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Inference, Arbitrage, and Asset Price Volatility

Tobias Adrian

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Abstract

Does the presence of arbitrageurs decrease equilibrium asset price volatility? I study an economy with arbitrageurs, informed investors, and noise traders. Arbitrageurs face a trade-off between arbitrage and inference: they would like to buy assets in response to temporary price declines (the arbitrage effect) but sell when prices decline permanently (the inference effect). In equilibrium, the presence of arbitrageurs increases volatility when the inference effect dominates the arbitrage effect. From a technical point of view, this paper offers closed-form solutions to a dynamic equilibrium model with asymmetric information and non-Gaussian priors.

Key words: asset pricing, learning, asymmetric information, limits to arbitrage

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Adrian: Federal Reserve Bank of New York (e-mail: tobias.adrian@ny.frb.org). The author would like to thank Olivier Blanchard, Xavier Gabaix, Augustin Landier, Markus Leippold, Sendhil Mullainathan, Stephen A. Ross, Dimitri Vayanos, Jiang Wang, Ivan Werning, and two anonymous referees as well as seminar participants at MIT, New York University, the Federal Reserve Bank of New York, and the European Finance Association for their comments and suggestions. The author also thanks Alexis Iwanisziw for outstanding research assistance.

1 Introduction

The importance of hedge funds as financial intermediaries has been growing in recent years. This growth of importance is primarily due to the increased size of hedge fund assets under management and the increased share of hedge funds' trading volume in many financial assets. In addition, capital allocated to quantitative trading strategies within the hedge fund sector has been rising. In this paper, I interpret hedge funds that employ quantitative trading strategies as arbitrageurs that trade aggressively on publicly available information, and ask whether their presence increases or decreases equilibrium asset price volatility.

I assume that arbitrageurs are unconstrained, infinitely lived, and risk neutral. Arbitrageurs trade against agents that have more information, an assumption that captures the growing importance of quantitative arbitrageurs who focus on the processing of publicly available information. Arbitrageurs' priors about the fundamental growth rates of assets are non-Gaussian, an assumption leading to richer pricing dynamics than the Gaussian set-ups of Wang (1993) and Vives (1994).

In the model, arbitrageurs exploit temporary deviations of prices from their fundamental value due to noise traders. The principal difficulty for arbitrageurs is to distinguish temporary from permanent shocks. In the absence of arbitrageurs, the presence of noise traders causes the equilibrium price to be predictable, presenting profit opportunities. Under complete information, the presence of arbitrageurs leads unambiguously to lower volatility of asset returns.

When arbitrageurs have limited information, their presence might increase or decrease equilibrium asset price volatility and the trading strategy of arbitrageurs can be upward-sloping in prices. Intuitively, arbitrageurs face a trade-off between an "inference effect" and an "arbitrage effect". When prices increase, arbitrageurs have an incentive to sell the risky asset as it becomes more expensive (holding constant the beliefs about the asset's fundamental value). This is the arbitrage effect. However, a higher price also makes it more likely that future payoffs are higher, which leads to an updating of beliefs. This is the inference effect. When the inference effect dominates the arbitrage effect, the arbitrageurs asset holdings can be upward sloping in prices, and the presence of arbitrageurs can increase equilibrium price volatility.

The sensitivity of asset holdings with respect to price depends on the forecast error of arbitrageurs about the fundamental value of assets. When the forecast error is small, arbitrageurs are relatively certain about the long-run growth rate of the risky asset's fundamentals, so that their trading strategy is downward-sloping and relatively insensitive to prices (they become contrarian investors). However, when the forecast error is large, an upward movement in prices can lead to a strong updating of their priors. As a result, the volatility of price changes is higher than it would be in the absence of arbitrageurs. Small disturbances make prices move very strongly in ranges of the price where the forecast error is large. When prices are very low or very high, the forecast error is small, not much is learned from new information, and the price reacts little to either noise or news.

The rest of the paper is organized as follows. The next Section 2 discusses related literature in more detail. In Section 3, I show that the presence of arbitrageurs unambiguously decreases asset price volatility in the benchmark economy with perfect information. In Section 4, it is assumed that there is a permanent shock to the drift of the dividend process at time 0 that is unobserved by arbitrageurs, leading to the possibility that the presence of arbitrageurs increases equilibrium price volatility. Section 5 concludes. All of the proofs are in the appendix.

2 Related Literature

In the literature on equilibrium pricing with asymmetric information, the current paper is most closely related to Wang (1993) and Vives (1994). Wang studies an economy with informed and uninformed investors as well as noise traders. As in Grossman and Stiglitz's (1980) static setting, noise traders prevent prices from being fully revealing in Wang's setting. In Wang (1993), all shocks and priors are Gaussian, so that the equilibrium can be solved in closed form with infinitely lived agents. In the current paper, priors are non-Gaussian, and closed form solutions require the assumption that informed investors are short-lived (i.e. they do not hedge the stochastic investment opportunity set). A similar assumption is made in Vives (1994), who examines equilibrium pricing in an economy with a non-hirarchical information structure (investors have differential information setting in Vives, but hirarchical information in Wang and this paper).

The result that the presence of arbitrageurs can lead to higher price volatility under some circumstances addresses the old question in finance whether speculation is stabilizing or destabilizing. Friedman (1953) argues that rational speculation generally reduces price volatility, whereas Hart and Kreps (1986) give a simple example that shows how speculation can destabilize prices.

In the case of perfect information, the model presented here is similar to DeLong et al. (1990a). As in DeLong et al. (1990a), investors are myopic and noise is priced. The first section of the paper shows that the introduction of risk-neutral, infinitely lived arbitrageurs into the (continuous time analogue) of DeLong et al. (1990a) noise trader economy stabilizes prices, as long as the arbitrageurs know the full structure of the model. However, when arbitrageurs have to learn about the drift of the dividend process, this result no longer holds: the present paper shows that volatility of returns can actually increase due to the presence of arbitrageurs. Furthermore, arbitrageurs can have asset holdings that are a positive function of both the current price, and of past prices. This finding gives a different interpretation to DeLong et al. (1990b), who argue that positive feedback trading is a pervasive but irrational behavior in financial markets.

The finding that learning is a limit to arbitrage is mentioned by Shleifer and Vishny (1997), but is not studied explicitly. Shleifer and Vishny alert to the fact that investors' learning leads to withdrawal of funds from arbitrageurs precisely during times when arbitrage opportunities are biggest. Since Shleifer and Vishny (1997), a number of authors have studied financial constraints as a limit to arbitrage. Xiong (2001) studies convergence traders that are wealth constrained. When prices drop sharply, the arbitrageurs wealth drops, which can amplify movements in prices. Gromb and Vayanos (2002) investigate the welfare implications of margin requirements in segmented markets. They show that financial constraints lead to inefficient risk taking behavior. Liu and Longstaff (2004) also study the effect of margin constraints on arbitrage, showing that margin constraints can severely limit arbitrageurs ability to take advantage of arbitrage opportunities. Attari and Mello (2006) study a model with financial constraints that leads to predictability of prices. Abreu and Brunnermeier (2002) study the coordination problem of arbitrageurs in the presence of a bubble. Even though rational arbitrageurs know that there is a bubble in asset prices, they do not know when the other arbitrageurs will start to trade against the market. Once coordination happens, the bubble disappears.

