e,jt})/(\varepsilon - 1)$.

When power capacity grows faster than area capacity, industries whose feasible regions are especially sensitive to the power constraint grow relatively quickly. When miniaturization proceeds faster, growth shifts toward industries whose feasible regions are especially sensitive to the area constraint. Unbalanced semiconductor progress can therefore change which industries lead aggregate productivity growth.

5.2 Harder Power-Efficiency Research and Slower Growth

The end of Dennard scaling illustrated in Section 2 motivates studying harder power-efficiency research. In the model, consider a permanent fall in χ_e , holding χ_s and the other primitives fixed. This allows power efficiency to keep improving, at greater research cost; it is a theoretical experiment, not an estimated historical shock.

Proposition 1 (Harder power-efficiency research) *Maintain the conditions of Theorem 2. A decrease in χ_e to a strictly positive value lowers the balanced-growth rate g^* , productivity growth g_A^* , and total research employment R^* . It raises the share R_e^*/R^* of researchers pursuing power efficiency and lowers stationary relative capacity $z^* = E/S$.*

Proof. A fall in χ_e raises $H = \chi_e^{-1/\alpha} + \chi_s^{-1/\alpha}$. To determine its effect on growth, start from the equilibrium research condition:

$$(\varepsilon - 1)\rho R^* = dg^*(1 - \varepsilon R^*).$$

Collecting the terms that contain R^* gives

$$R^*[(\varepsilon - 1)\rho + d\varepsilon g^*] = dg^*, \quad R^* = \frac{dg^*}{(\varepsilon - 1)\rho + d\varepsilon g^*}.$$

This expression is strictly increasing in g^* . Next, the research technology gives $g^* = (R^*/H)^\alpha$, or $H = R^*/(g^*)^{1/\alpha}$. Substituting for R^* yields

$$H = \frac{d(g^*)^{1-1/\alpha}}{(\varepsilon - 1)\rho + d\varepsilon g^*}. \quad (18)$$

Because $0 < \alpha < 1$, the numerator decreases strictly with g^* and the denominator increases. A higher H therefore requires a lower g^* . It also lowers R^* and $g_A^* = dg^*/(\varepsilon - 1)$.

The power-research share is

$$\frac{R_e^*}{R^*} = \frac{\chi_e^{-1/\alpha}}{\chi_e^{-1/\alpha} + \chi_s^{-1/\alpha}}.$$

It rises as χ_e falls. The ratio condition equates this share to $\mathcal{E}_e(z^*)/d$. The connected switching intervals make the latter strictly decreasing over its interior range. Its increase therefore requires a lower z^* . ■

The proposition distinguishes the power-research share from the number of power researchers. That number need not rise, because total research employment falls. The larger share does not keep growth at its old rate: both capacities grow more slowly in the new equilibrium, and the lower return to new applications supports fewer researchers.

If power-efficiency progress stops. A fixed power capacity gives the limiting case. Suppose $E_t = \bar{E} > 0$ while area capacity can continue to rise. Every feasible design must still lie within

$$\mathcal{E}_i(\bar{E}) = \{x \in \mathbb{R}_+^d : b_i \cdot x \leq \bar{E}\}.$$

This set is bounded because every $b_{ij} > 0$. Hence

$$\bar{N}_{it} \leq \frac{\mu_i \bar{E}^d}{d! \prod_j b_{ij}}, \quad \lim_{T \rightarrow \infty} \frac{\log \bar{A}_T - \log \bar{A}_0}{T} = 0.$$

Productivity is positive and nondecreasing, but the power budget puts a finite upper bound on feasible design mass. The bound gives the second statement and applies whenever power capacity is bounded above.

One can also find the area capacity beyond which miniaturization admits no new designs.

Among designs within the power budget, the largest area requirement is

$$\max_{x \in \mathcal{E}_i(\bar{E})} a_i \cdot x = \bar{E} \max_j \frac{a_{ij}}{b_{ij}}.$$

At this threshold for S_t , every design satisfying the power budget also satisfies the area budget: $\mathcal{F}_i(E_t, S_t) = \mathcal{E}_i(\bar{E})$. Making components smaller adds no designs in that industry. When all industries have reached their thresholds, $\eta_{s,it} = 0$ everywhere, so miniaturization earns no patent income. If power research is also unable to make progress, neither activity is profitable. Fixing area capacity instead gives the analogous result for power-efficiency research.

Proposition 1 studies harder power-efficiency research; a complete stop isolates the limiting logic. Moore’s law expands the designs that can fit. Power efficiency expands the designs that can operate. When power-efficiency improvements become harder, continued miniaturization delivers smaller productivity gains through new applications.

5.3 Gradual Adjustment of Research Employment and Eroom’s Law

Eroom’s law describes falling research productivity despite rising research effort, a pattern documented across industries by Bloom et al. (2020). Proposition 1 shows that harder power-efficiency research reduces both productivity growth and total research employment on the new balanced growth path. I now show that equilibrium can temporarily have more research and slower productivity growth while employment adjusts. The result follows from firms’ entry decisions when researchers remain committed to ongoing projects. I first establish it for a small shock with one industry and two-dimensional designs, then show that it persists with sufficiently similar industry constraints.

Existing research projects may take time to wind down when the direction of technical change shifts. To capture this possibility, suppose that a project commits one researcher until completion, which occurs at Poisson rate $\delta > 0$. The same restriction applies to power-efficiency research and miniaturization. New projects can be started in either direction without a setup cost, subject to the aggregate labor constraint. Completion releases the researcher; it does not terminate patents that the project has already generated.

