Federal Reserve Bank of New York
Staff Reports

Intermediary Balance Sheets

Tobias Adrian
Nina Boyarchenko

Staff Report No. 651
November 2013

Intermediary Balance Sheets
Tobias Adrian and Nina Boyarchenko
Federal Reserve Bank of New York Staff Reports, no. 651
November 2013
JEL classification: E02, E32, G00, G28

Abstract

We document the cyclical properties of the balance sheets of different types of intermediaries. While the leverage of the bank sector is highly procyclical, the leverage of the nonbank financial sector is acyclical. We propose a theory of a two-agent financial intermediary sector within a dynamic model of the macroeconomy. Banks are financed by issuing risky debt to households and face risk-based capital constraints, which leads to procyclical leverage. Households can also participate in financial markets by investing in a nonbank "fund" sector where fund managers face skin-in-the-game constraints, leading to acyclical leverage in equilibrium. The model also reproduces the empirical feature that the banking sector's leverage growth leads the financial sector's asset growth, while leverage in the fund sector does not precede growth in financial-sector assets. The procyclicality of the banking sector is due to its risk-based funding constraints, which give a central role to the time variation of endogenous uncertainty.

Key words: procyclical leverage, endogenous uncertainty, heterogeneous agents

---

Adrian, Boyarchenko: Federal Reserve Bank of New York (e-mail: tobias.adrian@ny.frb.org, nina.boyarchenko@ny.frb.org). The authors thank Evan Friedman for excellent research assistance. The views expressed in this paper are those of the authors and do not necessarily reflect the position of the Federal Reserve Bank of New York or the Federal Reserve System.

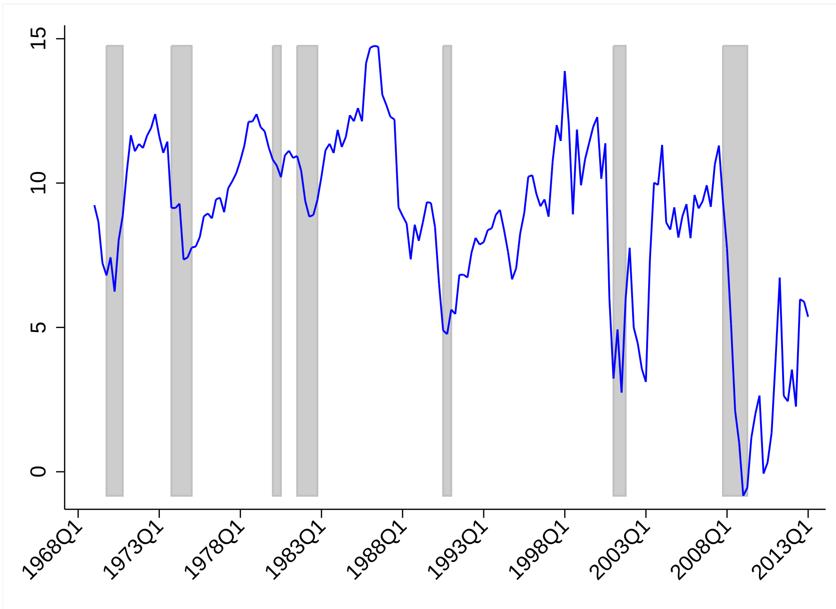
1 Introduction

The deleveraging in the 2008 financial crisis forcefully demonstrated the procyclicality of the financial sector. Figure I shows that financial sector procyclicality is a regular feature of business cycle fluctuations: total financial sector asset growth is significantly lower in recessions than in booms. The source of financial sector procyclicality is only starting to be understood. Adrian and Shin [2013] show the tight linkage between asset growth and leverage growth of banks and broker-dealers, identifying risk management constraints as the root cause of procyclicality. Figure II expands these findings by showing that leverage of the banking sector is also procyclical relative to total financial sector assets, while nonbank sector leverage is acyclical. In these plots as well as the remainder of the paper, the banking sector includes depository institutions and broker-dealers, while the nonbank sector primarily consists of investment funds, such as mutual funds, money market funds, pension funds, real estate investment trusts, and insurance funds. This evidence suggests a key role of the banking sector for procyclicality.

In this paper, we develop a theory of financial sector procyclicality that matches the empirical observations from Figures I and II. The theory relies on the different nature of financial constraints that banks and nonbanks face. Households delegate capital allocation decisions to financial institutions which screen, select, monitor and diversify across investment projects on households' behalf. This delegation of capital allocation decisions gives rise to principal-agent problems between households and intermediaries, which are solved by imposing constraints on financial institutions. The nature of these constraints varies across different types of intermediaries.

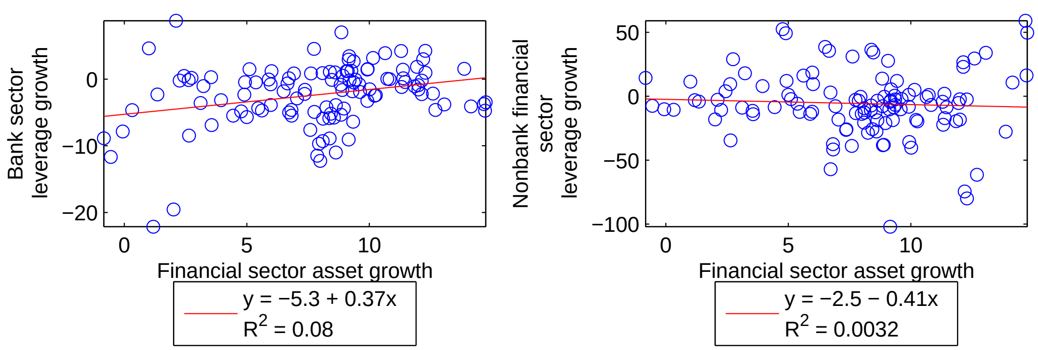
What distinguishes banks from other financial institutions are their risk based funding constraints. In the trading book, risk management constraints such as Value-at-Risk rules dictate balance sheet management, while in the banking book, the Vasicek [1984] model is the predominant analytical basis for risk management. Intuitively, such rules mitigate the principal-agent models between households and banks as risk shifting incentives are mitigated via risk based funding constraints. The risk based funding constraints put the evolution of endogenous uncertainty at the center stage of procyclicality. Periods of low uncertainty

Figure I: Procyclicality of Intermediary Financial Assets



NOTES: The annual growth rate (percent) of the total financial assets. Data on financial asset holdings of the financial sector comes from Flow of Funds Table L.107. NBER recessions are shaded in grey.

Figure II: Intermediary Leverage and Asset Growth



NOTES: Left panel: the annual growth rate (percent) of leverage of the bank sector versus the annual growth rate (percent) of the total financial assets of the financial sector; right panel: the annual growth rate (percent) of leverage of the nonbank financial sector versus the annual growth rate (percent) of the total financial assets of the financial sector. "Bank sector" refers to the aggregated balance sheets of the broker-dealer and commercial bank sectors; "Nonbank financial sector" refers to the remaining balance sheet of the financial sector after subtracting the aggregated balance sheets of the broker-dealer and commercial bank sectors. Data on the balance sheet of the financial sector comes from Flow of Funds Table L.107; data on the balance sheet of the broker-dealer sector comes from Flow of Funds Table L.128; data on the balance sheet of the commercial bank sector comes from FDIC.

imply loose funding constraints for banks, enabling balance sheet expansion.

The main difference between nonbank financial intermediaries and banks is their use of leverage. Nonbanks use profit sharing agreements with household investors, which can be interpreted as outside equity funding. The principal-agent problem between savers and the intermediaries is solved by skin-in-the-game constraints limiting the amount of outside equity financing (see Holmstrom and Tirole [1997]).

In our model, the banking and fund sectors compete for savings from the household sector. The price of capital and its volatility are endogenously determined. While the tightness of banks' leverage constraints directly depend on the equilibrium level of capital return volatility, the tightness of funds' constraint does not have a simple relationship with volatility or expected returns and can be characterized as acyclical.

The model generates business cycles due to the time varying tightness of constraints on the bank sector. In booms, volatility is endogenously low, the effective risk aversion of banks is low, leading to high leverage and a compressed pricing of risk. As a result, real activity increases, and bank sector leverage is procyclical. The leverage in the fund sector, on the other hand, has little correlation with total financial sector assets. Although expansions increase household wealth and household contributions to the funds, the value of assets held by the nonbank sector increases simultaneously, leading to acyclicality in leverage.

Our theory contributes to the growing literature on the role of financial intermediation in dynamic general equilibrium economies by merging two separate assumptions about the functioning of the financial sector into one model, and confronting the resulting dynamics with the data. The modeling of the fund sector is directly taken from He and Krishnamurthy [2012b, 2013], while the modeling of the bank sector is taken from Adrian and Boyarchenko [2012]. Modeling the equilibrium dynamics of He and Krishnamurthy [2012b, 2013] and Adrian and Boyarchenko [2012] within the same economy is relevant, as both banks and funds are important financial intermediaries, though their balance sheet behavior is very different. The empirical evidence presented here suggests that both bank sector dynamics and fund sector dynamics co-exist, and that bank sector dynamics are particularly important in understanding the evolution of pricing, volatility, and real activity.

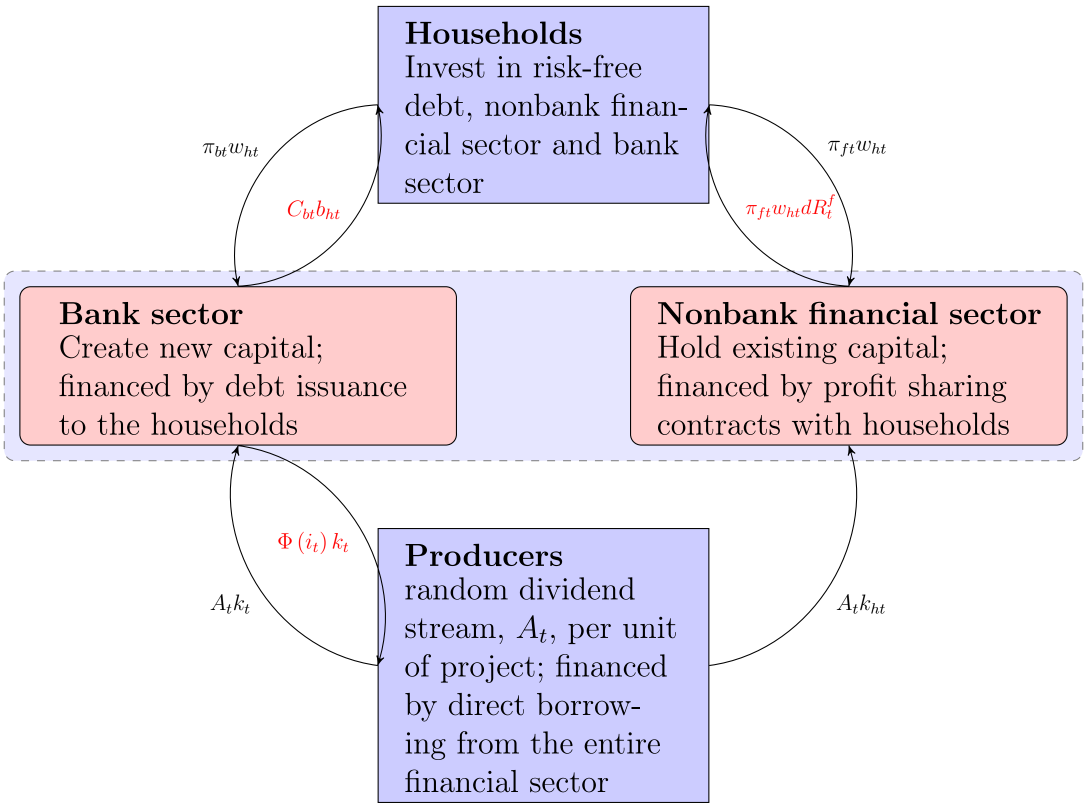
An important finding relative to the previous literature concerns the cyclicality of the nonbank sector. Leverage of the nonbank financial sector is acyclical: Expansions in leverage of the nonbank financial sector occur neither during growth of the financial sector nor during contractions of the financial sector. In contrast, He and Krishnamurthy [2012b, 2013] find that market leverage of the fund sector is countercyclical, and show that a skin-in-the-game constraint implies countercyclical leverage. Unlike their model, the households in our economy optimally choose to allocate capital between risk-free debt, risky debt of the bank sector and the equity contracts with funds. Thus, while the skin-in-the-game constraint becomes tighter for funds during contractions (since the sector's equity gets depleted), households are nonetheless unwilling to reduce their participation in funds. We find the acyclicality of the nonbank fund sector to be supported by data from the Flow of Funds.

The risk based leverage constraints of the bank sector can be viewed as a solution to a static principal agent contracting problem between banks and households, as presented by Adrian and Shin [2013]. Dynamically, these constraints are unlikely to be optimal since risk-based capital requirements tend to give rise to procyclical intermediary leverage. Nuño and Thomas [2012] derive a risk based leverage constraint within a dynamic macroeconomy; however, their contract is not necessarily optimal. The main reason for the suboptimality of such risk based leverage constraints is that they are indexed to contemporaneous volatility, while a dynamically optimal contract would likely make leverage constraints proportional to forward looking measures of risk, as is done with stress tests (see the discussion of stress tests in Adrian and Boyarchenko [2012]).

The findings presented in this paper are fully compatible with the empirical results of Adrian, Moench, and Shin [2010], who document that dealer leverage is forecasting returns, while asset growth or leverage growth of institutions in the fund sector do not have any forecasting power. We note that we consider broker-dealers to be part of the bank sector in the current paper.

The rest of the paper is organized as follows. We describe the model in Section 2. Section 3 investigates the cyclical properties of the financial system in the data and in the model. Conclusions are presented in Section 4. Technical details are relegated to the Appendix.

Figure III: Structure of the economy



2 THE MODEL

We consider a continuous time, infinite horizon economy. Uncertainty is described by a two-dimensional, standard Brownian motion $Z_t = [Z_{at},\ Z_{\xi t}]'$ for $t \geq 0$, defined on a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the augmented filtration generated by Z_t. There are three types of agents in the economy: (passive) producers, financially sophisticated intermediaries and unsophisticated households. The financial intermediation sector in the economy consists of two types of intermediaries: banks (including broker-dealers and commercial banks) and nonbank financial institutions (such as pension funds, insurance companies and mutual funds). The structure of the economy is illustrated in Figure III.

