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Abstract

This paper develops a growth model with land, housing services, and other goods that is capable of explaining a substantial portion of the movements in housing prices over the past forty years. Under certainty, the model exhibits a balanced aggregate growth, but with underlying sectoral change. The paper introduces a Markov regime-switching specification for productivity growth in the nonhousing sector and shows that such regime switches are a plausible candidate for explaining—both qualitatively and quantitatively—the large low-frequency changes in housing price trends. In particular, the model shows how housing prices can have a "bubbly" appearance in which housing wealth rises faster than income for an extended period, then collapses and experiences an extended decline. The paper also uses micro data to calibrate a key cross-elasticity parameter that governs the relationship between productivity growth and home price appreciation. Combined with a realistic model of learning about the productivity process, the model is able to capture the medium- and low-frequency fluctuations of both price and quantity from the residential sector. Finally, the model suggests that the current downturn in the housing sector was triggered by a productivity slowdown that may have begun in 2004, an event that could reasonably have been viewed as highly unlikely by investors and mortgage issuers in the early part of the decade.

Key words: housing prices, residential investment, productivity growth

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With the acceleration of housing prices since the mid-1990's in the United States, as well as the recent dramatic downturn, there has been increased attention given to the causes and effects of the fluctuations in housing prices and investment. From 1996 to its peak in early 2007, the real quality-adjusted price of new houses appreciated by approximately 33 percent in real terms. This was not the first time housing prices experienced such a sustained appreciation, as a similar episode took place beginning in the late 1960s. Between these two episodes was a 15-year period of real price sluggishness or even decline. Markets in particular geographical areas have exhibited still greater volatility (cf. Himmelberg et al, 2005). These sustained changes have been variously attributed to speculative bubbles, at least in specific localities (Case and Shiller, 2003), expansions and contractions in monetary policy (Iacoviello and Neri, 2006), and developments in financial markets—innovations such as new types of mortgage instruments, or breakdowns such as the recent subprime mortgage boom and bust. Such explanations suggest that market irrationality may have played an important role in housing sector fluctuations.

Figure 1 depicts the behavior of inflation-adjusted housing prices over the last few decades, according to several popular series. The one available going back the farthest (to 1963) is the quality-adjusted price index for new homes, published by the Census Bureau. An index of existing home prices based on a repeat-sales methodology, published by the Office of Federal Housing Enterprise Oversight (OFHEO) is available quarterly back to 1975, while the Case-Shiller index, also based on repeat sales, goes back to 1987. We can see that where they overlap, the series behave similarly except for having different trends. Both the Census and OFHEO indexes peak in 1979 or so, and decline until the mid-1980s. Then all three series have another small peak in the late 1980s, followed by flat or declining real prices until the mid-1990s. Then all three series take off and increase dramatically until around 2006.

This paper argues that trend productivity growth is a key driver of these medium to long-term movements in housing prices. It develops a stochastic growth model in which land, capital, and labor are inputs to production, and housing and non-housing consumption provide utility to consumers. Technical progress in output other than housing services is presumed to have a Markov regime-switching component as in Kahn and Rich (2007, hereafter KR). The calibrated model, together with a plausible learning process for trend producitivity, can explain the qualitative—and much of the quantitative—behavior of housing prices since the 1960s, including the recent slowdown. In particular, it shows that the regime-switching behavior of productivity growth, which KR showed was an accurate depiction of postwar data, can give housing prices a "bubbly" appearance in which housing wealth rises faster than income for an extended period, then collapses and experiences an extended decline. The model also provides a partial rationale for the beliefs of investors (and mortgage issuers) in the housing boom in the early part of this decade, as it suggests that the bust that occurred was a low-probability event viewed from the perspective of the early part of the decade.

A key parameter turns out to be the elasticity of substitution between housing and non-housing consumption. This parameter has been featured in many studies related to housing(e.g. Li et al, 2008; Piazzesi and Schneider, 2007; Flavin and Nakagawa, 2004.). At the same time, many studies of housing have assumed, presumably for convenience, a value of one for this elasticity (e.g. Iacoviello and Neri, 2006). We provide evidence based on both aggregate and microeconomic data that this elasticity is considerably less than one. This low elasticity plays a crucial role in the model's ability to explain both qualitatively and quantitatively the magnitude of housing price fluctuations.

1 Background

The real value of housing wealth, as measured by flow of funds data, has grown an average of 4.6 percent since 1952. This compares with 3.4 percent growth of private net worth excluding real estate, and 3.5 percent growth of personal consumption expenditures over the same time period. Figure 2 plots the ratio of nominal housing wealth to nominal consumption expenditure. This ratio nearly doubles between 1952 and 2005. Figure 3 plots the much more volatile ratio of housing wealth to total net worth. While the enormous volatility of non-real estate wealth hinders precise inferences about relative trends, this disparity is robust to different time periods, and is not just the result of the runup of the last 5-10 years in real estate wealth. The bottom line is that real estate has gone from 27 percent of net worth in 1952 to 42 percent by the end of 2005. Davis and Heathcote (2005), however, argue that there are problems with the Flow of Funds data, particularly over long periods of time, and construct their own measures of housing wealth (though only going back to 1975) that exhibit a less clearcut trend.

One possible explanation for the relative increase in housing prices is a simple income effect, or non-homogeneity in preferences. As people get wealthier, they may prefer to have more of their consumption coming from housing services, the price of which will tend to rise because of its being relatively intensive in land, a fixed factor. The (nominal) share of housing services in GDP has gone from 7.5 percent in 1952 to over 10 percent in 2005. The share of housing services in consumer expenditures has gone from 12.2 percent to 14.6 percent over the same period, but in fact has been without any meaningful trend since 1960 or so. In any case, evidence against this proposition will be presented below. Another driver of housing prices could be differing technical progress trends in housing services versus other goods, as in Baumol (1967). The relatively large share of land and structures, two inputs usually thought to be less amenable to embodied technical progress, in the value of housing makes this story plausible. This paper will argue that the timing of low-frequency changes in both housing prices and productivity suggests that this mechanism is important.

There is some evidence that in fact the increase in housing wealth does not stem from an increase in the value of houses per se, but rather from the increase in the value of the land upon which they are built. First, a price index that include the value of land, the Conventional Mortgage Home Price Index, has increased approximately 0.75% faster than indexes that do not, such as the Census's Composite Construction Cost index, on an annual basis. Davis and Heathcote (2004) compute a land price index based on this type of differential and find that land values have increased at an average annual rate of approximately 3.5% (inflation-adjusted) over the period 1975-2005. That price index may, however have an upward bias from not adequately adjusting for quality changes. Land price series available from the Bureau of Labor Statistics (see Figure 4) suggest behavior that is closer to the behavior of new home prices in Figure 1. We focus on the Census Bureau's quality-adjusted price of new homes in part because of concern over this bias, but also because it goes back to 1963.

Finally, Figure 5 depicts the behavior of the HP-trend component of productivity growth (relative to a linear trend) over the postwar period. While its pattern is similar to low-frequency movements in land and housing prices, the downturn in productivity clearly precedes the downturn in housing and land prices by several years. KR provide more detailed econometric estimates of a regime-switching model of the sort incorporated below into this paper, and find significant regime changes corresponding to the shifts depicted in the figure.

2 Related Literature

Research on aggregate housing prices has emphasized demographics, income trends, and government policy as fundamental drivers. In one well-known study, Mankiw and Weil (1989) argued that population demographics were the prime determininant, and predicted that prices would fall in the subsequent two decades with the maturation of the baby boom generation and resulting decline in the growth rate of the prime home-owning age group. While their prediction proved inaccurate, Martin (2005) renewed the argument for an important role for demographics. Glaeser et al (2005) argues that price increases since 1970 largely reflect artificial supply restrictions. Gyourko et al (2006) also cite inelastically supplied land as a key driver of the phenomenon they call "superstar cities." Van Nieuwerbergh and Weill (2006), however, argue that so long as there are regional markets in which such restrictions are not present, the aggregate impact of restrictions in some local markets is likely to be modest—in other words, they primarily affect the cross-sectional distribution of housing prices as opposed to the aggregate. Iacoviello and Neri (2006) examine the role of monetary policy with credit market frictions.

Consistent with the approach in this proposal, Attanasio et al (2005) find that "common causality" drives the comovement of house prices and consumption, as opposed to wealth or the collateral channels. Also consistent with the approach adopted here, Kiyotaki et al (2007) find that credit market frictions primarily affect own vs. rent decisions as opposed to prices. Piskorski and Tchistyi (2008) examine optimal mortgage lending in a setting where housing prices obey essentially the same type of regime-switching behavior assumed here, and find that "many features of subprime lending observed in practice are consistent with economic efficiency and rationality of both borrowers and lenders," though, as they point out, there may be negative externalities associated with massive defaults in a downturn.

Case and Shiller (2003) and Himmelberg et al (2005) investigate the bubble hypothesis, looking across a large number of cities, and both suggest that the phenomenon is limited to a few localities. As with the research above on inelastic land supplies, these papers emphasize the cross-sectional variation of house prices across metropolitan areas rather than aggregate time series variation.

One important innovation in this project is to allow for unbalanced sectoral growth. General equilibrium models with production have generally either assumed Cobb-Douglas preferences (e.g. Davis and Heathcote, 2005, Kiyotaki et al., 2007, Iacoviello and Neri, 2006) or have abstracted from longer-term growth issues (e.g. Van Nieuwerburgh and Weill, 2007). This is the first housing model (to my knowledge) with production that features balanced aggregate growth and systematically varying sectoral shares due to non-unit elastic preferences. The importance of this is that it is consistent with aggregate growth facts as well as with the evidence on substitution elasticities found by numerous authors (see the discussion below), and also enables the model to match the volatility of housing prices in a plausible and disciplined way. The modeling approach is based on recent work of Ngai and Pissarides (2007).

3 A Growth Model with Housing

This section presents a general equilibrium growth model that is capable of capturing the important stylized facts about housing and the economy. The model has two sectors, a "manufacturing" sector that produces non-housing related goods and services, as well as the capital (structures and durable goods) that go into housing services. A second sector uses capital, labor, and land to produce a flow of housing services. The model exhibits balanced aggregate growth, but with unequal growth across sectors. We then consider the behavior of the model under a regime-switching specification for productivity growth in the manufacturing sector.

3.1 Firms

Competitive final goods firms produce two types of goods: A "manufactured" good Y_m, and housing services Y_h. Under perfect competition the final goods firms make zero profits and have perfectly elastic supplies of

Y_m and Y_h at the above prices. The production functions for the two types of goods are

$$Y_j = A_j K_j^{\alpha} L_j^{\beta_j} (eN_j)^{1-\alpha-\beta_j}$$

for $j = m, h$, where K_j is capital allocated to j, L_j is land, and eN_j is labor input. The goods producers rent inputs in competitive markets. In particular, capital is rented from final goods producers of Y_m.

In the j sector, the representative firm's nominal profit in period t is given by

$$P_{jt}Y_{jt} - W_t e_t N_{mt} - R_{\ell t} L_{mt} - R_{kt} K_{mt} \tag{1}$$

where R_ℓ and R_k are nominal rental rates for land and capital respectively, and W_t is the nominal wage. Profit maximization implies

$$\alpha P_{jt} Y_{jt} / K_{jt} = R_{kt} \tag{2}$$

$$\beta_j P_{jt} Y_{jt} / L_{jt} = R_{\ell t} \tag{3}$$

$$(1 - \alpha - \beta_m) P_{jt} Y_{jt} / (eN_{jt}) = W_t \tag{4}$$

$$\frac{K_{jt}}{L_{jt}} = \frac{\alpha}{\beta_j} \frac{R_{\ell t}}{R_{kt}} \tag{5}$$

$$\frac{K_{jt}}{e_t N_{jt}} = \frac{\alpha}{1 - \alpha - \beta_j} \frac{W}{R_{kt}}. \tag{6}$$

This implies that marginal cost for all firms in sector $j = m, h$ is equal to

$$M_{jt} = \alpha^{-\alpha} \beta_j^{-\beta_j} \left(1 - \alpha - \beta_j\right)^{-\left(1-\alpha-\beta_j\right)} W_t^{1-\alpha-\beta_j} R_{\ell t}^{\beta_j} R_{kt}^{\alpha}. \tag{7}$$

Because firms are assumed to be price takers in factor markets, marginal cost is not directly a function of output. But because of the short-run fixity of capital and land, and imperfectly elastic labor supply, factor prices and hence marginal cost will move procyclically.