Let $h_{k,t} \geq 0$ denote entry into research direction $k \in \{e, s\}$. Between entry jumps, research employment satisfies

$$\dot{R}_{k,t} = h_{k,t} - \delta R_{k,t}, \quad h_{k,t} \geq 0, \quad k \in \{e, s\}. \quad (19)$$

Equivalently, both directions face the same lower bound,

$$\dot{R}_{e,t} \geq -\delta R_{e,t}, \quad \dot{R}_{s,t} \geq -\delta R_{s,t}. \quad (20)$$

Upward entry jumps are permitted, whereas downward jumps are not.⁶ The restriction limits contraction rather than expansion. In particular, the power-research constraint is strictly slack whenever $R_{e,t} > 0$ and $\dot{R}_{e,t} \geq 0$. It binds only when no new power-research projects enter and employment falls at rate δ .

Entry and balanced growth. Because projects entail continuing employment commitments, entry during a transition depends on their lifetime value rather than only on their current return. Let $Q_{k,t}$ denote the value of starting a project that employs one researcher. Selling newly generated patents as they become available gives net flow income $\chi_k R_{k,t}^{\alpha-1} V_{k,t} - w_t$. Therefore,

$$(r_t + \delta)Q_{k,t} = \chi_k R_{k,t}^{\alpha-1} V_{k,t} - w_t + \dot{Q}_{k,t}. \quad (21)$$

Free entry requires $Q_{k,t} \leq 0$, with equality whenever new projects enter. Existing projects can have negative value because their commitments cannot be terminated early. Patent values continue to satisfy the patent HJB with the time-varying interest rate; their balanced-growth formula is not imposed during a transition.⁷

On an interior BGP, research employment is constant and replacement entry is $h_k = \delta R_k > 0$ in both directions. Hence $Q_k = \dot{Q}_k = 0$, and Equation (21) reduces to the original free-entry conditions,

$$w = \chi_e R_e^{\alpha-1} V_e = \chi_s R_s^{\alpha-1} V_s.$$

The interior BGP calculation is therefore unchanged. In particular,

$$(\varepsilon - 1)\rho R^* = d \left(\frac{R^*}{H} \right)^\alpha (1 - \varepsilon R^*), \quad H = \chi_e^{-1/\alpha} + \chi_s^{-1/\alpha}. \quad (22)$$

The allocations, growth rates, and stationary relative capacity remain those characterized in Theorems 1–2. The parameter δ affects adjustment but does not enter these balanced-growth conditions.

⁶Formally, $dR_{k,t} = dJ_{k,t} - \delta R_{k,t} dt$, where $J_{k,t}$ is cumulative entry and is nondecreasing. Equation (19) applies where $dJ_{k,t} = h_{k,t} dt$.

⁷Outside a BGP, total household wealth includes ongoing-project values: $\tilde{B}_t = B_t + \sum_{k \in \{e,s\}} R_{k,t} Q_{k,t}$, where B_t retains its definition as patent wealth. Entry has zero value, so combining the project and patent valuation equations gives $\dot{\tilde{B}}_t = r_t \tilde{B}_t + w_t - C_t$. On a BGP, project values are zero and $\tilde{B}_t = B_t$.

An equilibrium transition. The next proposition determines entry after the shock. Research employment and patent values are solved jointly, with existing projects allowed to finish before their researchers can leave.

Proposition 2 (Equilibrium Eroom transition)

Consider one industry with $d = 2$, strictly positive nonproportional area and power requirement vectors, $\rho > 0$, $\varepsilon > 1$, $\alpha \in (0, 1)$, and strictly positive research productivities. Maintain the project-commitment and entry rules above, with fixed $\delta > 0$. Suppose the economy starts on its interior BGP. At date zero, an unanticipated permanent shock changes power-research productivity from χ_e^0 to $\chi_e^1 = \chi_e^0 e^{-h}$, leaving all other primitives unchanged.

For every sufficiently small $h > 0$, there exists an equilibrium transition converging to the new BGP and a nonempty open interval $I_h \subset (0, \infty)$ on which

$$R_{e,t} + R_{s,t} > R^0, \quad g_{A,t} < g_A^0, \quad t \in I_h, \quad (23)$$

where $R^0 = R_e^0 + R_s^0$ and g_A^0 are the pre-shock research employment and productivity growth rate. Miniaturization research jumps upward at the shock, while power research initially receives no new project entry. Both directions have positive entry after this initial phase.

Appendix B constructs the equilibrium and verifies project entry, patent valuation, and household optimality. Slower expected growth in the number of blueprints raises patent values in this local equilibrium and encourages miniaturization entry. The fall in power-research productivity makes new power projects initially unprofitable, but existing power researchers cannot leave immediately. Consequently, total research employment jumps above its pre-shock level.

During the initial phase, power-research employment declines at rate δ , while miniaturization research continues to expand. Total research declines from its initial jump and passes through an interval with more researchers than before the shock but slower productivity growth. The comparison in Equation (23) is with the pre-shock levels; employment need not be increasing during that interval. Productivity growth need not fall at the instant of the shock either. The result guarantees an early interval in which the two inequalities overlap.

Corollary 1 (Similar constraints across industries) *Fix a finite number of industries with positive expenditure weights and $d = 2$. Maintain the restrictions on preferences, research technology, and project commitments in Proposition 2. Around any configuration in which all industries share the same strictly positive nonproportional requirement vectors, there is an open neighborhood of requirement vectors such that the proposition's conclusions*

continue to hold. Each economy starts on its own interior BGP and experiences a sufficiently small permanent surprise decline in power-research productivity.

The proof is also in Appendix B. Thus the Eroom interval persists when industries have sufficiently similar constraints. It is a temporary equilibrium response: the transition converges to the new BGP, where both total research employment and productivity growth are lower.