2.1 Production

There is a single consumption good in the economy, produced continuously. We assume that physical capital is the only input into the production of the consumption good, so that the total output in the economy at date $t \geq 0$ is

$$Y_t = A_t K_t,$$

where K_t is the aggregate amount of capital in the economy at time t, and the stochastic productivity of capital $\{A_t = e^{a_t}\}_{t\geq 0}$ follows a geometric diffusion process of the form

$$da_t = \bar{a}dt + \sigma_a dZ_{at}.$$

The stock of physical capital in the economy depreciates at a constant rate λ_k, so that the total physical capital in the economy evolves as

$$dK_t = (I_t - \lambda_k) K_t dt,$$

where I_t is the reinvestment rate per unit of capital in place. There is a fully liquid market for physical capital in the economy, in which both types of financial intermediaries are allowed to participate. We denote by $p_{kt}A_t$ the price of one unit of capital at time $t \geq 0$ in terms of the consumption good.

2.2 *Financial intermediary sector*

There are two types of intermediaries in the economy: banks, subject to risk-based capital constraints, and nonbank financial institutions (or funds), subject to a skin-in-the-game constraint. The bank intermediaries represent the levered financial institutions, such as commercial and investment banks and broker-dealers, in the economy, while the nonbank intermediaries represent institutions such as hedge and mutual funds, pension funds and insurance companies.

2.2.1 Nonbank financial sector

The nonbank financial sector in our model corresponds to the financial sector of He and Krishnamurthy [2012a,b, 2013]. In particular, there is a unit mass of risk-averse fund managers that manage the nonbank intermediaries ("funds"), with each fund matched to a single agent. The fund managers are risk-averse, infinitely lived agents that evaluate consumption paths $\{c_{ft}\}_{t\geq 0}$ using

$$\mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}\log c_{ft}dt\right],$$

where ρ is the subjective discount rate.

As in He and Krishnamurthy [2013], at every date t, each fund manager is randomly matched with a household to form a fund, creating a continuum of identical bilateral relationships. The fund managers execute trades on behalf of the fund in the capital and risk-free debt markets, while the household only decides on the allocation between risk-free debt and the two intermediary sectors. The match is broken at date $t+dt$, and the fund managers and households are rematched.

Denote the fund manager wealth at time t by w_{ft}, and by H_t the household's wealth allocation to the nonbank financial sector. As in He and Krishnamurthy [2012b, 2013], we assume that the fund managers face a skin-in-the-game constraint when raising capital from the households. In particular, for every dollar of fund manager wealth invested in the fund, the households only contribute up to $m > 0$ dollars, so that

$$H_t \leq m w_{ft}. \tag{1}$$

The constant m measures the tightness of the capital constraint faced by the fund managers in the economy and can be micro founded from the moral hazard set-up of Holmstrom and Tirole [1997], as shown by He and Krishnamurthy [2012a].

Finally, the fund manager can allocate the total intermediary wealth freely between the risk-free debt and the existing capital stock in economy, with the total profits shared between the households and the fund managers in accordance to their relative contributions to the fund. Assuming for simplicity that the fund manager invests all of his post-consumption wealth in the fund, the representative fund manager solves

$$\max_{\{c_{ft},\theta_{ft}\}} \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}\log c_{ft}dt\right],$$

subject to the dynamic budget constraint

$$\frac{dw_{ft}}{w_{ft}} = \theta_{ft}\left(dR_{kt} - r_{ft}dt\right) + r_{ft}dt - \frac{c_{ft}}{w_{ft}}dt,$$

where θ_{ft} is the fraction of the fund's equity allocated to the risky capital, r_{ft} is the equilibrium risk-free rate and dR_{kt} is the return on the existing capital in the economy

$$dR_{kt} = \underbrace{\frac{A_t k_{ht}}{k_{ht}p_{kt}A_t}dt}_{\text{dividend}-\text{price ratio}} + \underbrace{\frac{d\left(k_{ht}p_{kt}A_t\right)}{k_{ht}p_{kt}A_t}}_{\text{capital gains}} \equiv \mu_{Rk,t}dt + \sigma_{ka,t}dZ_{at} + \sigma_{k\xi,t}dZ_{\xi t},$$

with k_{ht} the number of units of capital held by the funds. When $\theta_{ft} > 1$, the fund holds a levered position in risky capital. We have the following result.

Lemma 1. *The fund managers consume a constant fraction of their wealth*

$$c_{ft} = \rho w_{ft},$$

and allocate the fund's capital as a mean-variance investor

$$\theta_{ft} = \frac{\mu_{Rk,t} - r_{ft}}{\sigma^2_{ka,t} + \sigma^2_{k\xi,t}}.$$

Proof. See Appendix A.1. □

2.2.2 Bank sector

In addition to the fund managers that manage the institutions in the nonbank financial sector, there is also a unit mass of infinitely-lived, risk-averse bankers that manage the institutions ("banks") in the bank sector. The bank sector performs two functions in the economy. First, the bank is a financially constrained institution that has access to (a better) capital creation technology than the households and can thus intermediate between households and the productive sector, channeling funds from one to the other to the benefit of both parties. Since the intermediaries have wealth of their own, they serve a second important function in the economy by providing risk-bearing capacity to the households.

In particular, banks create new physical capital ("projects") through capital investment. Denote by $i_t A_t$ the investment rate and by $\Phi(i_t) A_t$ the new projects created per unit of capital held by the intermediaries. Here, $\Phi(\cdot)$ reflects the costs of (dis)investment. We assume that $\Phi(0) = 0$, so in the absence of new investment, capital depreciates at the economy-wide rate λ_k. Notice that the above formulation implies that costs of adjusting capital are higher in economies with a higher level of capital productivity, corresponding to the intuition that more developed economies are more specialized. We follow Brunnermeier and Sannikov [2012] in assuming that investment carries quadratic adjustment costs, so that Φ has the form

$$\Phi(i_t) = \phi_0 \left(\sqrt{1 + \phi_1 i_t} - 1 \right),$$

for positive constants ϕ_0 and ϕ_1.

Unlike the nonbank financial sector, the banks finance themselves by issuing long-term risky debt to the households. To keep the balance sheet structure of financial institutions time-invariant, we assume that the risky intermediary debt matures at a constant rate λ_b, so that the time t probability of a bond maturing before time $t + dt$ is $\lambda_b dt$. Notice that this corresponds to an infinite-horizon version of the "stationary balance sheet" assumption of Leland and Toft [1996]. The bonds pay a floating coupon $C_{dt} A_t$ until maturity, with the coupon payment determined in equilibrium to clear the risky bond market. Similarly to capital, risky bonds are liquidly traded, with the price of a unit of intermediary debt at time t in terms of the consumption good given by $p_{bt} A_t$.

We assume that the representative banker evaluates consumption paths $\{c_{bt}\}_{t \geq 0}$ using

$$\mathbb{E}\left[\int_0^{+\infty} e^{-\rho t} \log c_{bt} dt \right].$$

Denote by w_t the (inside) equity of the representative bank at date $t \geq 0$, by k_t the bank's holdings of capital and by b_t the bank's stock of debt outstanding. Introduce

$$\theta_t = \frac{k_t p_{kt} A_t}{w_t}$$

to be the total leverage of the representative bank, and

$$\theta_{bt} = \frac{b_t p_{bt} A_t}{w_t}$$

to be the ratio of long term debt to equity. Then, the value of inside equity evolves as

$$\frac{dw_t}{w_t} = \theta_t \left(dR_{kt} - r_{ft}dt + \left(\Phi\left(i_t\right) - \frac{i_t}{p_{kt}} \right) dt \right) - \theta_{bt}\left(dR_{bt} - r_{ft}dt\right) + r_{ft}dt - \frac{c_{bt}}{w_t}dt.$$

Here, dR_{bt} is the return to holding one unit of bank debt

$$dR_{bt} = \underbrace{\frac{\left(C_{dt} + \lambda_b - \text{б}_t p_{bt}\right) A_t b_t}{b_t p_{bt} A_t} dt}_{\text{dividend}-\text{price ratio}} + \underbrace{\frac{d\left(b_t p_{bt} A_t\right)}{b_t p_{bt} A_t}}_{\text{capital gains}} \equiv \mu_{Rb,t}dt + \sigma_{ba,t}dZ_{at} + \sigma_{b\xi,t}dZ_{\xi t},$$

where б_t is the new debt issuance at time t. We assume that the bank sector cannot participate in the instantaneous risk-free debt market, so that

$$\theta_t = 1 + \theta_{bt}.$$

Notice further that the excess return to holding capital from the viewpoint of the banks also includes the net gain from investing in a new project, $\Phi\left(i_t\right)$, rather than buying the corresponding capital stock in the market, i_t/p_{kt}.

The key distinction between the bank sector and the nonbank financial sector is in the funding constraint faced by banks. As in Adrian and Boyarchenko [2012], we assume that intermediary borrowing is restricted by a risk-based capital constraint, similar to the value at risk (VaR) constraint of Danielsson, Shin, and Zigrand [2011]. In particular, we assume that

$$w_t \geq \alpha \sqrt{\frac{1}{dt}\left\langle k_t d\left(p_{kt}A_t\right)\right\rangle^2}.$$

It should be noted that, since the banks are constrained, they create less new capital in equilibrium than they would without regulation. The risk-based constraint can be micro-founded using a moral hazard problem in a static setting, as in Adrian and Shin [2013]; however, we abstract from the question of why the risk-based constraint exists in our economy.[1] Notice that this risk-based funding constraint boils down to a time varying leverage constraint

$$\theta_t \leq \frac{1}{\alpha\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}}.$$

The value of inside equity of the representative banker evolves as

$$\frac{dw_t}{w_t} = \theta_t \left(dR_{kt} - r_{ft}dt + \left(\Phi\left(i_t\right) - \frac{i_t}{p_{kt}} \right) dt \right) - \theta_{bt}\left(dR_{bt} - r_{ft}dt\right) + r_{ft}dt - \frac{c_{bt}}{w_t}dt.$$

Thus, the representative banker solves

$$\max_{\theta_t, \theta_{bt}, i_t, c_{bt}} \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t} \log c_{bt} dt\right]$$

subject to the dynamic budget constraint

$$\frac{dw_t}{w_t} = \theta_t \left(dR_{kt} - r_{ft}dt + \left(\Phi\left(i_t\right) - \frac{i_t}{p_{kt}} \right) dt \right) - \theta_{bt}\left(dR_{bt} - r_{ft}dt\right) + r_{ft}dt - \frac{c_{bt}}{w_t}dt,$$

the risk-based capital constraint

$$\theta_t \leq \frac{1}{\alpha\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}},$$

and the adding-up constraint

$$\theta_{bt} = \theta_t - 1.$$

Lemma 2. *The representative banker optimally consumes at rate*

$$c_{bt} = \rho w_t$$

[1] An alternative interpretation would be in terms of a counterbalancing force to a government subsidy (such as access to a better investment technology than other sectors of the economy) provide to the bank sector. As pointed out in Kareken and Wallace [1978], government subsidies distort the risk-taking decisions of banks, precipitating the need for government regulation of risk taking.

and invests in new projects at rate

$$i_t = \frac{1}{\phi_1}\left(\frac{\phi_0^2\phi_1^2}{4}p_{kt}^2 - 1\right).$$

While the capital constraint in not binding, the banking system leverage is

$$\theta_t = \frac{\sigma_{ba,t}^2 - \sigma_{ka,t}\sigma_{ba,t} + \sigma_{b\xi,t}^2 - \sigma_{k\xi,t}\sigma_{b\xi,t} - (\mu_{Rb,t} - r_{ft}) + \left(\mu_{Rk,t} + \Phi\left(i_t\right) - \frac{i_t}{p_{kt}} - r_{ft}\right)}{\left((\sigma_{ba,t} - \sigma_{ka,t})^2 + (\sigma_{b\xi,t} - \sigma_{k\xi,t})^2\right)}.$$

Proof. See Appendix A.2. □

Thus, while the capital constraint does not bind, the banks act as standard log-utility investors. In particular, since the optimal portfolio allocation does not include an intertemporal hedging demand, the banking intermediaries do not self-insure against the possibility of the capital constraint binding.

2.3 Households

We model the household sector as a continuum of mass one of homogeneous, risk-averse agents. The representative household is matched with the representative fund manager to create the representative fund. The representative household is exposed to a preference shock, modeled as a change-of-measure variable in the household's utility function. In particular, we assume that the representative household evaluates possible consumption paths according to

$$\mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t - \rho_h t}\log c_t dt\right],$$

where c_t is consumption at date $t \geq 0$ and ρ_h is the subjective discount rate. Here, $\exp\left(-\xi_t\right)$ is the Radon-Nikodym derivative of the measure induced by households' time-varying preferences or beliefs with respect to the physical measure. For simplicity, we normalize $\xi_0 = 0$ and assume that $\left\{\xi_t\right\}_{t\geq 0}$ evolves as a Brownian motion, uncorrelated with the productivity shock,[2] Z_{at}

$$d\xi_t = \sigma_\xi dZ_{\xi t}.$$

The representative household finances its consumption through short-term risk-free borrowing and lending, an equity stake in the representative fund, and holdings of risky bank debt. Denote by π_{bt} the fraction of household wealth w_{ht} allocated to risky debt and by π_{kt} the fraction of household wealth allocated to the fund. The skin-in-the-game constraint for the fund managers implies that the household allocation to the fund is constrained by

$$\pi_{kt} w_{ht} \leq m w_{ft}.$$

The households are also constrained from shorting either the fund or the risky bank debt. Thus, the representative household solves

$$\max_{\substack{\pi_{kt},\pi_{bt} \\ c_t}} \; \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t - \rho_h t} \log c_t dt\right],$$

subject to the dynamic budget constraint

$$\frac{dw_{ht}}{w_{ht}} = \pi_{kt}\theta_{ft}\left(dR_{kt} - r_{ft}dt\right) + \pi_{bt}\left(dR_{bt} - r_{ft}dt\right) + r_{ft}dt - \frac{c_t}{w_{ht}}dt,$$

the skin-in-the-game constraint

$$\pi_{kt} w_{ht} \leq m w_{ft},$$

and no shorting constraints

$$\pi_{kt} \geq 0$$
$$b_{ht} \geq 0.$$

The household takes the portfolio choice θ_{ft} and the wealth of the fund managers w_{ft} as

[2] We allow for correlation between the productivity and preference shocks in Adrian and Boyarchenko [2012]. Since the correlation does not fundamentally affect the agents' equilibrium choices and shock amplification in the economy, we omit it here for simplicity.

given when solving for the optimal consumption plan and portfolio allocation strategy. We have the following result.