3 Equilibrium Price Volatility with Full Information

In this section, a model is developed in which time variation in expected returns is caused by noise-trader demand. The predictability of expected returns leads to profit opportunities for risk-neutral arbitrageurs. The presence of arbitrageurs eliminates return predictability and unambiguously yields lower equilibrium volatility.

There are three agents—arbitrageurs, investors, and noise traders—and two assets: a risky asset with a price p_t, that pays a dividend δ_t; and a riskless bond that pays a continuously compounded interest rate r. The risky asset yields dividends according to a mean-reverting process with growth rate μ, rate of mean reversion κ, and standard deviation σ^δ:

$$d\delta_t = \kappa\left(\mu - \delta_t\right)dt + \sigma^\delta dZ_t^\delta \tag{1}$$

The term Z_t^δ denotes a Brownian motion; σ^δ is the instantaneous standard deviation of the dividend. The initial value of the dividend is assumed to be normally distributed according to $\delta_0 \sim N\left(\bar{\delta}_0, \sigma_0^\delta\right)$, independently of all other shocks.

Arbitrageurs are risk neutral, and—in this section—are assumed to know the true drift μ. The focus of Section 4 is to relax the latter assumption. The arbitrageurs' information set evolves according to the filtration $\Im_t^A$:

$$\Im_t^A = \left\{ \Im_0^A, p_s, \delta_s, v_s \text{ for } 0 \leqslant s \leqslant t \right\} \text{ where } \Im_0^A = \{r, \mu, p_0, \delta_0, v_0\} \tag{2}$$

where v_s denotes trading volume. Arbitrageurs observe the price process, the dividend process, and trading volume. Furthermore, arbitrageurs have an infinite time horizon. The dynamic budget constraint of arbitrageurs is:

$$dw_t = rw_t dt + A_t \left(dp_t + \delta_t dt - rp_t \right) \tag{3}$$

where A_t denotes the amount of the risky asset that arbitrageurs hold at time t, and w_t is the wealth of the arbitrageurs at time t.

Investors have CARA utility over instantaneous consumption with constant absolute risk-aversion coefficient α and are assumed to invest myopically, meaning that they do not hedge changes in the investment opportunity set. Vives (1994) makes a similar assumption in a setting with non-hirarchical, differential information. The information set of investors is the filtration $\Im_t^I$:

$$\Im_t^I = \left\{ \Im_0^I, p_s, \delta_s, v_s \text{ for } 0 \leqslant s \leqslant t \right\} \text{ where } \Im_0^I = \{r, \mu, p_0, \delta_0, v_0\} \tag{4}$$

Noise traders are assumed to have irrational beliefs about μ. They believe that the drift of the dividend is time varying, according to an Ornstein-Uhlenbeck process:

$$du_t = \theta \left(\bar{u} - u_t \right) dt + \sigma^u dZ_t^u, \quad u_0 \sim N\left(0, \sigma_0^u\right) \tag{5}$$

where Z_t^u denotes a Brownian motion that is independent of Z_t^δ. The shock to u_0 is independent of all other shocks. The noise traders' belief about the drift of the dividend process is mean-reverting around $\bar{u}$, at rate θ. Like the investors, noise traders have CARA utility with risk-aversion coefficient α and are assumed to be myopic. The information set of the noise traders is the filtration $\Im_t^n$:

$$\Im_t^n = \{ \Im_0^n, p_s, \delta_s, u_s \text{ for } 0 \leqslant s \leqslant t \} \text{ where } \Im_0^n = \{r, p_0, \delta_0, u_0\} \tag{6}$$

The assumption that noise traders have incorrect beliefs about the drift of the dividend process will cause the equilibrium price to deviate from it's fundamental value, and to exhibit time-variation of expected returns when arbitrageurs are absent. The incorrect beliefs of the noise traders can be interpreted as non-common priors among the market participants. The noise traders' belief about μ is $E\left[\mu|\Im_t^n\right] = u_t$. However, noise traders agree with the arbitrageurs' expectation about μ so that $E\left[E\left[\mu|\Im_t^A\right]|\Im_t^n\right] = E\left[\mu|\Im_t^A\right]$. So the noise traders and arbitrageurs agree to disagree.

3.1 The benchmark economy without arbitrageurs

As the underlying shocks are Brownian motions, the equilibrium price of the risky asset can be written as:

$$dp_t = \eta_t^p dt + \sigma_t^p dZ_t^p \tag{7}$$

where η_t^p and $\sigma_t^p = \left[\sigma_t^{up}, \sigma_t^{\delta p}\right]'$ are possibly functions of the underlying shocks to δ_t and u_t, and $Z_t^p = \left[Z_t^u, Z_t^\delta\right]$. The drift and variance of the stock price process are determined in equilibrium.

As investors are myopic and have CARA utility with coefficient α, their demand is (see according Merton 1992, page 118, equation 4.65):

$$y_t^I = \frac{\eta_t^p\left(\mu\right) + \delta_t - rp_t}{r\alpha\left(\sigma_t^p\right)^2} \tag{8}$$

where y_t^I denotes the number of shares of the risky asset that investors demand and $\left(\sigma_t^p\right)^2 = \sigma_t^{p\prime}\sigma_t^p$. The noise traders' demand for the risky asset is:

$$y_t^n = \frac{\eta_t^p\left(u_t\right) + \delta_t - rp_t}{r\alpha\left(\sigma_t^p\right)^2} \tag{9}$$

The difference between the investors' demand for the risky asset and the noise traders' demand is that noise traders have distorted beliefs about the drift of the dividend. Instead of submitting demands as a function of the true drift of the price, $\eta_t^p\left(\mu\right)$, they are submitting demands as a function of their distorted beliefs about the drift of the price, $\eta_t^p\left(u_t\right)$.

The equilibrium results from market clearing. There is a fraction $1-\lambda$ of investors and a fraction λ of noise traders in the economy. The supply of the risky asset is S.

Definition 1 *The equilibrium in an economy without arbitrageurs is characterized by an equilibrium pricing function such that the demand for the risky asset equals the supply of the risky asset:*

$$(1-\lambda) y_t^I + \lambda y_t^n = S \tag{10}$$

From the market-clearing condition, it follows that the price is a function of δ_t and u_t:

Proposition 1 *The equilibrium pricing function without arbitrageurs is:*

$$p_t = \frac{\delta_t}{r+\kappa} + \frac{\lambda\kappa (u_t - \bar{u})}{(r+\kappa)(r+\theta)} + \frac{\lambda\kappa\bar{u} + (1-\lambda)\kappa\mu}{r(r+\kappa)} - S\alpha \left(\sigma_t^p\right)^2 \tag{11}$$

where

$$\left(\sigma^p\right)^2 = \left(\frac{\sigma^\delta}{r+\kappa}\right)^2 + \left(\frac{\lambda\kappa\sigma^u}{(r+\kappa)(r+\theta)}\right)^2 \tag{12}$$

The proof of this proposition, as well as the proofs of all other propositions, are in the appendix. Expression (11) shows that the price of the risky asset is the weighted average of the discounted present value of the dividend under the investors' and the noise traders' information set, minus a risk premium.