6 Conclusion

This paper studies how area and power constraints limit the electronic designs available to IT-using industries. I establish conditions for a unique balanced growth path on which miniaturization and power efficiency advance at the same rate. A decline in power-research productivity raises the share of researchers working on power efficiency but reduces total research employment and long-run productivity growth. The model thus provides a mechanism through which the end of Dennard scaling can contribute to the productivity slowdown.

Gradual adjustment of research employment also connects the model to Eroom’s law. For sufficiently small shocks in a two-dimensional design space, I construct an equilibrium transition with an interval of higher research employment and lower productivity growth than before the shock. The result holds for one industry and persists with sufficiently similar industry constraints. New miniaturization researchers enter while inherited power-research projects wind down. The initial employment increase is temporary, with total research eventually converging to the lower level on the new BGP. This gives a local equilibrium mechanism for Eroom’s law.

Online Appendix

A Proof of Lemma 1

Proof. The key is to apply Prékopa's theorem directly to feasible volume in logarithmic coordinates. It shows that log volume is concave in log power capacity. Its slope is the elasticity in the lemma. A geometric argument then establishes strict decrease inside the switching interval.

Suppress the industry index and set $r_j = b_j/a_j$. Rescaling the original design coordinate j by a_j/S gives the normalized feasible set

$$K(z) = \left\{ x \geq 0 : \sum_j x_j \leq 1, \quad \sum_j r_j x_j \leq z \right\}, \quad z = E/S.$$

Write $\phi(z) = |K(z)|$, where $|\cdot|$ denotes d -dimensional volume. The change of variables gives

$$\Phi(E, S) = \frac{S^d}{\prod_j a_j} \phi(E/S), \quad \eta(z) = \frac{z\phi'(z)}{\phi(z)}. \quad (24)$$

Thus the elasticity depends only on relative capacity. Let $\underline{z} = \min_j r_j$ and $\bar{z} = \max_j r_j$. Nonproportionality gives $\underline{z} < \bar{z}$.

Step 1. The endpoint values follow from containment. If $z \leq \underline{z}$, satisfying the power constraint automatically satisfies the area constraint, because

$$\underline{z} \sum_j x_j \leq \sum_j r_j x_j \leq z.$$

The feasible set is the entire power simplex, with volume

$$\phi(z) = \frac{z^d}{d! \prod_j r_j}.$$

Its elasticity is d . If $z \geq \bar{z}$, satisfying the area constraint automatically satisfies the power constraint, because

$$\sum_j r_j x_j \leq \bar{z} \sum_j x_j \leq \bar{z} \leq z.$$

The feasible set is then the fixed area simplex. Its volume is constant, so its elasticity is zero. The next step establishes continuity at the thresholds as well.

Step 2. Continuity follows by dominated convergence. Let $\Delta = \{x \geq 0 : \sum_j x_j \leq 1\}$ be the fixed area simplex, and let $n = r/\|r\|$. Describe each slice in the same hyperplane $n^\perp$ by writing $x = y + (z/\|r\|)n$. Its area is

$$A(z) = \int_{n^\perp} \mathbf{1}_\Delta \left(y + \frac{z}{\|r\|} n \right) dy,$$

where dy is $(d-1)$ -dimensional Euclidean measure. As z changes, this indicator converges pointwise except where the limiting slice meets a boundary facet of Δ . The power normal r is parallel to neither the area normal $(1, \dots, 1)$, by nonproportionality, nor any coordinate normal, since $r_j > 0$ and $d \geq 2$. These exceptional intersections therefore have zero $(d-1)$ -dimensional measure. The integrands are bounded by the indicator of the fixed, bounded orthogonal projection of Δ onto $n^\perp$. Dominated convergence makes $A(z)$ continuous. Integrating the slices gives

$$\phi(z) = \frac{1}{\|r\|} \int_0^z A(u) du, \quad \phi'(z) = \frac{A(z)}{\|r\|}. \quad (25)$$

Thus ϕ' is continuous. Since $\phi(z) > 0$ for $z > 0$, $\eta(z) = z\phi'(z)/\phi(z)$ is continuous, including at both thresholds.

Step 3. Apply Prékopa directly to log capacity. Prékopa's theorem supplies the following step: if a function is jointly log-concave in several variables, integrating over some of those variables leaves a log-concave function of the remaining variables. We define each term, construct the function to which the theorem applies, and connect its conclusion to the elasticity.

(a) *Identify what we need to prove.* Introduce log capacity $t = \log z$ and log volume $h(t) = \log \phi(e^t)$. The chain rule gives

$$h'(t) = \frac{\phi'(e^t)}{\phi(e^t)} e^t = \eta(e^t). \quad (26)$$

Thus the elasticity is the slope of log volume against log capacity. If h is concave, its slope is nonincreasing. It therefore suffices to prove concavity of h .

(b) *Write feasible volume as an integral.* The volume of a set is the integral of its indicator:

$$\phi(z) = \int_{[0, \infty)^d} \mathbf{1} \left\{ \sum_j x_j \leq 1, \quad \sum_j r_j x_j \leq z \right\} dx.$$

The indicator equals one at feasible designs and zero elsewhere. Integrating it measures the volume occupied by feasible designs.