Lemma 3. *The households' optimal consumption choice satisfies*

$$c_t = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) w_{ht}.$$

While the households are unconstrained in their wealth allocation, the households' optimal portfolio choice is given by

$$\begin{bmatrix} \pi_{kt} \\ \pi_{bt} \end{bmatrix} = \left(\begin{bmatrix} \theta_{ft}\sigma_{ka,t} & \theta_{ft}\sigma_{k\xi,t} \\ \sigma_{ba,t} & \sigma_{b\xi,t} \end{bmatrix} \begin{bmatrix} \theta_{ft}\sigma_{ka,t} & \sigma_{ba,t} \\ \theta_{ft}\sigma_{k\xi,t} & \sigma_{b\xi,t} \end{bmatrix}\right)^{-1} \begin{bmatrix} \theta_{ft}\left(\mu_{Rk,t} - r_{ft}\right) \\ \mu_{Rb,t} - r_{ft} \end{bmatrix}$$
$$- \begin{bmatrix} \theta_{ft}\sigma_{ka,t} & \sigma_{ba,t} \\ \theta_{ft}\sigma_{k\xi,t} & \sigma_{b\xi,t} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \sigma_\xi \end{bmatrix}.$$

The household becomes constrained in its allocation to the fund sector when

$$\mu_{Rk,t} - r_{ft} \geq 2m \frac{\theta_{ft}\omega_{ft}}{1 - \omega_{ft} - \omega_t} \frac{\left(\sigma_{ka,t}\sigma_{b\xi,t} - \sigma_{k\xi,t}\sigma_{ba,t}\right)^2}{\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2} + \frac{\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}}{\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2}\left(\mu_{Rb,t} - r_{ft}\right)$$
$$+ \frac{\left(\sigma_{ka,t}\sigma_{b\xi,t} - \sigma_{k\xi,t}\sigma_{ba,t}\right)\left(\sigma_{ba,t} - \sigma_{b\xi,t}\right)}{\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2}\sigma_\xi.$$

Proof. See Appendix A.3. □

Thus, the households consume a constant fraction of their wealth, but at a slower rate than they would if they were not exposed to the liquidity preference shocks, ξ_t. The households' optimal allocation to capital has a intratemporal hedging component to compensate them for exposure to the liquidity shocks.

2.4 Equilibrium

Definition 1. *An equilibrium in the economy is a set of price processes* $\{p_{kt}, p_{bt}, r_{ft}\}_{t\geq 0}$*, a set of household decisions* $\{\pi_{kt}, b_{ht}, c_t\}_{t\geq 0}$*, a set of fund managers' decisions* $\{k_{ft}, c_{ft}\}_{t\geq 0}$*, and a set of bankers' decisions* $\{k_t, i_t, b_t, c_{bt}\}_{t\geq 0}$ *such that the following apply:*

1. *Taking the price processes, the fund managers' decisions and the bankers' decisions as given, the household's choices solve the household's optimization problem, subject to the household budget constraint, the no shorting constraints and the skin-in-the-game constraint for the funds.*

2. *Taking the price processes, the fund managers' decisions and the household decisions as given, the bankers' choices solve the intermediary's optimization problem, subject to the intermediary budget constraint, and the regulatory constraint.*

3. *Taking the price processes, the household decisions and the bankers' decisions as given, the fund managers' choices solve the fund managers' optimization problem, subject to the fund managers' budget constraint.*

4. *The capital market clears at all dates*

$$k_t + k_{ft} = K_t.$$

5. *The risky bond market clears*

$$b_t = b_{ht}.$$

6. *The risk-free debt market clears*

$$w_t + w_{ft} + w_{ht} = p_{kt} A_t K_t.$$

7. *The goods market clears*

$$c_t + c_{bt} + c_{ft} + A_t k_t i_t = K_t A_t.$$

Notice that, in equilibrium, the stock of capital in the economy evolves as

$$dK_t = \left(\Phi\left(i_t\right) \frac{k_t}{K_t} - \lambda_k \right) K_t dt.$$

We characterize the equilibrium in terms of the evolutions of three state variables: the leverage of the banks, θ_t, the relative wealth of the banks in the economy, ω_t, and the relative wealth of the nonbank financial intermediaries, ω_{ft}. This representation allows us to characterize the equilibrium outcomes as a solution to a system of algebraic equations, which can easily be solved numerically. In particular, we represent the evolution of the state variables as

$$
\begin{aligned}
\frac{d\theta_t}{\theta_t} &= \mu_{\theta t} dt + \sigma_{\theta\xi,t} dZ_{\xi t} + \sigma_{\theta a,t} dZ_{at} \\
\frac{d\omega_t}{\omega_t} &= \mu_{\omega t} dt + \sigma_{\omega\xi,t} dZ_{\xi t} + \sigma_{\omega a,t} dZ_{at} \\
\frac{d\omega_{ft}}{\omega_{ft}} &= \mu_{\omega_f,t} dt + \sigma_{\omega_f\xi,t} dZ_{\xi t} + \sigma_{\omega_f a,t} dZ_{at}.
\end{aligned}
$$

We can then express all the other equilibrium quantities in terms of the state variables and the sensitivities of the return to holding capital to output and liquidity shocks, $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$. We solve for these last two equilibrium quantities numerically as solutions to the system of equations that

1. Equates θ_t to the solution to the optimal portfolio allocation choice of the representative bank;

2. Equates the equilibrium risk-free rate to the risk-free rate given by the fund managers' Euler equation.

The other equilibrium quantities can be expressed as follows.

1. Equilibrium price of capital, p_{kt}, (from goods market clearing) and optimal capital investment policy, i_t, (from bankers' optimization) as a function of the state variables only;

2. From capital market clearing, fund allocation to capital, θ_{ft}, as a function of the state variables only;

3. From debt market clearing, household allocation to debt, π_{bt}, as a function of state variables only;

4. From fund managers' optimal capital allocation, expected excess return to holding capital, $\mu_{Rk,t} - r_{ft}$, as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

5. From the equilibrium evolution of fund managers' wealth, the sensitivities of the funds' wealth share in the economy, ω_{ft}, to output and liquidity shocks, $\sigma_{\omega_f a,t}$ and $\sigma_{\omega_f \xi,t}$, as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

6. From the equilibrium evolution of the price of capital, the sensitivities of the bankers' leverage to output and liquidity shocks, $\sigma_{\theta a,t}$ and $\sigma_{\theta\xi,t}$, as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

7. From the equilibrium evolution of bankers' wealth, the sensitivities of the return to holding risky debt to output and liquidity shocks, $\sigma_{ba,t}$ and $\sigma_{b\xi,t}$, as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

8. From the households' optimal allocation to bank debt, expected excess return to holding capital, $\mu_{Rb,t} - r_{ft}$, as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

9. From the households' optimal portfolio choice, household allocation to funds, π_{kt}, as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

10. From the equilibrium evolution of fund managers' wealth, the expected growth rate of fund managers' wealth share, $\mu_{\omega_f t}$ as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

11. From the equilibrium evolution of bankers' wealth, the expected growth rate of bankers' wealth share, $\mu_{\omega t}$ as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

12. From the equilibrium evolution of capital, the expected growth rate of banks' leverage, $\mu_{\theta t}$ as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

13. From the equilibrium drift of the price of capital, the risk-free rate r_{ft} as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$.

The details of the solution are relegated to Appendix B.

Table I: Parameters

Parameter	Notation	Value
Expected growth rate of productivity	$\bar{a}$	0.0651
Volatility of growth rate of productivity	σ_a	0.388
Volatility of liquidity shocks	σ_ξ	0.0388
Discount rate of intermediaries	ρ	0.06
Effective discount rate of households	$\rho_h - \sigma_\xi^2/2$	0.05
Fixed cost of capital adjustment	ϕ_0	0.1
	ϕ_1	20
Depreciation rate of capital	λ_k	0.03
Tightness of risk-based capital constraint	α	2.5
Tightness of skin-in-the-game constraint	m	4

NOTES: Parameters used in simulations. The parameters of the productivity growth process ($\bar{a}$, σ_a), the parameters of the investment technology (ϕ_0, ϕ_1), subjective discount rates (ρ_h, ρ), and depreciation (λ_k) are taken from Brunnermeier and Sannikov [2012]; the tightness of the skin-in-the-game constraint m from He and Krishnamurthy [2013].

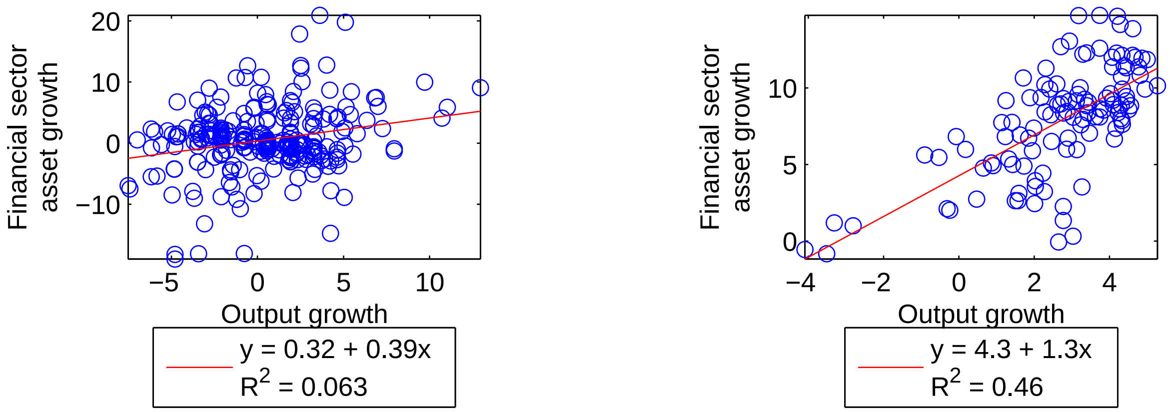
3 Equilibrium Outcomes

We present the equilibrium outcomes in terms of simulated paths of the model. We simulate 1000 paths of the economy at a monthly frequency for 80 years (roughly matching the length of the time series of post-war data in the US), using typical parameters from the literature (see Table I). We initialize the system to match the wealth of the bank sector relative to the nonbank financial sector for the US. We consider the outcomes along typical simulation paths, and then show that the cyclical properties of balance sheets of financial institutions hold across simulations.

We begin by showing that the assets of the financial sector are procyclical in our model. Figure IV plots the annual log growth rate of assets of the financial sector versus the log growth rate of the (real) output growth for a typical path (left panel) and the US data (right panel). While the relationship is weaker for the simulated data, both series exhibit a strong positive relationship between output growth and the growth rate of assets of the financial sector. Thus financial business is procyclical both in the data and in the model.

We now examine whether the procyclicality of the financial sector is driven by the bank

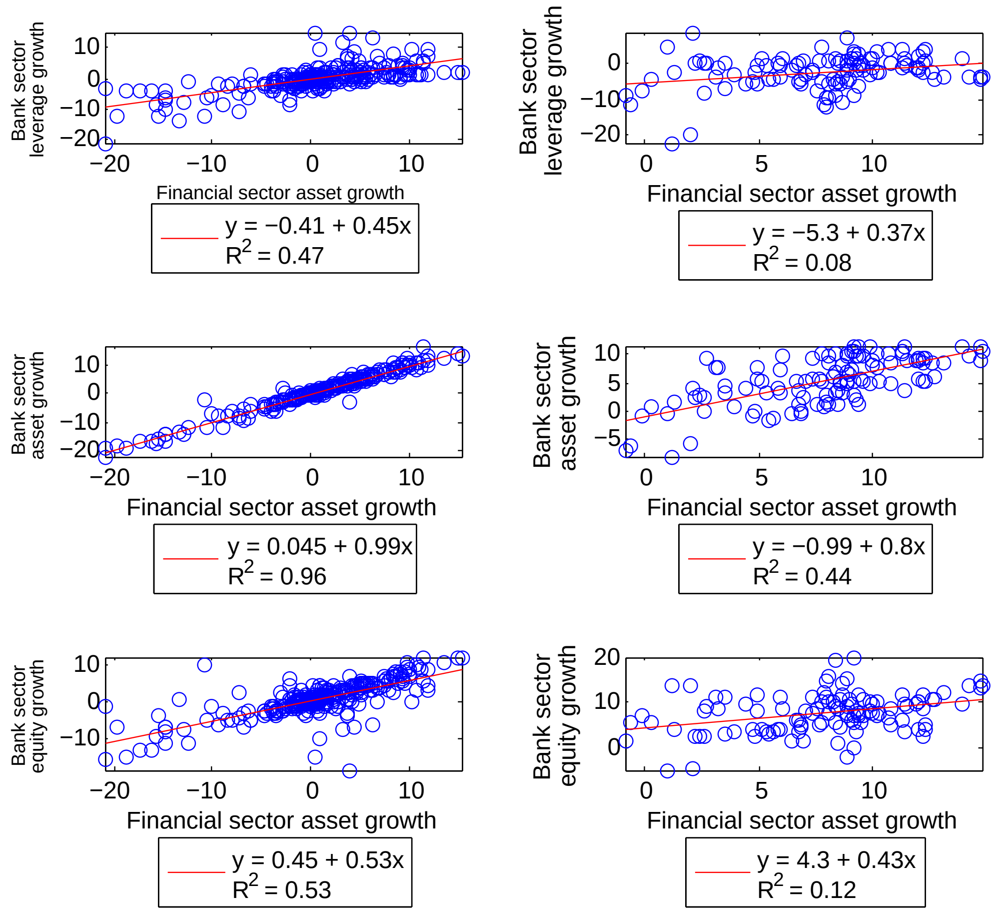
Figure IV: Procyclicality of Financial Business



NOTES: Annual growth rate (percent) of assets of the financial sector versus annual growth rate of output for a representative path of the simulated economy (left) and for the US (right). In the model, assets of the financial sector are measured by $(\omega_t + \omega_{ft})\, p_{kt} A_t K_t$ and output by $A_t K_t$. For the US, output is measured in real terms. Data on the balance sheet of the financial sector comes from Flow of Funds Table L.107; data on real GDP growth from US National Accounts. Data from the model is simulated using parameters in Table I at a monthly frequency for 80 years.

sector or the nonbank financial sector. Figure V plots the relationship between different features of the aggregated balance sheet of the bank sector and the growth rate of assets of the financial sector. The left panels plot the relationship for a representative simulated path, and the right panels plot the relationship for the US bank sector. For the US, "Bank sector" refers to the aggregated balance sheets of the broker-dealer and commercial bank sectors. The top row of Figure V shows that the leverage of the bank sector is strongly procyclical: Leverage of the bank sector expands as the financial system increases in size. This fact has been documented by Adrian and Shin [2010] for the broker-dealer sector and by Adrian, Colla, and Shin [2012] for the commercial bank sector. As we show in Adrian and Boyarchenko [2012], procyclical leverage is generated by the risk-based capital constraint: since volatility is low during expansions, the bank sector can increase its leverage and grow its balance sheet. The middle and bottom rows of Figure V show that, while both assets and mark-to-market equity of the bank sector are procyclical, total assets of the bank sector correlate much stronger with total assets of the financial system, resulting in procyclical leverage. The left panels of Table II shows that, in the model, the procyclicality of bank leverage, asset growth and equity growth holds across different paths of the model (we report

Figure V: Bank Balance Sheet Procyclicality



NOTES: Procyclicality of bank balance sheets. Top row: annual growth rate of leverage of the bank sector versus the annual growth rate of assets of the financial sector for a representative path of the simulated economy (left) and for the US (right). Middle row: annual growth rate of assets of the bank sector versus the annual growth rate of assets of the financial sector for a representative path of the simulated economy (left) and for the US (right). Bottom row: annual growth rate of equity of the bank sector versus the annual growth rate of assets of the financial sector for a representative path of the simulated economy (left) and for the US (right). In the model, assets of the financial sector are measured by $(\omega_t + \omega_{ft})\, p_{kt} A_t K_t$, leverage of the bank sector by θ_t and equity of the bank sector by $\omega_t p_{kt} A_t K_t$. For the US, "Bank sector" refers to the aggregated balance sheets of the broker-dealer and commercial bank sectors. Data on the balance sheet of the financial sector comes from Flow of Funds Table L.107; data on the balance sheet of the broker-dealer sector comes from Flow of Funds Table L.128; data on the balance sheet of the commercial bank sector comes from FDIC. Data from the model is simulated using parameters in Table I at a monthly frequency for 80 years.

the median, mean, 5% and 95% outcomes from the simulation).