3.2 Consumers

There are N_t representative agents at time t each supplying one unit of labor, where N is exogenous, growing exponentially at constant rate ν.[1]. Let C denote the aggregate non-housing consumption good,

[1] It is straightforward to endogenize labor supply, and the model exhibits constant hours of work along the balanced growth path under the usual restrictions on preferences (e.g. King et al, 1988).

and H aggregate housing services. We let c, h, and $\ell \equiv L/N$ denote per capita quantities. The representative consumer solves the problem

$$\max\ U = E_t \left\{ \sum_{s=0}^{\infty} (1+\rho)^{-s} \ln \left(\left[\omega_c c_{t+s}^{(\epsilon-1)/\epsilon} + \omega_h h_{t+s}^{(\epsilon-1)/\epsilon} \right]^{\epsilon/(\epsilon-1)} \right) \right\} \tag{8}$$

subject to

$$P_{m,t+s}\left(c_{t+s} + \iota_{t+s}\right) + P_{h,t+s}h_{t+s} + V_{t+s}\left[(1+\nu)\,\ell_{t+s} - \ell_{t+s-1}\right] + b_{t+s}/\left(1+R_{t+s}\right)$$

$$\begin{aligned} &\leq \quad b_{t+s-1} + W_{t+s} + (1+\nu)\,R_{k,t+s}P_{m,t+s-1}k_{t+s-1} + R_{\ell,t+s}V_{t+s-1}\ell_{t+s-1} && (9) \\ (1+\nu)\,k_{t+s} &= \quad (1-\delta)\,k_{t+s-1} + z\left(\iota_{t+s-1}/k_{t+s-1}\right)k_{t+s-1} \qquad (j = m,h) && (10) \end{aligned}$$

where ι_t denotes total capital investment at date t, from the profits of intermediate goods producers in sector j, b_t nominal one-period discount bonds, V_t the price of land at date t, W_t the wage, and ℓ_t land holdings at date t. k_t and ℓ_t denote aggregate per capita capital and land respectively. The function $z(x)$ reflects adjustment costs, which will be discussed in more detail below.

The first-order conditions for the consumer's problem, letting

$$\phi(c,h) \equiv \left[\omega_c c^{(\epsilon-1)/\epsilon} + \omega_h h^{(\epsilon-1)/\epsilon} \right]^{\epsilon/(\epsilon-1)}$$

are as follows:

$$\begin{aligned} \omega_c \phi^{-(\epsilon-1)/\epsilon} c_t^{-1/\epsilon} &= \tilde{\Lambda}_t P_{mt} && (11) \\ \omega_h \phi^{-(\epsilon-1)/\epsilon} h_t^{-1/\epsilon} &= \tilde{\Lambda}_t P_{ht} && (12) \\ \psi'(e_t) &= \tilde{\Lambda}_t W_t && (13) \\ \tilde{\Lambda}_t P_{mt}(1+\nu)(1+\rho) &= E_t \left\{ \tilde{\Lambda}_{t+1}\left[P_{mt}R_{kt+1} + P_{mt+1}(1-\delta)\right] \right\} && (14) \\ \tilde{\Lambda}_t V_t (1+\nu)(1+\rho) &= E_t \left\{ \tilde{\Lambda}_{t+1}\left[V_t R_{\ell t+1} + V_{t+1}\right] \right\} && (15) \\ & \tilde{\Lambda}_t P_{mt}???? && (16) \end{aligned}$$

The last two equations function as equilibrium conditions under which the consumer is indifferent between investing more or less capital or land at date t. In real terms (expressed in units of m sector output) we have

$$\omega_c \phi^{-(\epsilon-1)/\epsilon} c_t^{-1/\epsilon} = \Lambda_t \tag{17}$$

$$\omega_h \phi^{-(\epsilon-1)/\epsilon} h_t^{-1/\epsilon} = \Lambda_t p_{ht} \tag{18}$$

$$\psi'(e_t) = \Lambda_t w_t \tag{19}$$

$$\Lambda_t (1+\nu)(1+\rho) = E_t \{\Lambda_{t+1} [r_{kt+1} + 1 - \delta]\} \tag{20}$$

$$\Lambda_t v_t (1+\nu)(1+\rho) = E_t \{\Lambda_{t+1} [v_t r_{\ell t+1} + v_{t+1}]\} \tag{21}$$

where the lower-case prices are relative to P_m (e.g. $w_t = W_t/P_{mt}$, etc.).

3.3 Equilibrium Growth

Aggregating over producers in each sector, we have

$$\begin{aligned} C_t + I_t &= A_{mt} K_{mt}^{\alpha} L_{mt}^{\beta_m} (e_t N_{mt})^{1-\alpha-\beta_m} \\ K_t - (1-\delta) K_{t-1} &= z (I_t / K_{t-1}) K_{t-1} \\ H_t &= A_{ht} K_{ht}^{\alpha} L_{ht}^{\beta_m} (e_t N_{ht})^{1-\alpha-\beta_h} \end{aligned}$$

where

$$\begin{aligned} L_{mt} + L_{ht} &= \bar{L} \\ K_{mt} + K_{ht} &= K_{t-1} \\ N_{mt} + N_{ht} &= N_t \end{aligned}$$

The stocks of capital and land in the h sector would correspond to residential real estate. Labor in this sector would be partly non-market household labor, and partly service sector labor (particularly for apartment buildings). We assume (mainly for convenience) that capital's share is the same in both sectors, but labor's share is higher in manufacturing (implying of course that land's share is higher in the housing sector, i.e. $\beta_h > \beta_m$).

Let c and h denote per capita quantities of C and H, while $k,\ \ell,\ k_i,\ \ell_h$ refer to per worker quantities in sector i (e.g. $k_{ht} \equiv K_{ht}/N_{ht},\ k_t \equiv K_t/N_{t+1}$, i.e. no subscript refers to aggregates), while $n_{it} \equiv N_{it}/N_t$, $(i = m, h)$.[2] Given the assumption of perfect competition, we can assume the economy solves the following

[2] The derivations here draw on Ngai and Pissarides (2007), albeit in discrete time, and adding a fixed factor with heterogeneous technology.

planner's problem:

$$\max\ U = E_0 \left\{ \sum_{t=0}^{\infty} (1+\rho)^{-t} \ln \left(\left[\omega_c c_t^{(\epsilon-1)/\epsilon} + \omega_h h_t^{(\epsilon-1)/\epsilon} \right]^{\epsilon/(\epsilon-1)} \right) - \psi\left(e_t\right) \right\} \tag{22}$$

subject to resource constraints

$$c_t + i_t = A_{mt} k_{mt}^{\alpha} \ell_{mt}^{\beta_m} e_t^{1-\alpha-\beta_m} n_{mt} \tag{23}$$

$$(1+\nu) k_t - (1-\delta) k_{t-1} = z\left(i_t / k_{t-1}\right) k_{t-1} \tag{24}$$

$$h_t = A_{ht} k_{ht}^{\alpha} \ell_{ht}^{\beta_h} e_t^{1-\alpha-\beta_h} n_{ht} \tag{25}$$

$$k_{mt} n_{mt} + k_{ht} n_{ht} = k_{t-1} \tag{26}$$

$$\ell_{mt} n_{mt} + \ell_{ht} n_{ht} = \ell_t \tag{27}$$

$$n_{mt} + n_{ht} = 1. \tag{28}$$

Total land $\bar{L}$ is assumed fixed, so $\ell_t / \ell_{t-1} = (1+\nu)^{-1}$. Average technological progress in sector i, i.e. the average growth rate of A_i, is denoted γ_i $(i = m, h)$. We assume e is the same in the two sectors, and that $\beta_h \geq \beta_m$. Note that the timing assumptions in (26) and (27) are such that while aggregate capital k is chosen one period ahead of time, and total land and labor are exogenous, for simplicity the sectoral allocations are determined contemporaneously.

It is worth mentioning that technical progress in the h sector is unrelated to technological progress in construction. (In fact, home construction occurs in the m sector in this model.) Rather, it refers to an increase in the housing services from given stocks of K_h, L_h, and labor inputs eN_h. What this means in practice depends on exactly what the term "housing services" encompasses, and on how one measures K_h. In the model it is assumed for simplicity to be indistinguishable from K_m other than by its allocation to the h sector. In particular, it is assumed to have the same price as K_m and C. In principle it would include both residential structures and housing service-related consumer durables (home appliances). L_h would include both non-market and market labor involved in household production—time devoted to housework, food preparation, home and yard maintenance, and the like.

The model obviously abstracts from a number of potentially important factors. First and foremost, the housing and construction sectors are heavily affected by government intervention, both via distortionary taxation and regulations. In particular, much land in the U.S. (and in most other countries as well) is neither residential nor commercial, and is either owned or heavily restricted in its use by the government. Second, there is tremendous heterogeneity in land and housing values. Land near navigable bodies of water, or ports, or along coastlines is much more valuable than land that does not have these features. Obviously this model will have nothing directly to say about the cross-sectional distribution of land values or housing prices (though many of the factors that affect them over time undoubtedly come into play in the cross-section as well). Nonetheless if all of these factors remain relatively constant over time, then ignoring them in a model such as this should not be too great a sin.

Letting $\phi(c,h) \equiv \left[\omega_c c^{(\epsilon-1)/\epsilon} + \omega_h h^{(\epsilon-1)/\epsilon}\right]^{\epsilon/(\epsilon-1)}$, the first-order conditions are:

$$\begin{aligned}
\omega_c \phi^{-(\epsilon-1)/\epsilon} c_t^{-1/\epsilon} &= \mu_{mt} && (29)\\
\omega_h \phi^{-(\epsilon-1)/\epsilon} h_t^{-1/\epsilon} &= \mu_{ht} && (30)\\
\psi'(e_t) &= \mu_{mt}(1-\alpha-\beta_m) A_{mt} k_{mt}^{\alpha} \ell_{mt}^{\beta_m} e_t^{-\alpha-\beta_m} n_{mt} && (31)\\
&\quad + \mu_{ht}(1-\alpha-\beta_h) A_{ht} k_{ht}^{\alpha} \ell_{ht}^{\beta_h} e_t^{-\alpha-\beta_h} n_{ht} &&\\
\mu_{mt}\beta_m A_{mt} k_{mt}^{\alpha} \ell_{mt}^{\beta_m-1} e_t^{1-\alpha-\beta_m} &= \mu_{ht}\beta_h A_{ht} k_{ht}^{\alpha} \ell_{ht}^{\beta_h-1} e_t^{1-\alpha-\beta_h} && (32)\\
\mu_{mt} A_{mt} k_{mt}^{\alpha-1} \ell_{mt}^{\beta_m} e_t^{1-\alpha-\beta_m} &= \mu_{ht} A_{ht} k_{ht}^{\alpha-1} \ell_{h}^{\beta_h} e_t^{1-\alpha-\beta_h} && (33)\\
\mu_{mt} A_{mt} k_{mt}^{\alpha} \ell_{mt}^{\beta_m} e_t^{1-\alpha-\beta_m} &= \mu_{ht} A_{ht} e_t^{1-\alpha-\beta_h} \times && (34)\\
&\quad \left[\alpha k_{ht}^{\alpha-1} \ell_{ht}^{\beta_h} k_{mt} + \beta_h k_{ht}^{\alpha} \ell_{ht}^{\beta_h-1} \ell_{mt} + (1-\alpha-\beta_h) k_{ht}^{\alpha} \ell_{ht}^{\beta_h}\right] && (35)\\
\lambda_t \, z'(i_t/k_{t-1}) &= \mu_{mt} && (36)\\
\lambda_t (1+\nu)(1+\rho) &= E_t \left\{ \mu_{mt+1} A_{mt+1} \alpha k_{mt+1}^{\alpha-1} \ell_{mt+1}^{\beta_m} e_t^{1-\alpha-\beta_m} + \right. && (37)\\
&\quad \left. \lambda_{t+1}\left[z(i_{t+1}/k_t) - (i_{t+1}/k_t) z'(i_{t+1}/k_t) + 1 - \delta\right] \right\} &&
\end{aligned}$$

μ_{mt}, μ_{ht}, and λ_t are shadow prices on the resource constraints (23), (25), and (24). Note that in the absence of adjustment costs, i.e. when $z(x) = x$, we have $\lambda_t = \mu_{mt}$, and (37) becomes

$$\mu_{mt}(1+\nu)(1+\rho) = E_t \left\{ \mu_{mt+1} \left[A_{mt+1} \alpha k_{mt+1}^{\alpha-1} \ell_{mt+1}^{\beta_m} e_t^{1-\alpha-\beta_m} + 1 - \delta \right] \right\} \quad (38)$$

which is just the familiar condition that the intertemporal marginal rate of substitution equals the marginal product of capital.

The static first-order conditions can be shown to imply that

$$\begin{aligned}
\frac{k_m}{k_h} &= \frac{1-\alpha-\beta_h}{1-\alpha-\beta_m} && (39)\\
\frac{\ell_m}{\ell_h} &= \frac{\beta_m}{\beta_h}\frac{1-\alpha-\beta_h}{1-\alpha-\beta_m}, && (40)
\end{aligned}$$

Let p_t denote the relative price of housing services in terms of manufactured goods. We have

$$\begin{aligned} p_t &= \frac{\mu_{ht}}{\mu_{mt}} = \frac{A_{mt}k_{mt}^{\alpha-1}\ell_{mt}^{\beta_m}e_t^{1-\alpha-\beta_m}}{A_{ht}k_{ht}^{\alpha-1}\ell_{ht}^{\beta_h}e_t^{1-\alpha-\beta_h}} \\ &= \frac{A_{mt}}{A_{ht}}\left(\frac{\beta_m}{\beta_h}\right)^{\beta_m}\left(\frac{1-\alpha-\beta_h}{1-\alpha-\beta_m}\right)^{\alpha+\beta_m-1}\left(\frac{\ell_{ht}}{e_t}\right)^{-(\beta_h-\beta_m)} \end{aligned}$$

Thus growth in the price of housing services reflects both relative productivity growth in manufacturing and the increasing scarcity of land.