(c) *Put the integral into logarithmic coordinates.* Set $x_j = e^{y_j}$ and $z = e^t$. The constraints become

$$\sum_j e^{y_j} \leq 1, \quad \sum_j r_j e^{y_j - t} \leq 1.$$

The second expression comes from dividing the power constraint by e^t . We must also change the volume element. Since $dx_j = e^{y_j} dy_j$,

$$dx_1 \cdots dx_d = e^{y_1 + \cdots + y_d} dy_1 \cdots dy_d.$$

Consequently,

$$\phi(e^t) = \int_{\mathbb{R}^d} F(y, t) dy,$$

where

$$F(y, t) = e^{\sum_j y_j} \mathbf{1} \left\{ \sum_j e^{y_j} \leq 1, \quad \sum_j r_j e^{y_j - t} \leq 1 \right\}.$$

This is the function to which we will apply Prékopa. The exponential factor is the Jacobian of the change of coordinates. Coordinate hyperplanes have zero volume, so omitting designs with a zero coordinate does not change the integral.

(d) *Define log concavity.* A positive function f is log-concave when $\log f$ is concave. Equivalently, for $0 < \lambda < 1$,

$$f((1 - \lambda)\xi_0 + \lambda\xi_1) \geq f(\xi_0)^{1-\lambda} f(\xi_1)^\lambda.$$

At the midpoint this says

$$f\left(\frac{\xi_0 + \xi_1}{2}\right) \geq \sqrt{f(\xi_0)f(\xi_1)}.$$

The function's value between two points must be at least the corresponding geometric average of its endpoint values. The multiplicative definition also works when f can be zero: if either endpoint value is zero, the right-hand side is zero and the inequality holds automatically for a nonnegative function. For $F(y, t)$, joint log concavity treats the entire pair (y, t) as one point and allows both y and t to change.

(e) *Check that interpolating two feasible pairs preserves feasibility.* Suppose (y_0, t_0) and (y_1, t_1) satisfy both constraints, and define

$$y_\lambda = (1 - \lambda)y_0 + \lambda y_1, \quad t_\lambda = (1 - \lambda)t_0 + \lambda t_1.$$

The exponential function is convex, so

$$e^{(1-\lambda)a+\lambda b} \leq (1-\lambda)e^a + \lambda e^b.$$

Apply this inequality to each coordinate and sum:

$$\sum_j e^{y_{\lambda j}} \leq (1-\lambda) \sum_j e^{y_{0j}} + \lambda \sum_j e^{y_{1j}} \leq 1.$$

The area constraint therefore still holds. For the power constraint, observe that

$$y_{\lambda j} - t_{\lambda} = (1-\lambda)(y_{0j} - t_0) + \lambda(y_{1j} - t_1).$$

The same convexity inequality gives

$$\sum_j r_j e^{y_{\lambda j} - t_{\lambda}} \leq (1-\lambda) \sum_j r_j e^{y_{0j} - t_0} + \lambda \sum_j r_j e^{y_{1j} - t_1} \leq 1.$$

Both constraints remain satisfied. In other words, the set D of feasible pairs (y, t) is convex.

(f) *Verify joint log concavity of the integrand.* When both endpoints belong to D , part (e) shows that their interpolation belongs to D too. All three indicators then equal one, and

$$\begin{aligned} F(y_{\lambda}, t_{\lambda}) &= e^{\sum_j [(1-\lambda)y_{0j} + \lambda y_{1j}]} \\ &= (e^{\sum_j y_{0j}})^{1-\lambda} (e^{\sum_j y_{1j}})^{\lambda} \\ &= F(y_0, t_0)^{1-\lambda} F(y_1, t_1)^{\lambda}. \end{aligned}$$

If either endpoint is outside D , the right-hand side is zero and the inequality still holds. Thus, for every pair of endpoints,

$$F(y_{\lambda}, t_{\lambda}) \geq F(y_0, t_0)^{1-\lambda} F(y_1, t_1)^{\lambda}.$$

This is precisely joint log concavity of F .

(g) *Apply Prékopa's theorem.* Define the integral over design coordinates by

$$M(t) = \int_{\mathbb{R}^d} F(y, t) dy.$$

Prékopa's theorem says that the joint log concavity established in part (f) implies

$$M((1-\lambda)t_0 + \lambda t_1) \geq M(t_0)^{1-\lambda} M(t_1)^{\lambda}.$$

This passage from the function to its integral is the substantive theorem being used. It follows from Prékopa–Leindler applied to the three sections $F(\cdot, t_0)$, $F(\cdot, t_1)$, and $F(\cdot, t_\lambda)$ (Böröczky and De, 2021, Theorem 1.2). The sections here are nonnegative, measurable, and integrable: part (c) gives $M(t) = \phi(e^t)$, which is positive and bounded by the volume of the fixed area simplex Δ . All the hypotheses therefore hold, and the conclusion becomes

$$\phi(e^{t_\lambda}) \geq \phi(e^{t_0})^{1-\lambda} \phi(e^{t_1})^\lambda. \quad (27)$$

Taking logarithms gives

$$h(t_\lambda) \geq (1-\lambda)h(t_0) + \lambda h(t_1),$$

which is concavity of h .

(h) *Translate the conclusion back into the elasticity.* Because h is concave, its derivative is nonincreasing. Equation (26) identifies that derivative as $h'(t) = \eta(e^t)$. Since e^t increases with t , $\eta(z)$ is nonincreasing in z . This establishes weak monotonicity. The final step rules out affine segments of h to obtain strict decrease inside the switching interval.

Step 4. The fixed area boundary makes the volume inequality strict. Geometrically averaging two different designs leaves some area unused. At an intermediate power capacity, however, some newly feasible designs use the entire area budget. We show that this difference creates a strict volume gain.

