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Abstract

In the literature, bank runs take the form of withdrawals of real demand deposits that deplete a fixed reserve of goods in the banking system. This framework describes the type of bank run that has occurred historically in the United States and more recently in developing countries. However, in a modern banking system, large withdrawals take the form of electronic payments of inside money, with no analog of a depletion of a scarce reserve from the banking system. In a new framework of nominal demand deposits repayable in inside money, pure liquidity-driven bank runs do not occur. If there were excessive early withdrawals, nominal deposits would hedge the bank, and flexible monetary prices in the goods market would limit real consumption. The maturity mismatch of short-term liabilities and long-term assets is not sufficient for multiple equilibria bank runs without other frictions, such as problems in the interbank market. A key role of the bank is to ensure optimal real liquidity, allowing markets to optimally distribute consumption goods through the price mechanism.

Key words: bank runs, inside money, nominal contracts, demand deposits

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1 Introduction

In this paper, nominal demand deposits that are repayable in inside money are introduced to a Diamond and Dybvig (1983) framework. Pure liquidity-driven bank runs do not occur, and consumers receive the optimal consumption allocation. In a frictionless setting, these nominal deposits are Pareto improving to real demand deposits. The results show that with nominal deposits and inside money, the mismatch of short term liabilities and long term illiquid assets does not alone explain self-fulfilling bank runs. This implies that additional frictions are needed to produce runs. The results also show that a key role of the bank is to ensure the optimal amount of real liquidity, which a market does not do. But with the optimal real liquidity chosen by the bank, a market does distribute consumption goods optimally through the price mechanism to consumers.

A nominal deposits framework contrasts with the standard real deposits framework in the literature. Starting with Diamond-Dybvig, banks pay withdrawals of deposits in real goods. Excessive early withdrawals of deposits are initially triggered due to various causes in the different strands of the literature, including multiple equilibria following Diamond-Dybvig and asymmetric information about assets following Chari and Jagannathan (1988) and Calomiris and Kahn (1991). Regardless of the trigger, excessive withdrawals deplete a fixed reserve of liquid real goods available to be paid out from the banking system. Because demand deposits are modeled as fixed promises of goods, payments to withdrawers cannot be rationed. Long term investments have to be inefficiently liquidated to provide short term payouts. The bank will not be able to pay future withdrawals, so all depositors try to withdraw immediately. Even if the bank is otherwise fundamentally solvent, a run can occur in the real deposits framework because of the fragile liquidity structure of the bank's real short term liabilities and long term assets.

The literature describes bank runs that occurred historically in the U.S. and in recent decades in developing countries, where bank deposits were withdrawn in outside money (currency) or had a real value because of a pegged exchange rate or gold standard backing. Gorton (1988) and Friedman and Schwartz (1963) show that during banking crises in the U.S., the ratio of currency to deposits increased. This implies that depositors withdrew currency and stored it outside of the banking system, corresponding to bank run models in which real goods are withdrawn from the banking system. Allen and Gale (1998) and Smith (2003) cite the withdrawals of currency from the banking system during these

historical bank runs as what their models explain. Diamond and Rajan (2005) state that their real deposit model most closely resembles the gold standard era in the U.S., with an inflexible value of money, and the recent banking crisis in Argentina, with deposits repayable in dollars that could be withdrawn from the country. Contrasting the nominal and real models highlights two properties of bank deposits that are fundamental in addition to the asset-liability maturity mismatch for understanding traditional bank runs: bank deposits were redeemed for currency or had a real value.

Nominal deposits payable in inside money may better describe a modern banking system and economy, in which bank deposits and assets are primarily denominated in domestic currency with a floating exchange rate value. Large bank withdrawals are typically not converted to currency that is stored outside the banking system, but rather take the form of electronic payments of inside money for the purchase of goods or financial securities on markets.[1] There is no correspondence to the real-deposits bank run literature of a depletion from the banking system of a scarce reserve. This paper points out that unless currency is withdrawn and stored outside of the banking system, bank reserves are not drained from the banking system in a closed economy absent central bank intervention. To explain liquidity runs, frictions in addition to the asset-liability mismatch are needed, such as problems in the interbank market. These frictions need to be studied within a nominal deposits framework to understand modern banking liquidity crises.

In the model, the bank initially lends inside money to entrepreneurs who store and invest goods and then sell them on the goods market. If depositors run the bank by making excessive early withdrawals to purchase goods, an abundance of money to buy goods drives the price up in the early period. The bank's short term deposit withdrawals increase in nominal value, but the bank is hedged against the real cost of excessive early withdrawals since the price level increases. The bank borrows the excess funds it paid out that are received by the entrepreneur selling goods, requiring no liquidation of long term loans and illiquid investments. Higher prices in the goods market limit the real consumption received by early purchasers and save goods for those who wait to withdraw, regardless of

[1] An electronic withdrawal is much more practical and timely than a costly and risky physical withdrawal of a large sum of currency, especially if there is an imminent run on a bank. Demirguc-Kunt et al. (2006) show that in contemporary times, aggregate bank deposits do not significantly decline during times of financial distress, especially in developed countries and even in many less-developed economies. This suggests that currency is not withdrawn from the banking system in any critical amount in modern economies. Skeie (2004) examines nominal deposits allowing for withdrawals of currency, which may be stored outside of the banking system, and also withdrawals of inside money paid within a clearinghouse system of banks.

how many other late consumers withdraw early. Depositors face lower relative prices at the later period and prefer to purchase goods then. The unique equilibrium is no bank run, which is an equilibrium in weakly dominant strategies. Moreover, the price mechanism in the goods market provides the ex-ante optimal allocation of goods among consumers if the bank can choose the amount of liquidity that is stored. If instead the entrepreneur chooses the liquidity level, the consumption allocation is suboptimal. In addition, the no run and optimal allocation results extend from a centralized price-clearing market to a sequential goods market, in which price dynamics reflect the amount of early withdrawals over time.

Several papers in the bank run literature examine reduced form or partial equilibrium models of money paid for the withdrawal of real demand deposits.[2] Money paid for deposit withdrawals either receives utility directly, is withdrawn from the economy, or is used to buy goods without an endogenous monetary price level set by a goods market.[3] I show that with a general equilibrium model of money and a goods market, monetary prices imply nominal deposits hedge banks when there are excess withdrawals by depositors purchasing goods. Papers that show that the interbank market cannot prevent large liquidity runs typically depend on withdrawals of goods directly from the banking system to induce runs.[4]

Allen and Gale (1998) add nominal contracts through central bank loans of currency, which the bank pays depositors in addition to goods, in order to deflate the real value of deposit withdrawals during an equilibrium bank run. The currency is stored outside of the banking system by withdrawers until a later period, when they use it to purchase goods from the bank. The bank repays the currency to the central bank and pays remaining depositors in real goods for their withdrawals. This approach also provides the basis for Allen and Gale (2000b) and Gale and Vives (2002). In the current study, nominal deposits are contracted ex-ante to repay only in inside money, so that during a potential run prices would rise and real deposit values would fall endogenously, which prevents bank runs in equilibrium. Diamond and Rajan (2006) and Champ et al. (1996) examine bank runs

[2] Bryant (1980), Calomiris and Kahn (1991), Chang and Velasco (2000), Freixas et al. (2000, 2004), Freixas and Holthausen (2005), Peck and Shell (2004), Postlewaite and Vives (1987), Repullo (2000) and Rochet and Vives (2004).

[3] Martin (2005) shows that central bank lending in a real deposits model describes historical commodity money rather than modern fiat money as in the nominal deposits model of Martin (2006), which follows Allen and Gale (1998).

[4] Bhattacharya and Gale (1987), Bhattacharya and Fulghieri (1994), Allen and Gale (2000a, 2004), Aghion et al. (2000) and Diamond and Rajan (2005).

with nominal contracts and money in general equilibrium.[5] They are explicit in modeling how bank runs result if there is a large enough demand for currency withdrawals out of the banking system. Diamond and Rajan (2006) also examine the bank's asset side and show that nominal contracts cannot prevent bank runs caused by idiosyncratic delays in asset returns. I examine the bank's liability side and show that nominal contracts can prevent bank runs caused by depositor liquidity-driven withdrawals.

Jacklin (1987) shows that depositors purchasing assets with real bank deposits destroy the bank's optimal risk sharing. I show a very different point that depositors purchasing liquid goods with nominal bank deposits achieve the bank's optimal risk sharing. Jacklin (1987) also shows that an equity market can replicate the optimal risk sharing of a bank that issues real deposits, without bank runs. I show that a bank that issues nominal deposits can also replicate the optimal risk sharing of a bank that issues real deposits, without bank runs. A synthesis of the literature on real demand deposits, bank runs, liquidity provision, and the coexistence of banks and markets is given by von Thadden (1999).

Section 2 introduces the basic nominal bank deposits model with a centralized goods market. Section 3 gives results and makes comparisons to economies with various markets and no bank. Section 4 extends the basic results to a banking model with a sequential goods market, and Section 5 concludes. Proofs are in the Appendix.

2 Model

The model is based on that which has become common in the literature since Diamond and Dybvig (1983), to which entrepreneurs, goods markets and nominal deposit and loan contracts are added.

2.1 Real model

There are three periods, $t = 0, 1, 2$. A continuum of ex-ante identical consumers indexed by $i \in [0, 1]$ is endowed with good at $t = 0$. At $t = 1$, a fraction $\lambda \in (0, 1)$ of consumers receive an unverifiable liquidity shock, need to consume in that period and then die. These "early" consumers have utility given by $U = u(c_1)$, where c_1 is their consumption in $t = 1$. The remaining fraction $1 - \lambda$ are "late" consumers and have utility $U = u(c_2)$,

[5] See also Boyd et at. (2004a, 2004b) and McAndrews and Roberds (1995, 1999).

where c_2 is their consumption in $t = 2$. Consumption is expressed as goods per unit-sized consumer. The allocation consumed by early and late consumers is expressed as (c_1, c_2). Period utility functions $u(\cdot)$ are assumed to be twice continuously differentiable, increasing, strictly concave and satisfy Inada conditions $u'(0) = \infty$ and $u'(\infty) = 0$. I make the typical assumption following Diamond-Dybvig that the consumers' coefficient of relative risk aversion is greater than one, which implies that banks provide risk-decreasing insurance against liquidity shocks.

At $t = 0, 1$, any fraction of goods can be stored for a return of one in the following period. At $t = 0$, any fraction of goods can alternatively be invested by entrepreneurs for a return of $r > 1$ at $t = 2$. These invested goods can be liquidated for a salvage return of $s \in (0, 1)$ at $t = 1$ and zero return at $t = 2$, which reflects the inefficiency of liquidation. There is a unit continuum of entrepreneurs that are competitive, risk neutral, sell goods on a market at $t = 1, 2$, and maximize profits in terms of unsold goods they consume at the end of $t = 2$. Without loss of generality, I treat the continuum of entrepreneurs as a single price-taking entrepreneur.

2.2 Bank and nominal deposits

At $t = 0$, the bank issues loans and takes deposits denominated in a nominal unit of account, which is created using the following ad-hoc technique. At $t = 0$, a central bank creates fiat currency (outside money) with which it stands ready to buy and (to the extent feasible) sell goods at a fixed price of $P_0 = 1$. The central bank receives all currency back by the end of the $t = 0$ period and plays no role thereafter. Even though the supply of outside money is zero after $t = 0$, the role of central bank currency as a nominal unit of account carries over to later periods due to monetary contracts established at $t = 0$.

As illustrated in Figure 1, consumers sell their one unit of goods to the central bank for one unit of central bank currency at price $P_0 = 1$. The consumers deposit the currency in the bank in exchange for a nominal demand deposit contract. The deposit contract pays a nominal return of either $D_1 \geq 1$ or $D_2 \geq 1$ in inside money, described below, on demand when a depositor withdraws at either $t = 1$ or $t = 2$, respectively, subject to a sequential service constraint. Uppercase letters denote variables with nominal values and lowercase letters denote variables with real values. The bank lends the unit of central bank currency to the entrepreneur for the nominal loan contract repayments of K_1 and K_2, where K_t is payable in inside money. The entrepreneur pays the central bank the currency to buy

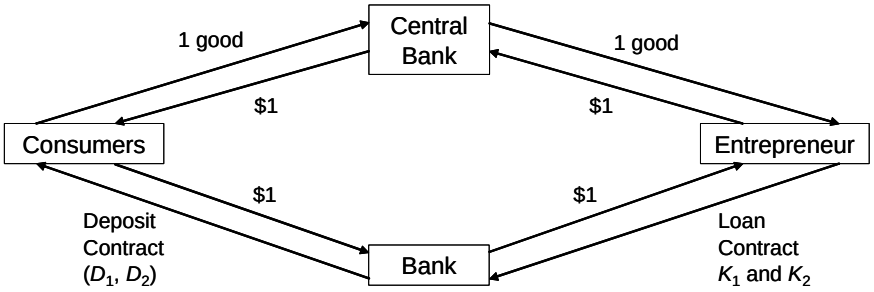

Figure 1: **Introduction of Nominal Contracts at $t = 0$.** *A central bank buys and sells goods with fiat currency at a fixed price of $P_0 = 1$ to establish a nominal unit of account. Consumers sell their good to the central bank for currency. They deposit the currency in the bank for a nominal demand deposit contract (D_1, D_2), which repays D_t upon withdrawal at either $t = 1$ or $t = 2$. The bank lends the currency to the entrepreneur for the nominal loan contract repayments of K_1 and K_2 due at $t = 1$ and $t = 2$, respectively. The entrepreneur buys the good from the central bank with the currency.*

the good, of which it invests $\alpha \in (0, 1)$ and stores $1 - \alpha$. At the end of this exchange at $t = 0$, the central bank holds all the currency, which it extinguishes, and does not hold any goods. The consumer holds the demand deposit account that repays D_1 or D_2. The entrepreneur holds $1 - \alpha$ stored goods and α invested goods. The bank holds the loan contract that repays K_1 and K_2.