Finally, from (31), (33) and (39) we have

$$\psi'(e_t) = \mu_{mt}(1-\alpha-\beta_h)A_{mt}k_{mt}^{\alpha-1}\ell_{mt}^{\beta_m}e_t^{-\alpha-\beta_m}k_{t-1} \tag{41}$$

which equates the marginal rate of substitution between consumption and leisure with the marginal product of labor expressed in terms of m sector output.

3.4 Balanced Aggregate Growth under Certainty

Let total expenditure $c+ph$ be denoted by x. It also turns out that $\mu_m = x^{-1}$, hence $\mu_{mt}/\mu_{mt-1} = x_{t-1}/x_t$. We can aggregate the two resource constraints as follows:

$$\begin{aligned} c_t + p_t h_t + i_t &= A_{mt}k_{mt}^{\alpha}\ell_{mt}^{\beta_m}e_t^{1-\alpha-\beta_m}n_{mt} + \qquad (42) \\ & \quad A_{mt}k_{mt}^{\alpha-1}\ell_{mt}^{\beta_m}k_{ht}e_t^{1-\alpha-\beta_h}n_{ht} \qquad (43) \\ x_t + i_t &= A_{mt}k_{mt}^{\alpha-1}\ell_{mt}^{\beta_m}e_t^{1-\alpha-\beta_m}k_{t-1} \qquad (44) \\ (1+\nu)k_t - (1-\delta)k_{t-1} &= z(i_t/k_{t-1})k_{t-1}. \qquad (45) \end{aligned}$$

For the dynamic equations describing the evolution of k_t, i_t, and x_t we then have

$$\begin{aligned} x_t + i_t &= A_{mt}k_{mt}^{\alpha-1}\ell_{mt}^{\beta_m}e_t^{1-\alpha-\beta_m}k_{t-1} \qquad (46) \\ (1+\nu)k_t &= z(i_t/k_{t-1})k_{t-1} + (1-\delta)k_{t-1} \qquad (47) \\ q_t(1+\rho)(1+\nu) &= E_t\Big\{(x_t/x_{t+1})\Big[A_{mt+1}\alpha k_{mt+1}^{\alpha-1}\ell_{mt+1}^{\beta_m}e_t^{1-\alpha-\beta_m} + \qquad (48) \\ & \quad q_{t+1}\left[z(i_{t+1}/k_t) + 1 - \delta\right] - i_{t+1}/k_t\Big]\Big\} \end{aligned}$$

where $q_t \equiv \lambda_t/\mu_{mt} = [z'(i_t/k_{t-1})]^{-1}$, the shadow value of capital in terms of m output. Note that in the absence of adjustment costs, $q_t = 1\ \forall t$, and $z(x) = x$, We will define aggregate balanced growth under certainty as an equilibrium path in which x and k both grow at a constant rate, and in which the interest rate (i.e. the marginal product of capital) is also constant. We will also assume that $z(i/k) = i/k$ and $z'(i/k) = 1$ at the steady state value of i/k, so that adjustment costs are zero on the balanced growth path. Balanced growth clearly requires that $A_{mt} k_{mt}^{\alpha-1} \ell_{mt}^{\beta_m} e_t^{1-\alpha-\beta_m}$ be constant, which we can express as

$$(k_{mt}/e_t) / (k_{mt-1}/e_{t-1}) = \left[(1+\gamma_m)\left[(\ell_{mt}/e_t) / (\ell_{mt-1}/e_{t-1})^{\beta_m}\right]\right]^{1/(1-\alpha)},$$

i.e. a relationship between the growth rates in the m sector of the capital-labor ratio, technological progress, and the land-labor ratio. Therefore, let

$$Z_t \equiv \left[A_{mt} (\ell_{mt}/e_t)^{\beta_m}\right]^{1/(1-\alpha)} e_t = \left[A_{mt} \ell_{mt}^{\beta_m} e_t^{1-\alpha-\beta_m}\right]^{1/(1-\alpha)} \tag{49}$$

and define variables with "~" over them to be deflated by Z_t, e.g. $\tilde{k}_{mt} \equiv k_{mt}/Z_t$. We then have

$$\begin{aligned}
x_t/k_{t-1} + i_t/k_{t-1} &= \tilde{k}_{mt}^{\alpha-1} && (50)\\
(1+\nu) k_t/k_{t-1} &= z(i_t/k_{t-1}) + 1 - \delta && (51)\\
(x_{t+1}/x_t)(1+\nu)(1+\rho) q_t &= \alpha \tilde{k}_{mt+1}^{\alpha-1} + && (52)\\
& \quad q_{t+1}\left[z(i_{t+1}/k_t) + 1 - \delta\right] - (i_{t+1}/k_t)
\end{aligned}$$

With $\tilde{k}_m$ constant under balanced growth, k and x both grow at the same constant rate.

From (26)-(28) and (39)-(40) we have

$$k_{mt}\left[\frac{1-\alpha-\beta_m}{1-\alpha-\beta_h} n_{ht} + n_{mt}\right] = k_{t-1}. \tag{53}$$

Now let

$$Q_t \equiv \frac{1-\alpha-\beta_m}{1-\alpha-\beta_h} n_{ht} + n_{mt} \tag{54}$$

Then we have $k_{mt} = k_{t-1}/Q_t$, and we can define $\hat{k}_t \equiv k_t/(Z_t Q_t)$. This gives a normalization of k_t that is constant on the balanced growth path. Note that if $\beta_m = \beta_h$, then $Q = 1$ and we would have $k_{mt} = k_{ht} = k_{t-1}$. But with $\beta_m > \beta_h$, $Q > 1$ and n_h and n_m are changing over time (unless $\epsilon = 1$). In particular, if $\epsilon < 1$ and $\gamma_m \geq \gamma_h$, then n_h (and hence Q) grows over time.

The model implies constant work effort along the balanced growth path. From (41) we have

$$\psi'(e_t) = \mu_{mt}(1-\alpha-\beta_h) A_{mt} k_{mt}^{\alpha-1} \ell_{mt}^{\beta_m} e_t^{-\alpha-\beta_m} k_{t-1}$$

which after normalization with Z yields

$$\begin{aligned}\psi'(e_t) e_t &= (1-\alpha-\beta_h) \tilde{k}_{mt}^{\alpha-1} k_{t-1}/x_t \\ &= (1-\alpha-\beta_h) \tilde{k}_{mt}^{\alpha}/\hat{x}_t\end{aligned}$$

Since $\tilde{k}_{mt}$ and $\hat{x}_t$ are constant along the balanced growth path, e_t is constant as well.

On the balanced aggregate growth path, ZQ grows at a constant rate. In fact it is straightforward to show that its growth rate g satisfies

$$\left[(1+\gamma_m)(1+\nu)^{-\beta_m}\right]^{1/(1-\alpha)} \equiv 1+g \tag{55}$$

We then have $\tilde{k}_{mt} = k_{mt}/Z_t = k_{t-1}/(Q_t Z_t) = \hat{k}_{t-1} Q_{t-1} Z_{t-1}/(Q_t Z_t) = \hat{k}_{t-1}/(1+g_t)$. Aggregate output per capita (in terms of manufactured goods), which we denote y_t, is $A_{mt} k_t^{\alpha} \ell_{mt}^{\beta_m} e_t n_{mt} + p_t A_{ht} k_{ht}^{\alpha} \ell_{ht}^{\beta_h} e_t n_{ht}$, or (after substituting for p_t and simplifying as before):

$$y_t = A_{mt} k_{mt}^{\alpha} \ell_{mt}^{\beta_m} Q_t = \tilde{k}_{mt}^{\alpha} Z_t Q_t, \tag{56}$$

so we can also define $\hat{y}_t = y_t/(Z_t Q_t) = \tilde{k}_{mt}^{\alpha} = \left[\hat{k}_{t-1}/(1+g_t)\right]^{\alpha}$ and $\hat{x}_t = x_t/(Z_t Q_t)$.

We can now characterize the dynamics in terms of stationary variables:

$$\hat{x}_t + \hat{\imath}_t = \left[\hat{k}_{t-1}/(1+g_t)\right]^{\alpha} \tag{57}$$

$$(1+g_t)(1+\nu)\hat{k}_t = z\left((1+g_t)\hat{\imath}_t/\hat{k}_{t-1}\right)\hat{k}_{t-1} + (1-\delta)\hat{k}_{t-1} \tag{58}$$

$$\begin{aligned}(\hat{x}_{t+1}/\hat{x}_t)(1+g_{t+1})(1+\nu)(1+\rho) q_t &= \alpha\left[\hat{k}_t/(1+g_{t+1})\right]^{\alpha-1} + \\ &\quad q_{t+1}\left[z\left((1+g_{t+1})\hat{\imath}_{t+1}/\hat{k}_t\right) + 1 - \delta\right] - (1+g_{t+1})\hat{\imath}_{t+1}/\hat{k}_t\end{aligned} \tag{59}$$

$$q_t = z'\left((1+g_t)\hat{\imath}_t/\hat{k}_{t-1}\right)^{-1} \tag{60}$$

which yields (letting variables without subscripts denote steady state variables, and using the assumption that $z(x) = x$ and $q = 1$ on the balanced growth path)

$$\hat{x} + \hat{\imath} = \left[\hat{k}/(1+g)\right]^{\alpha} \tag{61}$$
$$(1+\nu)(1+g) = \left[\hat{k}/(1+g)\right]^{\alpha-1} - (1+g)\,\hat{x}/\hat{k} + 1 - \delta \tag{62}$$
$$(1+\nu)(1+\rho)(1+g) = \alpha\left[\hat{k}/(1+g)\right]^{\alpha-1} + 1 - \delta \tag{63}$$

and

$$\hat{k} = (1+g)\left[\frac{\alpha}{(1+\nu)(1+\rho)(1+g) - (1-\delta)}\right]^{1/(1-\alpha)}. \tag{64}$$

This of course is just the standard neoclassical model. The innovation in this paper's model is to characterize the behavior of sectoral variables within the aggregate steady state, and also to specify a regime-switching model for the growth process.

3.5 Unbalanced Sectoral Growth

The sectoral variables (other than $\tilde{k}_m$ and $\tilde{k}_h$) will vary over time, but can be solved for directly as functions of the aggregates. Rewriting the relevant conditions in terms of p, n_m, ℓ_m, h, k_m, and c, we have

$$p_t h_t = \tilde{k}_{mt}^{\alpha} Z_t \frac{1-\alpha-\beta_m}{1-\alpha-\beta_h}(1-n_{mt}) \tag{65}$$
$$1 = \omega_c \phi(c_t, h_t)^{-(\epsilon-1)/\epsilon} c_t^{-1/\epsilon} \hat{x}_t Q_t Z_t \tag{66}$$
$$h_t/c_t = p_t^{-\epsilon}(\omega_h/\omega_c)^{\epsilon} \tag{67}$$
$$\ell_{mt} = \frac{\bar{L}}{N_t}\frac{\beta_m(1-\alpha-\beta_h)}{(1-n_{mt})\beta_h(1-\alpha-\beta_m) + n_{mt}\beta_m(1-\alpha-\beta_h)} \tag{68}$$
$$p_t = \frac{A_{mt}}{A_{ht}}\left(\frac{\beta_m}{\beta_h}\right)^{\beta_h}\left(\frac{1-\alpha-\beta_h}{1-\alpha-\beta_m}\right)^{\alpha+\beta_h-1}\left(\frac{\ell_{mt}}{e_t}\right)^{-(\beta_h-\beta_m)}. \tag{69}$$

It will be convenient to have normalized versions of h and c to eliminate Z from the above system. As with k and x, we denote by $\hat{h}$ and $\hat{c}$ the corresponding variables divided by ZQ. Then we have

$$p_t \hat{h}_t = \tilde{k}_{mt}^{\alpha}\frac{1-\alpha-\beta_m}{1-\alpha-\beta_h}(1-n_{mt})/Q_t \tag{70}$$
$$1 = \omega_c \phi\left(\hat{c}_t, \hat{h}_t\right)^{-(\epsilon-1)/\epsilon} \hat{c}_t^{-1/\epsilon}\hat{x}_t \tag{71}$$
$$\hat{h}_t/\hat{c}_t = p_t^{-\epsilon}(\omega_h/\omega_c)^{\epsilon}. \tag{72}$$

We also have

$$\tilde{k}_{ht} = \frac{1-\alpha-\beta_m}{1-\alpha-\beta_h}\tilde{k}_t \tag{73}$$

$$\ell_{ht} = \frac{\beta_h}{\beta_m}\frac{1-\alpha-\beta_m}{1-\alpha-\beta_h}\ell_{mt} \tag{74}$$

The six equations (54) and (68) – (72)provide solutions for the six sectoral variables $\hat{h}$, $\hat{c}$, n_m, ℓ_m, p, and Q as functions of the exogenous variables and the aggregate endogenous variables. Equations (**??**) – (74) along with $n_h = 1 - n_m$ then provide solutions for the remaining sectoral variables. Only in the knife-edge cases of $\epsilon = 1$ or $\gamma_m = \gamma_h - (\beta_h - \beta_m)\,\nu$ will these variables exhibit balanced growth in the sense of either being constant, or growing at the same rate as the aggregate economy.