The noise-trader risk changes the pricing function in two respects when compared to an economy with only investors ($\lambda = 0$). First, the larger the λ, the more the price depends on the deviation of noise traders' beliefs, $u_t - \bar{u}$. When $u_t > \bar{u}$, noise traders are overoptimistic about future dividends, and the price is above its fundamental value (where fundamental value refers to the price when no noise traders are present, $\lambda = 0$). Second, the risk premium increases because of the noise-trader risk, as both investors and noise-traders are compensated for noise trader risk: the volatility of price changes is proportional to the volatility of σ^u.

The pricing function (11) resembles the one derived by DeLong et al. (1990a) and Wang (1993). DeLong et al. are considering the case where $\sigma^\delta = 0$, and $\theta = 0$, and show that noise-trader risk is priced: the volatility of noise traders' beliefs σ^u enters into the risk premium $S\alpha \left(\sigma_t^p\right)^2$. Note that the pricing of noise-trader risk does not hinge on the assumption that investors are myopic or short lived. Wang (1993) shows that noise-trader risk is priced even if investors are infinitely lived. In Vives (1994), the pricing function is also linear, but depends on an average of private signals as market participants have differential information.

3.2 Equilibrium with arbitrageurs

The pricing function derived in Proposition 1 represents the equilibrium when no arbitrageurs are present. Suppose that arbitrageurs enter the economy. Would this pricing function still be an equilibrium? By Itô's lemma, the law of motion of the price under the arbitrageurs' information set $\Im_t^A$ is:

$$dp_t = \left(\frac{\kappa (\mu - \delta_t)}{r + \kappa} + \frac{\lambda \kappa \theta (\bar{u} - u_t)}{(r + \kappa)(r + \theta)} \right) dt + \frac{\sigma^\delta}{r + \kappa} dZ_t^\delta + \frac{\lambda \kappa \theta \sigma^u}{(r + \kappa)(r + \theta)} dZ_t^u \qquad (13)$$

The pricing of noise-trader risk leads to predictability of the equilibrium price process. The instantaneous expected return is defined as $E[dp_t] + (\delta_t - rp_t) dt$. The drift of the return process is a measure of the instantaneous expected excess return. Under the arbitrageurs' information set, it is:

$$\frac{\lambda \kappa}{r + \kappa} (\mu - u_t) + r\alpha (\sigma_t^p)^2 S \qquad (14)$$

The first component of the expected excess return in (14) is proportional to the deviation of the noise traders' beliefs about the dividend process from the true drift. If noise traders push up the price (u_t is high), expected returns are low: the expected return to the risky asset is thus negatively correlated with the level of the price. The second component of the expected return is constant and consists of the risk premium.

The time-variation of expected returns makes a contrarian investment strategy profitable for arbitrageurs:

Definition 2 *A profit opportunity for arbitrageurs at time t is an investment strategy $\{A_s\}_{s=t}^{\infty}$ that satisfies the dynamic budget constraint* (3)*, has zero initial cost, $w_t = 0$, and has positive instantaneous discounted expected payoff with respect to the arbitrageurs' information set: $E\left[e^{-r(s-t)} w_s | \Im_t^A\right] > 0$ for $s \to t$.*

The following investment strategy gives positive profits to arbitrageurs:

$$\begin{aligned} A_t > 0 \text{ if } u_t < u^* \equiv \mu + (r+\kappa) r\alpha \left(\sigma_t^p\right)^2 S/(\lambda\kappa) \\ A_t < 0 \text{ if } \ u_t > u^* \equiv \mu + (r+\kappa) r\alpha \left(\sigma_t^p\right)^2 S/(\lambda\kappa) \end{aligned} \tag{15}$$

Arbitrageurs can earn positive expected profits with this contrarian strategy, as noise-trader risk introduces a predictable component into the equilibrium price. Owing to their risk neutrality and infinite time horizon, arbitrageurs can exploit this predictability. The only state variable of arbitrageurs is the level of current belief, u_t. When noise traders push the price of the risky asset down ($u_t < u^*$), expected returns are high and arbitrageurs go long. Conversely, when noise traders push the price of the risky asset up ($u_t > u^*$), arbitrageurs go short (sell the asset). This contrarian strategy of arbitrageurs suggests that the presence of arbitrageurs is price stabilizing, which will be demonstrated in the next proposition.

The definition of a profit opportunity is slightly weaker than the statistical arbitrage opportunity of Bondarenko (2003). The additional requirement of Bondarenko is that the expected payoff of a statistical arbitrage is positive at a terminal date conditional on the terminal price.

The equilibrium with arbitrageurs is defined as follows:

Definition 3 *An equilibrium with arbitrageurs is a pricing function such that there are no profit opportunities for arbitrageurs, and the asset market clears.*

Proposition 2 *Under perfect information, the pricing function*

$$\tilde{p}_t = \frac{\delta_t}{\kappa + r} + \frac{\kappa\mu}{r(\kappa + r)} \tag{16}$$

constitutes an equilibrium with arbitrageurs. The volatility of price changes denoted $\left(\tilde{\sigma}^p\right)^2$ in the equilibrium with arbitrageurs corresponding to this pricing function is:

$$\left(\tilde{\sigma}^p\right)^2 = \left(\frac{\sigma^\delta}{\kappa + r}\right)^2 \tag{17}$$

Compared to the economy without arbitrageurs, there is one striking difference: the evolution of prices only depends on the evolution of the fundamental δ_t, and the beliefs of noise traders u_t do not enter to pricing function. Due to the risk neutrality of the arbitrageurs, the price of the risky asset is simply the present discounted value under the arbitrageurs' information set. Prices react only to dividend news, not to innovations in u_t.

Corollary 1 *Under perfect information, the volatility of returns is lower in an economy with arbitrageurs:*

$$\left(\tilde{\sigma}^p\right)^2 - \left(\sigma^p\right)^2 = \underbrace{-\left(\frac{\lambda\kappa\sigma^u}{\left(r+\kappa\right)\left(r+\theta\right)}\right)^2}_{\text{Arbitrage Effect}} \tag{18}$$

where $\left(\sigma^p\right)^2$ *and* $\left(\tilde{\sigma}^p\right)^2$ *denote the instantaneous variance of price changes in an economy with and without arbitrageurs, respectively.*

The volatility of price changes is unambiguously lower when arbitrageurs are present. The difference in volatilities is proportional to the variance of u_t: the higher the $\left(\sigma^u\right)^2$ the larger the reduction of volatility due to the presence of arbitrageurs. In the absence of arbitrageurs, u_t is priced and returns are time varying, presenting profit opportunities for arbitrageurs. Arbitrageurs take advantage of this predictability, and, in equilibrium, expected returns are constant once arbitrageurs are present in the economy. I label this decrease in the volatility of price changes due to the presence of arbitrageurs is the *arbitrage effect*.

4 Arbitrage and Volatility with Imperfect Information

The previous section shows that the presence of risk-neutral arbitrageurs reduces the volatility of prices when arbitrageurs have full information about the drift of the dividend process. Intuitively, arbitrageurs take advantage of the predictability in returns and thus, in equilibrium, reduce volatility. A contrarian trading strategy yields positive expected profits for arbitrageurs: they go long when prices are below fundamentals, and go short when prices are above fundamentals, after adjusting for risk aversion.