Take $\underline{z} < z_0 < z_1 < \bar{z}$ and $0 < \lambda < 1$, and set $z_\lambda = z_0^{1-\lambda} z_1^\lambda$. Let G_λ contain all coordinatewise geometric averages of designs feasible at the two endpoint capacities:

$$G_\lambda = \{g : g_j = u_j^{1-\lambda} v_j^\lambda \text{ for every } j, \quad u \in K(z_0), \quad v \in K(z_1)\}.$$

Every such average is feasible at z_λ . Indeed, the weighted arithmetic–geometric mean inequality gives

$$\begin{aligned} \sum_j g_j &\leq (1-\lambda) \sum_j u_j + \lambda \sum_j v_j \leq 1, \\ \frac{\sum_j r_j g_j}{z_\lambda} &\leq (1-\lambda) \frac{\sum_j r_j u_j}{z_0} + \lambda \frac{\sum_j r_j v_j}{z_1} \leq 1. \end{aligned}$$

For the second line, apply the inequality to u_j/z_0 and v_j/z_1 , then multiply by r_j and sum. Thus $G_\lambda \subseteq K(z_\lambda)$. The endpoint sets are compact, and geometric averaging is continuous on nonnegative coordinates. Hence G_λ is compact too.

Prékopa–Leindler gives a lower bound on the volume of these averaged designs. To see precisely which functions enter the inequality, retain F from Step 3 and define

$$H_\lambda(y) = e^{\sum_j y_j} \mathbf{1}\{(e^{y_1}, \dots, e^{y_d}) \in G_\lambda\}.$$

The definition of G_λ gives, for all $y_0, y_1 \in \mathbb{R}^d$,

$$H_\lambda((1 - \lambda)y_0 + \lambda y_1) \geq F(y_0, \log z_0)^{1-\lambda} F(y_1, \log z_1)^\lambda.$$

When both endpoint designs are feasible, the exponential factors agree and equality holds. Otherwise the right-hand side is zero. Compactness of the design sets ensures measurability and finite integrals. By the change of variables in Step 3(c), $\int H_\lambda(y) dy = |G_\lambda|$, since coordinate hyperplanes have zero volume. Prékopa–Leindler gives $|G_\lambda| \geq \phi(z_0)^{1-\lambda} \phi(z_1)^\lambda$.

Some designs feasible at z_λ cannot be obtained by geometric averaging. Choose a strictly positive design w satisfying

$$\sum_j w_j = 1, \quad z_0 < \sum_j r_j w_j < z_\lambda.$$

Such a design exists because positive weighted averages of the r_j attain every value strictly between $\underline{z}$ and $\bar{z}$. It uses the entire area budget and is feasible at z_λ , but not at z_0 . If $w \in G_\lambda$, its endpoint designs would satisfy

$$1 = \sum_j u_j^{1-\lambda} v_j^\lambda \leq (1 - \lambda) \sum_j u_j + \lambda \sum_j v_j \leq 1.$$

Equality must hold in every coordinate's arithmetic–geometric mean inequality, which forces $u_j = v_j$ for every j . Thus $u = v = w$, contradicting $u \in K(z_0)$. Hence $w \notin G_\lambda$.

The missing designs occupy positive volume. Since G_λ is compact, a neighborhood of w is disjoint from it. Move w slightly inward to $w_\varepsilon = (1 - \varepsilon)w$. For sufficiently small $\varepsilon > 0$, this point remains in that neighborhood, has positive coordinates, and satisfies both feasibility inequalities strictly. A small open ball around w_ε therefore lies in $K(z_\lambda) \setminus G_\lambda$. Consequently,

$$\phi(z_\lambda) > |G_\lambda| \geq \phi(z_0)^{1-\lambda} \phi(z_1)^\lambda.$$

Taking logarithms proves that $h(t) = \log \phi(e^t)$ is strictly concave on $(\log \underline{z}, \log \bar{z})$.

Finally, if the nonincreasing derivative h' had the same value at two distinct points, it would be constant between them, making h affine on that interval. Strict concavity rules this out. Hence $\eta(z) = h'(\log z)$ is strictly decreasing on $(\underline{z}, \bar{z})$, completing the proof. ■

B Proofs for the Equilibrium Eroom Transition

Proof of Proposition 2.

Notation and valuation. Throughout the proof, unadorned R_e, R_s, R, L, x, g, p denote pre-shock BGP values. Set

$$x = \log(E/S), \quad \eta(x) = \eta_e(x), \quad \eta_s(x) = d - \eta(x), \quad p(t) = \frac{\varepsilon \bar{N}(t)v(t)}{Y(t)}.$$

The one-industry expenditure share is one. Also define

$$\beta = 1 - \alpha, \quad L = 1 - R, \quad \zeta = \frac{R}{L}, \quad \theta = \frac{R_e}{R} = \frac{\eta(x)}{d}, \quad D = 1 - \varepsilon R > 0,$$

and

$$\tau = \frac{dg}{\rho + dg} = \frac{(\varepsilon - 1)R}{L}. \quad (28)$$

The final equality is the pre-shock aggregate free-entry condition.

The local regularity needed below follows directly from the two-dimensional geometry. Set $r_j = b_j/a_j$ and relabel coordinates so that $0 < r_1 < r_2$. In the normalized design coordinates used in the proof of Lemma 1, let $\phi(z)$ be the area of $\{u \geq 0 : u_1 + u_2 \leq 1, r_1 u_1 + r_2 u_2 \leq z\}$. On the interior of the switching interval,

$$\phi(z) = \frac{-z^2 + 2r_2 z - r_1 r_2}{2r_2(r_2 - r_1)}, \quad r_1 < z < r_2.$$

It follows that $\eta(x) = e^x \phi'(e^x)/\phi(e^x)$ is smooth and, writing $z = e^x$,

$$\eta'(x) = -\frac{2r_2 z[(z - r_1)^2 + r_1(r_2 - r_1)]}{(-z^2 + 2r_2 z - r_1 r_2)^2} < 0. \quad (29)$$

The interior BGP has $0 < \eta < d$ and hence lies strictly within this switching interval. Smoothness and a strictly negative derivative therefore hold on a neighborhood of the initial BGP without an additional curvature assumption.