Consider now the nonbank financial sector. Figure VI plots the relationship between different features of the aggregated balance sheet of the nonbank financial sector and the growth rate

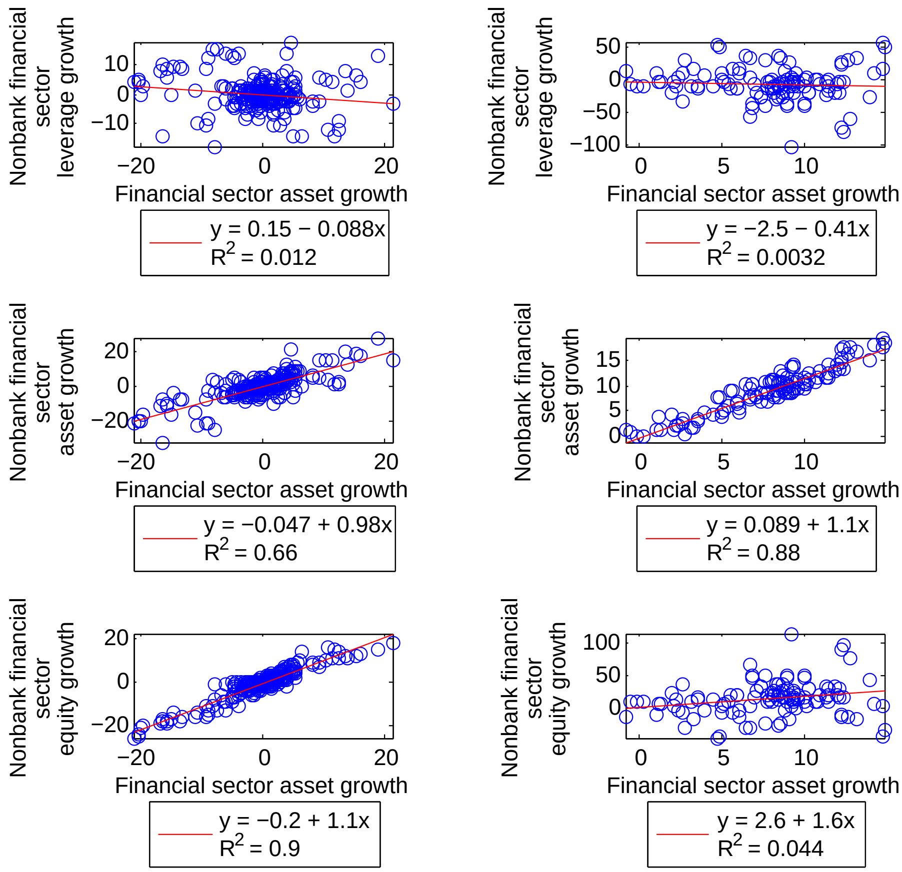
Table II: Cyclicality of Intermediary Balance Sheets

	Bank sector leverage growth						Nonbank financial sector leverage growth				
	Data	Mean	5%	Median	95%		Data	Mean	5%	Median	95%
β_0	-5.29	-0.33	-0.63	-0.32	-0.01	β_0	-2.54	0.22	-0.06	0.18	0.55
β_1	0.37	0.40	0.18	0.42	0.59	β_1	-0.41	-0.05	-0.27	-0.06	0.17
R^2	0.08	0.35	0.16	0.35	0.50	R^2	0.00	0.04	0.00	0.02	0.14
	Bank sector asset growth						**Nonbank financial sector asset growth**				
	Data	Mean	5%	Median	95%		Data	Mean	5%	Median	95%
β_0	-5.29	-0.33	-0.63	-0.32	-0.01	β_0	0.09	-0.09	-0.29	-0.09	0.11
β_1	0.37	0.40	0.18	0.42	0.59	β_1	1.15	1.01	0.84	1.01	1.18
R^2	0.08	0.35	0.16	0.35	0.50	R^2	0.88	0.67	0.48	0.68	0.82
	Bank sector equity growth						**Nonbank financial sector equity growth**				
	Data	Mean	5%	Median	95%		Data	Mean	5%	Median	95%
β_0	4.30	0.40	0.05	0.39	0.80	β_0	2.63	-0.31	-0.57	-0.27	-0.15
β_1	0.43	0.55	0.32	0.54	0.79	β_1	1.56	1.06	0.95	1.06	1.20
R^2	0.12	0.46	0.14	0.48	0.77	R^2	0.04	0.93	0.88	0.93	0.96

Notes: The relationship between balance sheets of two types of intermediaries and the growth rate of the assets of the financial sector. The "Data" column reports the coefficients estimated using data for the US financial sector. The "Mean", 5%, "Median" and 95% columns refer to moments of the distribution of coefficients estimated using 1000 simulated paths. β_0 is the constant in the estimated regression, β_1 is the loading on the explanatory variable, and R^2 is the percent variance explained. In the model, assets of the financial sector are measured by $(\omega_t + \omega_{ft})\, p_{kt} A_t K_t$, leverage of the bank sector by θ_t, equity of the bank sector by $\omega_t p_{kt} A_t K_t$, leverage of the nonbank financial sector by θ_{ft} and equity of the nonbank sector by $\omega_{ft} p_{kt} A_t K_t$. For the US, "Bank sector" refers to the aggregated balance sheets of the broker-dealer and commercial bank sectors; "Nonbank financial sector" refers to the remaining balance sheet of the financial sector after subtracting the aggregated balance sheets of the broker-dealer and commercial bank sectors. Data on the balance sheet of the financial sector comes from Flow of Funds Table L.107; data on the balance sheet of the broker-dealer sector comes from Flow of Funds Table L.128; data on the balance sheet of the commercial bank sector comes from FDIC. Data from the model is simulated using parameters in Table I at a monthly frequency for 80 years.

of assets of the financial sector. The left panels plot the relationship for a representative simulated path, and the right panels plot the relationship for the US nonbank financial sector. For the US, "Nonbank financial sector" refers to the remaining balance sheet of the financial sector after subtracting the aggregated balance sheets of the broker-dealer and commercial bank sectors. The top row of Figure VI shows that the leverage of the nonbank financial sector is acyclical: Expansions in leverage of the nonbank financial sector occur neither during growth of the financial sector nor during contractions of the financial sector. In contrast, He and Krishnamurthy [2012b, 2013] find that market leverage of the hedge fund

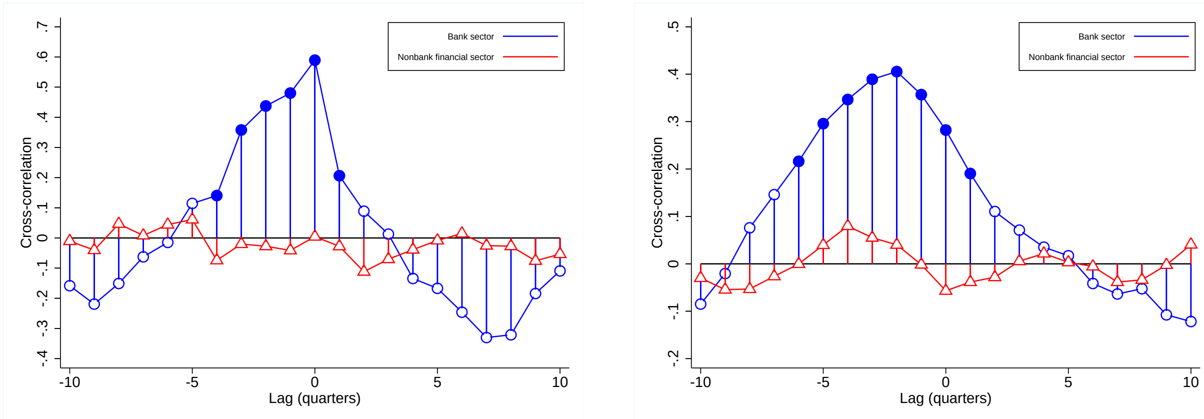
Figure VI: Nonbank Financial Sector Balance Sheet Acyclicality



NOTES: Acyclicality of nonbank financial sector balance sheets. Top row: annual growth rate of leverage of the nonbank financial sector versus the annual growth rate of assets of the financial sector for a representative path of the simulated economy (left) and for the US (right). Middle row: annual growth rate of assets of the nonbank financial sector versus the annual growth rate of assets of the financial sector for a representative path of the simulated economy (left) and for the US (right). Bottom row: annual growth rate of equity of the nonbank financial sector versus the annual growth rate of assets of the financial sector for a representative path of the simulated economy (left) and for the US (right). In the model, assets of the financial sector are measured by $(\omega_t + \omega_{ft})\, p_{kt} A_t K_t$, leverage of the nonbank financial sector by θ_{ft} and equity of the nonbank sector by $\omega_{ft} p_{kt} A_t K_t$. For the US, "Nonbank financial sector" refers to the remaining balance sheet of the financial sector after subtracting the aggregated balance sheets of the broker-dealer and commercial bank sectors. Data on the balance sheet of the financial sector comes from Flow of Funds Table L.107; data on the balance sheet of the broker-dealer sector comes from Flow of Funds Table L.128; data on the balance sheet of the commercial bank sector comes from FDIC. Data from the model is simulated using parameters in Table I at a monthly frequency for 80 years.

sector is countercyclical, and show that a skin-in-the-game constraint of the form (1) implies countercyclical leverage. Unlike their model, the households in our economy optimally choose to allocate capital between risk-free debt, risky debt of the bank sector and the equity

Figure VII: Cross-correlation of Intermediary Balance Sheets



Notes: The cross-correlation function of the annual growth rate (percent) of leverage versus the annual growth rate (percent) of assets of the financial sector for the bank sector (blue) and the nonbank financial sector (red), for a representative path of the simulated economy (left) and for the US (right). Significant correlations are denoted by filled-in points. Assets of the financial sector are measured by $(\omega_t + \omega_{ft})\, p_{kt} A_t K_t$, assets of the bank sector by $\omega_t p_{kt} A_t K_t$ and equity of the nonbank sector by $\omega_{ft} p_{kt} A_t K_t$. For the US, "Bank sector" refers to the aggregated balance sheets of the broker-dealer and commercial bank sectors; "Nonbank financial sector" refers to the remaining balance sheet of the financial sector after subtracting the aggregated balance sheets of the broker-dealer and commercial bank sectors. Data on the balance sheet of the financial sector comes from Flow of Funds Table L.107; data on the balance sheet of the broker-dealer sector comes from Flow of Funds Table L.128; data on the balance sheet of the commercial bank sector comes from FDIC. Data from the model is simulated using parameters in Table I at a monthly frequency for 80 years.

contracts with the nonbank financial sector. Thus, while the skin-in-the-game constraint becomes tighter for the nonbank financial sector during contractions (since the sector's equity gets depleted), households are nonetheless unwilling to reduce their participation in the nonbank financial sector and increase holdings of risky bank debt. The middle and bottom rows of Figure VI show that, just like the bank sector, both assets and mark-to-market equity of the nonbank financial sector are procyclical. For the nonbank financial sector, however, the correlation of the growth rate of assets with the growth rate of total assets of the financial system and the correlation of the growth rate of equity with the growth rate of total assets of the financial system are similar in magnitude, resulting in acyclical leverage. The right panels of Table II shows that, in the model, the acyclicality of nonbank financial sector leverage and the procyclicality of asset growth and equity growth holds across different paths of the model (we report the median, mean, 5% and 95% outcomes from the simulation).

Finally, we show in Figure VII that the growth rate of leverage of the bank sector leads the growth rate of assets of the financial sector as a whole in both the model (left panel) and US data (right panel). In particular, we plot the cross-correlation function between the growth rate of leverage of the bank sector and the growth rate of assets of the financial sector (in blue) and the cross-correlation function between the growth rate of leverage of the nonbank financial sector and the growth rate of assets of the financial sector (in red). The growth rate of leverage of the bank sector is positively and significantly correlated with the growth rate of assets of the financial sector, with the highest correlation at the one quarter lag, while the correlation between the growth rate of leverage of the nonbank financial sector and the growth rate of assets of the financial sector is never significant. Thus, growth of leverage of the bank sector Granger-causes growth for the rest of the financial system. In the model, this arises because the bank sector has access to a better technology for the production of new capital. When the bank sector grows, investment in the productive sector grows as well, increasing the value of capital projects held by the nonbank financial sector and thus the value of assets of the financial system.