In this section, uncertainty about the drift rate of the dividend process is introduced. In particular, it is assumed that there is a permanent shock to the drift of the dividend process at $t = 0$. Arbitrageurs then face a trade-off: on the one hand, they want to take advantage of the predictability introduced by noise-trader risk. On the other hand, the trading of investors is now informative for arbitrageurs, as investors know the true drift of the dividend process. Under certain conditions, the trading strategy of arbitrageurs can be upward-sloping, and the presence of arbitrageurs can increase the volatility of prices. In Section 4.1, it is assumed that the drift of the dividend μ has two states. In Section 4.2, no parametric assumption is made about the shock to μ.

4.1 An example with two states

In this subsection, I assume that μ has two states: $\mu = 0$ and $\mu = \bar{\delta}$. This assumption captures discrete economic events such as bankruptcies or recessions. For example, the expected long-term growth rate of Enron fell rapidly in a six-month period between June and December 2001. A parsimonious way to model this is with two states: a normal state and a bankruptcy state. Ex-ante, the decline of Enron's stock price in the summer of 2001 could have been a temporary deviation (representing a profit opportunity) or a permanent shock. Over time, it became clear that the shock to Enron's earnings was permanent (see Palepu and Healy, 2003, for detailed account of the fall of Enron).

Additional examples in which discrete states are natural assumptions include monetary policy regimes. The dividend of the arbitrage strategy could be interpreted as the carry (i.e. the relative interest rate) in a carry trade (i.e. short long-term bonds in a currency with low interest rates, long long-term bonds in a currency with high interest rates). The Brownian uncertainty then arises due to exchange rate fluctuations, and the two states correspond to different monetary regimes. The two states could also correspond to recessions and expansions that induce shifts in the expected returns to trading strategies (see Veronesi, 1999).

The information set of investors and noise traders is assumed to be the same as previously expressed in Equations (6) and (4). The filtration of arbitrageurs under imperfect information is denoted $\hat{\Im}_t^A$ and defined as:

$$\begin{gathered}\hat{\Im}_t^A = \left\{\hat{\Im}_0^A, \hat{p}_s, \delta_s, v_s \text{ for } s \leqslant t\right\} \text{ where } \hat{\Im}_0^A = \{\hat{p}_0, \delta_0, v_0\} \\ \text{Prior:} \quad \Pr\left(\mu = \bar{\delta}\right) = \hat{\pi}, \;\; \Pr\left(\mu = 0\right) = 1 - \hat{\pi}\end{gathered} \tag{19}$$

Arbitrageurs now face a learning problem, and they can make inferences from two sources: the observation of the dividend and the observation of trading volume. The arbitrageurs will learn the true drift of the dividend as time passes. Arbitrageurs learn from volume, as the demand function of the investors depends on the true μ. The probability that $\mu = \bar{\delta}$ under the arbitrageurs' information set will be denoted by π_t:

$$\pi_t = \Pr\left[\mu = \bar{\delta}|\hat{\Im}_t^A\right] \tag{20}$$

Definition 4 *An equilibrium with arbitrageurs under imperfect information is a pricing function such that there are no profit opportunities for arbitrageurs; the market for the risky asset clears; and the information sets of arbitrageurs $\hat{\Im}_t^A$, investors $\Im_t^I$, and noise traders $\Im_t^n$ are represented by* (19)*,* (4) *,and* (6)*.*

Arbitrageurs condition their expectation of the dividend drift on observations of the dividend and volume, and one of the determinants of volume is the shock u_t. Therefore, the equilibrium price is a function of both shocks Z_t^δ and Z_t^u. The proof to the proposition demonstrates that the following variable x_t is a sufficient statistic for trading volume:

$$x_t = (1 - \lambda)\,\mu + \lambda u_t \tag{21}$$

Observing trading volume v_t reveals x_t to arbitrageurs. As u_t is mean-reverting, they can learn μ over time from observing x_t.

Proposition 3 *The equilibrium pricing function with arbitrageurs under imperfect information is:*

$$\hat{p}_t = \frac{\delta_t}{\kappa + r} + \frac{\kappa \pi_t \bar{\delta}}{r\,(\kappa + r)} \tag{22}$$

where

$$d\pi_t = \pi_t (1-\pi_t) \frac{\kappa\bar{\delta}}{(\sigma^\delta)^2} \left[\kappa \left(\mu - \pi_t\bar{\delta}\right) dt + \sigma^\delta dZ_t^\delta\right] + \pi_t (1-\pi_t) \frac{\theta (1-\lambda)\bar{\delta}}{(\lambda\sigma^u)^2} \left[\theta (1-\lambda) \left(\mu - \pi_t\bar{\delta}\right) dt + \lambda\sigma^u dZ_t^u\right] \tag{23}$$

The volatility of the equilibrium price changes is

$$(\hat{\sigma}_t^p)^2 = \left(\frac{\sigma^\delta}{\kappa + r} + \frac{\pi_t (1-\pi_t) \kappa^2 \bar{\delta}^2}{\sigma^\delta r (\kappa + r)}\right)^2 + \left(\pi_t (1-\pi_t) \frac{\theta (1-\lambda) \kappa \bar{\delta}^2}{\lambda \sigma^u r (\kappa + r)}\right)^2 \tag{24}$$

The equilibrium price is the present discounted value of future dividends under the arbitrageurs' information set. As arbitrageurs do not know the true value of the drift μ, the second term in the pricing function is replaced with its expected value, $\pi_t\bar{\delta}$, when compared with the equilibrium pricing function under perfect information (16). The probability of the high state, π_t, evolves according to (23). Asymptotically, arbitrageurs learn the true state perfectly, and π_t converges to either 0 or 1. Once convergence to the true state is achieved, the variance of the equilibrium price converges to the one under perfect information (Proposition 2).

The arbitrageurs' expectation of μ depends on the innovations to Z_t^δ, and Z_t^u, as well as the forecast error $\left(\mu - \pi_t\bar{\delta}\right)$. When the forecast error is positive ($\mu > \pi_t\bar{\delta}$), the probability of the high state is revised upward, and vice versa. The volatility of the price process is now stochastic. The stochastic volatility is a result of the non-normality of the underlying shock to μ. The expectation of μ is a martingale under the arbitrageurs' information set.

Noise traders can observe both the equilibrium price and the dividend (see 2), and they are therefore able to induce the arbitrageurs beliefs π_t. However, we assume that noise traders do not use this information to update their own beliefs about the long-run mean of the dividend. Instead, they continue to believe that the drift μ equals u_t. Noise traders are therefore irrational. In the absence of this irrationality, the price would become fully revealing, and it would revert to the equilibrium price of Equation (16). As in Grossman and Stiglitz (1980), Wang (1993) and Vives (1994), the presence of noise traders is thus the key ingredient that prevents the full revelation of private information.