2.3 Discussion of inside money and nominal unit of account

The banking model with inside money and a nominal unit of account relates to the "ideal banking system" suggested by Wicksell (1906).[6] A single bank holds nominal deposits and loans and all payments in the economy are made internal to the bank. For simplicity,

[6] "If customers are in direct business contact the money need never leave the bank at all, but payment can be made by a simple transfer from one banking account to another... If we suppose for the sake of simplicity that all such business is concentrated in a single bank... All payments will be made by cheques drawn on the payer's banking account, but these cheques will never lead to any withdrawals of money from the bank, but only to a transfer to the payee's account in the books of the bank... The lending operations of the bank will consist rather in its entering in its books a fictitious deposit equal to the amount of the loan, on which the borrower may draw... [P]ayments...must naturally lead to a credit with another person's (seller's) account, either in the form of a deposit paid in or of a repayment of a debt..."

"Some authors, both earlier and more recent have leaned to the view...that mere 'bank cover', i.e. the holding of bills and securities in the portfolios of the banks as the sole basis of note issues and cheques would be the ideal... Indeed, our modern monetary system is afflicted by an imperfection, an inherent contradiction. The development of credit aims at rendering the holding of cash reserves unnecessary, and yet these cash reserves are a necessary, though far from sufficient guarantee of the stability of money values... Only by completely divorcing the value of money from metal...and by making...the unit employed in the accounts of the credit institutions, both the medium of exchange and the measure of value—only in this way can the contradiction be overcome and the imperfection be remedied."

Knut Wicksell (1906), *Lectures on Political Economy, Volume Two: Money*

no reserves are held by the bank, which implies an infinite money multiplier on deposits. This corresponds to a system of free banking, or laissez-faire, with the assumptions of no currency payments and a single bank. As Wicksell (1906) suggests, reserves are not fundamentally necessary if payments (his "medium of exchange") are made in a unit of account that is nominal, which corresponds to his "measure of value" of money divorced from metal (or goods).

This method for introducing nominal deposits is also a contribution to monetary theory by showing how inside money can be based on a unit of account established by a central bank, even if outside money originally issued by the central bank stops circulating, as envisioned by Woodford (2000). The fixed price $P_0 = 1$ may also be interpreted as an original commodity value for money such as a gold standard. Even though the fixed-value of the unit of account is known at $t = 0$ to be discontinued in future periods, the unit of account is still contracted upon, and the unit of account continues to exist with a flexible but stable nominal value. This model may provide insight into the historical process for money taking on over time a nominal unit of account, which is ultimately based on an original commodity value, as described by Kitson (1895), pages 6-8, and von Mises (1912), pages 108-123.

As an alternative modelling technique, inside money and nominal contracts could be introduced with more realistic transactions at $t = 0$. The central bank never actually has to exchange goods for currency if it just provides the guarantee that it is willing to do so at $t = 0$. The interpretation could be a gold standard, under which the central bank guarantees the issuance or redemption of currency for gold at P_0 but does not actively trade. This shows how a nominal unit of account can be established by central bank fiat even if there is no circulating central bank currency. The entrepreneur could borrow one nominal unit of inside money from the bank for nominal loan contract K_1 and K_2 to purchase the good at price $P_0 = 1$ from consumers to store and invest. The consumers would deposit the inside money at the bank for the nominal demand deposit paying D_1 or D_2.

2.4 Inside money payments and bank budget constraint

At $t = 1, 2$, inside money payments are made by debiting and crediting accounts internal to the bank. At $t = 1$, $\overline{\lambda} \in [\lambda, 1]$ is the fraction of consumers who withdraw to purchase goods, which includes early consumers and any late consumers who choose to withdraw. At $t = 2$,

any remaining fraction of late consumers $1-\overline{\lambda}$ withdraw to purchase goods. Depositor withdrawals at $t=1,2$ are made by debiting the depositor's account and crediting the entrepreneur's account at the bank. These withdrawals in aggregate are the demand schedule $q_t^D(P_t,\cdot)$ for purchases on the goods market, where P_t is the price of goods at $t=1,2$. The entrepreneur submits a supply schedule $q_t^S(P_t,\cdot)$ for sales on the goods market. The entrepreneur repays loans, which debits the entrepreneur's account and credits the bank's account. The entrepreneur's excess balance if any at $t=1$ is held on deposit at the bank for return D_{12} at $t=2$.

The bank's budget constraint for "settling" withdrawal payments each period is

$$t = 1: \qquad \overline{\lambda}\delta_1 D_1 = K_1 + \frac{\delta_K K_2}{K_{12}} + [P_1 q_1^S(\cdot) - K_1 - \frac{\delta_K K_2}{K_{12}}]^+ \tag{1}$$

$$t = 2: \qquad (1-\overline{\lambda})\delta_2 D_2 + [P_1 q_1^S(\cdot) - K_1 - \frac{\delta_K K_2}{K_{12}}]^+ \delta_2 D_{12} = (1-\delta_K)K_2, \tag{2}$$

where for $t=1,2$, $\delta_t \in [0,1]$ is the aggregate fraction of withdrawals that "settles" of the contracted return due; δ_K is the fraction of K_2 that the bank "calls" for repayment at $t=1$ at the discounted rate $\frac{1}{K_{12}}$ if needed to help prevent default at $t=1$. On the RHS of budget constraint (1), the first two terms are the $t=1$ loan repayment by the entrepreneur and the last term in brackets is the entrepreneur's $t=1$ deposit at the bank. The LHS term is the deposits repaid and settled by the bank for withdrawals at $t=1$. In constraint (2), on the RHS is the $t=2$ loan repayment from the entrepreneur. The LHS terms are deposits repaid and settled by the bank for withdrawals at $t=2$ by consumers and the entrepreneur, respectively. The entrepreneur delivers $q_t^S(P_t,\cdot)$ goods at period $t=1,2$ to consumers whose payments settle.

The maximum partial amount of withdrawals are settled and the minimal fraction of loans are called, such that the bank's budget constraints and a sequential service constraint hold as follows:

$$\max_{(\delta_1,\delta_2,\delta_K)\in[0,1]^3} \quad \delta_1+\delta_2-\delta_K \tag{3a}$$

$$\text{s.t.} \quad (1),(2) \tag{3b}$$

$$\delta_2 = 0 \text{ and } \delta_K = 1 \text{ if } \delta_1 < 1, \tag{3c}$$

where constraint (3c) is the sequential service constraint between periods. Settlement priority within periods is by the order of withdrawals at $t=1$, to satisfy a $t=1$ intraperiod

sequential service constraint, and is by the pro rata share of withdrawals at $t = 2$. The settlement value for the withdrawal of consumer i at $t = 1$ is $\mathbf{1}_{\delta_1^i \leq \overline{\lambda}\delta_1} D_1$, where δ_1^i is the fraction of consumers who have withdrawn before i and $\mathbf{1}_{[.]}$ is the indicator function. The settlement value for withdrawals at $t = 2$ is $\delta_2 D_2$ for consumers and $\delta_2 D_{12}$ for the entrepreneur.

2.5 Market equilibrium

Two different equilibrium concepts are used to determine (i) when consumers withdraw from the bank, and (ii) the price and quantity of goods that consumers purchase from the entrepreneur. In this subsection, I define the second concept as a market equilibrium, which is the competitive price and quantity equilibrium of the goods market for any given $\overline{\lambda} \in [\lambda, 1]$. In the following subsection, I define the first concept as a withdrawal equilibrium $\overline{\lambda}^*$ that is a Nash equilibrium of the noncooperative game played by individual late consumers who choose between the strategies of withdrawing at $t = 1$ or 2. I use two equilibrium concepts in order to have a competitive market approach for the trading of goods, and a strategic game approach following the bank run literature for the withdrawal decision. This allows me to show the result that a late consumer does not withdraw from the bank at $t = 1$ as a weakly dominant strategy—that is, regardless of whether other late consumers withdraw. For the withdrawal game, a late consumer i strategically chooses whether to withdraw at $t = 1$ or 2 without conditioning the decision on other late consumers' decisions, which are summarized by $\overline{\lambda} \in [\lambda, 1]$. This requires that the withdrawal decision cannot be conditioned on P_t.

First I examine the market equilibrium. For the goods market, consumer i withdrawing at period t does not choose the quantity to demand independently for each value of P_t. Instead, given that i chooses to withdraw at period t, i submits a "market order" individual demand schedule that spends the full amount of his withdrawal that settles to purchase goods at P_t. Late consumer i's individual demand schedule is either $q_1^{D,i}(P_1, \overline{\lambda}, \delta_1, \delta_1^i) = \frac{\mathbf{1}_{\delta_1^i \leq \overline{\lambda}\delta_1} D_1}{P_1}$ for a withdrawal at $t = 1$ or $q_2^{D,i}(P_2, \delta_2) = \frac{\delta_2 D_2}{P_2}$ for a withdrawal at $t = 2$.[7] For

[7] An alternative approach is to allow consumers to withdraw to purchase goods by submitting a "limit order" demand schedule to choose the amount $q_t^S(P_t)$ to demand independently for each value of P_t, and redeposit any excess withdrawal that is not used to purchase goods at the equilibrium price. This would correspond to consumers giving individual competitive demand schedules for given prices. This flexibility allows late consumers to condition their early withdrawal decision on the realized aggregate withdrawal decisions of all late consumers revealed through P_1. This would help to coordinate late depositor actions and reinforce the result of no bank runs.

a given $\overline{\lambda}$, the aggregate demand schedules are

$$q_1^D(P_1, \delta_1, \overline{\lambda}) = \begin{cases} \frac{\overline{\lambda}\delta_1 D_1}{P_1} & \text{if } P_1 > 0 \\ \infty & \text{if } P_1 = 0 \end{cases} \tag{4}$$

$$q_2^D(P_2, \delta_2, \overline{\lambda}) = \begin{cases} \frac{(1-\overline{\lambda})\delta_2 D_2}{P_2} & \text{if } P_2 > 0 \\ \infty & \text{if } P_2 = 0. \end{cases} \tag{5}$$

If $P_t = 0$, there is infinite demand at t, even if there are zero settled withdrawals when $\delta_t = 0$ or, for $t = 2$, $\overline{\lambda} = 1$.

In contrast, the entrepreneur submits a "limit order" supply schedule. The entrepreneur can choose the quantity to supply $q_t^S(P_t)$ at each value of P_t at $t = 1, 2$. I define the vectors $q^S(P, \delta) \equiv (q_1^S(P, \delta), q_2^S(P, \delta))$, $P \equiv (P_1, P_2)$ and $\delta \equiv (\delta_1, \delta_2, \delta_K)$. The entrepreneur's optimization is to find $q^S(P, \delta)$ and $\gamma(P, \delta)$ that maximize the entrepreneur's profits:

$$\max_{q_1^S, q_2^S, \gamma \geq 0} \quad \widehat{q}_2^S(q_1^S, \gamma) - q_2^S \tag{6a}$$

$$\text{s.t.} \quad q_1^S \leq \widehat{q}_1^S(\gamma) \tag{6b}$$

$$\gamma \leq \alpha \tag{6c}$$

$$q_2^S \leq \widehat{q}_2^S(q_1^S, \gamma) \tag{6d}$$

$$q_1^S \geq \frac{K_1 + \frac{\delta_K K_2}{K_{12}}}{P_1} \tag{6e}$$

$$q_2^S \geq \frac{(1-\delta_K)K_2 - [P_1 q_1^S - K_1 - \frac{\delta_K K_2}{K_{12}}]^+ \delta_2 D_{12}}{P_2}, \tag{6f}$$

where $\gamma(P, \delta)$ is the amount of invested goods that are liquidated at $t = 1$, and

$$\widehat{q}_1^S(\gamma) \equiv 1 - \alpha + \gamma s \tag{7}$$

$$\widehat{q}_2^S(q_1^S, \gamma) \equiv (\alpha - \gamma)r + \widehat{q}_1^S(\gamma) - q_1^S \tag{8}$$

are defined as the amount of goods available to sell at $t = 1, 2$, respectively. The objective function (6a) is the profit in goods that the entrepreneur consumes at $t = 2$. Constraints (6b), (6c) and (6d) are the maximum amounts of goods that can be sold at $t = 1$, liquidated at $t = 1$ and sold at $t = 2$, respectively. Constraints (6e) and (6f) are the entrepreneur's budget constraints to repay its loan at $t = 1$ and $t = 2$, respectively.