4 Conclusion

Intermediary balance sheet behaviors differ across financial sectors empirically. While banks' leverage is strongly procyclical and has forecasting power for intermediary sector asset growth, nonbank leverage is acyclical, and has no forecasting power. These empirical observations are qualitatively matched in a two intermediary sector macroeconomic model where the bank sector is subject to a risk-based capital constraint and the nonbank financial sector is subject to a skin-in-the-game constraint. In particular, the model gives rise to the following dynamics:

1. The financial sector is procyclical overall.
2. While total assets and mark-to-market equity of both types of intermediaries are procyclical, the leverage of the bank sector is procyclical and the leverage of the nonbank financial sector is acyclical.

3. Expansion of the balance sheet of the bank sector leads expansions of the balance sheets of other kinds of intermediaries.

The procyclicality of the bank sector arises due to the risk based leverage constraint, which mitigates the risk shifting problem. The fund sector, on the other hand, is subject to a skin in the game constraint on outside equity funding. The microfoundations of both types of constraints have been discussed in the literature by Adrian and Shin [2013] and Holmstrom and Tirole [1997], respectively. The contribution of the current paper is to show that these constraints give rise to aggregate intermediary balance sheet dynamics that match the data closely.

Our findings allow the reconciliation—both theoretically and empirically—of two previously alternative approaches to macroeconomic modeling of the financial sector as put forward by He and Krishnamurthy [2013] and Adrian and Boyarchenko [2012]. The approach by He and Krishnamurthy [2013] of modeling the financial sector as having a skin in the game constraint on outside equity should be viewed as a model of the nonbank financial sector, which primarily comprises investment funds. The risk based funding constraints on intermediaries of Adrian and Boyarchenko [2012], on the other hand, applies to the bank sector. In general equilibrium, the dynamic properties of the two sectors interact in such a way that the bank sector exhibits endogenously procyclical leverage, while the fund sector is acyclical. The dynamic properties of the nonbank sector are markedly different from He and Krishnamurthy [2013], as household allocate optimally between the bank and nonbank sectors in our setting. These findings matter for normative questions, as the cyclicality of leverage matters for policy conclusions.

References

Tobias Adrian and Nina Boyarchenko. Intermediary Leverage Cycle and Financial Stability. Federal Reserve Bank of New York Staff Reports 567, 2012.

Tobias Adrian and Hyun Song Shin. Liquidity and leverage. *Journal of Financial Intermediation*, 19(3):418 – 437, 2010.

Tobias Adrian and Hyun Song Shin. Procyclical Leverage and Value-at-Risk. *Review of Financial Studies*, *forthcoming*, 2013.

Tobias Adrian, Emanuel Moench, and Hyun Song Shin. Financial Intermediation, Asset Prices, and Macroeconomic Dynamics. Federal Reserve Bank of New York Staff Reports No. 442, 2010.

Tobias Adrian, Paolo Colla, and Hyun Song Shin. Which Financial Frictions? Parsing the Evidence from the Financial Crisis of 2007-09. In *NBER Macroeconomics Annual*, volume 27. University of Chicago Press, 2012.

Markus K. Brunnermeier and Yuliy Sannikov. A Macroeconomic Model with a Financial Sector. Unpublished working paper, Princeton University, 2012.

John Cox and Chi–fu Huang. Optimal Consumption and Portfolio Policies when Asset Prices follow a Diffusion Process. *Journal of Economic Theory*, 49:33–83, 1989.

Jakša Cvitanić and Ioannis Karatzas. Convex Duality in Constrained Portfolio Optimization. *The Annals of Applied Probability*, 2(4):767–818, 1992.

Jon Danielsson, Hyun Song Shin, and Jean-Pierre Zigrand. Balance sheet capacity and endogenous risk. Working Paper, 2011.

Zhiguo He and Arvind Krishnamurthy. A Model of Capital and Crises. *Review of Economic Studies*, 79(2):735–777, 2012a.

Zhiguo He and Arvind Krishnamurthy. A Macroeconomic Framework for Quantifying Systemic Risk. Unpublished working paper, 2012b.

Zhiguo He and Arvind Krishnamurthy. Intermediary Asset Pricing. *American Economic Review*, 103(2):732–770, 2013.

Bengt Holmstrom and Jean Tirole. Financial intermediation, loanable funds, and the real sector. *Quarterly Journal of Economics*, 112(3):663–691, 1997.

John H Kareken and Neil Wallace. Deposit insurance and bank regulation: A partial-equilibrium exposition. *Journal of Business*, 51(3):413–438, 1978.

Hayne E. Leland and Klaus Bjerre Toft. Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit Spreads. *Journal of Finance*, 51(3):987–1019, 1996.

Galo Nuño and Carlos Thomas. Bank Leverage Cycles. Unpublished Working Paper, 2012.

Oldrich Alfons Vasicek. Credit valuation. *KMV*, 1984.

A Portfolio allocation problems

A.1 Specialists' optimization

Recall that the fund managers solve

$$\max_{\{c_{ft},\theta_{ft}\}} \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}\log c_{ft}dt\right],$$

subject to the dynamic budget constraint

$$\frac{dw_{ft}}{w_{ft}} = \theta_{ft}\left(dR_{kt} - r_{ft}dt\right) + r_{ft}dt - \frac{c_{ft}}{w_{ft}}dt.$$

Instead of solving the dynamic optimization problem, we follow Cvitanić and Karatzas [1992] and rewrite the fund manager problem in terms of a static optimization. Cvitanić and Karatzas [1992] extend the Cox and Huang [1989] martingale method approach to constrained optimization problems, such as the one that the fund managers face in our economy. Notice that, even though the fund managers do not face no-shorting constraints, the market is incomplete from their point of view since they cannot invest in the risky debt of the bank sector. Introduce $\theta_{bf,t}$ to be the fraction of fund equity allocated to risky intermediary debt, and let $\vec{\theta}_{ft} \equiv [\theta_{ft}\ \theta_{bf,t}]'$ be the vector of portfolio choices of the fund at time t.
Define $\mathcal{K} = \mathbb{R}\times\{0\}$ to be the convex set of admissible portfolio strategies and introduce the support function of the set $-\mathcal{K}$ to be

$$\begin{aligned}\delta(x) = \delta(x|\,\mathcal{K}) &\equiv \sup_{\vec{\theta}_f\in\mathcal{K}}\left(-\vec{\theta}_f' x\right)\\ &= \begin{cases}0, & \text{if } x_1 = 0\\ +\infty, & \text{otherwise.}\end{cases}\end{aligned}$$

We can then define an auxiliary unconstrained optimization problem for the fund manager, with the returns in the auxiliary asset market defined as

$$\begin{aligned} r^v_{ft} &= r_{ft} + \delta(\vec{v}_t)\\ dR^v_{kt} &= \left(\mu_{Rk,t} + v_{1t} + \delta(\vec{v}_t)\right)dt + \sigma_{ka,t}dZ_{at} + \sigma_{k\xi,t}dZ_{\xi t}\\ dR^v_{bt} &= \left(\mu_{Rb,t} + v_{2t} + \delta(\vec{v}_t)\right)dt + \sigma_{ba,t}dZ_{at} + \sigma_{b\xi,t}dZ_{\xi t},\end{aligned}$$

for each $\vec{v}_t = [v_{1t}\ v_{2t}]'$ in the space $V(\mathcal{K})$ of square-integrable, progressively measurable processes taking values in $\mathcal{K}$. Corresponding to the auxiliary returns processes is an auxiliary state-price density

$$\frac{d\eta^v_t}{\eta^v_t} = -\left(r_{ft} + \delta(\vec{v}_t)\right)dt - \left(\vec{\mu}_{Rt} - r_{ft} + \vec{v}_t\right)'\left(\sigma'_{Rt}\right)^{-1}d\vec{Z}_t,$$

where

$$\vec{\mu}_{Rt} = \begin{bmatrix} \mu_{Rk,t} \\ \mu_{Rb,t} \end{bmatrix}; \quad \sigma_{Rt} = \begin{bmatrix} \sigma_{ka,t} & \sigma_{k\xi,t} \\ \sigma_{ba,t} & \sigma_{b\xi,t} \end{bmatrix}; \quad \vec{Z}_t = \begin{bmatrix} Z_{at} \\ Z_{\xi t} \end{bmatrix}.$$

The auxiliary unconstrained problem of the representative fund manager then becomes

$$\max_{c_{ft}} \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t} \log c_{ft} dt\right],$$

subject to the static budget constraint

$$w_{f0} = \mathbb{E}\left[\int_0^{+\infty} \eta_t^v c_{ft} dt\right].$$

The solution to the original constrained problem is then given by the solution to the unconstrained problem for the v that solves

$$\begin{aligned} \vec{v}_t^* &= \arg\min_{x_1=0} \left\{2\delta(x) + \left|\left|\sigma_{Rt}^{-1}\left(\vec{\mu}_{Rt} - r_{ft} + x\right)\right|\right|^2\right\} \\ &= \arg\min_{x_1=0} \left|\left|\sigma_{Rt}^{-1}\left(\vec{\mu}_{Rt} - r_{ft} + x\right)\right|\right|^2. \end{aligned}$$

Thus,

$$v_t^* = \begin{bmatrix} 0 \\ -\left(\mu_{Rb,t} - r_{ft}\right) \end{bmatrix}.$$

Consider now solving the auxiliary unconstrained problem. Taking the first order condition, we obtain

$$[c_t]: \quad 0 = \frac{e^{-\rho t}}{c_{ft}} - \lambda \eta_t^v,$$

or

$$c_{ft} = \frac{e^{-\rho t}}{\lambda \eta_t^v}.$$

Substituting into the static budget constraint, we obtain

$$\eta_t^v w_{ft} = \mathbb{E}_t\left[\int_t^{+\infty} \eta_s^v c_{fs} ds\right] = \mathbb{E}_t\left[\int_t^{+\infty} \frac{e^{-\rho s}}{\lambda} ds\right] = \frac{e^{-\rho t}}{\lambda \rho}.$$

Thus

$$c_{ft} = \rho w_{ft}.$$

To solve for the fund's optimal portfolio allocation, notice that:

$$\frac{d\left(\eta_t^v w_{ft}\right)}{\eta_t^v w_{ft}} = -\rho dt.$$

On the other hand, applying Itô's lemma, we obtain

$$\frac{d\left(\eta_t^v w_{ft}\right)}{\eta_t^v w_{ft}} = \frac{d\eta_t^v}{\eta_t^v} + \frac{dw_{ft}}{w_{ft}} + \frac{dw_{ft}}{w_{ft}}\frac{d\eta_t^v}{\eta_t^v}.$$

Equating the coefficients on the stochastic terms, we obtain

$$\vec{\theta}_{ft} = \left(\sigma_{Rt}\sigma_{Rt}'\right)^{-1}\left(\vec{\mu}_{Rt} - r_{ft} + \vec{v}_t\right).$$

A.2 Bankers' optimization

Recall that the representative banker solves

$$\max_{\theta_t,\theta_{bt},i_t,c_{bt}} \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}\log c_{bt} dt\right]$$

subject to the dynamic budget constraint

$$\frac{dw_t}{w_t} = \theta_t\left(dR_{kt} - r_{ft}dt + \left(\Phi\left(i_t\right) - \frac{i_t}{p_{kt}}\right)dt\right) - \theta_{bt}\left(dR_{bt} - r_{ft}dt\right) + r_{ft}dt - \frac{c_{bt}}{w_t}dt,$$

the risk-based capital constraint

$$\theta_t \leq \frac{1}{\alpha\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}},$$

and the adding-up constraint

$$\theta_{bt} = \theta_t - 1.$$

Notice first that the investment choice, i_t, only enters into the budget constraint directly. Thus, we can take the first order condition with respect to investment to obtain

$$\Phi\left(i_t\right)' = p_{kt}^{-1},$$

or

$$i_t = \frac{1}{\phi_1}\left(\frac{\phi_0^2\phi_1^2}{4}p_{kt}^2 - 1\right).$$

Consider now the optimal consumption and portfolio allocation choice of the representative banker. As with the fund managers, we rewrite the dynamic optimization as a static problem.

To make the derivation more concise, denote

$$\tilde{\mu}_{Rk,t} = \mu_{Rk,t} + \left(\Phi\left(i_t\right) - \frac{i_t}{p_{kt}} \right)$$
$$\tilde{\mu}_{Rt} = \left[\begin{array}{cc} \tilde{\mu}_{Rk,t} & \mu_{Rb,t} \end{array} \right]'.$$

As in the previous section, let

$$\mathcal{K}_t = \left[x \in \mathbb{R}^2 \middle| \ x_1 \leq \frac{1}{\alpha \sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}}, \ x_2 = 1 - x_1 \right]$$

be the convex set of admissible portfolio strategies as date t, with the support function of $-\mathcal{K}_t$ given by

$$\delta_t\left(x\right) = \begin{cases} -x_2 + \frac{x_2 - x_1}{\alpha \sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}} & \text{if} \quad x_1 \leq x_2 \\ +\infty & \text{otherwise} \end{cases}$$

and the barrier cone of $-\mathcal{K}_t$ by

$$\tilde{K} = \left[x \in \mathbb{R}^2 | x_1 \leq x_2 \right].$$

The auxiliary state-price density in this case is

$$\frac{d\eta_t^v}{\eta_t^v} = -\left(r_{ft} + \delta_t\left(\vec{v}_t\right)\right) dt - \left(\tilde{\mu}_{Rt} - r_{ft} + \vec{v}_t\right)' \left(\sigma'_{Rt}\right)^{-1} d\vec{Z}_t,$$

where the optimal $\vec{v}_t$ satisfies

$$\vec{v}_t^* = \arg\min_{x \in \tilde{K}} \left[2\delta\left(x\right) + \left\| \sigma_{Rt}^{-1}\left(\tilde{\mu}_{Rt} - r_{ft}\right) + \sigma_{Rt}^{-1} x \right\|^2 \right].$$

Denote the Lagrange multiplier on $x_1 \leq x_2$ by $2\phi_1$. Taking the first order conditions, we obtain

$$2 \left[\begin{array}{c} -\frac{1}{\alpha \sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}} \\ \frac{1}{\alpha \sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}} - 1 \end{array} \right] + 2\left(\sigma'_{Rt}\right)^{-1} \left(\sigma_{Rt}^{-1}\left(\tilde{\mu}_{Rt} - r_{ft}\right) + \sigma_{Rt}^{-1} x \right) + 2\phi_1 \left[\begin{array}{c} 1 \\ -1 \end{array} \right] = 0,$$

or

$$\vec{v}_t^* = \sigma_{Rt}\sigma'_{Rt} \left[\begin{array}{c} \frac{1}{\alpha \sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}} \\ -\frac{1}{\alpha \sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}} + 1 \end{array} \right] + \phi_1 \sigma_{Rt}\sigma'_{Rt} \left[\begin{array}{c} -1 \\ 1 \end{array} \right] - \left[\begin{array}{c} \mu_{Rk,t} + \Phi\left(i_t\right) - \frac{i_t}{p_{kt}} - r_{ft} \\ \mu_{Rb,t} - r_{ft} \end{array} \right].$$