The key result of this section is that the volatility of the equilibrium price can increase in the presence of arbitrageurs, when information is imperfect:

Corollary 2 *In an economy with arbitrageurs under imperfect information, the volatility of prices can be higher than in an economy without arbitrageurs if $\sigma^u > 0$. The difference in volatility is:*

$$\left(\hat{\sigma}^p\right)^2 - \left(\sigma^p\right)^2 = \underbrace{\pi_t\left(1-\pi_t\right)\frac{2\kappa^2\bar{\delta}^2}{r\left(\kappa+r\right)^2} + \pi_t^2\left(1-\pi_t\right)^2\left(\frac{\kappa\bar{\delta}^2}{r\left(\kappa+r\right)}\right)^2\zeta}_{\textit{Inference Effect}} \quad \underbrace{-\left(\frac{\lambda\kappa\sigma^u}{\left(r+\kappa\right)\left(r+\theta\right)}\right)^2}_{\textit{Arbitrage Effect}} \tag{25}$$

where $\zeta = \left(\frac{\kappa}{\sigma^\delta}\right)^2 + \left(\frac{\theta(1-\lambda)}{\lambda\sigma^u}\right)^2$. A necessary condition for the volatility of the return process to be higher in the presence of arbitrageurs under incomplete information is:

$$\left(\bar{\delta}/2\right)^2 > \sqrt{\left(\frac{r}{\zeta}\right)^2 + \frac{1}{\zeta}\left(\frac{r\lambda\sigma^u}{r+\theta}\right)^2} - \frac{r}{\zeta}$$

This proposition shows that equilibrium prices of the risky asset can be higher when arbitrageurs are present. If arbitrageurs know the true value of μ, the first term on the right-hand side of (25) disappears, and the result that the volatility of prices is lower when arbitrageurs are present holds. However, the imperfect information about μ introduces additional volatility. When dividend news is higher than expected, the arbitrageurs revise their estimate of μ upward. Similarly, a high demand from investors is more likely when $\mu = \bar{\delta}$.

The result that the volatility of prices can be higher if arbitrageurs are present is true only in finite time. Asymptotically, arbitrageurs learn the true drift, μ, perfectly, and the pricing function converges to the one under perfect information.

Figures 1 and 2 plot the price p_t as functions of x_t and δ_t, respectively, at a given point in time. The nonlinearity in the pricing function is striking. For sufficiently low and high x_t or δ_t, the expectation of μ is close to 0 or $\bar{\delta}$, respectively. However, for intermediate ranges of x_t or δ_t, arbitrageurs update their beliefs about the relative likelihood of the two states strongly.

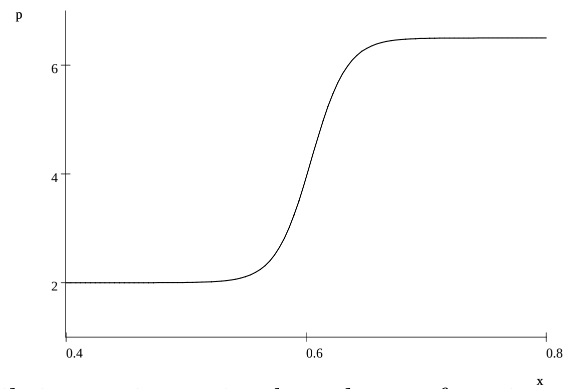

Figure 1: The equilibrium price p_t is plotted as a function of the demand shock $x_t = (1 - \lambda)\,\mu + \lambda u_t$, holding dividends δ_t constant.

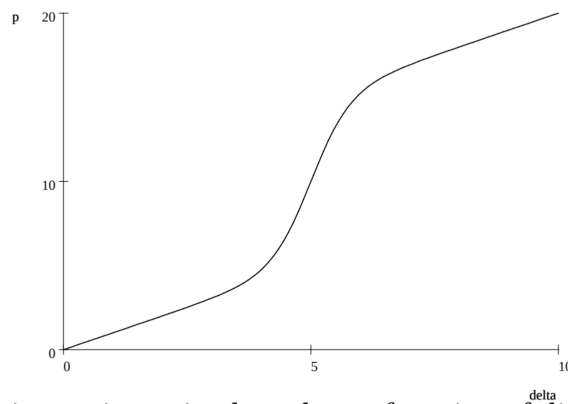

Figure 2: The equilibrium price p_t is plotted as a function of dividends δ_t, holding the demand shock $x_t = (1 - \lambda)\,\mu + \lambda u_t$ constant.

What is the mechanism that leads to the increased volatility of prices for intermediate values of x_t and δ_t? Figure 3 plots the arbitrageurs' equilibrium holdings of the risky asset as a function of its price. For relatively high and low values of p_t, the equilibrium asset holdings of arbitrageurs are downward-sloping with respect to the price. However, in an intermediate range, as variation in x_t and δ_t are very informative about μ, the equilibrium asset holdings are upward-sloping with respect to the price. This effect is the "inference effect": as p_t is increasing, arbitrageurs infer that the high state of μ is very likely, the price increases sharply, and hence the arbitrageurs' position in the risky asset increases. For high and low values of u, the demand is downward-sloping: arbitrageurs are absorbing shocks to beliefs about μ, and prices stabilize. For relatively high and low values of p_t, the demand curve is downward-sloping: arbitrageurs are effectively contrarians. However, for intermediate ranges of the price, the equilibrium trading strategy is upward-sloping. In this intermediate range, the inference effect dominates the arbitrage effect. In this range, higher values of x_t or δ_t lead to a strong updating of the arbitrageurs' beliefs about the value of μ.

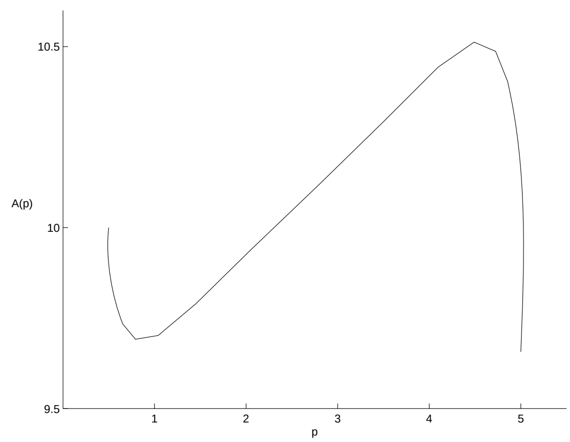

Figure 3: The figure plots the arbitrageurs' asset holdings $A(p_t)$ as a function of the price p_t.

4.2 General probability distributions of μ

In the previous section, explicit results for a specific distributional assumption about μ were obtained. In this section, this distributional assumption about μ will be relaxed and the equilibrium pricing function for general distributions will be obtained.