Let χ_e^0 denote the pre-shock power-research productivity and write $\hat{\chi}_e(h) = \chi_e^0 e^{-h}$ and $\hat{\chi}_s(h) = \chi_s$ for the post-shock productivities. Along a transition, write

$$g_e = \hat{\chi}_e(h) R_e^\alpha, \quad g_s = \chi_s R_s^\alpha, \quad n = \frac{\dot{\bar{N}}}{\bar{N}} = \eta(x) g_e + [d - \eta(x)] g_s.$$

The patent HJB, the household Euler equation, and $C = Y$ give

$$\dot{x} = g_e - g_s, \quad \dot{p} = (\rho + n)p - 1. \quad (30)$$

The fundamental solution for p is

$$p(t) = \int_t^\infty \exp \left[-\rho(u - t) - \int_t^u n(v) dv \right] du.$$

For projects, define $q_k = Q_k/Y$ and the normalized net flow

$$f_k = \hat{\chi}_k(h) R_k^{-\beta} \eta_k(x) p - \frac{\varepsilon - 1}{1 - R_e - R_s}.$$

Project values satisfy

$$\dot{q}_k = (\rho + \delta)q_k - \frac{f_k}{\varepsilon}, \quad q_k \leq 0, \quad q_k dJ_k = 0. \quad (31)$$

Here J_k is cumulative entry. Normalized patent and project values are continuous after the surprise shock, including at permitted employment jumps; the latter change flow returns but do not create impulses in their discounted-value integrals.

The proof first constructs the convergent interior phase. It then attaches a short phase with no power entry to the inherited research stocks and checks all entry conditions.

1. The interior system has a smooth stable manifold. For (x, p, h) near the pre-shock BGP, let $(\mathcal{R}_e, \mathcal{R}_s)(x, p, h)$ solve

$$\hat{\chi}_k(h) \mathcal{R}_k^{-\beta} \eta_k(x) p = \frac{\varepsilon - 1}{1 - \mathcal{R}_e - \mathcal{R}_s}, \quad k \in \{e, s\}. \quad (32)$$

These equations determine a unique smooth interior allocation. To see this, set

$$Z_k = \left[\frac{\hat{\chi}_k(h) \eta_k(x) p}{\varepsilon - 1} \right]^{1/\beta}, \quad \mathcal{L} = 1 - \mathcal{R}_e - \mathcal{R}_s.$$

Then $\mathcal{R}_k = Z_k \mathcal{L}^{1/\beta}$ and

$$\mathcal{L} + (Z_e + Z_s) \mathcal{L}^{1/\beta} = 1,$$

whose left side is strictly increasing in $\mathcal{L} > 0$.

Substitute this allocation into (30). At the BGP, the Jacobian of the resulting two-

dimensional system is diagonal, with eigenvalues

$$\lambda_x = \frac{\alpha g \eta'(x)}{\beta d \theta (1 - \theta)} < 0, \quad \lambda_p = \rho + dg + \frac{\alpha dg}{\beta + \zeta} > 0. \quad (33)$$

To see the zero off-diagonal terms, differentiate the logarithms of (32) at equal capacity-growth rates. Weighting the two equations by $(\theta, 1 - \theta)$ cancels the elasticity changes and gives

$$d \log(\mathcal{R}_e + \mathcal{R}_s) = \frac{d \log p}{\beta + \zeta}.$$

Thus total research and n have no first-order dependence on x at the BGP. Taking the difference of the two equations gives

$$d \log \mathcal{R}_e - d \log \mathcal{R}_s = \frac{\eta'(x)}{\beta d \theta (1 - \theta)} dx,$$

which has no p term. These identities give (33).

The stable-manifold theorem, with the shock size as a parameter, supplies a smooth local graph $p = P(x, h)$ converging to the post-shock BGP. It has

$$P(x^*(h), h) = p^*(h), \quad P_x(x^*(h), h) = 0. \quad (34)$$

Consequently, on this graph,

$$P(x, h) = p^*(h) + O((x - x^*(h))^2), \quad (35)$$

$$\mathcal{R}_e(x, P(x, h), h) + \mathcal{R}_s(x, P(x, h), h) = R^*(h) + O((x - x^*(h))^2). \quad (36)$$

2. Balanced-growth comparative statics determine the initial valuation change.

The BGP formulas give

$$\left. \frac{d \log g^*(h)}{dh} \right|_{h=0} = -\frac{\theta}{1 - \alpha D}, \quad (37)$$

$$\left. \frac{d \log R^*(h)}{dh} \right|_{h=0} = -\frac{D\theta}{1 - \alpha D}, \quad (38)$$

$$u \equiv \left. \frac{d \log p^*(h)}{dh} \right|_{h=0} = \frac{\tau \theta}{1 - \alpha D} > 0. \quad (39)$$

For completeness, $R = Hg^{1/\alpha}$ and $d \log H/dh = \theta/\alpha$, while the aggregate free-entry equation gives $d \log R/d \log g = D$. Combining them gives the first two derivatives; $p^* = 1/(\rho + dg^*)$

gives the third. Moreover,

$$0 < u < \theta < 1, \quad (40)$$

because $\tau < \varepsilon R < 1 - \alpha D$; the second strict inequality is $1 - \alpha D = \varepsilon R + \beta D$.