To get a more intuitive feel for how sectoral labor allocation evolves over time, following Ngai-Pissarides (2007), let σ_h denote the share of expenditure on housing services h relative to total expenditure on goods x, and $\sigma_c = 1 - \sigma_h$ the share of x spent on c:.

$$\sigma_{ht} \equiv \frac{p_t h_t}{x_t} = \left(\frac{\omega_h}{\omega_c}\right)^{\epsilon} p_t^{-(\epsilon-1)} \Big/ \left[1 + \left(\frac{\omega_h}{\omega_c}\right)^{\epsilon} p_t^{-(\epsilon-1)}\right] \tag{75}$$

Then $p_t h_t = \hat{k}_t^{\alpha} Z_t n_{ht} = y_t n_{ht}/Q_t = \sigma_{ht} x_t$, and we have

$$n_{ht} = \sigma_{ht}\frac{x_t Q_t}{y_t} \tag{76}$$

$$n_{mt} = 1 - \sigma_{ht}\frac{x_t Q_t}{y_t} \tag{77}$$

On the balanced growth path x/y is constant, over time we have (for $\epsilon < 1$ and $\gamma_m > \gamma_h$), $\sigma_h \to 1,\ \sigma_m \to 0$. Consequently, in the long-run $n_h \to xQ/y,\ n_m \to 1 - xQ/y$, and eventually all but an infinitesimal amount of labor is going toward producing housing services or structures. This is presumably not a realistic implication, but the model can still be a reasonable description of behavior over a very long time period.

Although land is not explicitly priced in the model, we can compute its shadow rental price v_t in terms of manufactured goods:

$$\begin{aligned} v_t &= \beta_m A_{mt} k_{mt}^{\alpha} \ell_{mt}^{\beta_m - 1} e_t^{1-\alpha-\beta_m} \\ &= \beta_m Z_t \tilde{k}_{mt}^{\alpha} / \ell_{mt} \end{aligned} \tag{78}$$

It will be convenient to define $\hat{v}_t \equiv v_t / (Q_t Z_t N_t)$, so that we have

$$\hat{v}_t = \beta_m \tilde{k}_{mt}^{\alpha} / (\ell_{mt} N_t Q_t) \tag{79}$$

which is expressed as a function of the sectoral variables for which the solution is described above. To a first approximation we can say that the land rental price grows at rate $g + \nu$ on the balanced growth path—exactly $g + \nu$ if $\epsilon = 1$, a bit faster if $\epsilon < 1$.

3.6 Stochastic Growth

We suppose that the growth rate of A_h is fixed at γ_h, but that of A_m follows a Markov regime-switching process:

$$A_{mt}/A_{mt-1} = (1 + \tilde{\gamma}_{mt}) \eta_t / \eta_{t-1} \tag{80}$$

where

$$\tilde{\gamma}_{mt} = \begin{cases} \gamma_m^1 & \text{if } \xi_t = 1 \\ \gamma_m^0 & \quad \xi_t = 0 \end{cases} \tag{81}$$

η_t is a transitory disturbance, and ξ_t is a state variable with Markov transition matrix Θ, where $\Theta[i,j] = \Pr\left(\xi_t = j | \xi_{t-1} = i\right)$. Since the columns of Θ must sum to one, we write it as

$$\Theta = \begin{bmatrix} \theta_1 & 1 - \theta_0 \\ 1 - \theta_1 & \theta_0 \end{bmatrix}. \tag{82}$$

If the diagonal elements of Θ are close to one, the growth states will be highly persistent, and a shift from one state to the other will carry with it a sizeable adjustment in the long-term level of A_m. Since the stationary distribtion of ξ is $\xi^* \equiv \begin{bmatrix} (1-\theta_0)/(2-\theta_1-\theta_0) & (1-\theta_1)/(2-\theta_1-\theta_0) \end{bmatrix}'$, the average growth rate of A_m is

$$\bar{\gamma}_m = \frac{1-\theta_0}{2-\theta_1-\theta_0}\gamma_m^1 + \frac{1-\theta_1}{2-\theta_1-\theta_0}\gamma_m^0. \tag{83}$$

For concreteness we will call $\xi = 1$ the “high-growth” regime, and $\xi = 0$ the "low-growth" regime, i.e. we assume $\gamma_m^1 > \gamma_m^0$.

Following Hamilton (1994), we can describe the above Markov chain as an AR(1) process. First, define

$$z_t = \begin{bmatrix} z_{1t} \\ z_{0t} \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \xi^* & \text{if } \xi_t = 1 \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \xi^* & \text{if } \xi_t = 0 \end{cases}, \tag{84}$$

which has an unconditional mean of zero. Then $z_t = \Theta z_{t-1} + v_t$, where $v_t = z_t - E_{t-1} z_t$, so that $E\left\{v_t | \xi_{t-1}\right\} = 0$. Thus, for example, if $\xi_{t-1} = 1$, we have

$$v_t = \begin{bmatrix} v_{1t} \\ v_{0t} \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 - \theta_1 \\ -(1 - \theta_1) \end{bmatrix} & \text{if } \xi_t = 1 \quad (\theta_1) \\ \begin{bmatrix} -\theta_1 \\ \theta_1 \end{bmatrix} & \text{if } \xi_t = 0 \quad (1 - \theta_1) \end{cases} \tag{85}$$

and if $\xi_{t-1} = 0$.

$$v_t = \begin{bmatrix} v_{1t} \\ v_{0t} \end{bmatrix} = \begin{cases} \begin{bmatrix} \theta_0 \\ -\theta_0 \end{bmatrix} & \text{if } \xi_t = 1 \quad (1 - \theta_0) \\ \begin{bmatrix} -(1 - \theta_0) \\ 1 - \theta_0 \end{bmatrix} & \text{if } \xi_t = 0 \quad (\theta_0) \end{cases} \tag{86}$$

where the terms in parenthesis are conditional probabilities. Note that while $E\left(v_t | \xi_{t-1}\right) = 0$, v_t is not identically distributed over time, as the conditional distribution depends on ξ_{t-1}.

The log deviation version of g_t satisfies

$$\begin{aligned} gg_t / (1 + g) &= \left(\frac{1}{1 + \bar{\gamma}_m} (\tilde{\gamma}_{mt} - \bar{\gamma}_m) + \eta_t - \eta_{t-1} \right) / (1 - \alpha) \\ &= \Gamma_m' z_t / (1 + \bar{\gamma}_m) + \eta_t - \eta_{t-1} / (1 - \alpha) \end{aligned} \tag{87}$$

where $\Gamma_m \equiv \begin{bmatrix} \gamma_m^1 & \gamma_m^0 \end{bmatrix}'$. We suppose that $\eta_t = \phi_1 \eta_{t-1} + \phi_2 \eta_{t-1} + v_t$, where v_t is i.i.d. with a zero mean. In what follows, we will first assume that economic agents observe both z_t and η_t before making their period t decisions. Later we will consider the possibility that they only observe g_t and must estimate z_t and η_t given the history of g_t.

3.7 Asset Prices

Thus far we have only described the behavior of the price of housing services and rental prices for land. The term "housing prices" generally refers to asset prices of homes and the land they are packaged with. In this model we can calculate the value of what might be called "real estate wealth," which would be the total value of capital and land allocated to the housing services sector. The value of the capital is just $K_h = k_h n_h$. The asset value of the land $L_h = \ell_h n_h$ requires some computation. Letting V_t denote the asset price of land in terms of m sector output, we must have

$$V_t = v_t + E_t \left\{ \Phi_{t,1} V_{t+1} \right\} = E_t \left\{ \sum_{\tau=0}^{\infty} \Phi_{t,\tau} v_{t+\tau} \right\} \tag{88}$$

where

$$\Phi_{t,\tau} = \frac{\mu_{m,t+\tau}}{\mu_{mt} \left(1+\nu\right)^{\tau} \left(1+\rho\right)^{\tau}} = \frac{x_t}{x_{t+\tau} \left(1+\nu\right)^{\tau} \left(1+\rho\right)^{\tau}} \tag{89}$$

is the stochastic discount factor. If we normalize the variables by dividing by $Q_t Z_t N_t$, we get

$$\hat{V}_t = \hat{v}_t + E_t \left\{ \hat{\Phi}_{t,1} \hat{V}_{t+1} \right\} = E_t \left\{ \sum_{\tau=0}^{\infty} \hat{\Phi}_{t,\tau} \hat{v}_{t+\tau} \right\}$$

where

$$\hat{\Phi}_{t,\tau} \equiv \frac{\hat{x}_t}{\hat{x}_{t+\tau} \left(1+\rho\right)^{\tau}}. \tag{90}$$

On the balanced growth path we have $\Phi^{-1} = (1+g)(1+\rho)(1+\nu)$, and $\hat{v}_t$, as mentioned previously, is (for plausible parameters) almost constant but technically a function of A_{mt}/A_{ht} and N_t (for $\epsilon < 1$ it is increasing in both arguments). While the capital stock and the economy grow at $g+\nu$, the price of land, and hence the price of "houses" (capital plus land in the h sector) grows at a rate (slightly) faster than $g+\nu$. We will examine the behavior of land prices off the steady state later after describing the model under stochastic growth.

3.8 Calibration

Most of the parameters take on standard values: $\alpha = 0.33$, $\nu = 0.01$, $\delta = 0.05$ (a compromise for structures and equipment). The parameters β_h and β_m should reflect the shares of land in the cost of housing services and non-housing output respectively. We set $\beta_h = 0.5$ and $\beta_m = 0.05$. Since housing services represent about 20 percent of overall consumer expenditures, we set $\omega_h = 0.2$, $\omega_c = 0.8$. We set the time preference rate ρ equal to 0.01. Finally, we choose the parameters of the regime-switching process for productivity to correspond roughly to the results in KR: $\left(\gamma_m^1 - \beta_m \nu\right)/(1-\alpha) = 0.029$, $\gamma_m^0 = 0.013$, $\theta_1 = 0.99$, $\theta_0 = 0.983$.

Thus high growth regimes are slightly more persistent than low-growth, and implied the overall mean growth rate of A_m, $\bar{\gamma}_m$, is 2.31 percent.

3.9 The Elasticity of Substitution between Housing and other Consumption

The first-order conditions of the model imply a relationship between the expenditure ratio for housing and non-housing consumption and the relative price.

$$\frac{\omega_c^\epsilon}{\omega_h^\epsilon} \frac{p_t h_t}{c_t} = p_t^{1-\epsilon} \tag{91}$$

The long-term behavior of aggregate expenditures on housing services suggests a unit income elasticity for such expenditures, but price inelastic (i.e. $\epsilon < 1$). This is because the ratio expenditures on housing services to non-housing consumption expenditures has no long-run trend, but is positively correlated with the relative price of housing services, at least as measured by NIPA. Figure 6 presents annual data going back to 1929 of the two series, which show a positive relationship for most of the sample, though recently (since roughly 1990) they have diverged. The magnitude of the elasticity, however, is difficult to infer from time series data, given that both the ratio and the price are endogenous variables. In addition, whereas the nominal expenditures on h and c may be measured accurately, there may be substantial error in measuring the true relative price. Whereas the Boskin Commission had estimated an upward bias in CPI rents, Gordon and and vanGoethem (2005) argue for a downward bias averaging roughly 0.5 percent annually, but varying over time.

As a consequence, we instead examine evidence from micro data. Specifically, we examine data from the Consumer Expenditure Survey (CEX) to gauge the extent of housing service expenditure share variability as a function the relative price of housing services. To do so we construct rent (or owner's equivalent rent) relative to other expenditures, and match this up with data on housing prices by region, total expenditures, and demographic controls. As the above condition suggests, we can obtain an estimate of ϵ by a suitable regression of the nominal expenditure ratio on relative price.

Looking at micro data solves several problems. First, it is reasonable to take the price, which is based on regional CPI measures of owner's equivalent rent relative to the CPI excluding shelter, as exogenous to individual households. Second, the price measurement issue alluded to above is arguably mitigated by reliance on relative prices across regions. Third, we can include specific demographic controls to account for variation in, for example, ω_c/ω_h.