The information sets of investors and noise traders are given by Equations (4) and (6). The information set of arbitrageurs is now assumed to be:

$$\begin{aligned} \bar{\Im}_t^A &= \left\{ \hat{\Im}_0^A, p_s, \delta_s, x_s \text{ for } s \leqslant t \right\} \quad \text{where } \bar{\Im}_0^A = \{p_0, \delta_0, x_0\} \qquad (26) \\ \text{prior:} \quad & \mu \sim f_\mu\left(\mu|\phi\right) \end{aligned}$$

The drift μ of the dividend process is thus assumed to be distributed according to a probability density function $f_\mu\left(\mu|\phi\right)$, where ϕ is a vector of parameters. The variance of μ is assumed to be finite. The conditional expectation and error of μ under the arbitrageurs' information set are denoted by:

$$m_t = E\left[\mu|\Im_t^A\right], \ \ \gamma_t = E\left[\mu^2 - m_t^2|\Im_t^A\right] \tag{27}$$

Under these assumptions, the equilibrium pricing function can be derived:

Proposition 4 *The equilibrium price $\bar{p}_t$ in the presence of arbitrageurs when the arbitrageurs' information set is $\bar{\Im}_t^A$ is given by:*

$$\bar{p}_t = \frac{\delta_t}{\kappa + r} + \frac{\kappa m_t}{r(\kappa + r)} \tag{28}$$

where

$$dm_t = \frac{\gamma_t}{(\sigma^\delta)^2}\left[\kappa(\mu - m_t)\,dt + \sigma^\delta dZ_t^\delta\right] + \frac{\theta(1-\lambda)\gamma_t}{(\lambda\sigma^u)^2}\left[\theta(1-\lambda)(\mu - m_t)\,dt + \lambda\sigma^u dZ_t^u\right] \tag{29}$$

The pricing function from Proposition 3 is a special case. The expectation of μ, m_t, is a function of shocks to δ_t and shocks to x_t, and of the forecast error γ_t. In order to derive a law of motion for γ_t, specific distributional assumptions about μ must be made. By Itô's lemma, the variance of the price process is:

$$(\bar{\sigma}_t^p)^2 = \left(\frac{\sigma^\delta}{\kappa + r} + \frac{\kappa\gamma_t}{\sigma^\delta r(\kappa + r)}\right)^2 + \left(\frac{\kappa\theta(1-\lambda)\gamma_t}{r(r+\kappa)\lambda\sigma^u}\right)^2 \tag{30}$$

Corollary 2 generalizes to the following:

Corollary 3 *In an economy with arbitrageurs under imperfect information and $\mu \sim f(\phi)$, the volatility of prices can be higher than in an economy without arbitrageurs if $\sigma^u > 0$. The difference in volatility is:*

$$(\bar{\sigma}_t^p)^2 - (\sigma^p)^2 = \underbrace{\left(\frac{\gamma_t}{r(\kappa + r)}\right)^2\zeta + \frac{2\kappa\gamma_t}{r(\kappa + r)^2}}_{\text{Inference Effect}} \ \underbrace{-\left(\frac{\lambda\kappa\sigma^u}{(r+\kappa)(r+\theta)}\right)^2}_{\text{Arbitrage Effect}} \tag{31}$$

where $\zeta = \left(\frac{\kappa}{\sigma^\delta}\right)^2 + \left(\frac{\theta(1-\lambda)}{\lambda\sigma^u}\right)^2$. A necessary condition for the volatility of the price process to be higher in the presence of arbitrageurs under incomplete information is:

$$\gamma_t > \sqrt{\left(\frac{\kappa r}{\zeta}\right)^2 + \frac{1}{\zeta}\left(\frac{\lambda\kappa\sigma^u r}{r+\theta}\right)^2} - \frac{\kappa r}{\zeta} \tag{32}$$

The variance of price changes from Corollary 3 is conditional on the realization of a particular value for μ. As an additional measure of the price volatility, the unconditional variance at time 0 is derived:

Proposition 5 *The difference in the unconditional variance of the price in time* 0 *in an economy with arbitrageurs compared to an economy without arbitrageurs is:*

$$var\left(\bar{p}_0\right) - var\left(p_0\right) = \underbrace{var\left(\frac{\kappa m_0}{r\left(r+\kappa\right)}\right)}_{\textit{Inference Effect}} \underbrace{-\left(\frac{\kappa\left(1-\lambda\right)\sigma^{\mu}}{r\left(r+\kappa\right)}\right)^2 - \left(\frac{\lambda\kappa\sigma_0^u}{\left(r+\kappa\right)\left(r+\theta\right)}\right)^2}_{\textit{Arbitrage Effect}} \tag{33}$$

Unconditionally, the arbitrage effect has two elements: a reduction in the variance of price due to the elimination of the noise-trader risk (the term proportional to $\lambda\sigma_0^u$) and the reduction in variance due to the uncertainty about the drift (the term proportional to $\left(1-\lambda\right)\sigma^{\mu}$). The inference effect is proportional to the variance of the expected value of μ at time 0.

5 Conclusion

This paper develops a mechanism for amplifying shocks that can cause an increase in asset price volatility. In the model, risk-neutral arbitrageurs learn the drift of the dividend process of a risky asset. Biased beliefs of noise traders lead to mis-pricing that translates into predictability of asset returns. The equilibrium trading strategy of arbitrageurs can be upward-sloping. Intuitively, arbitrageurs face a trade-off between an inference effect and an arbitrage effect. When arbitrageurs face little uncertainty about the drift of the dividend process, they are contrarian: relatively high prices lead arbitrageurs to sell the risky asset, and relatively low prices lead arbitrageurs to buy the risky asset. In an intermediate range, arbitrageurs' uncertainty is high, and the arbitrage effect is dominated by an inference effect: higher prices mean that a higher drift of the dividend is more likely, and arbitrageurs update their beliefs.

A Appendix

A.1 Proof of Proposition 1

Proof. Let's start with a linear price:

$$p_t = F\delta_t + Gu_t + H \tag{34}$$

Using Itô's lemma, the instantaneous variance of the price process is:

$$\left(\sigma_t^p\right)^2 = \left(F\sigma^\delta\right)^2 + \left(G\sigma^u\right)^2 \tag{35}$$

The drift of the price process under the information set of investors, $\Im_t^I$, is:

$$\eta_t^P = F\kappa\left(\mu - \delta_t\right) + G\theta\left(\bar{u} - u_t\right) \tag{36}$$

The drift under the noise traders' information set, $\Im_t^n$, is:

$$\eta_t^P = F\kappa\left(u_t - \delta_t\right) + G\theta\left(\bar{u} - u_t\right) \tag{37}$$

Replacing the drift and the variance of the price process into the demand function of investors and noise traders gives:

$$y_t^I = \frac{F\kappa(\mu-\delta_t)+G\theta(\bar{u}-u_t)+\delta_t-rp_t}{r\alpha\left(\sigma_t^p\right)^2} \qquad y_t^n = \frac{F\kappa(u_t-\delta_t)+G\theta(\bar{u}-u_t)+\delta_t-rp_t}{r\alpha\left(\sigma_t^p\right)^2} \tag{38}$$

Together with the market-clearing condition $\left(1-\lambda\right)y_t^I + \lambda y_t^n = S$, and matching coefficients, this equation gives the following pricing function:

$$p_t = \frac{\delta_t}{r+\kappa} + \frac{\lambda\kappa\left(u_t - \bar{u}\right)}{\left(r+\kappa\right)\left(r+\theta\right)} + \frac{\lambda\kappa\bar{u} + \left(1-\lambda\right)\kappa\mu}{r\left(r+\kappa\right)} - S\alpha\left(\sigma_t^p\right)^2 \tag{39}$$

where the coefficients are:

$$F = \frac{1}{r+\kappa} \qquad G = \frac{\lambda\kappa}{(r+\kappa)(r+\theta)} \qquad H = \frac{\lambda\kappa\theta\bar{u}}{r(r+\kappa)(r+\theta)} + \frac{(1-\lambda)\kappa\mu}{r(r+\kappa)} - S\alpha\left(\sigma_t^p\right)^2 \tag{40}$$