For any given $\overline{\lambda} \in [\lambda, 1]$, a *market equilibrium* is defined as $(P^*(\overline{\lambda}), q^*(\overline{\lambda}), \gamma^*(\overline{\lambda}), \delta^*(\overline{\lambda}))$

that solve market clearing conditions

$$q_1^D(P_1, \delta_1, \overline{\lambda}) = q_1^S(P, \delta) \tag{9a}$$

$$q_2^D(P_2, \delta_2, \overline{\lambda}) = q_2^S(P, \delta), \tag{9b}$$

the entrepreneur's zero profit condition

$$q_2^S(P, \delta) = \widehat{q}_2^S(q_1^S(P, \delta), \gamma(P, \delta)), \tag{10}$$

and bank budget and sequential service constraint (3), where $q_1^D(\cdot)$ and $q_2^D(\cdot)$ are given by the consumers' aggregate demand (4) and (5), respectively, and $(q_1^S(\cdot), q_2^S(\cdot), \gamma(\cdot))$ is a solution to the entrepreneur's optimization (6). I assume that if money is worthless, consumers pay their withdrawals to the entrepreneur even though they receive no goods. Written as

$$q_t^*\left(\overline{\lambda}\right) P_t^*\left(\overline{\lambda}\right) \equiv \begin{cases} \overline{\lambda}\delta_1^*\left(\overline{\lambda}\right) D_1 & \text{for } t = 1 \\ \left(1 - \overline{\lambda}\right) \delta_2^*\left(\overline{\lambda}\right) D_2 & \text{for } t = 2 \end{cases} \tag{11}$$

if $q_t^*\left(\overline{\lambda}\right) = 0$ and $P_t^*\left(\overline{\lambda}\right) = \infty$ for $t = 1, 2$, this assumption is made since the product is not otherwise well defined.

2.6 Withdrawal equilibrium

I define the *withdrawal game* as a noncooperative game, where the players are the late consumers and the possible strategies of late consumer i are withdrawing at $t = 1$ or withdrawing at $t = 2$. The aggregate strategies, represented by $\overline{\lambda}$, of all late consumers not i are $\overline{\lambda} - \lambda$ for those withdrawing at $t = 1$, and $1 - \overline{\lambda}$ for those withdrawing at $t = 2$. The payoff for late consumer i in the withdrawal game, as a function of consumer i's strategy to withdraw at $t = 1, 2$, and other late consumers' strategies given by $\overline{\lambda} \in [\lambda, 1]$, is

$$c_2^i(t, P_t^*(\overline{\lambda}), \overline{\lambda}, \delta_t^*(\overline{\lambda}), \delta_t^i) = q_t^{D,i}(P_t^*(\overline{\lambda}), \overline{\lambda}, \delta_t^*(\overline{\lambda}), \delta_t^i), \tag{12}$$

where $P_t^*(\overline{\lambda})$ and $\delta_t^*(\overline{\lambda})$ are part of a market equilibrium for $t = 1, 2$. For a given $\overline{\lambda} \in [\lambda, 1]$, withdrawing at $t = 2$ is a best response for late consumer i if

$$\frac{\delta_2^*(\overline{\lambda}) D_2}{P_2^*(\overline{\lambda})} \geq \frac{\mathbf{1}_{\delta_1^i \leq \overline{\lambda}\delta_1^*(\overline{\lambda})} D_1}{P_1^*(\overline{\lambda})} \tag{13}$$

for all $\delta_1^i \in [0, \overline{\lambda}]$ and all $(P^*(\overline{\lambda}), \delta^*(\overline{\lambda}))$ that are part of a market equilibrium.

I define a *withdrawal equilibrium* $\overline{\lambda}^*$ to be any Nash equilibrium of the withdrawal game, and call any $\overline{\lambda}^* > \lambda$ a bank run equilibrium. Withdrawing at $t = 2$ is a weakly dominant strategy for late consumer i if

$$\begin{aligned} &\text{condition (13) holds for all } \overline{\lambda} \in [\lambda, 1] \\ &\text{and holds with strict inequality for some } \overline{\lambda} \in [\lambda, 1]. \end{aligned} \tag{14}$$

Since late consumers are identical, if condition (14) holds, then $\overline{\lambda}^* = \lambda$ is a unique Nash equilibrium, and an equilibrium in weakly dominant strategies: each late consumer has the greatest consumption by withdrawing at $t = 2$ regardless of $\overline{\lambda}$. Consumption for early consumer i is

$$c_1^i(P_1^*(\overline{\lambda}), \overline{\lambda}, \delta_1^*(\overline{\lambda}), \delta_1^i) = q_1^{D,i}(P_1^*(\overline{\lambda}), \overline{\lambda}, \delta_1^*(\overline{\lambda}), \delta_1^i). \tag{15}$$

3 Results

I initially take α, D_1 and D_{12} as exogenous and show that there are no bank runs. I also explain that there would be no bank runs with multiple banks. I then endogenize α to examine consumption allocations. I examine the choice of α by the the bank and by the entrepreneur, and I explain why the bank can likely determine α. I also discuss D_t as an exogenous deposit rate and K_t as a competitive lending rate. I initially make assumptions of limited participation. Only the bank lends to the entrepreneur, only the entrepreneur invests goods, and invested goods cannot be traded. I relax limited participation when I make comparisons to alternate economies with no bank. These include a market economy in which consumers can invest goods and trade investments, and a consumer-lending economy in which consumers lend directly to the entrepreneur and trade nominal loans.

3.1 Withdrawal results

The bank chooses $D_2 = \frac{D_{12}}{D_1}$, $K_1 = \lambda D_1$, $K_2 = (1 - \lambda)D_2$, and $K_{12} = D_{12}$. I first describe the market equilibrium for any $\overline{\lambda} \in [\lambda, 1]$. The equilibrium quantities can be written as $q_1^*(\overline{\lambda}) = q_1^D(P_1^*(\overline{\lambda}), \delta_1^*(\overline{\lambda}), \overline{\lambda})$ and $q_2^*(\overline{\lambda}) = q_2^D(P_2^*(\overline{\lambda}), \delta_2^*(\overline{\lambda}), \overline{\lambda})$, which requires

that equilibrium values satisfy

$$q_1^*(\overline{\lambda}) = \frac{\overline{\lambda}\delta_1^*(\overline{\lambda})D_1}{P_1^*(\overline{\lambda})} \tag{16a}$$

$$q_2^*(\overline{\lambda}) = \frac{(1-\overline{\lambda})\delta_2^*(\overline{\lambda})D_2}{P_2^*(\overline{\lambda})} \tag{16b}$$

$$P_t^*(\overline{\lambda}) > 0 \text{ for } t = 1, 2. \tag{16c}$$

$P_t = 0$ is not an equilibrium for any $\overline{\lambda} \in [\lambda, 1]$. If it were an equilibrium, it would imply $q_t = \infty$ according to the consumers' demand (4) and (5), which is not feasible according to the entrepreneur's constraints (6b) and (6d). Equations (16a), (16b) and (16c) imply that equilibrium prices can be expressed as

$$P_1^*(\overline{\lambda}) = \frac{\overline{\lambda}\delta_1^*(\overline{\lambda})D_1}{q_1^*(\overline{\lambda})} \text{ if } \delta_1^*(\overline{\lambda}) > 0 \tag{17}$$

$$P_2^*(\overline{\lambda}) = \frac{(1-\overline{\lambda})\delta_2^*(\overline{\lambda})D_2}{q_2^*(\overline{\lambda})} \text{ if } \overline{\lambda} < 1 \text{ and } \delta_2^*(\overline{\lambda}) > 0, \tag{18}$$

and there may be multiple market equilibrium values for $P_2^*(1)$.

Lemma 1. *For a given $\overline{\lambda} \in [\lambda, 1]$, the market equilibrium is characterized by*

$$P_1^*(\overline{\lambda}) \geq \frac{P_2^*\left(\overline{\lambda}\right)}{D_{12}} \tag{19}$$

$$P_1^*(1) > \frac{P_2^*(1)}{D_{12}} \tag{20}$$

$$sP_1^*\left(\overline{\lambda}\right) \geq r\frac{P_2^*\left(\overline{\lambda}\right)}{D_{12}} \text{ if } \gamma^*\left(\overline{\lambda}\right) > 0 \tag{21}$$

$$\delta_1^*\left(\overline{\lambda}\right) = \delta_2^*\left(\overline{\lambda}\right) = 1 \tag{22}$$

$$\delta_K^* = 0. \tag{23}$$

Proof. See Appendix. ■

Result (19) reflects that the entrepreneur attempts to equate discounted marginal revenues from sales at $t = 1$ and at $t = 2$ in order to maximize profits. The LHS of (19) is the entrepreneur's marginal revenue P_1 of selling additional goods at $t = 1$, whereas the RHS is the entrepreneur's marginal revenue P_2 discounted to $t = 1$ by D_{12} of selling additional goods at $t = 2$. The inequality is due to the asymmetry between selling an

additional good at $t = 1$ versus at $t = 2$. Suppose $P_1 < \frac{P_2}{D_{12}}$. The entrepreneur would store all goods at $t = 1$ to sell at $t = 2$, but this would imply $P_1 = \infty$. Hence, $P_1 < \frac{P_2}{D_{12}}$ is not an equilibrium. Conversely, if $P_1 > \frac{P_2}{D_{12}}$, the entrepreneur would have to liquidate invested goods at a loss to sell more goods at $t = 1$. Thus, $P_1 > \frac{P_2}{D_{12}}$ is a potential equilibrium. Result (21) shows that there is liquidation only if P_1 is greater than discounted P_2 by the marginal rate of transformation $\frac{r}{s}$ from not liquidating an invested good, implying (19) does not bind if $\gamma > 0$.

If there is a complete bank run ($\overline{\lambda} = 1$), the entrepreneur liquidates all invested goods to sell at $t = 1$ ($\gamma^*(1) = \alpha$), giving result (20). $P_2^*(1)$ is a meaningful price at which a marginal late consumer considers buying goods for his strategic withdrawal decision (13). If the zero-mass infinitesimal late consumer withdraws at $t = 2$, he spends a zero-mass amount of money to buy goods. The entrepreneur is indifferent to selling him a zero-mass amount of goods at $P_2^*(1) < P_1^*(1)D_{12}$.

The bank's intertemporal budget constraint for $t = 1, 2$ in discounted $t = 1$ terms based on constraints (1) and (2) is

$$\overline{\lambda}\delta_1 D_1 + \frac{(1-\overline{\lambda})D_2}{D_{12}} = K_1 + \frac{K_2[1-(1-\frac{\delta_2 D_{12}}{K_{12}})\delta_K]}{\delta_2 D_{12}}. \tag{24}$$

The LHS gives the bank's total discounted liabilities, which are non-increasing in $\overline{\lambda}$, and the RHS assets are constant. If there is an increase in $t = 1$ withdrawals $\overline{\lambda}$, the bank can borrow from the entrepreneur against the bank's long term loan revenues. For $\overline{\lambda} \leq 1$, the bank never defaults or has to call the long term loan and the sequential service constraint does not bind, giving results (22) and (23).

Proposition 1. *The unique Nash equilibrium of the withdrawal game is* $\overline{\lambda}^* = \lambda$*, no bank run, which is an equilibrium in weakly dominant strategies.*

Proof. See Appendix. ■

A late consumer only wants to run the bank if, according to condition (14), $P_1 < \frac{P_2}{D_{12}}$, but Lemma 1 shows that these cannot be equilibrium prices. This result shows that late consumers never have to run the bank at $t = 1$ to purchase goods because the market always provides goods for depositors withdrawing at $t = 2$. The no bank run result is a unique equilibrium, and moreover is an equilibrium in weakly dominant strategies, since $\frac{D_2}{P_2^*(\overline{\lambda})} \geq \frac{D_1}{P_1^*(\overline{\lambda})}$ for all $\overline{\lambda}$ and $\frac{D_2}{P_2^*(1)} \geq \frac{D_1}{P_1^*(1)}$. There are no assumptions required regarding

symmetry of late consumers' beliefs or actions.