Consider the following model for individual i at date t in region j:

$$\ln\left[p_{jt}h_{it}/\left(x_{it}-p_{jt}h_{it}\right)\right]=a_j+b\ln x_{it}+(1-\epsilon)\ln p_{jt}+z_{it}'\theta+u_{itj}.$$

Here a_j reflects some constant region-specific factor that might affect expenditure shares. For example, if living in a region provides some amenities such as inexpensive public transportation or moderate weather that substitute for other expenditures (automobiles, heating oil), a_j might be positive. Note that we have to assume that a_j is constant over time or else we would not be able to identify the price effect. The coefficient b would reflect a wealth effect to the extent it differs from zero, and the coefficient on p_{jt} has the interpretation indicated: if $\epsilon < 1$, relative expenditures on housing services increase with their cost. The model also includes a set of demographic variables z_{it}, which would include things like family size and number of wage-earners. Note that we would expect a negative coefficient on an indicator of two adults working, as this would typically result in less household production and more expenditures on non-housing goods and services. Finally, the error term u_{ijt} represents idiosyncratic variation in preferences for housing services (in the model represented by ω_h/ω_c), measurement error of the dependent variable, and other omitted variables.

If the CEX had a true panel structure we could difference out the a_j. We could also allow for individual fixed effects. Unfortunately each individual household observation is present for at most four quarterly observations, so the panel aspect is probably useless (the other explanatory variables are not likely to change in a meaningful way over the course of a household's participation). Consequently we choose to pool the sample in levels and use regional dummies to capture the region effects. We also just use one (the first) observation per household, to avoid giving more weight to households simply because they remain in the sample.

The total expenditure variable x_{it} is likely to be measured with error. Such error would tend to produce a potentially large downward bias in the estimate of b because x_{it} enters the denominator of the dependent variable. Fortunately we have candidates for instruments: demographic variables such as race and education that are plausibly unrelated to the u_{itj}.

For nominal expenditures on housing services we use rent for renters and owner's equivalent rent (OER) on the primary residence for owners.[3] Consequently we subtract mortgage and home equity interest from total expenditures, as well as property taxes and expenditures on various categories of home repairs and maintenance that would normally be included in rent. We assume that OER is not intended to include utilities, so for owners utility expenditures are included in total but not housing expenditures. For renters,

[3] We do not include rent or OER on vacation or second homes. Though it might be desirable to do so, especially for second homes, it is difficult to distinguish, for example, someone who owns a vacation home but rents it out for 48 weeks of the year and uses it for four weeks, on the one hand, from someone who uses it every weekend. The former should be treated as equivalent to someone who rents a vacation home for four weeks, which we regard as something qualitatively different from housing services.

reported utility expenditures are also included only in total expenditures, but a dummy variable is included for renters that report zero utilities expenditures (on the assumption that they are included in rent).

The results of this estimation exercise are shown in Table 1. Because of an unexplained break in the level of the dependent variable that occurs in 1993 (presumably because of some change in variable definitions), all of our estimates included a dummy variable for post-1993 data. We present results with and without constraining $b = 0$, and with and without instrumenting for x. Without instrumenting for x, the estimate of b is an implausibly large negative number, and the estimate of ϵ is only 0.134. Instrumenting brings $\hat{b}$ down to a more reasonable -0.254, and increases $\hat{\epsilon}$ to around 0.2. Imposing $b = 0$ results in $\hat{\epsilon} = 0.284$.

The fact that the estimates of ϵ are relatively insensitive to the treatment of other variables in the equation provides some reassurance that it is well below one. This is consistent with a long history of studies that find housing demand to have a low price elasticity (e.g. Hanushek and Quigley, 1980; Polinsky and Ellwood, 1979), or other estimates of ϵ based on micro data. For example, Flavin and Nakagawa (2004) use PSID data and estimate ϵ to be 0.13. Using a different methodology based on asset prices, Lustig and Van Niewerburgh (2007) set $\epsilon = 0.05$ in their calibration. The significant negative estimate of b is problematic for the assumption of homotheticity, but is hard to square with the relatively flat expenditure share in Figure 6. If true it would tend to mute the effects described in the simulation, as growth in the demand for housing services would not be as responsive to expected productivity growth. For this reason we will be conservative in our choice of ϵ and set it equal to 0.3 for our benchmark case.

3.10 Model Simulations

The model separates conveniently into its dynamic aggregate component, which is essentially the neoclassical growth model, and the sectoral variables, which do not have a simple steady state representation, but are static functions of the aggregate state variables. Thus we can use standard methods (e.g. Uhlig, 1997) to obtain a solution for the linearized aggregate model of the form

$$\begin{aligned} \hat{k}_t &= \mathbf{P}\hat{k}_{t-1} + \mathbf{Q}\Lambda_t \\ \begin{bmatrix} \hat{x}_t \\ \hat{\imath}_t \\ q_t \end{bmatrix} &= \mathbf{R}\hat{k}_{t-1} + \mathbf{S}\Lambda_t \\ \Lambda_t &= N\Lambda_{t-1} + \Xi_t. \end{aligned}$$

where

$$\Lambda_t = \begin{bmatrix} z_{1t} & z_{0t} & \hat{\eta}_t & \hat{\eta}_{t-1} \end{bmatrix}'$$

$$\mathbf{N} = \begin{bmatrix} \theta_1 & 1-\theta_0 & 0 & 0 \\ 1-\theta_1 & \theta_0 & 0 & 0 \\ 0 & 0 & \phi_1 & \phi_2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\Xi_t = \begin{bmatrix} v_{1t} \\ v_{2t} \\ u_t \\ 0 \end{bmatrix}$$

and where $\hat{k}_t$, $\hat{x}_t$, etc. are now logarithmic deviations from their steady state values. In this case with only one state variable, $\mathbf{P}$ is a scalar, $\mathbf{R}$ is 3×1, $\mathbf{Q}$ is 1×4 and $\mathbf{S}$ is 3×4. It should be noted that parameters related to the sectoral dimension of the model do not enter into this part of the problem. These include β_h, ϵ, γ_h, ω_c, and ω_h.

For given realizations of the exogenous disturbances, time paths for the aggregates can be computed, they can be converted back to levels and become inputs to solving the for the sectoral variables (e.g. p, n_m, n_h, etc.) period-by-period using (68) – (74) and the definition of Q (54).

The key to doing interesting simulations is to take the peculiar error structure of the disturbance process into account. Even though the conditional expectation of the errors in the z_t process (the regime states) is zero, actual realizations of zero are not possible, and in fact given the values of q_1 and q_0, a small error (of absolute value $1 - q_0$ or $1 - q_1$) that leaves z unchanged is highly likely in any given time period. So rather than consider a one-time shock to v, it makes sense to consider a single large shock (a regime-switch) followed by a sequence of identical small shocks that leave the regime unchanged for an extended period of time. Such a path is more like a modal outcome rather than the improbable mean.

Figure 7 gives an example of this type of simulation. The economy is in the low growth regime in periods 1 to 11, and then switches to the high growth regime, where it remains. The figure plots the behavior of the asset price of a house (a fixed-weight combination of capital and land—see the Appendix) against per capita income, for $\epsilon = 0.3$ and 0.9. House prices are clearly much more responsive, both at impact and during the regimes, in the $\epsilon = 0.3$ case. When the regime shift occurs, the price jumps about 5 percent if $\epsilon = 0.3$ versus around 2 percent if $\epsilon = 0.9$. During the high-growth regime the growth rate of the price is about 0.5 percent (annualized) faster if $\epsilon = 0.3$ versus $\epsilon = 0.9$. Prices actually accelerate as long as the economy remains in the high-growth regime, the more so the lower is ϵ.

3.11 Regime Uncertainty

The model solution and simulations above assume perfect knowledge about the growth regime. This is unlikely to be a realistic assumption. Fortunately, the framework in KR provides a natural mechanism for extracting what economic agents know about the growth regime from the behavior of other economic variables. It is a dynamic factor model with Markov regime-switching in the stochastic growth component. Suppose a set of J economic series related to each other as follows:

$$Y_{it} = \mu_i X_t + \lambda_i x_t + \zeta_{it}$$

where X_t is a common permanent component, x_t a common transitory component, and ζ_{it} an idiosyncratic component. Assume

$$\begin{aligned} (1-\phi(L))X_t &= \mu(S_t) + \upsilon_t \\ (1-\theta(L))x_t &= \epsilon_t \\ (1-\psi(L))\zeta_{it} &= \eta_{it} \end{aligned}$$

and

$$\mu(S_t) = \begin{cases} \gamma_1 & \text{if } S_t = 1 \\ \gamma_0 & \quad S_t = 0 \end{cases}.$$

The $\{\mu_i\}$ and $\{\lambda_i\}$are factor loadings on the permanent and transitory components respectively, and the variances of υ and ϵ are normalized to unity.

Figure 8 provides an updated estimate of the so-called smoothed (incorporating all available data through 2007:Q3) and zero-lag (incorporating data for each observation only up to that date) estimates of regime probabilities, incorporating information about the growth regime from data on productivity, labor compensation, aggregate consumption, and aggregate hours of work. The zero-lag estimates provide a one basis for what economic agents might have thought at the time. Note in particular the recent signs of a shift back to the low-growth regime, roughly coincident with the sudden end of the housing boom.

As an extreme benchmark, we can simulate the model under the assumption of perfect information, i.e. that agents actually observe the regime shifts and level shocks η in real time. Figure 9 illustrates the behavior of housing prices according to the model under this extreme assumption, where the simulation assumes, based on Figure 8, that regime shifts occurred in 1973, 1996, and 2004. As suggested by the discussion in the previous section, a regime shift triggers both a level and growth rate change in housing prices. Transitory shocks have relatively little impact since they are not confused with permanent shocks. Note that for these and subsequent simulations, because the linear trend depends on the unobservable γ_h, we just assume a value for γ_h such that the trends line up exactly, and judge the model by its ability to match deviations from the linear trend. By that standard, the perfect information assumption clearly misses on both timing and amplitude.

There are, however, two potential sources of incomplete information. First, agents cannot disentangle η_t, the persistent but transitory disturbance to A_{mt}, from v_t, the regime-dependent error. This is a standard inference problem addressed by Hamilton (1994) and others. Agents observe A_{mt} and update their assessment of the current regime. This sort of rational expectations updating makes sense in a stationary environment where agents know (or have had sufficient time to learn) the underlying parameters of the stochastic processes. But in the early 1970s there was no experience in the United States with a sustained productivity slowdown other than during the Great Depression. To believe that the low productivity growth beginning in 1973 was a change in the trend would have required vivid imagination. Other than considering the experience of other countries, there would have been no realistic way form estimates of either the alternative mean growth rate or the transition probabilities.

One way to approach this is to suppose that expectations are formed by estimating the regime-switching dynamic factor model in "real time"—that is, based on data only available at each point in time.[4] Kahn and Rich (2006) do this for the 1990's and find that the model detects the productivity boom relatively quickly, within a year or two of when, with hindsight, it apparently began. Here we adopt a similar approach for the entire period for which the housing price series is available. While estimation of a regime-switching model when there is no regime switch (yet) in the data is potentially problematic, as some parameters are not identified, the point in real time at which the estimation begins to detect a regime switch provides a plausible mechanism for dating when economic agents might have done so (apart from the fact that the econometric techniques had not been developed yet!) It turns out that rolling estimation of the model annually through the 1970s confirms that it fails even to detect a second regime until around 1978, and only puts high probability on it in 1979 (see Figure 10a). After that the estimation converges quickly to close to the full sample estimates. It is worth noting that 1979 is also the year when a number of studies about "the productivity slowdown" began to appear (e.g. Denison, 1979; Norsworthy et al., 1979), though even these were primarily retrospective and did not take clear stands on how likely the slowdown was to persist. Edge et al (2007) also report that official forecasts of long-run productivity growth drifted down slowly in the 1970s, but remained above 2 percent through 1978, and then plummeted in 1979 to 1.5 percent.

An additional complication is illustrated by Figure 10b. Whereas the KR model estimated with data

[4] Edge et al (2007) consider a similar question in a linear context using a Kalman filter.

available as of the end of July 2007 show the current productivity slowdown beginning (with high probability) in 2004, data available prior to July 2007 had not indicated a slowdown with any substantial probability. Even as of June 2007 the low-growth regime probabilities were in the vicinity of 0.1. What happened was that benchmark revisions that came out in July revised productivity growth downward over the previous three years. So in addition to the difficulties presented by learning the parameters of the process, and by—given the parameters—assessing the regime probabilities in real time as the data arrive, there is the third difficulty that the data are regularly revised.

The general equilibrium model in this paper does not have the complexity that would allow the methods of KR to be applied or simulated directly; in particular, there is only one type of shock, so there would be no point to dynamic factor analysis. To get the spirit of that analysis, while maintaining the simplicity of the model, we simulate the productivity process as in (80) – (87), imposing regime switches that correspond to those found in the data, i.e. in 1973, 1996, and 2004. The simulated data are then used as inputs to estimation of the parameters of the process along with regime probabilities as of each date. We set $\phi_1 = 1.4$ and $\phi_2 = -0.5$, and the standard deviation of the shock to the η process σ_v is set to 0.4. The idea is to have a process that when estimated in the full sample generates estimated regime probabilities similar to those of Figure 9.