From Itô's lemma, the variance of the price process is:

$$\left(\sigma^p\right)^2 \equiv \left(\sigma_t^p\right)^2 = \left(\frac{\sigma^\delta}{r+\kappa}\right)^2 + \left(\frac{\lambda\kappa\sigma^u}{\left(r+\kappa\right)\left(r+\theta\right)}\right)^2 \tag{41}$$

■

A.2 Proof of Proposition 2

Proof. When arbitrageurs are present in the economy, the price must be the present discounted value of future dividends under the arbitrageurs' information set:

$$\tilde{p}_t = E\left[\int_t^{\infty} e^{-r(v-t)}\delta_v dv | \Im_t^A\right] \tag{42}$$

Solving this expectation yields:

$$\tilde{p}_t = \frac{\delta_t}{\kappa + r} + \frac{\kappa\mu}{r(\kappa + r)} \tag{43}$$

The volatility in the proposition follows by applying Itô's lemma. ■

A.3 Proof of Proposition 3

Proof. As arbitrageurs are risk-neutral, the equilibrium price is the expected discounted value of future dividends, under the arbitrageurs' information set.

$$\hat{p}_t = E\left[\int_t^{\infty} e^{-r(v-t)}\delta_v dv | \hat{\Im}_t^A\right] \tag{44}$$

Solving this expectation gives:

$$\hat{p}_t = \frac{\delta_t}{\kappa + r} + \frac{\kappa E\left[\mu | \hat{\Im}_t^A\right]}{r(\kappa + r)} \tag{45}$$

Arbitrageurs condition their expectations on two variables: the observation of the dividend δ_t and the observation of volume v_t. As there are only two other agents in the economy, observing volume is equivalent to observing the total demand of investors and noise traders, y_t:

$$y_t = (1 - \lambda) y_t^I + \lambda y_t^n \tag{46}$$

This demand reveals the following variable x_t in equilibrium:

$$x_t = (1 - \lambda)\mu + \lambda u_t \tag{47}$$

The stochastic process of x_t is:

$$\begin{aligned} dx_t &= \theta\left(\bar{x} - x_t\right) dt + \lambda \sigma^u dZ^u \\ \text{where } \bar{x} &= \lambda \bar{u} + \left(1 - \lambda\right) \mu \end{aligned} \tag{48}$$

Arbitrageurs have two ways to learn about μ: the observation of x_t and the observation of δ_t. At time 0, the observation of the dividend does not reveal any information about μ, as its distribution is assumed to be independent from μ. However, the observation of y_0 is informative, as it reveals x_0. Denote the probability of $\mu = \bar{\delta}$ based on information revealed at time period 0:

$$\pi_0 = \Pr\left(\mu = \bar{\delta} | \delta_0, x_0\right) \tag{49}$$

By Bayes rule, and due to the normality of u_0:

$$\pi_0 = \frac{\hat{\pi} \exp\left(\bar{\delta}\left(1 - \lambda\right)\left(x_0 - \left(1 - \lambda\right)\bar{\delta}/2\right) / \left(\lambda \sigma_0^u\right)^2\right)}{1 - \hat{\pi} + \hat{\pi} \exp\left(\bar{\delta}\left(1 - \lambda\right)\left(x_0 - \left(1 - \lambda\right)\bar{\delta}/2\right) / \left(\lambda \sigma_0^u\right)^2\right)}$$

where $\hat{\pi}$ is the prior probability that $\mu = \bar{\delta}$. The conditional distribution function for the dividend process is (see Küchler and Sørensen, 1997):

$$f_\delta\left(\delta_t | \delta^t, \mu\right) = \exp\left[\frac{\kappa}{2\left(\sigma^\delta\right)^2}\left(\left(\delta_0 - \mu\right)^2 - \left(\delta_t - \mu\right)^2 - \kappa \int_0^t \left(\delta_s - \mu\right)^2 ds + t\right)\right] \tag{50}$$

where $\delta^t = \{\delta_\tau : 0 \leqslant \tau \leqslant t\}$ denotes the history of δ_t. The Radon-Nikodym derivative of changing the measures of δ_t from $\mu = \bar{\delta}$ to $\mu = 0$ is then:

$$\psi_t \equiv \frac{f_\delta\left(\delta_t | \delta^t, \mu = \bar{\delta}\right)}{f_\delta\left(\delta_t | \delta^t, \mu = 0\right)} = \exp\left[\frac{\bar{\delta}\kappa}{\left(\sigma^\delta\right)^2}\left(\delta_t - \delta_0 + \kappa \int_0^t \delta_s ds - \kappa \bar{\delta} t/2\right)\right]$$

The likelihood function of x_t conditional on μ and a path of $x^t = \{x_\tau : 0 \leqslant \tau \leqslant t\}$, is:

$$f_x\left(x_t | x^t, \mu\right) = \exp\left[\frac{\theta}{2\left(\lambda \sigma^u\right)^2}\left(\left(x_0 - \bar{x}\right)^2 - \left(x_t - \bar{x}\right)^2 - \theta \int_0^t \left(x_s - \bar{x}\right)^2 ds + t\right)\right] \tag{51}$$

We find:

$$\phi_t = \frac{f_x\left(x_t | x^t, \mu = \bar{\delta}\right)}{f_x\left(x_t | x^t, \mu = 0\right)} = \exp\left[\frac{\theta \bar{\delta}\left(1 - \lambda\right)}{\left(\lambda \sigma^u\right)^2}\left(x_t - x_0 + \theta \int_0^t x_s ds - \theta t\left(\lambda \bar{u} + \frac{1}{2}\bar{\delta}\left(1 - \lambda\right)\right)\right)\right] \tag{52}$$

By Bayes rule:

$$\pi_t = \Pr\left(\mu = \bar{\delta} | \delta^t, x^t\right) = \frac{\pi_0 \phi_t \psi_t}{1 - \pi_0 + \pi_0 \phi_t \psi_t} \tag{53}$$

Applying Itô's lemma gives:

$$d\pi_t = \pi_t (1 - \pi_t) \left(\frac{d\phi_t}{\phi_t} + \frac{d\psi_t}{\psi_t} - \pi_t \frac{d \langle \phi_t \psi_t \rangle^2}{(\phi_t \psi_t)^2} \right) \tag{54}$$

As $d\psi_t/\psi_t = \left(\bar{\delta}\kappa / \left(\sigma^\delta\right)^2\right)(d\delta_t + \kappa\delta_t dt)$,$d\phi_t/\phi_t = \left(\theta\bar{\delta}(1-\lambda)/(\lambda\sigma^u)^2\right)(dx_t + \theta(x_t - \lambda\bar{u})\,dt)$, and $d\langle\phi_t\psi_t\rangle^2/(\phi_t\psi_t)^2 dt = \bar{\delta}^2\kappa^2/\left(\sigma^\delta\right)^2 + \theta^2\bar{\delta}^2(1-\lambda)^2/(\lambda\sigma^u)^2$, we find the evolution of beliefs:

$$\begin{aligned} d\pi_t &= \pi_t (1 - \pi_t) \frac{\kappa\bar{\delta}}{\left(\sigma^\delta\right)^2} \left[\kappa\left(\mu - \pi_t\bar{\delta}\right) dt + \sigma^\delta dZ_t^\delta\right] \\ &\quad + \pi_t (1 - \pi_t) \frac{\theta(1-\lambda)\bar{\delta}}{(\lambda\sigma^u)^2} \left[\theta(1-\lambda)\left(\mu - \pi_t\bar{\delta}\right) dt + \lambda\sigma^u dZ_t^u\right] \end{aligned} \tag{55}$$

