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Abstract

We propose a new interest rate rule that implements the optimal equilibrium and eliminates all indeterminacy in a canonical New Keynesian model in which the zero lower bound on nominal interest rates (ZLB) is binding. The rule commits to zero nominal interest rates for a length of time that increases in proportion to how much past inflation has deviated—either upward or downward—from its optimal level. Once outside the ZLB, interest rates follow a standard Taylor rule. Following the Taylor principle outside the ZLB is neither necessary nor sufficient to ensure uniqueness of equilibria. Instead, the key principle is to respond strongly enough to deviations of past inflation from optimal levels by sufficiently increasing the amount of time interest rates are promised to be kept at zero.

Key words: zero lower bound, ZLB, liquidity trap, New Keynesian model, indeterminacy, monetary policy, Taylor rule, Taylor principle, interest rate rule, forward guidance

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1 Introduction

Short-term nominal interest rates in many developed economies —including Japan, the US and Europe— have by now been against their effective zero lower bound (ZLB) for several years. For Japan and most of Europe, liftoff from the ZLB is nowhere in sight, and expectations of an increase in the federal funds rate in the US have been shifting into the future ever since the ZLB was first reached in 2008. Despite the many insightful ideas offered by economists on how to manage a liquidity trap,[1] and the concomitant unprecedented efforts by policymakers, one thing is clear: Exiting the ZLB is not easy.

In this paper, we put forth a new class of interest rate rules for an economy in a liquidity trap. The crucial ingredient is to make the date of liftoff from the ZLB depend on past economic conditions. One concrete example is to keep the policy rate pegged at zero for a period of time that increases at a fast enough rate with deviations of past inflation, either upwards or downwards, from its optimal level. Once interest rates become positive, they then follow a standard Taylor rule[2]. The essence of this example is quite dovish: Monetary accommodation, given by the length of time spent at the ZLB, increases whether past inflation turns out to be lower *or higher* than desired. More generally, all rules within the class we propose call for generating higher future inflation and output by increasing the time spent at the ZLB when inflation and output at the beginning of the liquidity trap are higher than what is socially optimal.

For our analysis, we adopt the deterministic continuous time version of the canonical New Keynesian model of Werning (2011). A binding ZLB arises because the exogenous natural rate of interest is negative for some initial period of time. We now discuss properties of the rule and how it compares to previous research.

First, we show the rule implements the socially optimal "forward guidance" equilibrium of

[1] We use the term "liquidity trap" to refer to times in which the natural rate of interest is negative, as in Werning (2011). Liquidity trap episodes may or may not coincide with periods in which the nominal interest rate is at the ZLB.

[2] Nothing would change if we used an inflation targeting regime instead of a Taylor rule outside the ZLB. We do not seek in this paper to contribute to the research on the relative merits of inflation targeting versus Taylor rules.

the kind characterized by Werning (2011); Eggertsson and Woodford (2003); Jung, Teranishi, and Watanabe (2005) as a globally determinate equilibrium (i.e. the optimal equilibrium is guaranteed to always be the unique equilibrium). While indeterminacy is an important issue in all New Keynesian models, its economic implications and the difficulties eliminating it are amplified in the presence of a binding ZLB. In models that ignore the ZLB, the central bank can eliminate indeterminacy and either achieve or get close to the optimal monetary policy by following an appropriate interest rate or inflation targeting rule. For example, a Taylor rule in which the policy rate reacts more than one-for-one with inflation —the so-called Taylor principle— can guarantee uniqueness of equilibria. Instead, when a ZLB is introduced, Benhabib, Schmitt-Grohe, and Uribe (2001) show that indeterminacy is a robust feature of Taylor-style feedback rules, *especially* when the Taylor principle is satisfied. Indeterminacy in this case takes the form of stable self-fulfilling deflationary equilibria that can hamper the return to a more desirable equilibrium. Using the same framework of Werning (2011) that we consider in this paper, Cochrane (2013) shows that an economy in a liquidity trap will exhibit indeterminacy for *any* given path of nominal interest rates. One immediate implication is that a central bank that pursues calendar-based forward guidance by announcing and committing to a fixed future liftoff date will not eliminate indeterminacy. Furthermore, the different equilibria that are consistent with the same given path of nominal interest rates can be arbitrarily far from the socially optimal equilibrium and exhibit radically different economic behavior. In the "standard" equilibrium, in which inflation and output are optimal from a social welfare point of view, forward guidance and government spending are powerful stimulative tools, and more price stickiness is helpful in getting out of the ZLB. In contrast, in the "local-to-frictionless" equilibrium in which inflation and the output gap do not explode backwards in time, forward guidance and fiscal stimulus are not expansionary while lower price stickiness improves outcomes. Many other models on how the economy behaves at the ZLB and what the right policy prescriptions are rely on the presence of multiple steady states and other forms of indeterminacy.[3]