We then simulate both types of incomplete information. Under the more conventional rational expectations assumption, we assume agents know the parameters of the model, but cannot distinguish between level and growth rate shocks to the productivity process. Under learning, we assume agents estimate the parameters of the model at each date in real time, and use the estimated state vector and current parameter estimates to forecast the model going forward.

More formally, the idea here is to assume that agents observe $A_{mt} = A_{mt-1}\left(1+\tilde{\gamma}_{mt}\right)\eta_t/\eta_{t-1}$, but whereas the true process, and accordingly the values of $\tilde{\gamma}_{mt}$ and η_t, are as described earlier, agents instead have time-varying estimates of the parameters $\hat{\gamma}^1_{mt}$, $\hat{\gamma}^0_{mt}$, $\hat{\theta}_{1t}$, $\hat{\theta}_{0t}$, probabilities $\hat{\pi}_{t|t} = \Pr\left(\xi_t = 1|\Omega_t\right)$, and transitory terms $\hat{\eta}_{s|t} = E\left(\eta_s|\Omega_t\right)$ where Ω_t represents data observed through period t. (We assume only that they know the production function parameters α, β_h, and β_m, and the parameters of the utility function.) They obtain estimates of the parameters by using maximum likelihood (or, rather, an approximation[5] based on Kim, 1994). At each date they update their estimates upon observing the latest realization of A_{mt}, and revise their expectations of future exogenous and endogenous variables accordingly. In linearized form, we have, for example

$$gg_t/\left(1+g\right) = \left[\hat{\Gamma}'_m\hat{z}_t/\left(1+\bar{\gamma}_{mt}\right) + \hat{\eta}_{t|t} - \hat{\eta}_{t-1|t}\right]/\left(1-\alpha\right)$$

[5] The approximation arises because the inferences about η_t and η_{t-1} should involve expectations conditional on not only the current regime, but on the whole history of regimes. Instead these are collapsed into single unconditional values.

where a variable such as $\hat{\eta}_{t-1|t}$ is the value at time $t-1$ estimated with data through time t, and

$$\bar{\gamma}_{mt} = \frac{1-\hat{\theta}_{0t}}{2-\hat{\theta}_{1t}-\hat{\theta}_{0t}}\hat{\gamma}^{1}_{mt} + \frac{1-\hat{\theta}_{1t}}{2-\hat{\theta}_{1t}-\hat{\theta}_{0t}}\hat{\gamma}^{0}_{mt}.$$

In other words, by construction agents' estimates of the various components of g_t add up to the actual g_t, which is observed, but have different implications for expectations compared to the complete information case where the parameters are known and the shocks observed. (See the Appendix.)

Figure 11 provides the results of a model simulation of housing prices under the real-time learning assumption, in comparison with the Census and OFHEO indexes. All series are detrended, since the model does not have any prediction about the price trend. Transitory shocks are derived from those estimated by the KR model. While the model misses on the full amplitude of the price fluctuations, it gets most of it. With regard to timing, the tends to anticipate the actual price increases, especially in the 1960s, but gets the timing of the peaks and subsequent declines reasonably well. Overall, considering the model's parsimony, and the fact that key parameters are calibrated to micro data or expenditure shares, and has essentially only one shock, it fits the data very well.

Another check on the model should be its predictions about investment. Clearly the lack of adjustment costs at the sectoral level will mean that sectoral investment will be excessively volatile. With relative price changes, the model makes it too easy to flip capital from one sector to the other. Nonetheless we can still examine the model's predictions and focus on the lower frequency behavior of investment. Figure 12 displays the behavior of residential investment (defined as the change in residential capital), along with actual series. Both series are detrended, the model's by the steady state growth trend, and the data by a Hodrick-Prescott trend. The model overpredicts residential investment in the early part of the sample, perhaps because of the unusual demographics arising from the baby boom: There had been an explosion of family formation and residential investment following World War II, and the offspring in those families had for the most part not yet come of age. This may have held down residential investment in the 1960s and early 1970s, and boosted it in the late 1970s and 1980s. Nonetheless the model gets the medium and low frequency behavior of residential investment almost dead on from the early 1970s until the present, especially the boom and bust over the last decade. Again, while the share parameters would get the average level of the series approximately right, the medium frequency movements are largely dictated by the calibration of ϵ.

4 Conclusions

This paper has developed a growth model with land, housing services, and other goods and shown that it is capable both qualitatively and quantitatively of explaining a substantial portion of the movements in housing prices over the past 40 years, including the recent downturn. The paper also uses micro data to calibrate a key cross-elasticity parameter that governs the relationship between productivity growth and home price appreciation. The matching of the model to the data relies not on fitting the overall trend (which depends on an unobservable), but on the deviations from that trend as a function of productivity growth. The calibrated model under rational expectations can explain some of the acceleration in housing prices that occurred both in the 1960s and since the mid-1990s, and also suggests a contributing explanation for the recent downturn, but fails to get the full amplitude and timing of the fluctuations. When a realistic model of learning is added in lieu of rational expectations, however, the model does much better on both. In particular, the continued boom in housing prices in the 1970s is largely explained by the time it took for agents to figure out that the productivity slowdown was quasi-permanent. Finally, the paper also has some success in matching the low frequency behavior of housing investment, in particular the boom that began in the late 1990s.

Considering the general lack of success that DSGE models have had in capturing asset price behavior (e.g. Jermann, 1998), and considering that the model has only one driving force, exogenous productivity growth, it is reasonable to call it a success. Even so, future work will incorporate adjustment costs, and other sources of shocks to capture more accurately the high frequency behavior of both prices and quantities.

Another implication of this analysis is that it puts the recent subprime mortgage crisis in a somewhat different light. If we accept the idea that the housing downturn was triggred by a productivity slowdown, and that this was a low probability event in light of what was known prior to 2007 about productivity growth, it may be that subprime mortgage lending was, ex ante, economically rational. The probability of a productivity slowdown—and therefore of a reversal in housing prices—such as the one that occurred in 1973 could reasonably have been viewed as small over the relevant horizon for mortgage lenders.. Indeed, a recent paper by Piskorski and Tchistyi (2008) makes precisely this point. They examine optimal mortgage lending in a setting where housing prices obey essentially the same type of regime-switching behavior, and find that "many features of subprime lending observed in practice are consistent with economic efficiency and rationality of both borrowers and lenders," though, as they point out, there may be negative externalities associated with massive defaults in a downturn.

5 Appendix

5.1 Solving the Closed Economy Model under Complete Information

First we linearize the system

$$\hat{x}_t + \hat{\imath}_t = \left[\hat{k}_{t-1}/\left(1+g_t\right)\right]^{\alpha} \tag{92}$$

$$\left(1+g_t\right)\left(1+\nu\right)\hat{k}_t = z\left(\left(1+g_t\right)\hat{\imath}_t/\hat{k}_{t-1}\right)\hat{k}_{t-1} + \left(1-\delta\right)\hat{k}_{t-1} \tag{93}$$

$$\begin{aligned}\left(1+\nu\right)\left(1+\rho\right)q_t = E_t\Big\{&\left(\hat{x}_t/\hat{x}_{t+1}\right)\left(1+g_{t+1}\right)^{-1}\Big[\alpha\left[\hat{k}_t/\left(1+g_{t+1}\right)\right]^{\alpha-1}+\\ &q_{t+1}\left[z\left(\left(1+g_{t+1}\right)\hat{\imath}_{t+1}/\hat{k}_t\right)+1-\delta\right]-\left(1+g_{t+1}\right)\hat{\imath}_{t+1}/\hat{k}_t\Big]\Big\}\end{aligned} \tag{94}$$

$$q_t = z'\left(\left(1+g_t\right)\hat{\imath}_t/\hat{k}_{t-1}\right)^{-1} \tag{95}$$

around the steady state values $\hat{k}$ and $\hat{x}$. After some rearranging, and letting $R \equiv \alpha\left[\hat{k}/\left(1+g\right)\right]^{\alpha-1}+1-\delta = \left(1+\rho\right)\left(1+\nu\right)\left(1+g\right)$, the linearized versions of the four equations can be expressed as

$$\left(1+g\right)\left(1+\nu\right)\hat{k}_t = R\left(\hat{k}_{t-1} - gg_t/\left(1+g\right)\right) - \hat{x}_t\left[\left(1+g\right)\hat{x}/\hat{k}\right] \tag{96}$$

$$\hat{x}_t\left[\left(1+g\right)\hat{x}/\hat{k}\right] + \hat{\imath}_t\left[\left(1+g\right)\hat{\imath}/\hat{k}\right] = \left[R-\left(1-\delta\right)\right]\left(\hat{k}_{t-1} - gg_t/\left(1+g\right)\right) \tag{97}$$

$$Rq_t = E_t\left\{\left[\hat{x}_t - \hat{x}_{t+1} - \frac{gg_{t+1}}{1+g}\right]R\right. \tag{98}$$

$$\left. -\alpha\left(1-\alpha\right)\left(\frac{\hat{k}}{1+g}\right)^{\alpha-1}\left(\hat{k}_t - \frac{gg_{t+1}}{1+g}\right) + q_{t+1}\left(\left(1+g\right)\hat{\imath}/\hat{k}+1-\delta\right)\right\} \tag{99}$$

$$0 = q_t + z''\left(\left(1+g\right)\hat{\imath}/\hat{k}\right)\left(\frac{gg_t}{1+g} + \hat{\imath}_t - \hat{k}_{t-1}\right) \tag{100}$$

Note that

$$\begin{aligned}\alpha\left(\frac{\hat{k}}{1+g}\right)^{\alpha-1}+1-\delta &= \left(1+\rho\right)\left(1+\nu\right)\left(1+g\right)\\ \left(1+g\right)\hat{x}/\hat{k} &= \left(\frac{\hat{k}}{1+g}\right)^{\alpha-1}+1-\delta-\left(1+\nu\right)\left(1+g\right)\end{aligned}$$

So

$$\left(\frac{\hat{k}}{1+g}\right)^{\alpha-1} = \frac{\left(1+\rho\right)\left(1+\nu\right)\left(1+g\right)-\left(1-\delta\right)}{\alpha}$$

This system is of the form

$$
\begin{aligned}
0 &= \mathbf{A}\hat{k}_t + \mathbf{B}\hat{k}_{t-1} + \mathbf{C}\begin{bmatrix} \hat{x}_t \\ \hat{\imath}_t \\ q_t \end{bmatrix} + \mathbf{D}\Lambda_t && (101) \\
0 &= E_t\left\{\mathbf{F}\hat{k}_{t+1} + \mathbf{G}\hat{k}_t + \mathbf{H}\hat{k}_{t-1} + \mathbf{J}\begin{bmatrix} \hat{x}_{t+1} \\ \hat{\imath}_{t+1} \\ q_{t+1} \end{bmatrix} + \mathbf{K}\begin{bmatrix} \hat{x}_t \\ \hat{\imath}_t \\ q_t \end{bmatrix} + \mathbf{L}\Lambda_{t+1} + \mathbf{M}\Lambda_t\right\} && (102) \\
\Lambda_{t+1} &= \mathbf{N}\Lambda_t + \Xi_{t+1} && (103)
\end{aligned}
$$

where

$$
\Lambda_t = \begin{bmatrix} z_{1t} & z_{0t} & \hat{\eta}_t & \hat{\eta}_{t-1} \end{bmatrix}'
$$

$$
\begin{aligned}
0 &= (1+g)(1+\nu)\hat{k}_t - R\left(\hat{k}_{t-1} - gg_t/(1+g)\right) + \hat{x}_t\left[(1+g)\hat{x}/\hat{k}\right] && (104) \\
0 &= \hat{x}_t\left[(1+g)\hat{x}/\hat{k}\right] + \hat{\imath}_t\left[(1+g)\hat{\imath}/\hat{k}\right] - [R-(1-\delta)]\left(\hat{k}_{t-1} - gg_t/(1+g)\right) && (105) \\
0 &= E_t\left\{\left[\hat{x}_t - \hat{x}_{t+1} - \frac{gg_{t+1}}{1+g}\right]R\right. && (106) \\
&\quad \left. -\alpha(1-\alpha)\left(\frac{\hat{k}}{1+g}\right)^{\alpha-1}\left(\hat{k}_t - \frac{gg_{t+1}}{1+g}\right) + q_{t+1}\left((1+g)\hat{\imath}/\hat{k} + 1 - \delta\right)\right\} - Rq_t && (107) \\
0 &= q_t + z''\left((1+g)\hat{\imath}/\hat{k}\right)\left(\frac{gg_t}{1+g} + \hat{\imath}_t - \hat{k}_{t-1}\right) && (108)
\end{aligned}
$$

and

$$\mathbf{A} = \begin{bmatrix} (1+\nu)(1+g) \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} -R \\ -[R-(1-\delta)] \\ -z''\left((1+g)\hat{\imath}/\hat{k}\right) \end{bmatrix} \tag{109}$$