This expression is an extension of theorem 9.1. of Liptser and Shiryaev (2000) to two conditioning variables. The drift and volatility of the price process are:

$$\hat{\eta}_t^p = \frac{\kappa(\mu - \delta_t)}{\kappa + r} + \pi_t(1-\pi_t)\frac{\kappa\left(\mu - \pi_t\bar{\delta}\right)}{r(\kappa + r)}\left[\left(\frac{\kappa\bar{\delta}}{\sigma^\delta}\right)^2 + \left(\frac{\theta(1-\lambda)\bar{\delta}}{\lambda\sigma^u}\right)^2\right] \tag{56}$$

$$\left(\hat{\sigma}_t^p\right)^2 = \left(\frac{\sigma^\delta}{\kappa + r} + \frac{\pi_t(1-\pi_t)\kappa^2\bar{\delta}^2}{\sigma^\delta r(\kappa + r)}\right)^2 + \left(\pi_t(1-\pi_t)\frac{\theta(1-\lambda)\kappa\bar{\delta}^2}{\lambda\sigma^u r(\kappa + r)}\right)^2 \tag{57}$$

The demand from the investors and noise traders is then:

$$(1-\lambda)\,y^I + \lambda y^n = \frac{x_t - \bar{\delta}\pi_t}{r\alpha k(\pi_t)} \tag{58}$$

where:

$$k(\pi_t) = \frac{\kappa\left(r\sigma^\delta + \pi_t(1-\pi_t)\kappa\bar{\delta}^2/\sigma^\delta\right)^2 + \left(\pi_t(1-\pi_t)\theta(1-\lambda)\bar{\delta}^2/(\lambda\sigma^u)\right)^2}{r(\kappa + r)\left(r + \pi_t(1-\pi_t)\left(\left(\kappa\bar{\delta}/\sigma^\delta\right)^2 + \left(\theta(1-\lambda)\bar{\delta}/(\lambda\sigma^u)\right)^2\right)\right)} \tag{59}$$

Trading volume reveals x_t, so that

$$E\left[\mu | \hat{\Im}_t^A\right] = E\left[\mu | \hat{\Im}_0^A, \delta_s, x_s \text{ for } s \leqslant t\right] \tag{60}$$

■

A.4 Proof of Proposition 4

Proof. As in Proposition 2, the no-profit opportunities for arbitrageurs imply that the price is the present discounted value under the arbitrageurs' information set:

$$\bar{p}_t = \frac{\delta_t}{\kappa + r} + \frac{\kappa m_t}{r(\kappa + r)} \tag{61}$$

From theorem 8.1. in Liptser and Shiryaev (2000) we find:

$$\begin{aligned} dm_t &= \frac{\gamma_t}{(\sigma^\delta)^2}\left[\kappa(\mu - m_t)\,dt + \sigma^\delta dZ_t^\delta\right] \\ &\quad + \frac{\theta(1-\lambda)\gamma_t}{(\lambda\sigma^u)^2}\left[\theta(1-\lambda)(\mu - m_t)\,dt + \lambda\sigma^u dZ_t^u\right] \end{aligned} \tag{62}$$

$$dm_t = \gamma_t\left(\frac{d\delta_t - \kappa(m_t - \delta_t)\,dt}{(\sigma^\delta)^2} + \theta(1-\lambda)\frac{dx_t - \theta(1-\lambda)(m_t - x_t)\,dt}{(\lambda\sigma^u)^2}\right) \tag{63}$$

By Itô's lemma, the drift and variance of the price process are:

$$\bar{\eta}_t^P = \frac{\kappa(\mu - \delta_t)}{\kappa + r} + \frac{\kappa\gamma_t(\mu - m_t)}{r(\kappa + r)}\left(\left(\frac{1}{\sigma^\delta}\right)^2 + \left(\frac{\theta(1-\lambda)}{\lambda\sigma^u}\right)^2\right) \tag{64}$$

$$(\bar{\sigma}_t^p)^2 = \left(\frac{\sigma^\delta}{\kappa + r} + \frac{\kappa\gamma_t}{\sigma^\delta r(\kappa + r)}\right)^2 + \left(\frac{\kappa\theta(1-\lambda)\gamma_t}{r(r+\kappa)\lambda\sigma^u}\right)^2 \tag{65}$$

Total demand of investors and noise traders is:

$$\bar{y}_t = \frac{x_t - m_t}{r\alpha q(\gamma_t)} \tag{66}$$

where

$$q(\gamma_t) = \frac{\left(r\sigma^\delta + \gamma_t/\sigma^\delta\right)^2 + \left(\theta(1-\lambda)\gamma_t/(\lambda\sigma^u)\right)^2}{\kappa r(\kappa + r)\left(r + \gamma_t((1/\sigma^\delta)^2 + (\theta(1-\lambda)(\lambda\sigma^u))^2)\right)} \tag{67}$$

Trading volume thus reveals x_t, which concludes the proof. ■

A.5 Proof of Proposition 5

Proof. From Proposition 1, the price at time 0 without arbitrageurs is:

$$p_0 = \frac{\delta_0}{r + \kappa} + \kappa\frac{\lambda\bar{u} + (1-\lambda)\mu}{r(r+\kappa)} + \frac{\lambda\kappa(u_0 - \bar{u})}{(r+\kappa)(r+\theta)} - S\alpha(\sigma_0^p)^2 \tag{68}$$

The unconditional variance of p_0 is then:

$$var\left(p_0\right) = \left(\frac{\sigma_0^\delta}{r+\kappa}\right)^2 + \left(\frac{\kappa\left(1-\lambda\right)\sigma^\mu}{r\left(r+\kappa\right)}\right)^2 + \left(\frac{\lambda\kappa\sigma_0^u}{\left(r+\kappa\right)\left(r+\theta\right)}\right)^2 \tag{69}$$

With arbitrageurs, the price at time 0 is from Proposition 4:

$$\bar{p}_0 = \frac{\delta_0}{r+\kappa} + \frac{\kappa m_0}{r\left(r+\kappa\right)} \tag{70}$$

Computing the unconditional variance gives:

$$var\left(\bar{p}_0\right) = var\left(\frac{\delta_0}{r+\kappa}\right) + var\left(\frac{\kappa m_0}{r\left(\kappa+r\right)}\right) = \left(\frac{\sigma_0^\delta}{r+\kappa}\right)^2 + var\left(\frac{\kappa m_0}{r\left(r+\kappa\right)}\right) \tag{71}$$

The difference in the variance with and without arbitrageurs is then:

$$var\left(\bar{p}_0\right) - var\left(p_0\right) = var\left(\frac{\kappa m_0}{r\left(r+\kappa\right)}\right) - \left(\frac{\kappa\left(1-\lambda\right)\sigma^\mu}{r\left(r+\kappa\right)}\right)^2 - \left(\frac{\lambda\kappa\sigma_0^u}{\left(r+\kappa\right)\left(r+\theta\right)}\right)^2 \tag{72}$$

■

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