This no run result is an important contrast to the multiple equilibria in Diamond-Dybvig. In Diamond-Dybvig, the consumers deposit goods at the bank, which stores and invests the goods, and owes fixed real deposit payouts. If all depositors withdraw at $t = 1$, the bank has to liquidate invested goods, and so runs occur in equilibrium. Banks are fragile because the no run outcome is a Nash equilibrium that depends on coordinated beliefs that other late consumers do not run. A shift in beliefs that other late consumers will run triggers the bank run equilibrium. In Diamond-Dybvig, the greater the number of late consumers that run the bank, the less the bank can pay out in goods at $t = 2$, and so the greater the desire for a marginal late consumer to run. In this paper, since goods are sold by the market, a late consumer prefers to withdraw at $t = 2$ even if other late consumers run the bank. The price mechanism rations goods to depositors who run the bank, ensuring greater consumption for withdrawing at $t = 2$. In fact, the greater the number of late consumers who run the bank, the higher is P_1 (lower is P_2) and the lower is consumption $\frac{D_1}{P_1}$ from withdrawing at $t = 1$ (higher is consumption $\frac{D_2}{P_2}$ from withdrawing at $t = 2$).

There is no problem of a coordination failure, in which the entrepreneur expects a run and prefers to liquidate and sell all goods at $t = 1$, and late consumers expect all goods to be sold at $t = 1$ and prefer to run. This is because the entrepreneur conditions $q_1^S(\cdot)$ and $\gamma(\cdot)$ on P_1. As a rough illustration, consider if $\overline{\lambda} = 1$. The entrepreneur would liquidate and sell all goods at $t = 1$ at $P_1^*(1) \geq \frac{rP_2^*(1)}{sD_{12}}$. If instead $\overline{\lambda} = 1 - \epsilon$, this would be reflected in market clearing prices $P_1^*(1-\epsilon) < P_2^*(1)$ and signal to the entrepreneur to save some goods to sell to ϵ at $t = 2$. The entrepreneur and the marginal late consumer who buys at $t = 2$ are all better off because there is no inefficient liquidation. Thus, no late consumers prefer to run.

Even if the model was modified such that prices were sticky and could not adjust to clear markets, there would not necessarily be a bank run in equilibrium. Consider $P_1^* = \frac{D_1}{\widetilde{c}_1}$ and $P_2^* = \frac{D_2}{\widetilde{c}_2}$ for all $\overline{\lambda}$. If the entrepreneur could choose $q_1^S(\cdot)$ conditional on $\overline{\lambda}$, the entrepreneur would save goods to sell at $t = 2$ to any consumers who do not withdraw at $t = 1$, and there would be no run. If instead the entrepreneur had to choose $q_1^S(\cdot)$ not conditional on $\overline{\lambda}$, there could be rationing at $t = 1$. If $q_1^* < \frac{\overline{\lambda}D_1}{P_1^*}$, then $\frac{q_1^* P_1^*}{D_1} < \overline{\lambda}$ withdrawing consumers would purchase goods and the remaining withdrawals could not purchase anything. If deposits could be redeposited for withdrawing consumers who could

not purchase, or if deposits were not debited from withdrawers' accounts until purchases were settled, then the entrepreneur would choose $q_1^S = 1 - \widetilde{\alpha}$, and there would be no runs. If deposits that are withdrawn but unspent could not be redeposited, then there would be multiple equilibria including a bank run equilibrium. For instance, if the late consumers and the entrepreneur expected $\overline{\lambda} = 1$, and late consumers expected the entrepreneur to liquidate and sell all goods at $t = 1$ ($q_1^S = 1 - \alpha + \alpha s$), then $\overline{\lambda}^* = 1$ would be an equilibrium. Finally, if prices were flexible to clear the market, but the quantity of goods sold was not conditional on P_1 or $\overline{\lambda}$, there also would be multiple equilibria with bank runs. For example, if the entrepreneur had to submit a "market order," choosing q_1^S not conditional on P_1, then $\overline{\lambda}^* = 1$ and $q_1^* = 1 - \alpha + \alpha s$ would be an equilibrium.

Under the basic model assumptions of market clearing prices and $q_1^S(\cdot)$ as a function of prices, the no bank run result would extend to a model with multiple banks and efficient interbank markets. A potential coordination problem is that all late consumers withdraw from the original bank to redeposit at a second bank, and the entrepreneur deposits any revenues it receives in excess of its $t = 1$ loan repayment at the second bank. If the inside money withdrawals that the original bank has to pay at $t = 1$ are in excess of what it receives at $t = 1$, the bank would default. This scenario would cause a self-fulfilling bank run.

However, the second bank would always prefer to lend back to the original bank through the interbank market to receive an interbank lending rate equal D_{12}, which is greater than the next-best alternative of storing the funds or buying goods to store for one period. Inside money withdrawals with an efficient interbank market allow the original bank to pay any amount of withdrawals up to $\overline{\lambda} = 1$ without needing to call the K_2 portion of the loan or prompt the entrepreneur to liquidate invested goods. The second bank could efficiently lend through a clearinghouse system for payments and settling of inside money between banks operated on either an end-of-period netting basis or a real-time gross basis. There is no bank default and no bank run equilibrium.

3.2 Liquidity and allocation results

The (ex-ante) optimal allocation for consumers is the well-known result of what a benevolent planner could provide based on observing consumer types to maximize a consumer's

expected utility. The planner's problem is:

$$\max_{c_1,c_2,\alpha,\gamma} \quad \lambda u(c_1) + (1-\lambda)\, u(c_2) \tag{25}$$

$$\text{s.t.} \quad \lambda c_1 \leq 1 - \alpha + \gamma s \tag{26}$$

$$(1-\lambda)\, c_2 \leq (\alpha - \gamma) r + 1 - \alpha + \gamma s - \lambda c_1. \tag{27}$$

The two constraints are the physical quantities of goods available for consumption at $t = 1$ and $t = 2$, respectively. The first-order conditions and binding constraints give the well-known optimal allocation $(\widetilde{c}_1, \widetilde{c}_2)$ and optimal choice of $\widetilde{\alpha}$ and $\widetilde{\gamma}$, defined by

$$\frac{u'(\widetilde{c}_1)}{u'(\widetilde{c}_2)} = r \tag{28}$$

$$\lambda \widetilde{c}_1 = 1 - \widetilde{\alpha} \tag{29}$$

$$(1-\lambda)\, \widetilde{c}_2 = \widetilde{\alpha} r \tag{30}$$

$$\widetilde{\gamma} = 0. \tag{31}$$

Optimal consumption requires that early consumers only consume from goods stored at $t = 0$, and that late consumers only consume from the returns of invested goods. This ensures no inefficient liquidation and no underinvestment of goods. Equation (28) shows that the ratio of marginal utilities between $t = 1$ and $t = 2$ is equal to the marginal rate of transformation r.

If the bank can choose the investment amount α_b and liquidity storage amount $1 - \alpha_b$, where I designate α_b as the bank's choice of α, the allocation for consumers is the optimal $(\widetilde{c}_1, \widetilde{c}_2)$. The bank chooses optimal $\alpha_b^* = \widetilde{\alpha}$ optimally.

Proposition 2. *With bank choice of liquidity, $\alpha_b^* = \widetilde{\alpha}$ and the consumers' allocation in the market equilibrium is the optimal consumption $(\widetilde{c}_1, \widetilde{c}_2)$.*

Proof. See Appendix. ■

All liquid goods are sold at $t = 1$, $q_1 = 1 - \widetilde{\alpha}$, and all returns from invested goods are sold at $t = 2$, $q_2 = \widetilde{\alpha} r$. Substituting these quantities and $\overline{\lambda}^* = \lambda$ into prices in (17) and (18), $P_1^*\left(\overline{\lambda}\right) = \frac{D_1}{\widetilde{c}_1}$ and $P_2^*\left(\overline{\lambda}\right) = \frac{D_2}{\widetilde{c}_2}$. These prices satisfy first-order condition (19) because $\widetilde{c}_2 \geq \widetilde{c}_1$ based on (28). The entrepreneur does not store goods at $t = 1$ to sell at $t = 2$ because the relative discounted return is $\frac{P_2}{P_1 D_{12}} < 1$. The entrepreneur also does not liquidate invested goods to sell at $t = 1$ because the relative discounted return is

$\frac{sP_1}{rP_2D_{12}} = \frac{s\widetilde{c}_2}{r\widetilde{c}_1}$. This is less than one because the assumption that the coefficient of relative risk aversion is greater than one implies $\widetilde{c}_1 > 1$ and $\widetilde{c}_2 < r$.[8] Liquidating goods would contradict the first-order condition (21).

Nominal deposits payable in inside money are a Pareto optimal improvement over real deposits. Consumers receive optimal consumption with no risk of bank runs when they hold nominal deposits. If consumers were to hold real deposits, they would be exposed to the risk of a bank run and suboptimal consumption. Consider a second Diamond-Dybvig type bank that competes and takes real deposits from consumers at $t = 0$, stores and invests the goods itself, and pays a withdrawal in goods with a sequential service constraint of $\widetilde{c}_1$ at $t = 1$ or $\widetilde{c}_2$ at $t = 2$. Consumers would all deposit at the original bank, and nominal deposits would be the unique equilibrium. Nominal deposits eliminate suboptimal allocations that occur with real deposits.

If the bank cannot choose α, but instead the entrepreneur can, designated by α_e, the entrepreneur will choose suboptimal liquidity that gives a suboptimal consumption allocation. I define the *entrepreneur's optimization with liquidity choice* as the optimization problem (6) with the addition of $\alpha_e \geq 0$ as a maximizer and the additional constraint

$$\alpha_e \leq 1. \tag{32}$$

The market equilibrium is modified to be $(P^*(\overline{\lambda}), q^*(\overline{\lambda}), \delta^*(\overline{\lambda}), \gamma^*(\overline{\lambda}), \alpha_e^*(\overline{\lambda}))$ for any given $\overline{\lambda} \in [\lambda, 1]$, with $(q_1^S(\cdot), q_2^S(\cdot), \gamma(\cdot), \alpha_e(\cdot))$ as a solution to the entrepreneur's optimization with liquidity choice.

Proposition 3. *With entrepreneur choice of liquidity,* $\overline{\lambda}^* = \lambda$ and $\alpha_e^*(\overline{\lambda}^*) = 1 - \lambda > \widetilde{\alpha}$. *The consumer's allocation in the market equilibrium is* $(1, r)$.

Proof. See Appendix. ■

Under the bank's choice of liquidity, the relative discounted return of marginally increasing the investment amount from $\widetilde{\alpha}$ is $\frac{P_2 r}{P_1 D_{12}} > 1$. Thus, the entrepreneur increases investment to $\alpha_e^*(\overline{\lambda}^*) = 1 - \lambda$, at which point the relative discounted return of marginally increasing $\alpha_e^*(\overline{\lambda}^*)$ is $\frac{P_2 r}{P_1 D_{12}} = 1$. This demonstrates that a key role for a bank is to ensure the ex-ante welfare optimal amount of liquidity storage, rather than allow the entrepreneur

[8] The coefficient of relative risk aversion $\frac{-cu''(c)}{u'(c)} > 1$ implies that $cu'(c)$ is decreasing in c. Hence, $u'(1) > ru'(r)$, so from (28), $c_1^* > 1$, $c_2^* < r$.

to whom it lends to increase investment up to the level that will maximize profits based on ex-post market prices. However, when the bank does ensure optimal liquidity storage, the entrepreneur does distribute liquidity allocations to consumers optimally.

There are several reasons why the bank may be able to determine α. One is that the bank may be able to actively monitor the entrepreneur's investment and liquidity storage decision. The ability of a bank to monitor it's borrowers is a standard distinction made between bank and capital market financing. If the bank cannot monitor and enforce the entrepreneur to store $1-\widetilde{\alpha}$, Skeie (2004) shows that the bank could achieve the optimal outcome by lending $1-\widetilde{\alpha}$ for return K_1 to a "short term" entrepreneur who does not have the ability to invest, and lending $\widetilde{\alpha}$ for return K_2 to a "long term" entrepreneur who invests. Alternatively, the bank could play the role of the short term entrepreneur itself, and buy and store the $1-\widetilde{\alpha}$ goods to sell on the market at $t=1$. This could be interpreted as the bank holding "liquidity reserves."