Thus, if $\phi_1 = 0$, so that banks are at the risk-based capital constraint,

$$\vec{v}_t^* = \sigma_{Rt}\sigma'_{Rt} \left[\begin{array}{c} \frac{1}{\alpha \sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}} \\ -\frac{1}{\alpha \sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}} + 1 \end{array} \right] - \left[\begin{array}{c} \mu_{Rk,t} + \Phi\left(i_t\right) - \frac{i_t}{p_{kt}} - r_{ft} \\ \mu_{Rb,t} - r_{ft} \end{array} \right]$$

$$
= \begin{bmatrix} -\left(\widetilde{\mu}_{Rk,t} - r_{ft}\right) - \left(\frac{1}{\alpha\sqrt{\sigma_{ka,t}^2+\sigma_{k\xi,t}^2}} - 1\right)\left(\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}\right) + \frac{1}{\alpha}\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2} \\ -\left(\mu_{Rb,t} - r_{ft}\right) - \left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right)\left(\frac{1}{\alpha\sqrt{\sigma_{ka,t}^2+\sigma_{k\xi,t}^2}} - 1\right) + \frac{1}{\alpha\sqrt{\sigma_{ka,t}^2+\sigma_{k\xi,t}^2}}\left(\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}\right) \end{bmatrix}.
$$

Otherwise, $v_1^* = v_2^*$. Solving for ϕ_1, we obtain

$$
\phi_1 = \frac{-\left(\widetilde{\mu}_{Rk,t} - r_{ft}\right) + \left(\mu_{Rb,t} - r_{ft}\right)}{\left(\left(\sigma_{ba,t} - \sigma_{ka,t}\right)^2 + \left(\sigma_{b\xi,t} - \sigma_{k\xi,t}\right)^2\right)} + \frac{1}{\alpha\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}} + \frac{\left(\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}\right) - \left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right)}{\left(\left(\sigma_{ba,t} - \sigma_{ka,t}\right)^2 + \left(\sigma_{b\xi,t} - \sigma_{k\xi,t}\right)^2\right)}.
$$

Substituting into the equation for v_2^*, we obtain

$$
\begin{aligned}
v_1 = v_2 = &-\left(\widetilde{\mu}_{Rk,t} - r_{ft}\right)\frac{\left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right) - \left(\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}\right)}{\left(\left(\sigma_{ba,t} - \sigma_{ka,t}\right)^2 + \left(\sigma_{b\xi,t} - \sigma_{k\xi,t}\right)^2\right)} \\
&- \left(\mu_{Rb,t} - r_{ft}\right)\frac{\left(\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2 - \left(\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}\right)\right)}{\left(\left(\sigma_{ba,t} - \sigma_{ka,t}\right)^2 + \left(\sigma_{b\xi,t} - \sigma_{k\xi,t}\right)^2\right)} + \left(\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}\right) \\
&- \frac{\left(\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}\right) - \left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right)}{\left(\left(\sigma_{ba,t} - \sigma_{ka,t}\right)^2 + \left(\sigma_{b\xi,t} - \sigma_{k\xi,t}\right)^2\right)}\left(\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2 - \left(\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}\right)\right)
\end{aligned}
$$

Turn now to the solution of the augmented unconstrained problem. Analogously to the fund managers, the representative banker solves

$$
\max_{c_{bt}} \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}\log c_{bt}dt\right]
$$

subject to the static budget constraint

$$
w_0 = \mathbb{E}\left[\int_0^{+\infty} \eta_t^v c_{bt}dt\right].
$$

Taking the first order condition, we obtain

$$
[c_{bt}]: \quad 0 = \frac{e^{-\rho t}}{c_{bt}} - \lambda\eta_t^v,
$$

or

$$
c_{bt} = \frac{e^{-\rho t}}{\lambda\eta_t^v}.
$$

Substituting into the static budget constraint, we obtain

$$
\eta_t^v w_t = \mathbb{E}_t\left[\int_t^{+\infty} \eta_s^v c_{bs}ds\right] = \mathbb{E}_t\left[\int_t^{+\infty} \frac{e^{-\rho s}}{\lambda}ds\right] = \frac{e^{-\rho t}}{\lambda\rho}.
$$

Thus

$$c_{bt} = \rho w_t.$$

To solve for the bank's optimal portfolio allocation, notice that:

$$\frac{d\left(\eta_t^v w_t\right)}{\eta_t^v w_t} = -\rho dt.$$

On the other hand, applying Itô's lemma, we obtain

$$\frac{d\left(\eta_t^v w_t\right)}{\eta_t^v w_t} = \frac{d\eta_t^v}{\eta_t^v} + \frac{dw_t}{w_t} + \frac{dw_t}{w_t}\frac{d\eta_t^v}{\eta_t^v}.$$

Equating the coefficients on the stochastic terms, we obtain

$$\vec{\theta}_t = \left(\sigma_{Rt}\sigma_{Rt}'\right)^{-1}\left(\tilde{\mu}_{Rt} - r_{ft} + \vec{v}_t\right).$$

When the bank is unconstrained, $v_1^* = 0$ and the optimal portfolio allocation is

$$\begin{aligned}
\begin{bmatrix} \theta_t \\ -\theta_{bt} \end{bmatrix} &= \left(\begin{bmatrix} \sigma_{ka,t} & \sigma_{k\xi,t} \\ \sigma_{ba,t} & \sigma_{b\xi,t} \end{bmatrix}\begin{bmatrix} \sigma_{ka,t} & \sigma_{ba,t} \\ \sigma_{k\xi,t} & \sigma_{b\xi,t} \end{bmatrix}\right)^{-1}\begin{pmatrix} \mu_{Rk,t} + \Phi\left(i_t\right) - \frac{i_t}{p_{kt}} - r_{ft} + v_1 \\ \mu_{Rb,t} - r_{ft} + v_2 \end{pmatrix} \\
&= \begin{bmatrix} \frac{\sigma_{ba,t}^2 - \sigma_{ka,t}\sigma_{ba,t} + \sigma_{b\xi,t}^2 - \sigma_{k\xi,t}\sigma_{b\xi,t} - \left(\mu_{Rb,t} - r_{ft}\right) + \left(\mu_{Rk,t} + \Phi(i_t) - \frac{i_t}{p_{kt}} - r_{ft}\right)}{\left(\left(\sigma_{ba,t} - \sigma_{ka,t}\right)^2 + \left(\sigma_{b\xi,t} - \sigma_{k\xi,t}\right)^2\right)} \\ \frac{\sigma_{ka,t}^2 - \sigma_{ba,t}\sigma_{ka,t} + \sigma_{k\xi,t}^2 - \sigma_{b\xi,t}\sigma_{k\xi,t} + \left(\mu_{Rb,t} - r_{ft}\right) - \left(\mu_{Rk,t} + \Phi(i_t) - \frac{i_t}{p_{kt}} - r_{ft}\right)}{\left(\left(\sigma_{ba,t} - \sigma_{ka,t}\right)^2 + \left(\sigma_{b\xi,t} - \sigma_{k\xi,t}\right)^2\right)} \end{bmatrix}.
\end{aligned}$$

A.3 Households' optimization

Recall that the representative household solves

$$\max_{\substack{\pi_{kt}, \pi_{bt} \\ c_t}} \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t - \rho_h t}\log c_t dt\right],$$

subject to the dynamic budget constraint

$$\frac{dw_{ht}}{w_{ht}} = \pi_{kt}\theta_{ft}\left(dR_{kt} - r_{ft}dt\right) + \pi_{bt}\left(dR_{bt} - r_{ft}dt\right) + r_{ft}dt - \frac{c_t}{w_{ht}}dt,$$

the skin-in-the-game constraint

$$\pi_{kt} \leq m\frac{w_{ft}}{w_{ht}},$$

and no shorting constraints

$$\begin{aligned} \pi_{kt} &\geq 0 \\ \pi_{bt} &\geq 0. \end{aligned}$$

Notice that the representative household takes the skin-in-the-game constraint as given. In particular, denote the relative wealth of the fund manager to be

$$\omega_{ft} = \frac{w_{ft}}{w_t + w_{ft} + w_{ht}}$$

and the relative wealth of the bankers to be

$$\omega_t = \frac{w_t}{w_t + w_{ft} + w_{ht}},$$

so that the constraint can be represented as

$$\pi_{kt} \leq m \frac{\omega_{ft}}{1 - \omega_{ft} - \omega_t}.$$

The representative household then takes $\omega_{ft}/(1-\omega_{ft}-\omega_t)$ as given in solving for the optimal consumption and portfolio allocation rules.
The time t convex set of admissible portfolio strategies is then $\mathcal{K}_t = \mathbb{R}_+ \times [0, m\omega_{ft}/(1-\omega_{ft}-\omega_t)]$, with the support function

$$\begin{aligned}\delta_t(x) = \delta_t(x|\mathcal{K}_t) &\equiv \sup_{\vec{\pi}\in\mathcal{K}_t}(-\vec{\pi}'x) \\ &= m\frac{\omega_{ft}}{1-\omega_{ft}-\omega_t}x_1\mathbf{1}_{x_1\leq 0} + \infty x_2\mathbf{1}_{x_2\leq 0}.\end{aligned}$$

For future use, introduce also the barrier cone $\tilde{\mathcal{K}}_t$ of $-\mathcal{K}_t$ to be

$$\tilde{\mathcal{K}}_t = \left\{x \in \mathbb{R}^2 : \ x_2 \geq 0\right\}.$$

We can then define an auxiliary unconstrained optimization problem for the household, with the returns in the auxiliary asset market defined as

$$\begin{aligned} r_{ft}^v &= r_{ft} + \delta(\vec{v}_t) \\ dR_{kt}^v &= (\mu_{Rk,t} + v_{1t} + \delta(\vec{v}_t))\,dt + \sigma_{ka,t}dZ_{at} + \sigma_{k\xi,t}dZ_{\xi t} \\ dR_{bt}^v &= (\mu_{Rb,t} + v_{2t} + \delta(\vec{v}_t))\,dt + \sigma_{ba,t}dZ_{at} + \sigma_{b\xi,t}dZ_{\xi t}, \end{aligned}$$

for each $\vec{v}_t = [v_{1t}\ v_{2t}]'$ in the space $V(\mathcal{K})$ of square-integrable, progressively measurable processes taking values in $\mathcal{K}$. Corresponding to the auxiliary returns processes is an auxiliary state-price density

$$\frac{d\eta_t^v}{\eta_t^v} = -\left(r_{ft} + \delta(\vec{v}_t)\right)dt - \left(\vec{\mu}_{Rt} - r_{ft} + \vec{v}_t\right)'\left(\sigma_{Rt}'\right)^{-1}d\vec{Z}_t,$$

where

$$\vec{\mu}_{Rt} = \begin{bmatrix}\mu_{Rk,t}\\ \mu_{Rb,t}\end{bmatrix}; \quad \sigma_{Rt} = \begin{bmatrix}\sigma_{ka,t} & \sigma_{k\xi,t}\\ \sigma_{ba,t} & \sigma_{b\xi,t}\end{bmatrix}; \quad \vec{Z}_t = \begin{bmatrix}Z_{at}\\ Z_{\xi t}\end{bmatrix}.$$

The auxiliary unconstrained problem of the representative household then becomes

$$\max_{c_t} \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t-\rho_h t}\log c_t dt\right],$$

subject to the static budget constraint:

$$w_{h0} = \mathbb{E}\left[\int_0^{+\infty} \eta_t^v c_t dt\right].$$

The solution to the original constrained problem is then given by the solution to the unconstrained problem for the v that solves the dual problem

$$\min_{v\in V(\mathcal{K})} \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t-\rho_h t}\tilde{u}\left(\lambda\eta_t^v\right)dt\right],$$

where $\tilde{u}\left(x\right)$ is the convex conjugate of $-u\left(-x\right)$

$$\tilde{u}\left(x\right) \equiv \sup_{z>0}\left[\log\left(zx\right) - zx\right] = -\left(1+\log x\right)$$

and λ is the Lagrange multiplier of the static budget constraint. Extending the result of Cvitanić and Karatzas [1992] for logarithmic utility to the case of a random rate of time discounting, the optimal choice of v satisfies

$$\begin{aligned}
\vec{v}_t^* &= \arg\min_{x\in\tilde{\mathcal{K}}_t}\left\{2\delta(x) + \left|\left|\sigma_{Rt}^{-1}\left(\vec{\mu}_{Rt} - r_{ft} + x\right) + \Sigma_\xi\right|\right|^2\right\} \\
&= \arg\min_{x\in\tilde{\mathcal{K}}_t}\left\{2m\frac{\omega_{ft}}{1-\omega_{ft}-\omega_t}x_1\mathbf{1}_{x_1\leq 0} + \left|\left|\sigma_{Rt}^{-1}\left(\vec{\mu}_{Rt} - r_{ft} + x\right) + \Sigma_\xi\right|\right|^2\right\},
\end{aligned}$$

where

$$\Sigma_\xi = \left[\begin{array}{c} 0 \\ \sigma_\xi \end{array}\right].$$

Taking the first order conditions and solving for the variables of interest, we obtain

$$\begin{aligned}
x_1 &= \begin{cases} 2m\frac{\omega_{ft}}{1-\omega_{ft}-\omega_t} - \sigma_{Rt}^{-1}\left(\vec{\mu}_{Rt} - r_{ft}\right), & \text{if } \mu_{Rk,t} - r_{ft} \geq 2m\frac{\omega_{ft}}{1-\omega_{ft}-\omega_t} \\ 0, & \text{otherwise} \end{cases}, \\
x_2 &= -\left(\mu_{Rb,t} - r_{ft}\right) + \frac{\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}}{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}\left(\mu_{Rk,t} - r_{ft} + x_1\right).
\end{aligned}$$

Consider now solving the auxiliary unconstrained problem. Taking the first order condition, we obtain

$$\left[c_t\right]: \quad 0 = \frac{e^{-\xi_t-\rho_h t}}{c_t} - \lambda\eta_t^v,$$

or

$$c_t = \frac{e^{-\xi_t - \rho_h t}}{\lambda \eta_t^v}.$$

Substituting into the static budget constraint, we obtain

$$\eta_t^v w_{ht} = \mathbb{E}_t \left[\int_t^{+\infty} \eta_s^v c_s ds \right] = \mathbb{E}_t \left[\int_t^{+\infty} \frac{e^{-\xi_s - \rho_h s}}{\lambda} ds \right] = \frac{e^{-\xi_t - \rho_h t}}{\lambda \left(\rho_h - \sigma_\xi^2 / 2 \right)}.$$