$$\mathbf{C} = \begin{bmatrix} (1+g)\hat{x}/\hat{k} & 0 & 0 \\ (1+g)\hat{x}/\hat{k} & (1+g)\hat{\imath}/\hat{k} & 0 \\ 0 & z''\left((1+g)\hat{\imath}/\hat{k}\right) & 1 \end{bmatrix} \tag{110}$$

$$\mathbf{D} = \begin{bmatrix} R\frac{1}{1-\alpha} \\ [R-(1-\delta)]\frac{1}{1-\alpha} \\ z''\left((1+g)\hat{\imath}/\hat{k}\right)\frac{1}{1-\alpha} \end{bmatrix} \begin{bmatrix} \frac{\gamma_m^1}{1+\bar{\gamma}_m} & \frac{\gamma_m^0}{1+\bar{\gamma}_m} & 1 & -1 \end{bmatrix} \tag{111}$$

$$\begin{aligned}
\mathbf{F} &= 0 \\
\mathbf{G} &= -\alpha(1-\alpha)\left(\frac{\hat{k}}{1+g}\right)^{\alpha-1} \\
\mathbf{H} &= 0 \\
\mathbf{J} &= \begin{bmatrix} -R & 0 & (1+g)\hat{\imath}/\hat{k}+1-\delta \end{bmatrix} \\
\mathbf{K} &= \begin{matrix} R & 0 & -R \end{matrix} \\
\mathbf{L} &= -\left[\alpha^2\left(\frac{\hat{k}}{1+g}\right)^{\alpha-1}+1-\delta\right]\frac{1}{1-\alpha}\begin{bmatrix} \frac{\gamma_m^1}{1+\bar{\gamma}_m} & \frac{\gamma_m^0}{1+\bar{\gamma}_m} & 1 & -1 \end{bmatrix} \\
\mathbf{M} &= 0
\end{aligned}$$

$$\mathbf{N} = \begin{bmatrix} \theta_1 & 1-\theta_0 & 0 & 0 \\ 1-\theta_1 & \theta_0 & 0 & 0 \\ 0 & 0 & \phi_1 & \phi_2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\Xi_t = \begin{bmatrix} v_{1t} \\ v_{2t} \\ u_t \\ 0 \end{bmatrix}$$

where v_{1t} and v_{2t} are as defined earlier. We can then use the method of undetermined coefficients outlined by Uhlig (1997) to find the solution of the model in the form

$$\begin{aligned} \hat{k}_t &= \mathbf{P}\hat{k}_{t-1} + \mathbf{Q}\Lambda_t \\ \begin{bmatrix} \hat{x}_t \\ \hat{\imath}_t \\ q_t \end{bmatrix} &= \mathbf{R}\hat{k}_{t-1} + \mathbf{S}\Lambda_t \\ \Lambda_t &= N\Lambda_{t-1} + \Xi_t. \end{aligned}$$

where in this case, of course, $\mathbf{P}$ and $\mathbf{R}$ are scalars.

To solve for the work effort e_t and sectoral variables, which we can denote by the vector

$$\hat{\boldsymbol{s}}_t = \left(e_t, \hat{h}_t, \hat{c}_t, n_{mt}, n_{ht}, \ell_{mt}, \ell_{ht}, p_t, \hat{q}_t, Q_t, \tilde{k}_{mt}, \tilde{k}_{ht}\right),$$

we pick initial values for A_m, A_h, and N, which then evolve exogenously. We start with the steady state levels $\hat{k}$ and $\hat{x}$, which do not depend on the levels of the exogenous variables. We then construct an entire path of $\left\{\hat{k}_t, \hat{x}_t, \Lambda_t\right\}$ from the above solution, exponentiated and multipled by the steady state levels. These can be used first to construct at each t

$$\hat{s}_t = \Psi\left(\hat{k}_{t-1}, \Lambda_t; A_{m0}/A_{h0}, N_0\right).$$

Finally, we compute the path of $Q_t Z_t$, which depends on A_{mt}, ℓ_{mt}, and n_{mt}. From that we can compute the path of the non-normalized variables, multiplying the "^" variables by $Q_t Z_t$ and the "~" variables by Z_t.

The nonlinearity of Ψ implies that in general the sectoral variables (except for $\tilde{k}_m$ and $\tilde{k}_h$, which are linear in $\hat{k}$) do not grow at a constant rate along the balanced growth path. This does not present problems for computing the time paths of these variables, but it does make computing something like V, the price of land, considerably more difficult—mainly because the levels of A_m/A_h and N have nonlinear effects on V.

We have

$$\hat{V}_t = \hat{q}_t + E_t\left\{\hat{\Phi}_{t,1}\hat{V}_{t+1}\right\} = E_t\left\{\sum_{\tau=0}^{\infty}\hat{\Phi}_{t,\tau}\hat{q}_{t+\tau}\right\}$$

where

$$\hat{\Phi}_{t,\tau} \equiv \frac{\hat{x}_t}{\hat{x}_{t+\tau}(1+\rho)^{\tau}} \tag{112}$$

$$\hat{q}_t = \beta_m\hat{k}_{mt}^{\alpha}/\left(\ell_{mt}Q_t\right). \tag{113}$$

What about the price of a "house"? The total value of land and capital in housing services is $V_tL_{ht} + K_{ht}$. Given a path $\{K_{ht}, L_{ht}\}$, we can define a "constant-quality" house price index P_{ht} by choosing a base year, say $t = 0$, and defining $P_{ht} = 100\left(V_tL_{h0} + K_{h0}\right)/\left(V_0L_{h0} + K_{h0}\right)$.

5.2 Learning

Now the system describing the equilibrium is a time-dependent version of what we had under complete information:

$$0 = \mathbf{A}_t\hat{k}_t + \mathbf{B}_t\hat{k}_{t-1} + \mathbf{C}_t\hat{x}_t + \mathbf{D}_t\Lambda_t \tag{114}$$

$$0 = E_t\left\{\mathbf{F}_t\hat{k}_{t+1} + \mathbf{G}_t\hat{k}_t + \mathbf{H}_t\hat{k}_{t-1} + \mathbf{J}_t\hat{x}_{t+1} + \mathbf{K}_t\hat{x}_t + \mathbf{L}_t\Lambda_{t+1} + \mathbf{M}_t\Lambda_t\right\}. \tag{115}$$

The exogenous state vector Λ_t is reassessed at each date according to the procedure described in the text. The solution will be similarly time-dependent:

$$\hat{k}_t = \mathbf{P}_t\hat{k}_{t-1} + \mathbf{Q}_t\Lambda_t$$

$$\hat{x}_t = \mathbf{R}_t\hat{k}_{t-1} + \mathbf{S}_t\Lambda_t.$$

As with many learning models, agents do not take into account the fact that there is parameter uncertainty and that their beliefs about the parameter values and shocks will change over time.

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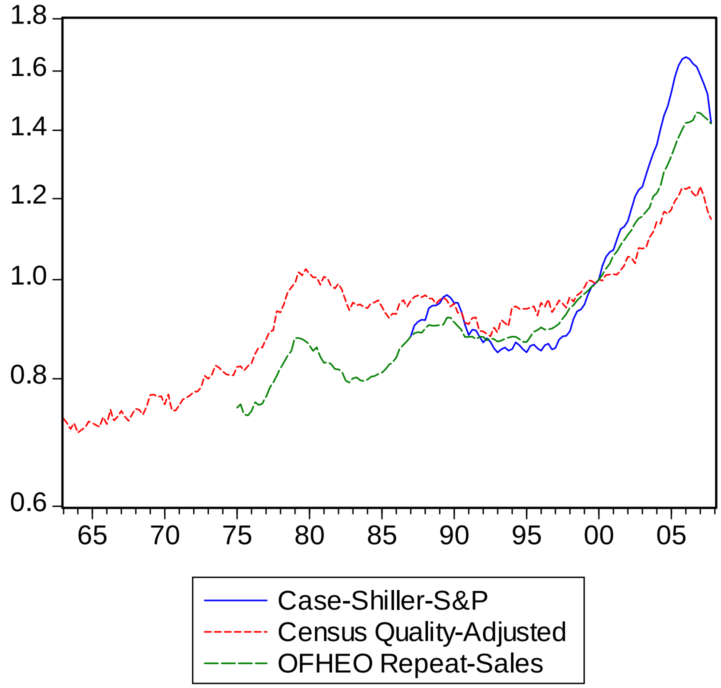
Figure 1: Alternative Home Price Indexes (Inflation-Adjusted)



Note: Logarithmic scale, 2000:Q1 = 1.00

Table 1: Parameter Estimates from CEX Household Data

Parameter			
$\hat{\epsilon}$	0.134	0.195	0.284
	(0.042)	(0.046)	(0.052)
$\hat{b}$	-0.743	-0.254	—
	(0.003)	(0.009)	
Instruments for x	N	Y	—
R^2	0.575	0.464	0.317

Figure 2: Ratio of Housing Wealth to Consumption

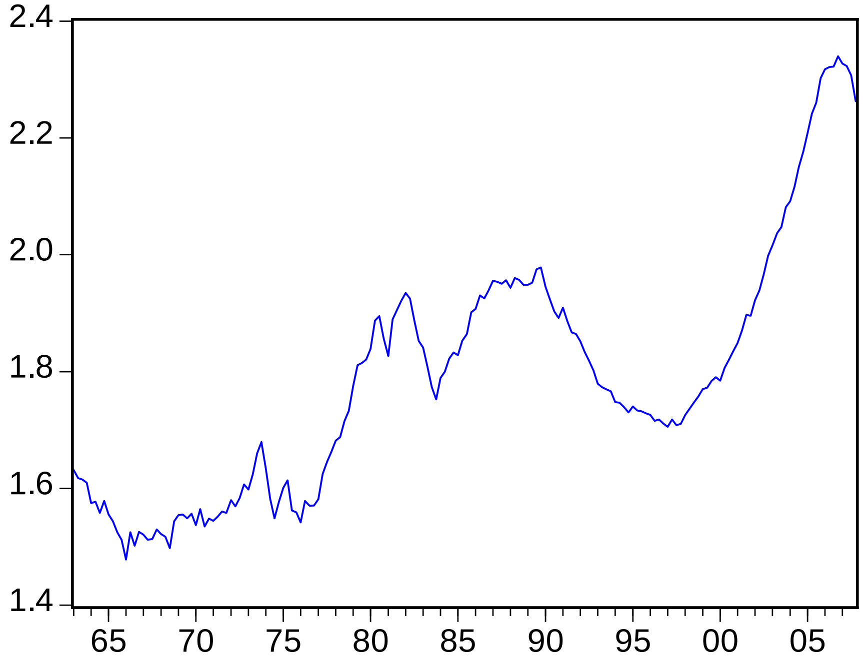


Figure 3: Ratio of Housing Wealth to Total Net Worth

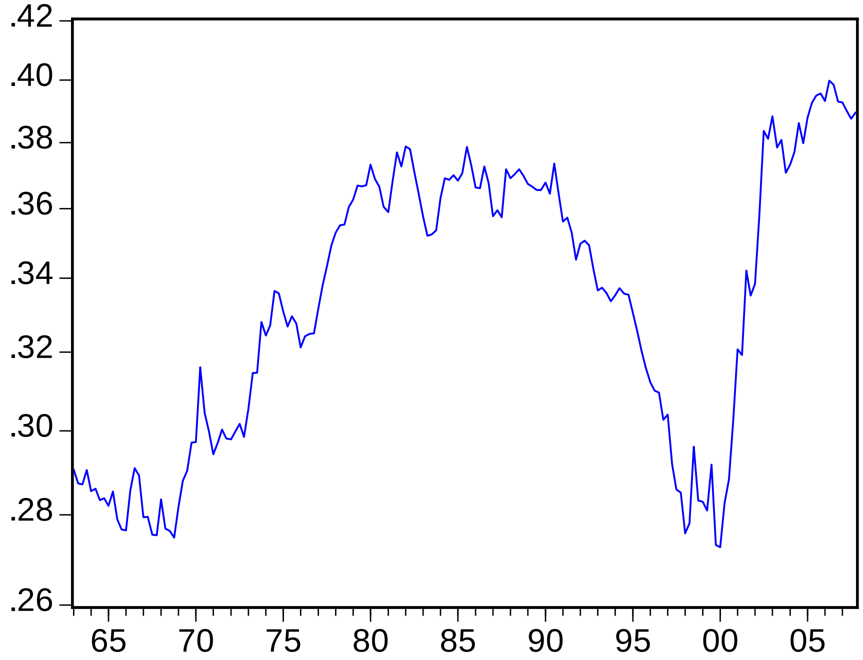
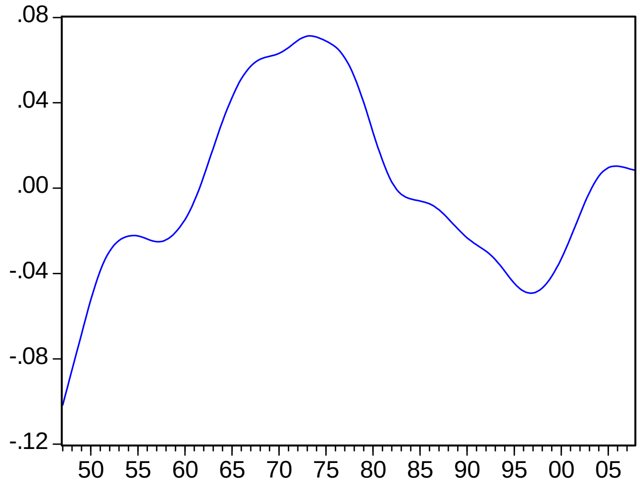
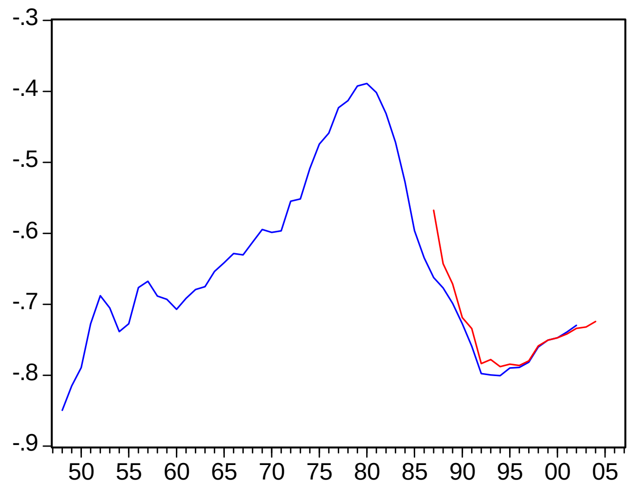