Results from Propositions 1, 2 and 3 hold if the bank can choose one or more of the rates D_1, D_2, and D_{12}, such that the bank can set $D_2 = D_1 D_{12}$. If all three rates exist exogenously and $D_{12} > \frac{D_2}{D_1}$ ($D_{12} < \frac{D_2}{D_1}$), the bank would offer $D_2' = D_1 D_{12} > D_2$ ($D_1' = \frac{D_2}{D_{12}} > D_1$) as a competitive part of its deposit contract and D_2 (D_1) is irrelevant. If offering an alternative higher rate D_2' (D_1') is not possible, then for $D_{12} > \frac{D_2}{D_1}$, depositors would always run the bank to deposit at the outside rate D_{12}. Thus, no consumers would deposit at $t=0$. For $D_{12} < \frac{D_2}{D_1}$, there are no runs. But for $D_{12} < \frac{\widetilde{c}_1 D_2}{\widetilde{c}_2 D_1}$, there is an inefficient allocation (c_1, c_2), where $c_1 < \widetilde{c}_1$ and $c_2 > \widetilde{c}_2$. The entrepreneur inefficiently stores liquid goods at $t=1$ to sell at $t=2$. The entrepreneur discounts P_2 at a rate D_{12} that is less than the implicit intertemporal rate received by late consumers $\frac{D_2}{D_1}$, shown by first-order condition (73). Thus, the entrepreneur wants to sell additional goods at $t=2$ but the late consumers do not want to withdraw early.

The gross returns on the loan K_1 and K_2 are set by the bank; however, these rates also reflect the marginal product of capital in a competitive market. The equilibrium real rate of return on the loan amount $1-\alpha^*$ for goods that are stored is $\frac{\frac{K_1}{1-\alpha^*}}{P_1^*(\overline{\lambda}^*)} = 1$, and on the loan amount α^* for goods that are invested is $\frac{\frac{K_2}{\alpha^*}}{P_2^*(\overline{\lambda}^*)} = r$, reflecting the marginal product of capital. These equilibrium rates hold regardless of whether the bank or the entrepreneur chooses α. The bank and entrepreneur each make zero profit, so the loan rates are the competitive rates that would occur in a contestable loan market.

3.3 Market economy

The level of liquidity storage and consumption allocations in the banking economy with entrepreneur chosen liquidity equals that of the well-known market economy, in which there is no bank, entrepreneur or contracts, and consumers individually can invest goods and trade them at $t = 1$. However, I show that in a market economy in which all consumers are forced to store the optimal liquidity $1 - \widetilde{\alpha}$, the optimal consumption allocation does obtain. This further demonstrates that a key role of the bank is to enforce the optimal liquidity holding, and that markets more generally are not able to choose the optimal liquidity ex-ante but are able to efficiently distribute ex-post liquidity that is ex-ante enforced.

In a market economy, consumers each invest α_m goods and store $1 - \alpha_m$ goods at $t = 0$. At $t = 1$, λ early consumers trade their α_m invested goods for the $1 - \alpha_m$ stored goods of $1 - \lambda$ late consumers at price p_m, expressed as liquid goods per invested good. In equilibrium, $p_m^* = \frac{(1-\lambda)(1-\alpha_m^*)}{\lambda \alpha_m^*} = 1$. After trading in the market, consumption for early and late types is the standard result

$$c_1 = 1 - \alpha_m^* + \frac{\alpha_m^*}{p_m^*} = 1 \tag{33}$$

$$c_2 = [\alpha_m^* + (1 - \alpha_m^*)\, p_m^*]r = r. \tag{34}$$

The market allocation and $\alpha_m^* = 1 - \lambda$ equal the levels from the banking economy with an entrepreneur choice of liquidity.

A novel result is that if consumers are forced to store $1 - \widetilde{\alpha}_m^* = 1 - \widetilde{\alpha}$ goods, the market achieves the optimal outcome. Early consumers trade $\widetilde{\alpha}_m^* = \widetilde{\alpha}$ invested goods for $1 - \widetilde{\alpha}_m^*$ stored goods from late consumers at $t = 1$ at a market price of $\widetilde{p}_m^* = \frac{\widetilde{c}_1}{\widetilde{c}_2} r > 1$.

Proposition 4. *The allocation in the market economy equilibrium when consumers are required to store* $1 - \widetilde{\alpha}_m^* = 1 - \widetilde{\alpha}$ *is the optimal consumption outcome* $(\widetilde{c}_1, \widetilde{c}_2)$.

Proof. See Appendix. ■

3.4 Consumer-lending economy

To further show the bank's role for optimal liquidity, I show that consumers choose suboptimal liquidity even if they can make nominal loans directly to the entrepreneur. Consider

a consumer-lending economy in which the bank does not exist, each consumer makes a nominal loan directly to the entrepreneur, and the nominal rate of return on loans is that which prevails in the bank liquidity choice economy: $K_1' \equiv \frac{K_1}{1-\widetilde{\alpha}}$ for a short-term loan due at $t = 1$ and $K_2' \equiv \frac{K_2}{\widetilde{\alpha}}$ for a long-term loan due at $t = 2$. At $t = 0$, a consumer lends its central bank currency directly to the entrepreneur rather than deposit it in the bank. A consumer lends $1 - \alpha_c$ as a short term loan and α_c as a long term loan. At $t = 1$, there is a resale market for long term loans among consumers, entrepreneurs repay loans and late consumers offer to borrow excess funds from the entrepreneur at rate D_{12}. At $t = 1, 2$, the goods market trades, and there is a net clearing for consumers and the entrepreneur of the nominal unit of account, in which all transactions are paid. I will show that each consumer chooses an underprovision of storage, even if he can enforce for his loan the amount the entrepreneur stores, $1 - \alpha_c$, and invests, α_c. The liquidity and allocation results of this consumer-lending economy with nominal prices are the same as the market economy with real prices and the bank-lending economy with entrepreneur choice of liquidity.

The early consumers purchase $c_1(\alpha_c)$ goods at $t = 1$ and the late consumers purchase $c_2(\alpha_c)$ goods at $t = 2$, where

$$c_1(\alpha_c) = \frac{(1-\alpha_c)K_1' + \alpha_c P_L^*}{P_1^*} \tag{35}$$

$$c_2(\alpha_c) = \frac{\alpha_c K_2' + \frac{(1-\alpha_c)K_1'K_2'}{P_L^*}}{P_2^*}, \tag{36}$$

and where P_L^* is the $t = 1$ equilibrium price of an amount of long term loans that pay a gross return of K_2' at $t = 2$. Equation (35) states that at $t = 1$, a consumer can buy $c_1(\alpha_c)$ goods at price P_1^*, using return $(1 - \alpha_c)K_1'$ from his short term loan plus proceeds $\alpha_c P_L^*$ from sales of his long term loan. Equation (36) states that at $t = 2$, a consumer can alternatively buy $c_2(\alpha_c)$ at price P_2^* using return $\alpha_c K_2'$ from his long term loan held from $t = 0$ plus return $(1 - \alpha_c)K_1'K_2'$ from long term loans purchased at $t = 1$. Aggregate demand for goods is

$$q_1^D(P_1) = \frac{\lambda[(1-\alpha_c^*)K_1' + \alpha_c^* P_L]}{P_1} \tag{37}$$

$$q_2^D(P_2) = \frac{(1-\lambda)[\alpha_c^* K_2' + \frac{(1-\alpha_c^*)K_1'K_2'}{P_L}]}{P_2}. \tag{38}$$

At $t = 1$, aggregate demand and supply for long term loans are

$$q_L^D(P_L) = \frac{(1-\lambda)(1-\alpha_c)K_1'}{P_L} \tag{39}$$

$$q_L^S(P_L) = \lambda\alpha_c. \tag{40}$$

I do not treat the consumer's decision about when to trade the nominal loan and goods as a strategic game in this economy since there is no bank. Incentive compatibility for late consumers requires $c_2(\alpha_c^*) > c_1(\alpha_c^*)$ and $\frac{K_2'}{P_2^* P_L^*} > \frac{1}{P_1^*}$, where the latter implies that late consumers prefer buying long term loans over goods at $t = 1$. I define the *entrepreneur's optimization for a consumer-lending economy* as optimization (6) with the modifications that K_1 is replaced by $K_1^C \equiv (1-\alpha_c)K_1'$, K_2 is replaced by $K_2^C \equiv \alpha_c K_2'$ and $\delta_K = 0$.

I define a market equilibrium for a consumer-lending economy as $(P^*, P_L^*, q^*, \gamma^*)$ that solves market clearing conditions

$$q_t^D(P) = q_t^S(P) \qquad t = 1, 2 \tag{41}$$

$$q_L^D(P_L) = q_L^S(P_L), \tag{42}$$

and the entrepreneur's zero profit condition $q_2^S(P) = \widehat{q}_2^S(q_1^S(P), \gamma(P))$, where q_2^D is given by (37) and (38) , q_L^D and q_L^S are given by (39) and (40), and $(q_1^S(P), q_2^S(P), \gamma(P))$ is a solution to the entrepreneur's optimization for a consumer-lending economy. I define a *consumer-liquidity equilibrium* in a consumer-lending economy as

$$\alpha_c^* \equiv \text{argmax}_{\alpha_c \in [0,1]}\ u(c_1(\alpha_c)) + (1-\lambda)u(c_2(\alpha_c)). \tag{43}$$

Equations (37), (38), (39) and (40) imply equilibrium prices satisfy

$$P_1^* = \frac{\lambda[(1-\alpha_c^*)K_1' + \alpha_c^* P_L^*]}{q_1^*} \tag{44}$$

$$P_2^* = \frac{(1-\lambda)[\alpha_c^* K_2' + \frac{(1-\alpha_c^*)K_1' K_2'}{P_L^*}]}{q_2^*} \tag{45}$$

$$P_L^* = \frac{(1-\lambda)(1-\alpha_c^*)K_1'}{\lambda\alpha_c^*}. \tag{46}$$

Proposition 5. *In a consumer-lending economy, the consumer-liquidity equilibrium is* $\alpha_c^* = 1 - \lambda$. *The consumer's allocation in the market equilibrium* $(c_1, c_2) = (1, r)$.

Proof. See Appendix. ■

4 Bank with Sequential Market

An important reason why real demand deposit contracts allow bank runs in Diamond-Dybvig is that the contract is not contingent on the realized state of $\overline{\lambda}$. The assumption of a sequential-service constraint implies that the deposit contract cannot depend on $\overline{\lambda}$ as depositors are initially withdrawing at $t = 1$, because $\overline{\lambda}$ is not known until $\overline{\lambda}$ depositors have already withdrawn. Thus, real deposits repay a non-contingent fixed amount of consumption goods $\widetilde{c}_1$ to each depositor who withdraws at $t = 1$ until all goods are paid out.

In the nominal deposits model, nominal payouts by the bank are also not contingent on $\overline{\lambda}$. The fixed payout (in inside money) of the deposit contract and the sequential service constraint are strictly adhered to. However, real consumption $\frac{\delta_{1t} D_1}{P_1} = \frac{q_1}{\overline{\lambda}}$ to the depositor withdrawing at $t = 1$ is contingent on $\overline{\lambda}$ through P_1. This is because the market for goods is assumed to be a centralized one-price Walrasian market. The price in the goods market is based on aggregate withdrawals at $t = 1$ and is not determined until $\overline{\lambda}$ is realized. Although the payout by the bank does not settle until $\overline{\lambda}$ and P_1 are realized, the sequential service constraint is formally upheld by the bank because the monetary amount paid and settled on a withdrawal for a depositor does not depend on the withdrawals after him. The inside money payment from the bank for a depositor withdrawal settles in full before the payment for the next depositor withdrawal settles.

However, this may not reflect the spirit of the sequential service constraint as interpreted by Wallace (1988). He requires that a withdrawer receives and consumes his real consumption before the next depositor withdraws, due to an assumption that consumers are physically isolated from each other and a liquidity shock at $t = 1$ means that consumers need to consume immediately within the period rather than at the end of the period.

4.1 Sequential market model

The Wallace (1988) requirement for what I call his "sequential consumption constraint" may be easily met in the nominal bank deposits model with the introduction of a sequential goods market that replaces the centralized goods market as follows. During period $t = 1$, depositors labeled $i \in [0, 1]$ sequentially discover their early or late type and have

the opportunity to sequentially withdraw and consume in that order. A consumer who discovers being an early type needs to withdraw and consume before the next consumer in the sequential order discovers his type and has an opportunity to withdraw. The late consumer's consumption c_2 is modified to equal goods he consumes in both $t = 1$ and $t = 2$. In the centralized market model in Sections 2 and 3, I take as given that bank deposits are paid on demand under a sequential service constraint, following Diamond-Dybvig. In the physical environment of the sequential market model, where consumers need to consume sequentially during a period, deposits paid on demand under a sequential service constraint are optimal features of the deposit contract, as in Wallace (1988).