[3] In addition to the aforementioned papers, a necessarily incomplete list includes Mertens and Ravn (2014),

Analogously to the principle that the short-term nominal interest rate must react strongly enough to inflation in order to prevent indeterminacy in models without the ZLB, the rules we propose require that the amount of time spent at the ZLB increases fast enough as a function of past inflation or output. This feature eliminates indeterminacy because, unless initial inflation and output (at time $t = 0$) are socially optimal, the stimulus due to an extended period at the ZLB is always large enough to make agents expect economic recovery and a corresponding liftoff from the ZLB sooner than the central bank has promised. The discrepancy between agents' expectations and the future actions of a committed central bank cannot support a rational expectations equilibrium.

Because the crucial expectations are about when the policy rate will exit the ZLB, what happens after the ZLB is less important for determinacy. Whether the Taylor principle holds once the nominal rate turns positive is inconsequential: we give examples in which a unique optimal equilibrium is achieved with and without the Taylor principle. While the difficulties in eliminating indeterminacy explained by Benhabib et al. (2001) and Cochrane (2013) still apply in our setup, we show that allowing for a broader class of history-dependent interest rate rules can restore global determinacy and preserve optimality even in the presence of Taylor-type feedback rules outside the ZLB.

Second, our rules are simple along several dimensions. Once the optimal path for the economy that the central bank would like to implement is known, the rule only needs to additionally reference realized inflation[4]. In fact, the rules require knowledge of initial inflation no earlier than when the liquidity trap is over and no other information about inflation, the output gap or the natural rate of interest is needed until exit from the ZLB. While the initial level of inflation in the model is common knowledge at $t = 0$, it might be helpful in the real world to only have to know inflation with a lag. Use of real-time estimates of inflation that may be revised or forecasts that can create further sources of indeterminacy and uncertainty

Hursey, Wolman, and Hornstein (2014), Armenter (2014), Aruoba and Schorfheide (2013), Richter and Throckmorton (2013), Schmitt-Grohé and Uribe (2012), Sims (2004).

[4] In this model, after observing realized inflation, the optimal path, the output gap and the natural rate of interest provide the same information; knowledge of any one of them implies knowledge of all three of them.

can then be reduced.

Communicating the rule should also be relatively easy. The rule involves a familiar Taylor rule and a form of uncomplicated rule-based forward guidance that requires interest rates to stay at zero for longer in the presence of higher past inflation. Using a standard calibration, we show that the behavior of the central bank need not be extreme for indeterminacy to disappear. In the example described earlier, in which liftoff time increases with deviations of past inflation from optimal, it is enough to extend the ZLB period from 2.6 to 2.8 years when initial inflation deviates from optimal by two percentage points.

The policy parameters of the rule are also straightforward to choose and communicate. After initial inflation is realized, the amount of time that will be spent at the ZLB and all Taylor rule coefficients are decided and remain constant over time. The magnitude of the policy response to inflation both inside and outside the ZLB can be made independent of the parameters of the model by having a "strong enough" policy response function. Although knowing the numeric value of the deep parameters of the model can give policymakers bounds on what "strong enough" actually requires, it is always possible to find a policy rule that eliminates indeterminacy for any ex-ante set of parameters we want to consider. We provide necessary and sufficient conditions that characterize what "strong enough" means. In addition, the Taylor rule coefficients can be made completely independent of inflation, the output gap and all other economic variables if we endow the central bank with knowldege of some of the parameters of the model, making policy memoryless (not path-dependent) after liftoff from the ZLB.