Figure 4: Non-Farm Productivity
(HP-smoothed, relative to linear trend)



Source: BLS

Figure 5: Inflation-Adjusted Land Prices
(relative to linear trend)



Source: BLS

Note: Both series are in logarithms. The land series are from different vintages of BLS data

Figure 6: Housing Services: Expenditures and Prices

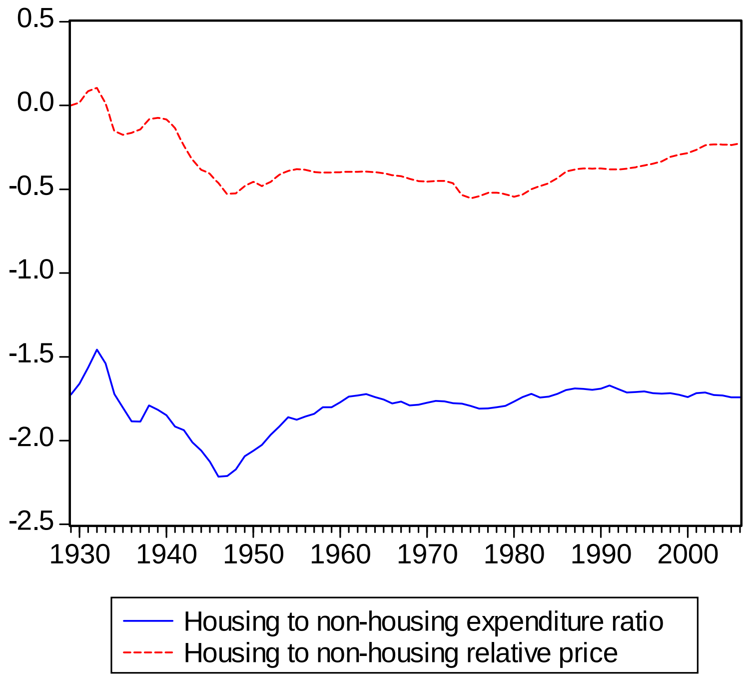


Figure 7: Housing Price Response to
Low-to-High Growth Regime Switch

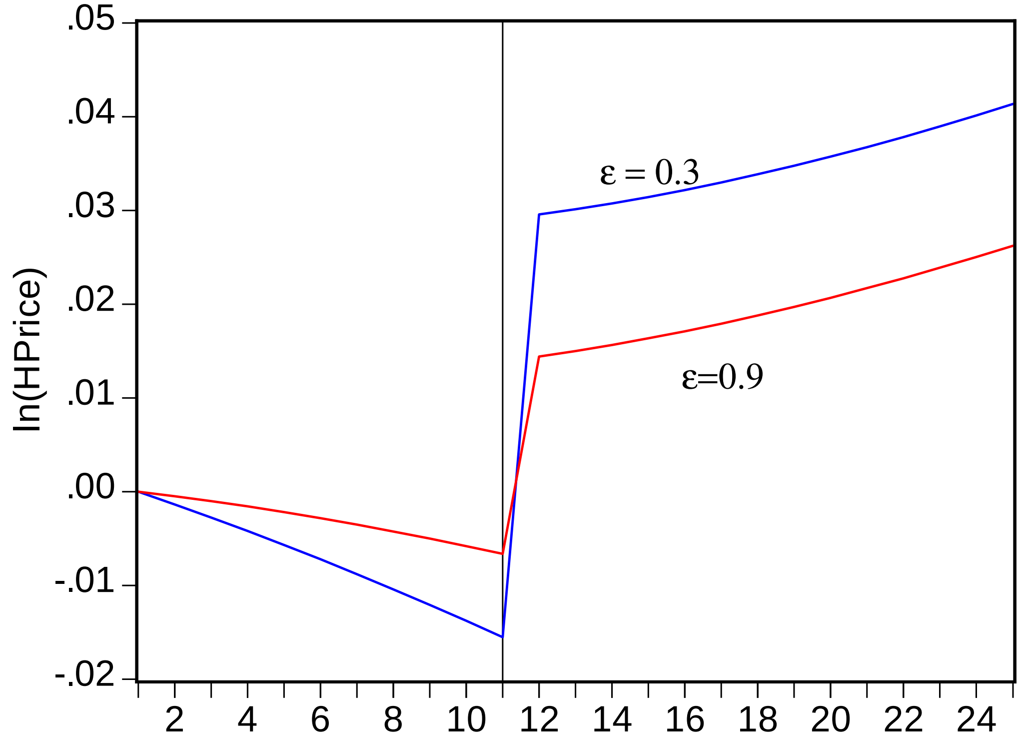
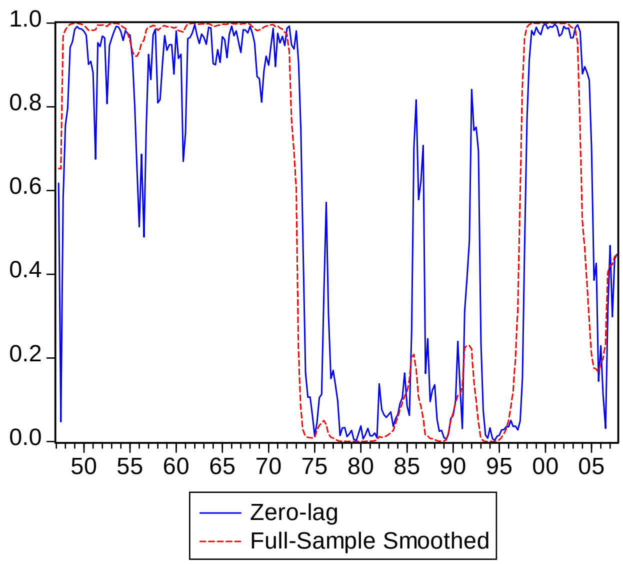


Figure 8: High-Growth Regime Probabilities



Calculations based on Kahn and Rich (2007)

Figure 9: Model Simulation of Housing Prices

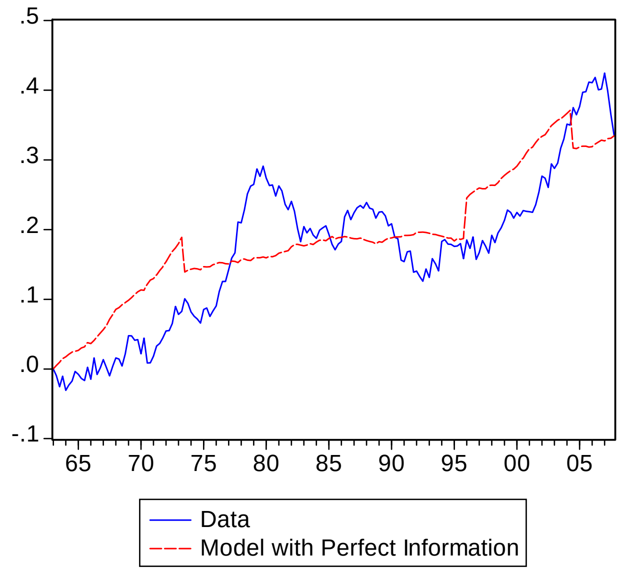


Figure 10a: Real-Time Probabilities of Low-Growth Regimes

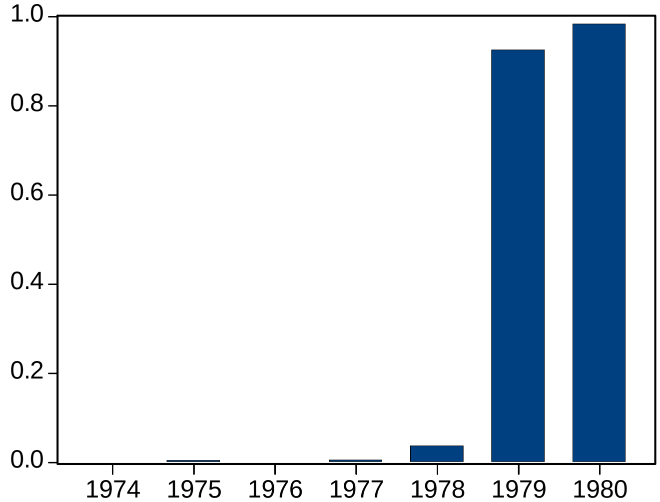


Figure 10b: Real-Time Low-Growth Regime Probabilities since 2005

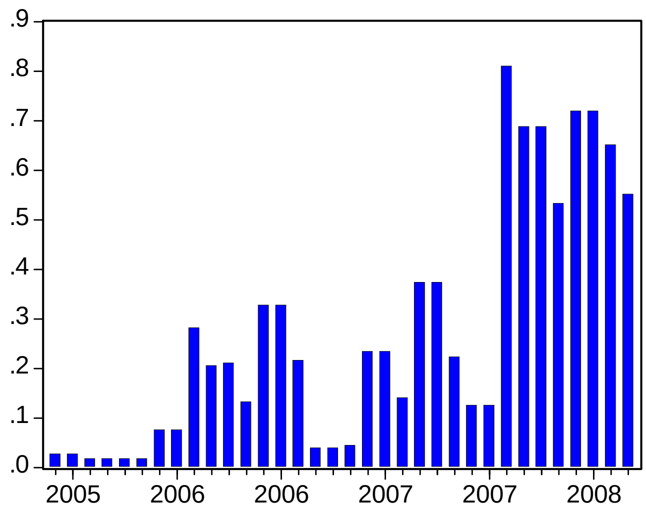


Figure 11: Model Simulation vs. Detrended Data

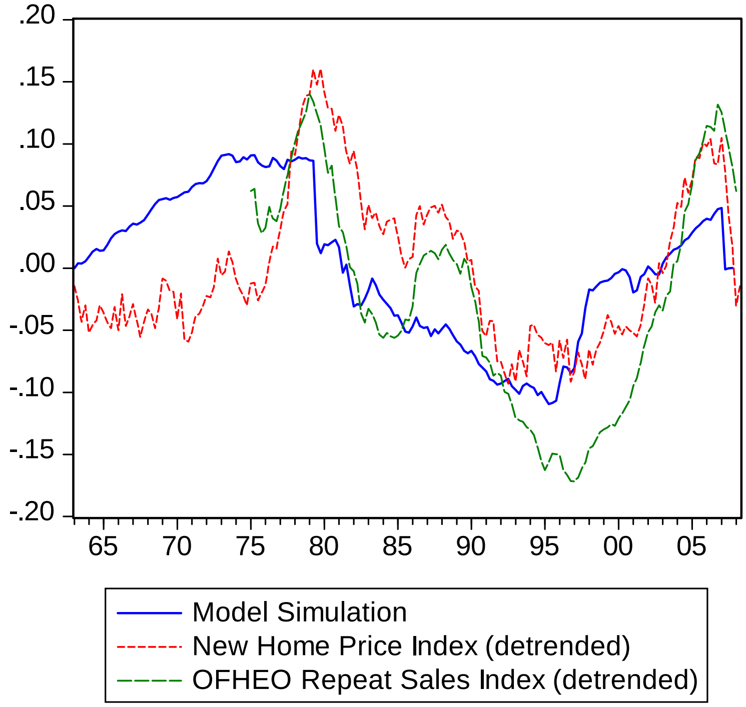


Figure 12: Residential Investment (Detrended)