3. A free-boundary matching problem constructs the initial constrained phase.

During this phase, set

$$R_e(t) = R_e e^{-\delta t}$$

and impose miniaturization free entry. Write $R_s(t) = S(R_e(t), x(t), p(t), h)$ for the unique smooth solution of

$$\chi_s S^{-\beta}[d - \eta(x)]p = \frac{\varepsilon - 1}{1 - R_e - S}. \quad (41)$$

Integrate (30) using this allocation, with $x(0) = x$ and unknown initial value $p(0) = p_0$. The two unknowns are p_0 and the release time T . They solve

$$M_1(p_0, T, h) = p(T) - P(x(T), h) = 0, \quad M_2(p_0, T, h) = f_e(T) = 0. \quad (42)$$

At $(p_0, T, h) = (p, 0, 0)$, the first row of the Jacobian with respect to (p_0, T) is $(1, 0)$. The second row satisfies

$$\partial_T M_2 = \frac{\varepsilon - 1}{L} \delta \beta \left(1 + \frac{R_e}{\beta L + R_s} \right) > 0. \quad (43)$$

For this derivative, $\dot{x} = \dot{p} = 0$ at the initial BGP, whereas $\dot{R}_e = -\delta R_e$. Differentiating (41) gives $\dot{R}_s = \delta R_e R_s / (\beta L + R_s)$ there. Substitution into the derivative of f_e gives (43).

The smooth implicit-function theorem therefore constructs $p_0(h)$ and $T(h)$. For applying that theorem, the smooth ODE may be mathematically extended to signed T near zero. Define

$$j = \frac{u}{\beta + R_s/L} > 0.$$

Differentiating (42) at zero, using (34) and (39), gives

$$\frac{p_0(h)}{p} = 1 + uh + O(h^2), \quad (44)$$

$$\frac{R_s(0+)}{R_s} = 1 + jh + O(h^2), \quad (45)$$

$$T(h) = T_1 h + O(h^2), \quad T_1 = \frac{1 - \beta j}{\delta \beta [1 + R_e / (\beta L + R_s)]} > 0. \quad (46)$$

The positivity of T_1 follows from $\beta j < u < 1$. Thus miniaturization receives an upward entry jump and the constrained phase has strictly positive duration. At release, (41) and $f_e(T) = 0$ identify exactly the interior allocation (32); hence employment is continuous at

release. Continue the economy along the convergent stable manifold after T .

4. The constructed path satisfies project complementarity and entry feasibility.

Uniformly on $[0, T(h)]$, differentiation along the constrained path gives the strictly positive value in (43) plus $O(h)$ for $\dot{f}_e$. Since $f_e(T) = 0$, it follows that $f_e(t) < 0$ for $0 \leq t < T$. Set

$$q_e(t) = \frac{1}{\varepsilon} \int_t^{T(h)} e^{-(\rho+\delta)(v-t)} f_e(v) dv < 0, \quad 0 \leq t < T(h), \quad (47)$$

and $q_e(t) = 0$ thereafter. Set $q_s(t) = 0$ at all dates. These are the fundamental project values because both net flows are zero after release. They satisfy (31), including $q_e(T) = \dot{q}_e(T) = 0$.

There is no power entry before T . Miniaturization entry is strictly positive, since differentiating (41) along the short constrained phase gives, uniformly,

$$\dot{R}_s(t) = \frac{\delta R_e R_s}{\beta L + R_s} + O(h) > 0. \quad (48)$$

Thus $\dot{R}_s + \delta R_s > 0$. After release, the stable path has $\dot{R}_k = O(h)$ and positive stocks bounded away from zero, so $\dot{R}_k + \delta R_k > 0$ in both directions when h is sufficiently small. Entry rates may jump at release, which is permitted. There are no downward employment jumps. Production labor remains positive throughout, because allocations are uniformly $O(h)$ from the initial interior BGP.

5. The equilibrium passes through an Erooom interval. Write

$$r(t) = \frac{R_e(t) + R_s(t) - R}{R}$$

for the proportional research-employment deviation. Its initial value is

$$r(0+) = (1 - \theta)jh + O(h^2) \equiv Kh + O(h^2), \quad K > 0. \quad (49)$$

During the constrained phase,

$$\dot{R}_e + \dot{R}_s = -\frac{\delta R_e \beta L}{\beta L + R_s} + O(h) < 0. \quad (50)$$

Also, $\dot{x} = O(h)$ and $\dot{p} = O(h)$ uniformly during this phase. Since $T = O(h)$, its changes in x and p are $O(h^2)$. In particular, $x(T) - x^*(h) = O(h)$. Equations (36) and (38) then give

$$r(T) = -\frac{D\theta}{1 - \alpha D}h + O(h^2) < 0. \quad (51)$$

Total research therefore crosses its old level strictly inside the constrained phase.

The growth decomposition has a particularly simple uniform expansion on that phase:

$$\frac{n(t) - dg}{dg} = -\theta h + \alpha r(t) + O(h^2). \quad (52)$$

To derive it, expand $\chi_e^0 e^{-h} R_e(t)^\alpha$ and $\chi_s R_s(t)^\alpha$ around their common initial value g . The elasticity changes cancel to first order because their sum is d . The pre-shock shares $\eta/d = R_e/R = \theta$ then convert the weighted research changes into $\alpha r(t)$.