Thus

$$c_t = \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) w_{ht}.$$

To solve for the household's optimal portfolio allocation, notice that:

$$\frac{d\left(\eta_t^v w_{ht}\right)}{\eta_t^v w_{ht}} = -\rho_h dt - d\xi_t + \frac{1}{2} d\xi_t^2 = \left(-\rho_h + \frac{1}{2}\sigma_\xi^2 \right) dt - \sigma_\xi dZ_{\xi t}.$$

On the other hand, applying Itô's lemma, we obtain

$$\frac{d\left(\eta_t^v w_{ht}\right)}{\eta_t^v w_{ht}} = \frac{d\eta_t^v}{\eta_t^v} + \frac{dw_{ht}}{w_{ht}} + \frac{dw_{ht}}{w_{ht}} \frac{d\eta_t^v}{\eta_t^v}.$$

Equating the coefficients on the stochastic terms, we obtain

$$\vec{\pi}_t' = \left(\vec{\mu}_{Rt} - r_{ft} + \vec{v}_t \right)' \left(\sigma_{Rt}' \sigma_{Rt} \right)^{-1} - \sigma_\xi \left[0 \quad 1 \right] \sigma_{Rt}^{-1}.$$

B Solving for the equilibrium

B.1 *Equilibrium capital price evolution*

To derive the equilibrium evolution of the price of capital, recall that the market clearing condition for capital gives

$$K_t = k_t + k_{ft}.$$

Dividing both sides by K_t, we obtain

$$1 = \frac{p_{kt} A_t k_t}{p_{kt} A_t K_t} + \frac{p_{kt} A_t k_{ft}}{p_{kt} A_t K_t} = \frac{w_t p_{kt} A_t k_t}{w_t p_{kt} A_t K_t} + \frac{p_{kt} A_t k_{ft}}{w_{ft}} \frac{w_{ft}}{p_{kt} A_t K_t} = \theta_t \omega_t + \theta_{ft} \omega_{ft},$$

so that

$$\theta_{ft} = \frac{1 - \theta_t \omega_t}{\omega_{ft}}.$$

Similarly, the debt market clearing condition gives

$$0 = b_t + b_{ht} = -\theta_{bt}\omega_t + \pi_{bt}\left(1 - \omega_t - \omega_{ft}\right),$$

or

$$\pi_{bt} = \frac{\theta_{bt}\omega_t}{1 - \omega_t - \omega_{ft}}.$$

Recall further that the price of capital satisfies

$$0 = p_{kt}^2\theta_t\omega_t + 2p_{kt}\left(\rho_1 + \rho_0\left(\omega_t + \omega_{ft}\right)\right) - \frac{4}{\phi_0^2\phi_1^2}\left(\phi_1 + \theta_t\omega_t\right),$$

and, in terms of moments of the excess return to holding capital, we can represent

$$\frac{dp_{kt}}{p_{kt}} = \left(\mu_{Rk,t} - \frac{1}{p_{kt}} + \lambda_k - \bar{a} - \frac{\sigma_a^2}{2} - \sigma_a\left(\sigma_{ka,t} - \sigma_a\right)\right)dt + \left(\sigma_{ka,t} - \sigma_a\right)dZ_{at} + \sigma_{k\xi,t}dZ_{\xi t}.$$

Then, applying Itô's lemma, we obtain

$$\begin{aligned}
0 &= 2p_{kt}^2\theta_t\omega_t\frac{dp_{kt}}{p_{kt}} + p_{kt}^2\theta_t\omega_t\left\langle\frac{dp_{kt}}{p_{kt}}\right\rangle^2 + \left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\right)\theta_t\omega_t\frac{d\left(\theta_t\omega_t\right)}{\theta_t\omega_t} \\
&\quad + 2p_{kt}\left(\rho_1 + \rho_0\left(\omega_t + \omega_{ft}\right)\right)\frac{dp_{kt}}{p_{kt}} + 2\rho_0 p_{kt}\omega_t\frac{d\omega_t}{\omega_t} + 2\rho_0 p_{kt}\omega_t\left\langle\frac{dp_{kt}}{p_{kt}}, \frac{d\omega_t}{\omega_t}\right\rangle \\
&\quad + 2\rho_0 p_{kt}\omega_{ft}\frac{d\omega_{ft}}{\omega_{ft}} + 2\rho_0 p_{kt}\omega_{ft}\left\langle\frac{dp_{kt}}{p_{kt}}, \frac{d\omega_{ft}}{\omega_{ft}}\right\rangle.
\end{aligned}$$

Hence, equating coefficients, we obtain

$$\begin{aligned}
[dt:] \quad 0 &= \left(2\theta_t\omega_t p_{kt}^2 + 2p_{kt}\left(\rho_0\left(\omega_t + \omega_{ft}\right) + \rho_1\right)\right)\left(\mu_{Rk,t} - \frac{1}{p_{kt}} + \lambda_k - \bar{a} - \frac{\sigma_a^2}{2} - \sigma_a\left(\sigma_{ka,t} - \sigma_a\right)\right) \\
&\quad + \theta_t\omega_t p_{kt}^2\left(\sigma_{ka,t} - \sigma_a\right)^2 + \theta_t\omega_t p_{kt}^2\sigma_{k\xi,t}^2 + \left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\right)\theta_t\omega_t\Phi\left(i_t\right)\left(1 - \theta_t\omega_t\right) \\
&\quad + 2\rho_0 p_{kt}\omega_t\mu_{\omega t} + 2\rho_0\omega_t p_{kt}\left(\sigma_{\omega a,t}\left(\sigma_{ka,t} - \sigma_a\right) + \sigma_{k\xi,t}\sigma_{\omega\xi,t}\right) \\
&\quad + 2\rho_0 p_{kt}\omega_{ft}\mu_{\omega_f t} + 2\rho_0\omega_{ft}p_{kt}\left(\sigma_{\omega_f a,t}\left(\sigma_{ka,t} - \sigma_a\right) + \sigma_{k\xi,t}\sigma_{\omega_f\xi,t}\right) \\
[dZ_{at}]: \quad 0 &= \left(2\theta_t\omega_t p_{kt}^2 + 2p_{kt}\left(\rho_0\left(\omega_t + \omega_{ft}\right) + \rho_1\right)\right)\left(\sigma_{ka,t} - \sigma_a\right) - 2\rho_0 p_{kt}\omega_t\sigma_{\theta a,t} + 2\rho_0 p_{kt}\omega_{ft}\sigma_{\omega_f a,t} \\
[dZ_{\xi t}]: \quad 0 &= \left(2\theta_t\omega_t p_{kt}^2 + 2p_{kt}\left(\rho_0\left(\omega_t + \omega_{ft}\right) + \rho_1\right)\right)\sigma_{k\xi,t} - 2\rho_0 p_{kt}\omega_t\sigma_{\theta\xi,t} + 2\rho_0 p_{kt}\omega_{ft}\sigma_{\omega_f\xi,t}.
\end{aligned}$$

Recall from the equilibrium evolution of the wealth of the fund manager

$$\begin{aligned}
\sigma_{\omega_f a,t} &= \left(\theta_{ft} - 1\right)\sigma_{ka,t} \\
\sigma_{\omega_f\xi,t} &= \left(\theta_{ft} - 1\right)\sigma_{k\xi,t}.
\end{aligned}$$

Then, using $[dZ_{at}]$ and $[dZ_{\xi t}]$, we can represent

$$\sigma_{\theta a,t} = \frac{\left(\theta_t \omega_t p_{kt} + \rho_0 \left(1 - \omega_t \theta_t + \omega_t\right) + \rho_1\right)}{\rho_0 \omega_t} \left(\sigma_{ka,t} - \sigma_a\right) + \frac{1 - \theta_t \omega_t - \omega_{ft}}{\omega_t} \sigma_a$$
$$\sigma_{\theta \xi,t} = \frac{\left(\theta_t \omega_t p_{kt} + \rho_0 \left(1 - \omega_t \theta_t + \omega_t\right) + \rho_1\right)}{\rho_0 \omega_t} \sigma_{k\xi,t}.$$

B.2 Equilibrium prices of risk

Recall that, in equilibrium, the prices of the fundamental risks a and ξ are given, respectively, by

$$\eta_{at} = \pi_{kt} \theta_{ft} \sigma_{ka,t} + \pi_{bt} \sigma_{ba,t}$$
$$\eta_{\xi t} = \pi_{kt} \theta_{ft} \sigma_{k\xi,t} + \pi_{bt} \sigma_{b\xi,t} + \sigma_\xi,$$

and the prices of leverage and output risk by

$$\eta_{\theta t} = \eta_{\xi t} \sqrt{1 + \frac{\sigma_{\theta a,t}^2}{\sigma_{\theta \xi,t}^2}}$$
$$\eta_{yt} = \eta_{at} - \eta_{\xi t} \frac{\sigma_{\theta a,t}}{\sigma_{\theta \xi,t}}.$$

We begin by deriving the prices of fundamental risk in the case when households are unconstrained. Then, we can express the expected excess return on bank debt as

$$\mu_{Rb,t} - r_{ft} = \frac{1 - \theta_t \omega_t}{\omega_{ft}} \left(\sigma_{ba,t} \sigma_{ka,t} + \sigma_{b\xi,t} \sigma_{k\xi,t}\right) + \frac{\theta_{bt} \omega_t}{1 - \omega_t - \omega_{ft}} \frac{\left(\sigma_{ba,t} \sigma_{k\xi,t} - \sigma_{ka,t} \sigma_{b\xi,t}\right)^2}{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}$$
$$+ \sigma_\xi \sigma_{ka,t} \frac{\sigma_{b\xi,t} \sigma_{ka,t} - \sigma_{k\xi,t} \sigma_{ba,t}}{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2},$$

so that

$$\eta_{at} = \frac{1 - \theta_t \omega_t}{\omega_{ft}} \sigma_{ka,t} + \frac{\theta_{bt} \omega_t}{1 - \omega_t - \omega_{ft}} \frac{\left(\sigma_{ba,t} \sigma_{k\xi,t} - \sigma_{ka,t} \sigma_{b\xi,t}\right) \sigma_{k\xi,t}}{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2} - \sigma_\xi \sigma_{ka,t} \frac{\sigma_{k\xi,t}}{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}$$
$$\eta_{\xi t} = \frac{1 - \theta_t \omega_t}{\omega_{ft}} \sigma_{k\xi,t} - \frac{\theta_{bt} \omega_t}{1 - \omega_t - \omega_{ft}} \frac{\left(\sigma_{ba,t} \sigma_{k\xi,t} - \sigma_{ka,t} \sigma_{b\xi,t}\right) \sigma_{ka,t}}{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2} + \sigma_\xi \sigma_{ka,t} \frac{\sigma_{ka,t}}{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}.$$

Next, recall from the equilibrium evolution of the wealth share of bankers that

$$\theta_{bt} \sigma_{ba,t} = \left(\theta_t - 1\right) \sigma_{ka,t} + \sigma_{\theta a,t}$$
$$\theta_{bt} \sigma_{b\xi,t} = \left(\theta_t - 1\right) \sigma_{k\xi,t} + \sigma_{\theta \xi,t},$$

and from the equilibrium evolution of the price of capital that

$$\sigma_{\theta a,t} = \frac{\left(\theta_t \omega_t p_{kt} + \rho_0 \left(1 - \omega_t \theta_t - \omega_{ft}\right) + \rho_1\right)}{\rho_0 \omega_t} \left(\sigma_{ka,t} - \sigma_a\right) + \frac{1 - \theta_t \omega_t - \omega_{ft}}{\omega_t} \sigma_a$$

$$\sigma_{\theta\xi,t} = \frac{(\theta_t\omega_t p_{kt} + \rho_0 (1 - \omega_t\theta_t - \omega_{ft}) + \rho_1)}{\rho_0\omega_t}\sigma_{k\xi,t}.$$

Hence, we can represent

$$\begin{aligned}
\theta_{bt}\sigma_{ba,t} &= \left(\frac{\theta_t\omega_t p_{kt} + \rho_0 (1 - \omega_t - \omega_{ft}) + \rho_1 + \rho_0}{\rho_0\omega_t}\right)(\sigma_{ka,t} - \sigma_a) + \left(\frac{1 - \omega_{ft} - \omega_t}{\omega_t}\right)\sigma_a \\
\theta_{bt}\sigma_{b\xi,t} &= \left(\frac{\theta_t\omega_t p_{kt} + \rho_0 (1 - \omega_t - \omega_{ft}) + \rho_1 + \rho_0}{\rho_0\omega_t}\right)\sigma_{k\xi,t}.
\end{aligned}$$

Substituting, the equilibrium prices of leverage and output risk are given, respectively, by

$$\begin{aligned}
\eta_{\theta t} &= \sqrt{1 + \frac{\sigma^2_{\theta a,t}}{\sigma^2_{\theta\xi,t}}}\left(\frac{1 - \theta_t\omega_t}{\omega_{ft}}\sigma_{k\xi,t} + \sigma_\xi\frac{\sigma^2_{ka,t}}{\sigma^2_{ka,t} + \sigma^2_{k\xi,t}} + \sigma_a\frac{\omega_t}{1 - \omega_t - \omega_{ft}}\frac{\sigma_{ka,t}\sigma_{k\xi,t}}{\sigma^2_{ka,t} + \sigma^2_{k\xi,t}}\left(\frac{\theta_t\omega_t p_{kt} + \rho_1 + \rho_0}{\rho_0\omega_t}\right)\right) \\
\eta_{yt} &= \frac{1 - \theta_t\omega_t}{\omega_{ft}}\sigma_{ka,t} - \sigma_\xi\sigma_{ka,t}\frac{\sigma_{k\xi,t}}{\sigma^2_{ka,t} + \sigma^2_{k\xi,t}} - \sigma_a\frac{\omega_t}{1 - \omega_t - \omega_{ft}}\frac{\sigma^2_{k\xi,t}}{\sigma^2_{ka,t} + \sigma^2_{k\xi,t}}\left(\frac{\theta_t\omega_t p_{kt} + \rho_1 + \rho_0}{\rho_0\omega_t}\right) \\
&\quad - \frac{\sigma_{\theta a,t}}{\sigma_{\theta\xi,t}}\left(\frac{1 - \theta_t\omega_t}{\omega_{ft}}\sigma_{k\xi,t} + \sigma_\xi\frac{\sigma^2_{ka,t}}{\sigma^2_{ka,t} + \sigma^2_{k\xi,t}} + \sigma_a\frac{\omega_t}{1 - \omega_t - \omega_{ft}}\frac{\sigma_{ka,t}\sigma_{k\xi,t}}{\sigma^2_{ka,t} + \sigma^2_{k\xi,t}}\left(\frac{\theta_t\omega_t p_{kt} + \rho_1 + \rho_0}{\rho_0\omega_t}\right)\right).
\end{aligned}$$