I assume for simplicity in this section that late consumers take symmetric actions, so $\overline{\lambda} \in \{\lambda, 1\}$, but this does not effect the results. At the start of period $t = 1$, the entrepreneur chooses a belief $\underline{\lambda} \in \{\lambda, 1\}$ of the realization of $\overline{\lambda}$. For each depositor $i \in [0, 1]$ in sequence, i learns his type and decides whether to withdraw and purchase goods by submitting his individual demand schedule $q_1^{D,i}(P_1, \overline{\lambda}, \delta_1, \delta_1^i)$. I assume that i does not observe his place in line or any amount of previous withdrawals or prices, such that all depositors have the same information set and decision at $t = 1$, to be consistent with the assumption of symmetric actions. The competitive entrepreneur submits a supply schedule. For a withdrawal, the bank makes a credit to the entrepreneur's account for the withdrawal amount. The entrepreneur uses the funds to repay a portion of his loan due if it is outstanding or else holds the funds as a deposit for return D_{12}. The goods market clears and payments are netted out and settled. The bank's payment must settle before the bank makes a payment for the next depositor withdrawal, otherwise it defaults. If a purchase payment settles, the entrepreneur delivers goods to the purchaser and the depositor immediately consumes. The next depositor decides whether to withdraw, and the entrepreneur submits a new supply schedule. If $\underline{\lambda} = \lambda$ and more than λ consumers withdraw, the entrepreneur at that time realizes that $\overline{\lambda} = 1$. At $t = 2$, the remaining $1 - \overline{\lambda}$ late consumers simultaneously withdraw, purchase and consume equal pro rata amounts as in the centralized market model.

For a given $\underline{\lambda}$ and $\overline{\lambda}$, a market equilibrium is defined as

$$(\underline{P}^*(\underline{\lambda}), \overline{P}^*(\underline{\lambda}, \overline{\lambda}), \underline{q}^*(\underline{\lambda}), \overline{q}^*(\underline{\lambda}, \overline{\lambda}), \underline{\gamma}^*(\underline{\lambda}), \gamma^*(\underline{\lambda}, \overline{\lambda}), \delta^*(\underline{\lambda}, \overline{\lambda})) \tag{47}$$

that solves (i) market clearing conditions

$$\begin{aligned}
\underline{q}_1^D(\underline{P}_1, \delta, \overline{\lambda}) &= \underline{q}_1^S(\underline{P}, \delta) && (48)\\
\overline{q}_1^D(\overline{P}_1, \delta, \underline{\lambda}, \overline{\lambda}) &= \overline{q}_1^S(\overline{P}, \delta) && (49)\\
\underline{q}_2^D(\underline{P}_2, \delta, \underline{\lambda}) &= \underline{q}_2^S(\underline{P}, \delta) && (50)\\
q_2^D(P_2, \delta, \overline{\lambda}) &= q_2^S(\overline{P}, \delta), && (51)
\end{aligned}$$

(ii) the entrepreneur's zero profit conditions $\underline{q}_2^S(\underline{P}, \delta) = \widehat{q}_2^S(\cdot)$ and $q_2^S(\overline{P}, \delta) = \widehat{q}_2^S(\cdot)$, and (iii) bank budget and sequential service constraint (3); where $\underline{P} \equiv (\underline{P}_1, \underline{P}_2)$, $\overline{P} \equiv (\overline{P}_1, P_2)$, $\overline{q} \equiv (\overline{q}_1, q_2)$; $\underline{P}_1 \underline{q}_1^S(\cdot) + \overline{P}_1 \overline{q}_1^S(\cdot)$ replaces $P_1 q_1^S$ in (1) and (2);

$$\begin{aligned}
\underline{q}_1^D(\underline{P}_1, \delta_1, \underline{\lambda}) &= \frac{\underline{\lambda}\delta_1 D_1}{\underline{P}_1} && (52)\\
\overline{q}_1^D(\overline{P}_1, \delta_1, \underline{\lambda}, \overline{\lambda}) &= \frac{(\overline{\lambda} - \underline{\lambda})\delta_1 D_1}{\overline{P}_1} && (53)\\
\underline{q}_2^D(\underline{P}_2, \delta_2, \underline{\lambda}) &= \frac{(1 - \underline{\lambda})\delta_2 D_2}{\underline{P}_2} && (54)\\
q_2^D(P_2, \delta_2, \overline{\lambda}) &= \frac{(1 - \overline{\lambda})\delta_2 D_2}{P_2} && (55)\\
\overline{q}_1^S(\cdot) &= q_1^S(\cdot) - \underline{q}_1^S(\cdot) && (56)\\
\overline{\gamma}(\cdot) &= \gamma(\cdot) - \underline{\gamma}(\cdot); && (57)
\end{aligned}$$

$(\underline{q}_1^S(\cdot), \underline{q}_2^S(\cdot), \underline{\gamma}(\cdot))$ is a solution to (6), in which $\underline{P}$ replaces P; $(q_1^S(\cdot), q_2^S(\cdot), \gamma(\cdot))$ a solution to (6), in which (6e) and (6f) are replaced respectively by

$$\begin{aligned}
&\underline{P}_1 \underline{q}_1^S + \overline{P}_1 \overline{q}_1^S \geq K_1 + \frac{\delta_K K_2}{K_{12}} && (58)\\
&q_2^S \geq \frac{(1 - \delta_K)K_2 - [\underline{P}_1 \underline{q}_1^S + \overline{P}_1 \overline{q}_1^S - K_1 - \frac{\delta_K K_2}{K_{12}}]^+ \delta_2 D_{12}}{P_2}; && (59)
\end{aligned}$$

and

$$\begin{aligned}
\underline{q}_1^*(\cdot)\,\underline{P}_1^*(\cdot) &\equiv \underline{\lambda}\delta_1^*(\cdot) D_1 \qquad \text{if } \underline{q}_1^*(\cdot) = 0 \text{ and } P_1^*(\cdot) = \infty && (60)\\
\underline{q}_2^*(\cdot)\,\underline{P}_2^*(\cdot) &\equiv (1 - \underline{\lambda})\delta_2^*(\cdot) D_2 \qquad \text{if } \underline{q}_2^*(\cdot) = 0 \text{ and } P_2^*(\cdot) = \infty && (61)\\
\overline{q}_1^*(\cdot)\,\overline{P}_1(\cdot) &\equiv (\overline{\lambda} - \underline{\lambda})\delta_1^*(\cdot) D_1 \qquad \text{if } \overline{q}_1^*(\cdot) = 0 \text{ and } \overline{P}_1^*(\cdot) = \infty && (62)\\
q_2^*(\cdot)\,P_2^*(\cdot) &\equiv (1 - \overline{\lambda})\delta_2^*(\cdot) D_2 \qquad \text{if } q_2^*(\cdot) = 0 \text{ and } P_2^*(\cdot) = \infty. && (63)
\end{aligned}$$

For the withdrawal game, withdrawing at $t = 2$ is a strictly dominant strategy for late consumer i if for each $\underline{\lambda} \in \{\lambda, 1\}$ and $\overline{\lambda} \in \{\lambda, 1\}$,

$$\frac{\delta_2^*(\cdot)\, D_2}{P_2^*(\underline{\lambda}, \overline{\lambda})} > \frac{\mathbf{1}_{\delta_1^i \leq \overline{\lambda}\delta_1^*(\cdot)} D_1}{\underline{P}_1^*(\underline{\lambda})} \tag{64}$$

$$\frac{\delta_2^*(\cdot)\, D_2}{P_2^*(\underline{\lambda}, \overline{\lambda})} > \frac{\mathbf{1}_{\delta_1^i \leq \overline{\lambda}\delta_1^*(\cdot)} D_1}{\overline{P}_1^*(\underline{\lambda}, \overline{\lambda})} \tag{65}$$

$$\text{for all } (\underline{P}^*(\underline{\lambda}), \overline{P}^*(\underline{\lambda}, \overline{\lambda}), \delta^*(\underline{\lambda}, \overline{\lambda})) \text{ that are part of a market equilibrium} \tag{66}$$

$$\text{and for all } \delta_1^i \in [0, \overline{\lambda}]. \tag{67}$$

As in the centralized market, I define a withdrawal equilibrium $\overline{\lambda}^*$ to be any Nash equilibrium of the withdrawal game. Finally, I define the entrepreneur's equilibrium belief choice $\underline{\lambda}^* \in \{\overline{\lambda}^* : \overline{\lambda}^* \text{ is a withdrawal equilibrium}\}$ to be any equilibrium value of $\overline{\lambda}$.

4.2 Sequential market results

The entrepreneur starts $t = 1$ anticipating to sell $\underline{q}_1^*(\underline{\lambda})$ goods at $\underline{P}_1^*(\underline{\lambda})$ and $\underline{q}_2^*(\cdot)$ goods at $\underline{P}_2^*(\underline{\lambda})$. The entrepreneur sells in actuality to the first $\min\{\underline{\lambda}, \overline{\lambda}\}$ consumers who withdraw at $\underline{P}_1^*(\underline{\lambda})$. If $\underline{\lambda} = \lambda$, then after λ fraction have withdrawn and purchased, the entrepreneur sells at $\overline{P}_1^*(\lambda, 1)$ an amount $\overline{q}_1^*(\lambda, \overline{\lambda})$, which equals zero if $\overline{\lambda} = \lambda$ and is positive if $\overline{\lambda} = 1$. If $\underline{\lambda} = 1$, the entrepreneur sells at $\underline{P}_1^*(1)$ throughout $t = 1$. In the case of $\overline{\lambda} = \lambda$, $\overline{q}_1^*(1, \lambda) < 0$ reflects goods the entrepreneur anticipated selling at $t = 1$ but does not. The realized total amount of goods the entrepreneur sells at period $t = 1, 2$ is $q_t^*(\underline{\lambda}, \overline{\lambda})$, and the realized price at $t = 2$ is $P_2^*(\underline{\lambda}, \overline{\lambda})$.

The market equilibrium, depositor equilibrium and allocation in the sequential market model is that of the nominal bank deposits model with a centralized market for either a bank or entrepreneur choice of α. For simplicity and concreteness of discussion, I show the results for a bank choice of $\alpha_s = \widetilde{\alpha}$. If $\underline{\lambda} = \lambda$, the entrepreneur sells at $\underline{P}_1(\lambda) = \frac{D_1}{\widetilde{c}_1}$ the $1 - \widetilde{\alpha}$ stored goods to the first λ purchasers at $t = 1$. First, consider the case of $\overline{\lambda} = \lambda$. $P_2^*(\lambda, \overline{\lambda}) = \frac{D_2}{\widetilde{c}_2}$, so prices and consumption equal those in the centralized market model. Next, consider the case of $\overline{\lambda} = 1$. The entrepreneur applies the first-order condition for liquidation as in (21), $s\overline{P}_1^*(\lambda, 1) \geq \frac{rP_2^*(\lambda,1)}{D_{12}}$. This implies $\overline{P}_1^*(\lambda, 1) \geq \frac{(1-\lambda)D_1}{\widetilde{\alpha}s} > \underline{P}_1^*(\lambda)$ and $P_2^*(\lambda, 1) = P_2^*(\lambda, \lambda)$. Prices at $t = 2$ are unaffected by whether an unanticipated run occurs. The loss from liquidating invested goods is fully borne by early purchasers who buy liquidated invested goods after the unanticipated run is realized by the entrepreneur. This

is because $\overline{P}_1(\lambda, 1)$ equalizes the entrepreneur's discounted revenues between a marginal amount of invested goods sold in liquidation at $t = 1$ versus at full return for at $t = 2$. Hence, no late consumer prefers to purchase goods at $t = 1$ after λ fraction of depositors have purchased. Moreover, a late consumer who purchases goods among the first λ fraction is also worse off than purchasing goods at $t = 2$, because a late purchaser's consumption at $t = 2$ is unaffected by whether other late consumers run at $t = 1$.

If $\underline{\lambda} = 1$, the entrepreneur's first-order condition under liquidation applies analogously to result (21), $s\underline{P}_1^*(\underline{\lambda}) \geq \frac{r\underline{P}_2^*(\underline{\lambda})}{D_{12}}$. $\underline{P}_1^*(1) \geq \frac{D_1}{1-\widetilde{\alpha}+\widetilde{\alpha}s} > D_1$, while $P_2^*(1,1) \leq \frac{sD_2}{r(1-\widetilde{\alpha}+\widetilde{\alpha}s)} < \frac{D_2}{\widetilde{c}_2}$ and $P_2^*(1,\lambda) \leq \frac{(1-\lambda)D_2}{\widetilde{\alpha}(r-\lambda s)+(1-\lambda)(1-\widetilde{\alpha})} < \frac{D_2}{r}$. Prices at the start of $t = 1$ are higher if a run is anticipated than if a run is not anticipated. This implies that prices at $t = 2$ are even lower for $\underline{\lambda} = 1$ than for $\underline{\lambda} = \lambda$. In sum, for $\underline{\lambda} \in \{\lambda, 1\}$ and $\overline{\lambda} \in \{\lambda, 1\}$, $\underline{P}_1^*(\underline{\lambda}) \geq \frac{D_1}{\widetilde{c}_1}$ and $\overline{P}_1^*(\underline{\lambda}, \overline{\lambda}) \geq \frac{D_1}{\widetilde{c}_1}$, while $P_2(\underline{\lambda}, \overline{\lambda}) \leq \frac{D_2}{\widetilde{c}_2}$. A late consumer prefers to withdraw at $t = 2$ regardless of $\overline{\lambda} \in \{\lambda, 1\}$. Thus, $\overline{\lambda}^* = \lambda$ is the unique equilibrium and implies $\underline{\lambda}^* = \lambda$.