Choose any constant

$$0 < \kappa < \min\{K, \theta/(2\alpha)\}.$$

For every sufficiently small $h > 0$, continuity and (49)–(51) give a time $t_h \in (0, T(h))$ with $r(t_h) = \kappa h > 0$. Equation (52) implies $n(t_h) < dg$, since its first-order coefficient is at most $-\theta/2$. Both inequalities are strict and hence persist on a nonempty open interval around t_h . Finally $g_A = n/(\varepsilon - 1)$, so this interval is the asserted I_h . This argument imposes no further condition on α .

6. Fundamental valuation and household optimality complete equilibrium. The constructed p is positive, bounded, and converges to the new BGP value. Its ODE therefore has the stated fundamental integral solution; the terminal term vanishes at least as fast as $e^{-\rho t}$. Equation (47) verifies fundamental project valuation. Set $C = Y$ and $r_{\text{interest}} = \rho + \dot{Y}/Y$. Production and labor markets clear by construction. The production, patent, and project accounting identities aggregate to the household budget, including the value of ongoing commitments.

After release, all project values are zero and household patent wealth divided by output equals p/ε , which is bounded. During the finite initial phase, project values and wealth divided by output are also bounded. Consequently

$$\lim_{t \rightarrow \infty} e^{-\rho t} \frac{\tilde{B}_t}{C_t} = 0, \quad \lim_{t \rightarrow \infty} e^{-\int_0^t r_{\text{interest}}(u) du} \tilde{B}_t = 0.$$

Thus the household transversality and no-Ponzi conditions hold. Log utility, the Euler equation, and these budget conditions deliver household optimality and finish the equilibrium construction. ■

Proof of Corollary 1. Let $m = |\mathcal{I}|$. For industry i , use the normalized patent value $p_i = \varepsilon \bar{N}_i v_i / (\omega_i Y)$. The interior transition system has states $(x, p_1, \dots, p_m)$. Its patent

equations and research returns are

$$\dot{p}_i = (\rho + n_i)p_i - 1, \quad \hat{\chi}_k(h)R_k^{-\beta} \sum_i \omega_i \eta_{ki}(x)p_i.$$

At common geometry, the weighted mean $\sum_i \omega_i p_i$ has the positive eigenvalue λ_p in (33), while each of the $m - 1$ independent patent-value differences has the positive eigenvalue $\rho + dg$. The remaining eigenvalue is $\lambda_x < 0$. Thus the equilibrium is hyperbolic, with a one-dimensional stable manifold that is a smooth graph $(p_1, \dots, p_m) = P(x, h)$.

Hyperbolicity, this graph representation, and smooth dependence on parameters persist under small perturbations of the requirement vectors. The constrained-phase matching problem has m initial patent values and one release time. At common geometry its Jacobian has block form

$$\begin{pmatrix} I_m & 0 \\ * & \partial_T f_e \end{pmatrix}, \quad \partial_T f_e > 0,$$

so it remains invertible nearby. The strictly positive derivative of f_e during the constrained phase persists. Together with $f_e(T) = 0$, it continues to imply $f_e < 0$ and $q_e < 0$ before release. The coefficients of the positive initial miniaturization jump and positive release time, the positive entry rates after release, and the strict first-order research and growth margins on an interior portion of the Eroom interval are continuous in the primitives. They therefore persist in a sufficiently small neighborhood. Fundamental valuation and the household conditions follow as before, with patent wealth divided by output equal to $\sum_i \omega_i p_i / \varepsilon$ after release. ■

C Substitution Between Area and Power Capacity

Write aggregate productivity as $\bar{A}(E, S)$ and let $z = E/S$. Consider a point where $\mathcal{E}_e(z)$ is differentiable and $0 < \mathcal{E}_e(z) < d$, so both capacities have positive marginal effects on productivity. Since $\bar{A} = \prod_i [\mu_i \Phi_i(E, S)]^{\omega_i/(\varepsilon-1)}$, the aggregate productivity elasticities are

$$\frac{\partial \log \bar{A}}{\partial \log E} = \frac{\mathcal{E}_e(z)}{\varepsilon - 1}, \quad \frac{\partial \log \bar{A}}{\partial \log S} = \frac{d - \mathcal{E}_e(z)}{\varepsilon - 1}.$$

Consequently, the ratio of marginal products is

$$\frac{\bar{A}_E}{\bar{A}_S} = \frac{\mathcal{E}_e(z)}{d - \mathcal{E}_e(z)} \frac{1}{z},$$

where subscripts denote partial derivatives. This ratio gives the area capacity needed to compensate for a marginal loss of power capacity while holding productivity fixed. It depends only on relative capacity.

The local Hicks elasticity measures the proportional change in the capacity ratio associated with a proportional change in the inverse marginal-product ratio, along a constant-productivity curve. Thus

$$\sigma_{ES}(z) = - \left[\frac{d \log(\bar{A}_E/\bar{A}_S)}{d \log z} \right]^{-1}.$$

Taking logarithms of the marginal-product ratio gives

$$\log \frac{\bar{A}_E}{\bar{A}_S} = \log \mathcal{E}_e(z) - \log[d - \mathcal{E}_e(z)] - \log z.$$

Differentiating and substituting into the definition yields

$$\sigma_{ES}(z) = \left[1 - \frac{d}{\mathcal{E}_e(z)[d - \mathcal{E}_e(z)]} \frac{d\mathcal{E}_e(z)}{d \log z} \right]^{-1}. \quad (53)$$

Lemma 1 implies $d\mathcal{E}_e/d \log z \leq 0$. Hence $0 < \sigma_{ES}(z) \leq 1$, with strict inequality whenever this derivative is strictly negative. Given industry expenditure weights, component requirements determine the function $\sigma_{ES}(z)$. Research determines the path of z and therefore the elasticity realized along that path. The expression applies where both aggregate marginal products are positive; it does not assign an elasticity at saturation corners.

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