Consider now the pricing kernel when the households are constrained in their allocation to the fund sector. In this case, we can represent the expected excess return to bank debt as

$$\mu_{Rb,t} - r_{ft} = \frac{m(1 - \theta_t\omega_t)}{1 - \omega_t - \omega_{ft}}(\sigma_{ba,t}\sigma_{ka,t} + \sigma_{b\xi,t}\sigma_{k\xi,t}) + \frac{\theta_{bt}\omega_t\left(\sigma^2_{ba,t} + \sigma^2_{b\xi,t}\right)}{1 - \omega_t - \omega_{ft}} + \sigma_\xi\sigma_{b\xi,t},$$

so that

$$\begin{aligned}
\eta_{at} &= \frac{m(1 - \theta_t\omega_t)}{1 - \omega_t - \omega_{ft}}\sigma_{ka,t} + \frac{\theta_{bt}\omega_t\sigma_{ba,t}}{1 - \omega_t - \omega_{ft}} \\
\eta_{\xi t} &= \frac{m(1 - \theta_t\omega_t)}{1 - \omega_t - \omega_{ft}}\sigma_{k\xi,t} + \frac{\theta_{bt}\omega_t\sigma_{b\xi,t}}{1 - \omega_t - \omega_{ft}} + \sigma_\xi.
\end{aligned}$$

Substituting once again for $\theta_{bt}\sigma_{ba,t}$ and $\theta_{bt}\sigma_{b\xi,t}$, the equilibrium prices of leverage and output risk are given, respectively, by

$$\begin{aligned}
\eta_{\theta t} &= \sqrt{1 + \frac{\sigma^2_{\theta a,t}}{\sigma^2_{\theta\xi,t}}}\left(\frac{m(1 - \theta_t\omega_t)}{1 - \omega_t - \omega_{ft}}\sigma_{k\xi,t} + \sigma_{k\xi,t} + \frac{1}{\rho_0}\frac{(\theta_t\omega_t p_{kt} + \rho_1 + \rho_0)\,\sigma_{k\xi,t}}{1 - \omega_t - \omega_{ft}} + \sigma_\xi\right) \\
\eta_{yt} &= \frac{m(1 - \theta_t\omega_t)}{1 - \omega_t - \omega_{ft}}\sigma_{ka,t} + \sigma_{ka,t} + \left(\frac{\theta_t\omega_t p_{kt} + \rho_1 + \rho_0}{\rho_0(1 - \omega_t - \omega_{ft})}\right)(\sigma_{ka,t} - \sigma_a) \\
&\quad - \frac{\sigma_{\theta a,t}}{\sigma_{\theta\xi,t}}\left(\frac{m(1 - \theta_t\omega_t)}{1 - \omega_t - \omega_{ft}}\sigma_{k\xi,t} + \sigma_{k\xi,t} + \frac{1}{\rho_0}\frac{(\theta_t\omega_t p_{kt} + \rho_1 + \rho_0)\,\sigma_{k\xi,t}}{1 - \omega_t - \omega_{ft}} + \sigma_\xi\right).
\end{aligned}$$

B.3 Risk free rate

The Euler equation of the funds gives an expression for the risk free rate

$$r_{ft} = \frac{1}{dt}\mathbb{E}_t\left[\frac{dc_{ft}}{c_{ft}} - \frac{1}{2}\left\langle\frac{dc_{ft}}{c_{ft}}\right\rangle^2 + \rho\right].$$

Recall that the goods market clearing condition gives

$$c_t + c_{ft} + c_{bt} + i_t A_t k_t = A_t K_t.$$

Solving for c_{ft}, we can express

$$c_{ft} = A_t K_t\left(1 - i_t\theta_t\omega_t\right) - \rho\left(p_{kt}A_tK_t\right)\omega_t - \left(\rho_h - \frac{\sigma_\xi^2}{2}\right)\left(p_{kt}A_tK_t\right)\left(1 - \omega_t - \omega_{ft}\right).$$

Applying Itô's lemma, we obtain

$$\begin{aligned}
dc_{ft} = {} & A_tK_t\left(1 - i_t\theta_t\omega_t\right)\frac{d\left(A_tK_t\right)}{A_tK_t} - A_tK_ti_t\theta_t\omega_t\frac{d\left(\theta_t\omega_t\right)}{\theta_t\omega_t} - A_tK_t\theta_t\omega_tdi_t \\
& - A_tK_t\theta_t\omega_t\left\langle di_t, \frac{d\left(A_tK_t\right)}{A_tK_t}\right\rangle - \rho\left(p_{kt}A_tK_t\right)\omega_t\left[\frac{d\omega_t}{\omega_t} + \frac{d\left(p_{kt}A_tK_t\right)}{p_{kt}A_tK_t} + \left\langle\frac{d\omega_t}{\omega_t}, \frac{d\left(p_{kt}A_tK_t\right)}{p_{kt}A_tK_t}\right\rangle\right] \\
& - \left(\rho_h - \frac{\sigma_\xi^2}{2}\right)\left(p_{kt}A_tK_t\right)\left(1 - \omega_t - \omega_{ft}\right)\frac{d\left(p_{kt}A_tK_t\right)}{p_{kt}A_tK_t} \\
& + \left(\rho_h - \frac{\sigma_\xi^2}{2}\right)\left(p_{kt}A_tK_t\right)\omega_t\left[\frac{d\omega_t}{\omega_t} + \left\langle\frac{d\omega_t}{\omega_t}, \frac{d\left(p_{kt}A_tK_t\right)}{p_{kt}A_tK_t}\right\rangle\right] \\
& + \left(\rho_h - \frac{\sigma_\xi^2}{2}\right)\left(p_{kt}A_tK_t\right)\omega_{ft}\left[\frac{d\omega_{ft}}{\omega_{ft}} + \left\langle\frac{d\omega_{ft}}{\omega_{ft}}, \frac{d\left(p_{kt}A_tK_t\right)}{p_{kt}A_tK_t}\right\rangle\right].
\end{aligned}$$

From the bankers' optimal investment choice, we have

$$\begin{aligned}
di_t = {} & \frac{\phi_0^2\phi_1p_{kt}^2}{2}\left(\mu_{Rk,t} - r_{ft} - \frac{1}{p_{kt}} - \bar{a} - \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a\left(\sigma_{ka,t} - \sigma_a\right) + r_{ft}\right)dt \\
& + \frac{\phi_0^2\phi_1p_{kt}^2}{4}\left(\left(\sigma_{ka,t} - \sigma_a\right)^2 + \sigma_{k\xi,t}^2\right)dt + \frac{\phi_0^2\phi_1p_{kt}^2}{2}\left(\sigma_{ka,t} - \sigma_a\right)dZ_{at} + \frac{\phi_0^2\phi_1p_{kt}^2}{2}\sigma_{k\xi,t}dZ_{\xi t}.
\end{aligned}$$

Notice also that

$$\frac{d\left(A_tK_t\right)}{A_tK_t} = \frac{dA_t}{A_t} + \frac{dK_t}{K_t} = \left(\bar{a} + \frac{\sigma_a^2}{2} + \Phi\left(i_t\right)\theta_t\omega_t - \lambda_k\right)dt + \sigma_a dZ_{at}.$$

Thus

$$\frac{1}{dt}\mathbb{E}_t\left[\frac{dc_{ft}}{c_{ft}}\right] \propto \left(1 - i_t\theta_t\omega_t\right)\left(\bar{a} + \frac{\sigma_a^2}{2} + \Phi\left(i_t\right)\theta_t\omega_t - \lambda_k\right) - i_t\theta_t\omega_t\Phi\left(i_t\right)\left(1 - \theta_t\omega_t\right)$$

$$
\begin{aligned}
&- \theta_t \omega_t \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \left(\mu_{Rk,t} - r_{ft} - \frac{1}{p_{kt}} - \bar{a} - \frac{\sigma_a^2}{2} + \lambda_k + r_{ft} + \frac{\left(\sigma_{ka,t} - \sigma_a\right)^2}{2} + \frac{\sigma_{k\xi,t}^2}{2} \right) \\
&- \rho p_{kt} \omega_t \left[\mu_{Rk,t} - r_{ft} - \frac{1}{p_{kt}} + r_{ft} + \theta_t \omega_t \Phi\left(i_t\right) + \mu_{\omega t} + \sigma_{ka,t} \sigma_{\omega a,t} + \sigma_{k\xi,t} \sigma_{\omega\xi,t} \right] \\
&- \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) p_{kt} \left(1 - \omega_t - \omega_{ft}\right) \left[\mu_{Rk,t} - r_{ft} - \frac{1}{p_{kt}} + r_{ft} + \theta_t \omega_t \Phi\left(i_t\right) \right] \\
&+ \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) p_{kt} \omega_t \left[\mu_{\omega t} + \sigma_{ka,t} \sigma_{\omega a,t} + \sigma_{k\xi,t} \sigma_{\omega\xi,t} \right] \\
&+ \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) p_{kt} \omega_{ft} \left[\mu_{\omega_f t} + \sigma_{ka,t} \sigma_{\omega_f a,t} + \sigma_{k\xi,t} \sigma_{\omega_f \xi,t} \right],
\end{aligned}
$$

with the constant of proportionality given by

$$
\left(1 - i_t \theta_t \omega_t - \rho p_{kt} \omega_t - \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) p_{kt} \left(1 - \omega_t - \omega_{ft}\right) \right)^{-1}.
$$

Similarly,

$$
\begin{aligned}
\frac{1}{dt} \mathbb{E}_t \left[\left\langle \frac{dc_{ft}}{c_{ft}} \right\rangle^2 \right] \propto & \left(\left(1 - i_t \theta_t \omega_t\right) \sigma_a - \theta_t \omega_t \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \left(\sigma_{ka,t} - \sigma_a\right) - \rho p_{kt} \omega_t \left(\sigma_{ka,t} + \sigma_{\omega a,t}\right) - \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) p_{kt} \left(1 - \omega_t - \omega_{ft}\right) \sigma_{ka,t} + \right. \\
& \left. + \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) p_{kt} \omega_t \sigma_{\omega a,t} + \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) p_{kt} \omega_{ft} \sigma_{\omega_f a,t} \right)^2 \\
& + \left(\theta_t \omega_t \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \sigma_{k\xi,t} + \rho p_{kt} \omega_t \left(\sigma_{k\xi,t} + \sigma_{\omega\xi,t}\right) + \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) p_{kt} \left(1 - \omega_t - \omega_{ft}\right) \sigma_{k\xi,t} - \right. \\
& \left. - \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) p_{kt} \omega_t \sigma_{\omega\xi,t} - + \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) p_{kt} \omega_{ft} \sigma_{\omega_f \xi,t} \right)^2,
\end{aligned}
$$

with the constant of proportionality given by

$$
\left(1 - i_t \theta_t \omega_t - \rho p_{kt} \omega_t - \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) p_{kt} \left(1 - \omega_t - \omega_{ft}\right) \right)^{-2}.
$$

B.4 Boundaries

Recall that the household is constrained in its allocation to the fund sector if

$$
\begin{aligned}
\mu_{Rk,t} - r_{ft} \geq & \frac{1}{\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2} \left(\left(\mu_{Rb,t} - r_{ft}\right) \left(\sigma_{ka,t} \sigma_{ba,t} + \sigma_{k\xi,t} \sigma_{b\xi,t}\right) + \frac{2m\left(1 - \theta_t \omega_t\right)}{1 - \omega_{ft} - \omega_t} \left(\sigma_{ka,t} \sigma_{b\xi,t} - \sigma_{k\xi,t} \sigma_{ba,t}\right)^2 \right) \\
& + \frac{1}{\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2} \left(\sigma_{ka,t} \sigma_{b\xi,t} - \sigma_{k\xi,t} \sigma_{ba,t}\right) \left(\sigma_{ba,t} - \sigma_{b\xi,t}\right) \sigma_\xi,
\end{aligned}
$$

while the bank is constrained in its leverage decision if

$$
\begin{aligned}
\left(\mu_{Rk,t} + \Phi\left(i_t\right) - \frac{i_t}{p_{kt}} - r_{ft} \right) \geq & \left(\mu_{Rb,t} - r_{ft}\right) + \frac{1}{\alpha \sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}} \left(\left(\sigma_{ba,t} - \sigma_{ka,t}\right)^2 + \left(\sigma_{b\xi,t} - \sigma_{k\xi,t}\right)^2 \right) \\
& + \left(\sigma_{b\xi,t} \sigma_{k\xi,t} + \sigma_{ba,t} \sigma_{ka,t}\right) - \left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right)
\end{aligned}
$$

We can thus have four cases:

Case 1. Neither the fund nor the bank constraint is binding

Case 2. The fund constraint is binding but not the bank

Case 3. The bank constraint is binding but not the fund

Case 4. Both constraints are binding.

In this Section, we are only interested in Cases 2 and 3, when only one of the constraints is binding. Consider first Case 2. Then we must have

$$\mu_{Rk,t} - r_{ft} \geq \frac{1}{\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2}\left((\mu_{Rb,t} - r_{ft})(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}) + \frac{2m(1-\theta_t\omega_t)}{1-\omega_{ft}-\omega_t}(\sigma_{ka,t}\sigma_{b\xi,t} - \sigma_{k\xi,t}\sigma_{ba,t})^2 \right)$$
$$+ \frac{1}{\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2}(\sigma_{ka,t}\sigma_{b\xi,t} - \sigma_{k\xi,t}\sigma_{ba,t})(\sigma_{ba,t} - \sigma_{b\xi,t})\sigma_\xi$$
$$\mu_{Rk,t} - r_{ft} \leq -\left(\Phi(i_t) - \frac{i_t}{p_{kt}}\right) + (\mu_{Rb,t} - r_{ft}) + \frac{1}{\alpha\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}}\left((\sigma_{ba,t} - \sigma_{ka,t})^2 + (\sigma_{b\xi,t} - \sigma_{k\xi,t})^2\right)$$
$$+ (\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}) - \left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right).$$