Proposition 6. *The unique Nash equilibrium of the withdrawal game of the sequential market model is $\overline{\lambda}^* = \lambda$, no bank run, which is an equilibrium in strictly dominant strategies. The entrepreneur's equilibrium belief is $\underline{\lambda}^* = \lambda$, and the allocation with bank-chosen liquidity $\alpha_s = \widetilde{\alpha}$ is the optimal consumption $(\widetilde{c}_1, \widetilde{c}_2)$.*

Proof. See Appendix. ■

These results show that the real consumption of each withdrawer at $t = 1$ does not need to depend on the number of early withdrawers following him, revealed by a centralized market price for runs to be avoided. Rather, the sequential market ensures that the real consumption of each early withdrawer depends on only the number of withdrawers before him to ration excess early consumption and dissuade a run. Although the market price for each transaction at $t = 1$ is based on perfect information of the number of prior purchasers through the entrepreneur's supply schedule, this could be relaxed. Withdrawers at $t = 1$ could each purchase goods at $t = 1$ from a random draw of a competitive entrepreneur out of a continuum of informationally isolated entrepreneurs. Even if a run is not anticipated, for each individual entrepreneur $\overline{P}_1^*(\lambda, 1) > \frac{D_1}{\widetilde{c}_1}$ for any liquidated goods it sells and $P_2^*(\lambda, 1) = \frac{D_2}{\widetilde{c}_2}$ for $t = 2$ sales. Since $\frac{D_2}{P_2^*(\lambda,1)} > \frac{D_1}{\overline{P}_1^*(\lambda,1)}$, a late consumer would always be better off withdrawing at $t = 2$.

5 Conclusion

In a frictionless model of nominal bank deposits repayable in inside money, I show that there are no bank runs. In this setting, these nominal deposits are Pareto improving over real deposits. The asset-liability maturity mismatch is not enough to produce bank runs, implying that further frictions are needed to explain banking liquidity crises. In particular, there is no bank run regardless of whether goods are sold to withdrawing depositors though a centralized market price that is based on aggregate withdrawals or a sequential market price that evolves and reflects the amount of early withdrawals revealed over time. The no bank run result is an equilibrium in dominant strategies because late consumers are better off waiting to withdraw regardless of how many others withdraw early.

A key role of the bank is to ensure the optimal amount of real liquidity, similar to the planner's role in a market economy with no bank. Markets do not provide for optimal liquidity, as shown in the nominal deposits model, if the entrepreneur can choose the liquidity storage level. The underprovision of liquidity induced by various types of markets is also shown by the market economy and the consumer-lending economy, in which there is no bank and consumers choose liquidity levels. A bank does not distribute real liquidity optimally, since a bank paying real deposits is subject to bank runs. When optimal liquidity is stored, markets do distribute liquidity efficiently, as shown by the market economy with forced optimal storage, and the nominal bank model with bank-chosen liquidity in either the centralized market or sequential market.

Appendix: Proofs

Proof of Lemma 1: First, I show that in any market equilibrium, results (22) and (23) obtain. Suppose instead that $\delta_1^*\left(\overline{\lambda}\right) < 1$, which requires by (3c) that $\delta_K^*\left(\overline{\lambda}\right) = 1$. The bank's budget constraint (1) requires $\overline{\lambda}\delta_1^*\left(\overline{\lambda}\right) D_1 \geq D_1$, or $\delta_1^*\left(\overline{\lambda}\right) \geq 1$, a contradiction to the supposition. Thus, $\delta_1^*\left(\overline{\lambda}\right) = 1$. Next, constraint (1) requires $\overline{\lambda}\delta_1^*\left(\overline{\lambda}\right) D_1 \geq K_1 + \frac{\delta_K^*\left(\overline{\lambda}\right)K_2}{K_{12}}$, which simplifies to

$$\delta_K^*\left(\overline{\lambda}\right) \leq \frac{\overline{\lambda} - \lambda}{1 - \lambda}. \tag{68}$$

Solving for $[P_1 q_1^S(\cdot) - K_1 - \frac{\delta_K K_2}{K_{12}}]^+$ in constraint (1), substituting for it into (2) and simplifying,

$$\delta_2^*(\overline{\lambda})(1 - \delta_K^*(\overline{\lambda}))\,(1 - \lambda) = (1 - \delta_K^*(\overline{\lambda}))\,(1 - \lambda)\,. \tag{69}$$

The solutions $\delta_2^*(\overline{\lambda}) = 1$ and $\delta_K^*(\overline{\lambda}) = 0$ maximize the objective function in (3a) and satisfy (68) and (69). Thus, (22) and (23) obtain.

Second, I show results (19), (20) and (21). The entrepreneur has no value for revenues at $t = 2$ beyond repaying the loan, which implies that his $t = 2$ budget constraint (6f) always binds. His $t = 1$ budget constraint (6e) implies that the entrepreneur's deposits at $t = 1$ are nonnegative. Thus, constraint (6f) can be replaced by

$$q_2^S = \frac{(1 - \delta_K)K_2 - (P_1 q_1^S - K_1 - \frac{\delta_K K_2}{K_{12}})\delta_2 D_{12}}{P_2}. \tag{70}$$

I define the entrepreneur's *relaxed optimization* as (6), with constraint (6f) replaced by constraint (70) and constraints (6d) and (6e) relaxed. I define a *relaxed market equilibrium* as a market equilibrium in which (6) is replaced by the entrepreneur's relaxed optimization. I show that in a relaxed market equilibrium, results (19), (20) and (21) obtain and constraints (6d) and (6e) are satisfied. Thus, results (19), through (23) obtain in the market equilibrium.

To show results (19), (20) and (21) for a relaxed market equilibrium, I start by finding the first-order conditions for the entrepreneur's relaxed optimization. The nonnegative Lagrange multipliers associated with constraints (6b) and (6c) are θ_1 and θ_2. The Kuhn-Tucker conditions for the entrepreneur's relaxed optimization with respect to q_1^S

and γ, respectively, are:

$$D_{12}\tfrac{P_1}{P_2} \leq 1+\theta_1 \qquad (= \text{ if } q_1^S > 0) \tag{71}$$

$$(1+\theta_1)s \leq r+\theta_2 \qquad (= \text{ if } \gamma > 0). \tag{72}$$

Equilibrium conditions then imply that $q^S(P^*\left(\overline{\lambda}\right), \delta^*\left(\overline{\lambda}\right)) = q^*\left(\overline{\lambda}\right)$.

I establish that in a relaxed market equilibrium, $P_2^*\left(\overline{\lambda}\right)$ is finite and $q_2^*(\overline{\lambda} < 1) > 0$. Suppose not, that either i) $P_2^*\left(\overline{\lambda}\right) = \infty$, which by (16b) implies that $q_2^*\left(\overline{\lambda}\right) = 0$, or ii) $q_2^*(\overline{\lambda} < 1) = 0$, which implies by (18) that $P_2^*(\overline{\lambda} < 1) = \infty$. Equilibrium condition (10) implies that $\widehat{q}_2^S(q_1^*\left(\overline{\lambda}\right), \gamma^*\left(\overline{\lambda}\right)) = 0$, hence $\gamma^*\left(\overline{\lambda}\right) = \alpha$ and $q_1^*\left(\overline{\lambda}\right) > 0$, so $P_1^*\left(\overline{\lambda}\right)$ is finite by (16a). But (71) binds and implies $P_1^*\left(\overline{\lambda}\right) = \frac{P_2^*\left(\overline{\lambda}\right)(1+\theta_1)}{D_{12}} = \infty$, a contradiction. Thus, $P_2^*\left(\overline{\lambda}\right)$ is finite and $q_2^*(\overline{\lambda} < 1) > 0$.

Next, I show $q_1^*(\overline{\lambda}) > 0$, $P_1^*(\overline{\lambda})$ is finite and $P_2^*(\overline{\lambda}) \geq \frac{P_2^*(\overline{\lambda})}{D_{12}}$. Suppose either $q_1^*(\overline{\lambda}) = 0$ or $P_1^*(\overline{\lambda}) = \infty$, which by (17) implies the other, and by complementary slackness implies $\theta_1 = 0$. By (71), $P_2^*(\overline{\lambda}) = \infty$, a contradiction to $P_2^*(\overline{\lambda})$ as finite shown above. thus, $q_1^*(\overline{\lambda}) > 0$, $P_1^*(\overline{\lambda})$ is finite and hence

$$P_1^*(\overline{\lambda}) = \frac{(1+\theta_1)P_2^*(\overline{\lambda})}{D_{12}}, \tag{73}$$

hence $P_1^*(\overline{\lambda}) \geq \frac{P_2^*(\overline{\lambda})}{D_{12}}$.

Next, I show that given $\overline{\lambda} \leq 1$, if $\gamma^*\left(\overline{\lambda}\right) > 0$, then $q_1^*\left(\overline{\lambda}\right) = \widehat{q}_1^S(\gamma^*\left(\overline{\lambda}\right))$ and $sP_1^*\left(\overline{\lambda}\right) \geq \frac{rP_2^*\left(\overline{\lambda}\right)}{D_{12}}$. Suppose $\gamma^*\left(\overline{\lambda}\right) > 0$ and $q_1^*\left(\overline{\lambda}\right) < \widehat{q}_1^S(\gamma^*\left(\overline{\lambda}\right))$, which implies that $\theta_1 = 0$. Inequality (72) binds and implies that $s \geq r$, a contradiction to the assumption that $s < r$. Thus, $\gamma^*\left(\overline{\lambda}\right) > 0$ implies that $q_1^*\left(\overline{\lambda}\right) = \widehat{q}_1^S(\gamma^*\left(\overline{\lambda}\right))$. If $\gamma^*\left(\overline{\lambda}\right) > 0$, then (72) binds and $q_1^*\left(\overline{\lambda}\right) > 0$; thus (71) binds. These together imply $\frac{sD_{12}P_1^*\left(\overline{\lambda}\right)}{P_2^*\left(\overline{\lambda}\right)} = r+\theta_2$, hence $sP_1^*\left(\overline{\lambda}\right) \geq \frac{rP_2^*\left(\overline{\lambda}\right)}{D_{12}}$.

Following, I show that $\gamma^*(1) = \alpha$ and $P_1^*\left(1\right) > \frac{P_2^*(1)}{D_{12}}$. Suppose that $\gamma^*(1) < \alpha$, which implies $\widehat{q}_2^S(q_1^*\left(1\right), \gamma^*\left(1\right)) > 0$ by (8). By (10), $q_2^*(1) > 0$. But by (16c), $P_2^*(1) > 0$, so (16b) implies $q_2^*(1) = 0$, a contradiction. Thus, $\gamma^*(1) = \alpha > 0$, which implies $\frac{sD_{12}P_1^*(1)}{P_2^*(1)} = r+\theta_2$. From above, $P_2^*\left(1\right)$ is finite, and $r > s$, thus $P_1^*\left(1\right) > \frac{P_2^*(1)}{D_{12}}$.

Thus, I have shown that (19), (20) and (21) obtain in a relaxed market equilibrium. Finally, I show that constraints (6d) and (6e) are satisfied in a relaxed market equilibrium.

Constraint (6d) holds from equilibrium condition (10). From equation (16a),

$$q_1^S(P^*, \delta^*) = \frac{\overline{\lambda} D_1}{P_1^*(\overline{\lambda})} \tag{74}$$

$$\geq \frac{K_1}{P_1^*(\overline{\lambda})}, \tag{75}$$

which satisfies constraint (6e). Therefore, results (19) through (23) obtain in the market equilibrium. Q.E.D.

Proof of Proposition 1: The late consumer's condition (14) to withdraw at $t = 2$ holds for all $\delta_1^i \in [0, \overline{\lambda}]$ and all market equilibria for every $\overline{\lambda} \in [\lambda, 1]$, based on results (19) and (22), and for $\overline{\lambda} = 1$ based on (20). Thus, $\overline{\lambda}^* = \lambda$ is the unique Nash equilibrium and is an equilibrium in weakly dominant strategies. Q.E.D.

Proof of Proposition 2: First, I show that $q_1^*(\lambda) = \widehat{q}_1^S(\gamma^*(\lambda))$. Suppose not: $q_1^*(\lambda) < \widehat{q}_1^S(\gamma^*(\lambda))$, which implies $\theta_1 = 0$ by complementary slackness and $\gamma^*(\lambda) = 0$ from the proof for Lemma 1. Thus, $q_1^*(\lambda) < 1 - \widetilde{\alpha}$ and from (10), $q_2^*(\lambda) > \widetilde{\alpha} r$. From (17) and (18), $P_2^*(\lambda) < 1 < P_1^*(\lambda)$. But by (73), $P_1^*(\lambda) = \frac{P_2^*(\lambda)}{D_{12}} \leq P_2^*(\lambda)$, a contradiction. Thus, $q_1^*(\lambda) = \widehat{q}_1^S(\gamma^*(\lambda))$.

Second, I show that $\gamma^*(\lambda) = 0$. Suppose not. By substituting into (21) for prices from (17) and (18), then substituting with $q_1^*(\lambda) = \widehat{q}_1^S(\gamma^*(\lambda))$ and with $q_2^*(\lambda) = \widehat{q}_2^S(q_1^*(\lambda), \gamma^*(\lambda))$ by (10):

$$\frac{s\lambda D_1}{\lambda D_1 + \gamma^*(\lambda) s} \geq \frac{r\,(1-\lambda)\,D_2}{D_{12}(\widetilde{\alpha} - \gamma^*(\lambda)) r}. \tag{76}$$

The LHS is less than s, while the RHS is greater than one. This implies $s > 1$, a contradiction to the assumption of $s < 1$. Thus, $\gamma^*(\lambda) = 0$. Hence, the unique market equilibrium is $q_1^*(\lambda) = 1 - \widetilde{\alpha}$, $q_2^*(\lambda) = \widehat{q}_2^S(q_1^*(\lambda), \gamma^*(\lambda)) = \widetilde{\alpha} r$, $P_1^*(\lambda) = \frac{D_1}{\widetilde{c}_1}$, $P_2^*(\lambda) = \frac{D_2}{\widetilde{c}_2}$ and consumption is $c_1 = \frac{D_1}{P_1^*(\lambda)} = \widetilde{c}_1$ and $c_2 = \frac{D_2}{P_2^*(\lambda)} = \widetilde{c}_2$. Q.E.D.

Proof of Proposition 3: The nonnegative Lagrange multiplier associated with constraint (32) is θ_3. Following the proof for Lemma 1, I relax constraints (6d) and (6e) and replace (6f) with (70). The Kuhn-Tucker condition for α_e is

$$r \leq 1 + \theta_1 - \theta_2 + \theta_3 \qquad (= \text{ if } \alpha_e > 0). \tag{77}$$

First, suppose $\alpha_e^*(\overline{\lambda}) > 0$ and that $\gamma^*(\overline{\lambda}^*) > 0$, which implies by (72) binding that $r < 1 + \theta_1 - \theta_2$. But (32) implies $r \geq 1 + \theta_1 - \theta_2$, a contradiction. Thus, $\alpha_e^*(\overline{\lambda}) > 0$ implies

$\gamma^*(\overline{\lambda}^*) = 0$. Next, suppose $\alpha_e^*(\overline{\lambda}) = 1$, so $q_1^*(\overline{\lambda}) = 0$ and $P_1^*(\overline{\lambda}^*) = \infty$. By (10), $P_2^*(\overline{\lambda}^*)$ is finite. (77) implies $1 + \theta_1 = r - \theta_3$, but (71) implies $1 + \theta_1 \geq \infty$, a contradiction. Thus, $\alpha_e^*(\overline{\lambda}) \neq 1$.

Next, I show that $\overline{\lambda}^* = \lambda$. Consider $\alpha_e^*(\overline{\lambda}) = 0$, which implies $\theta_3 = 0$ by complementary slackness. By (77), $\theta_1 \geq r - 1 > 0$, so by complementary slackness $q_1^*(\overline{\lambda}) = \widehat{q}_1^S(\gamma^*(\overline{\lambda})) = 1$ and $P_1^*(\overline{\lambda})$ is finite by (17). Condition (71) binds, so (73) holds, which with $\theta_1 \geq r - 1$ implies $P_1^*(\overline{\lambda}) \geq \frac{(r+\theta_2)P_2^*(\overline{\lambda})}{D_{12}} > \frac{P_2^*(\overline{\lambda})}{D_{12}}$. For $\alpha_e^*(\overline{\lambda}) \in (0,1)$, Lemma 1 holds and Proposition 1 holds. Thus, condition (14) holds and $\overline{\lambda}^* = \lambda$.

Now I find $\alpha_e^*(\overline{\lambda}^*)$. Suppose $\alpha_e^*(\overline{\lambda}^*) = 0$, which implies $\widehat{q}_2^S(q_1^*(\overline{\lambda}^*), \gamma^*(\overline{\lambda}^*)) = 0$, so by (18), $P_2^*(\overline{\lambda}^*) = \infty$. This implies $P_1^*(\overline{\lambda}^*) = \infty$, a contradiction to $P_1^*(\overline{\lambda}^*)$ being finite. Thus, $\alpha_e^*(\overline{\lambda}^*) \neq 0$. Consider $\alpha_e^*(\overline{\lambda}^*) \in (0,1).\overline{\lambda}^* = \lambda$. (77) implies $1 + \theta_1 = r$, since $\theta_2 = \theta_3 = 0$ by complementary slackness. Suppose $q_1^*(\overline{\lambda}^*) < \widehat{q}_1^S(\gamma^*(\overline{\lambda}^*))$. This implies $\theta_1 = 0$, a contradiction to $\theta_1 = r - 1 > 0$. Thus, $q_1^*(\overline{\lambda}^*) = 1 - \alpha_e^*(\overline{\lambda}^*)$ and by (10) $q_2^*(\overline{\lambda}^*) = \alpha_e^*(\overline{\lambda}^*)r$. (73) gives $P_1^*(\overline{\lambda}^*) = \frac{rP_2^*(\overline{\lambda}^*)}{D_{12}}$. Substituting from (16a) and (16b) and for $q_1^*(\overline{\lambda}^*) = 1 - \alpha_e^*(\overline{\lambda}^*)$ and $q_2^*(\overline{\lambda}^*) = \alpha_e^*(\overline{\lambda}^*)r$, $\frac{\lambda D_1}{1-\alpha_e^*(\overline{\lambda}^*)} = \frac{r(1-\lambda)D_2}{\alpha_e^*(\overline{\lambda}^*)rD_{12}}$. Simplifying, $\alpha_e^*(\overline{\lambda}^*) = 1 - \lambda$. Finally, $P_1^*(\overline{\lambda}^*) = D_1$, $P_2^*(\overline{\lambda}^*) = \frac{D_2}{r}$, $\widetilde{c}_1 = 1$ and $\widetilde{c}_2 = r$. Q.E.D.

Proof of Proposition 4: The market clearing price of invested goods in terms of stored goods at $t = 1$ is $\widetilde{p}_m^* = \frac{(1-\lambda)(1-\widetilde{\alpha}_m^*)}{\lambda\widetilde{\alpha}_m^*}$. Consumption is given by

$$c_1 = 1 - \widetilde{\alpha}_m^* + \frac{(1-\lambda)(1-\widetilde{\alpha}_m^*)}{\lambda} = \frac{1-\widetilde{\alpha}_m^*}{\lambda} = \widetilde{c}_1 \tag{78}$$

$$c_2 = \widetilde{\alpha}_m^* r + \frac{\lambda\widetilde{\alpha}_m^* r}{1-\lambda} = \frac{\widetilde{\alpha}_m^* r}{1-\lambda} = \widetilde{c}_2. \tag{79}$$

The trade is incentive compatible for late consumers since the value of invested goods received from trading is greater than the value of liquid goods paid,

$$\frac{(1-\widetilde{\alpha}_m^*)r}{\widetilde{p}_m^*} = \lambda\widetilde{c}_2 > \lambda\widetilde{c}_1 = (1-\widetilde{\alpha}_m^*). \tag{80}$$

The trade is incentive compatible for early consumers if the value of liquid goods received from trading is greater than the value of liquidating invested goods,

$$\widetilde{p}_m^*\widetilde{\alpha}_m^* = \frac{\widetilde{c}_1\widetilde{\alpha}_m^* r}{\widetilde{c}_2} > s\widetilde{\alpha}_m^*, \tag{81}$$

which holds since the coefficient of relative risk aversion greater than one implies $\frac{\widetilde{c}_2}{\widetilde{c}_1} < r$.

Q.E.D.

Proof of Proposition 5: The result $\alpha_c^* = 1-\lambda$ is derived from the consumers' first-order condition for (43):

$$\frac{\lambda(K_1' - P_L^*)u'(c_1(\alpha_c))}{P_1^*} = \frac{(1-\lambda)\left(K_2' - \frac{K_1'K_2'}{P_L^*}\right)u'(c_2(\alpha_c))}{P_2^*}. \tag{82}$$

Substituting for prices from (44), (45) and (46) into (82) and simplifying,

$$\frac{q_1^*[\overline{\lambda}\alpha_c^* - (1-\overline{\lambda})(1-\alpha_c^*)]u'(c_1(\alpha_c))}{[\overline{\lambda}\alpha_c^*(1-\alpha_c^*) + \alpha_c^*(1-\overline{\lambda})(1-\alpha_c^*)} = \frac{q_2^*[(1-\overline{\lambda})(1-\alpha_c^*) - \overline{\lambda}\alpha_c^*]u'(c_2(\alpha_c))}{[\overline{\lambda}\alpha_c^*(1-\alpha_c^*) + \alpha_c^*(1-\overline{\lambda})(1-\alpha_c^*)]}. \tag{83}$$

The unique solution is $\alpha_c^* = 1-\overline{\lambda}$. Substituting for α_c^* in (44) and (45) gives $P_1^* = \frac{\lambda D_1^C}{q_1^*}$ and $P_2^* = \frac{(1-\lambda)D_2^C}{q_2^*}$, where $D_1^C \equiv \frac{\lambda D_1}{1-\widetilde{\alpha}}$ and $D_2^C \equiv \frac{(1-\lambda)D_2}{\widetilde{\alpha}}$. Substituting for α_c^* in (35) and (36), $c_1(\alpha_c^*) = \frac{q_1^*}{\overline{\lambda}}$ and $c_2(\alpha_c^*) = \frac{q_2^*}{1-\overline{\lambda}}$. The result $\alpha_c^* = 1-\lambda$ implies $K_1^C = \lambda D_1^C$ and $K_2^C = (1-\lambda)\,D_2^C$. Substituting D_t^C for D_t, $(q_1^*, q_2^*, \gamma^*) = (1-\alpha_c^*, \alpha_c^* r, 0)$ is the solution to the entrepreneur's optimization, following from Proposition 2, so $(c_1(\alpha_c^*), c_2(\alpha_c^*)) = (1, r)$. Thus, $c_2(\alpha_c^*) > c_1(\alpha_c^*)$. Substituting for P_L into (35) and (36) and simplifying shows that $\frac{K_2'}{P_2^* P_L^*} > \frac{1}{P_1^*}$ is equivalent to $c_2(\alpha_c^*) > c_1(\alpha_c^*)$. Q.E.D.

Proof of Proposition 6: Results are analogous to those of Lemma 1 and Proposition 1 and follow from their proofs. The entrepreneur's optimization holds ex-ante for the expectation $\underline{\lambda}$ with the solution $(\underline{q}_1^S(\cdot), \underline{q}_2^S(\cdot), \underline{\gamma}(\cdot))$ and ex-post after the realization of $\overline{\lambda}$ for the solution $(q_1^S(\cdot), q_2^S(\cdot), \gamma(\cdot))$. The sequential payments for purchases and entrepreneur loan repayments aggregate up in each period $t = 1, 2$ and are equal to the equivalent lump-sum aggregate payments in the centralized market model, so (21) holds for the realization of $\overline{\lambda} \in \{\lambda, 1\}$. For $\underline{\lambda} \in \{\lambda, 1\}$ and $\overline{\lambda} \in \{\lambda, 1\}$, $\frac{D_2}{P_2^*(\underline{\lambda},\overline{\lambda})} > \frac{D_1}{\underline{P}_1^*(\underline{\lambda})}$ and $\frac{D_2}{P_2^*(\underline{\lambda},\overline{\lambda})} > \frac{D_1}{\overline{P}_1^*\{\underline{\lambda},\overline{\lambda}\}}$, so $\overline{\lambda}^* = \lambda$, $\underline{\lambda}^* = \lambda$ and $(c_1, c_2) = (\frac{D_1}{\underline{P}_1^*(\lambda)}, \frac{D_2}{P_2^*(\lambda,\lambda)}) = (\widetilde{c}_1, \widetilde{c}_2)$. Q.E.D.

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