Third, our proposal requires little change in the institutional arrangements and anlytical framworks of most central banks, an advantage when putting it into practice. The policy instrument of our rule is the short-term nominal interest rate, already the predominant instrument of choice. There is no need to make reference to new or time-varying monetary or price aggregates, price-level or inflation targets, "shadow" rates, exchange rates, the central bank's balance sheet, or the quantity or price of other assets. Furthermore, our rule can be made to have history-dependence that ends as soon as the policy rate exits the ZLB. At that point, central banks can return to the standard policy regime they had in place before they entered the liquidity trap without having to take into consideration their prior actions while at the ZLB. Finally, our proposal does not rely on fiscal policy being "active" or "non-Ricardian" to obtain global determinacy. Our results are obtained under the assumption that the fiscal authority always adjusts taxes or spending ex-post to validate any path of the endogenous variables that may arise. This shows that interest-rate based monetary policy need not be "passive" at the ZLB, as is usually supposed.[5]

Eggertsson and Woodford (2003) implement the same optimal path we consider as a unique equilibrium by means of an "output-gap adjusted" price-level target. Their rule has the same informational requirements as ours and also calls for a history-dependent commitment. In contrast to our proposal, in order to be globally determinate, their rule must be accompanied by either a commitment of fiscal policy to an appropriate non-Ricardian rule, or a commitment to a monetary-base supply rule accompanied by the milder fiscal commitment that the government will asymptotically be neither a creditor nor a debtor. Furthermore, the price-level target remains path-dependent after exiting the ZLB in all cases. Cochrane (2013) shows that having a time-varying inflation target or an appropriately designed "stochastic intercept" in the Taylor rule can also implement the optimal path as a unique equilibirum if the Taylor principle is followed outside the ZLB. Svensson (2004) advocates an intentional currency depreciation combined with a calibrated crawling peg. He shows optimality of this scheme in a two-country model, although he does not address whether the resulting equilibrium is determinate.Without formally analyzing optimality or determinacy issues, Hall and Mankiw (1994), McCallum (2011), Sumner (2014) and Romer (2011) recommend nominal GDP targeting, while Blanchard, DellAriccia, and Mauro (2010) and Ball (2014) advocate increasing the inflation target.

Section 2 presents the model. Section 3 briefly reviews the relevant elements from Werning (2011). Section 4 replicates and generalizes some of the results from Cochrane (2013).

[5] Of course, this does not imply that fiscal rules or the fiscal theory of the price level are unimportant in theory or not relevant in practice. For issues related to our results that analyze the monetary-fiscal interaction, see Sims (1994), Benhabib, Schmitt-Grohé, and Uribe (2001) and Woodford (2001).

Section 5 describes rules that implement the socially optimal path as the unique equilibrium of the economy by allowing the liftoff date to depend on inflation and the output gap. Section 6 concludes.

2 The Canonical New Keynesian Model with a ZLB

We use the framework of Werning (2011), which is a standard deterministic New Keynesian model in continuous time, log-linearized around a zero-inflation steady state.[6] The economy is described by:

$$\dot{x}_t = \sigma^{-1}(i_t - r_t - \pi_t), \tag{1}$$

$$\dot{\pi}_t = \rho\pi_t - \kappa x_t, \tag{2}$$

$$i_t \geq 0. \tag{3}$$

The "over-dot" notation represents partial derivatives with respect to time. The variables x_t and π_t are the output gap and the inflation rate, respectively. The output gap is the log-deviation of actual output from the hypothetical output that would prevail in the flexible price, efficient equilibrium. Henceforth, for brevity, we refer to the output gap simply as "output". The central bank's policy instrument is the path for the nominal short-term interest rate i_t, which must remain non-negative at all times. The variable r_t is the natural rate of interest, defined as the real interest rate that would prevail in the flexible price, efficient economy with $x_t = 0$ for all t.

Equation (1) is the IS curve, the log-linearized Euler equation of the representative consumer. The constant $\sigma^{-1} > 0$ is the elasticity of intertemporal substitution. Equation (2) is

[6] See Woodford (2003) or Galí (2009) for details.

Although essentially all analysis of determinacy in New Keynesian models is done in log-linearized models, Braun, Körber, and Waki (2012) contend that conclusions would differ in the full non-linear model. On the other hand, Christiano and Eichenbaum (2012) show that the additional equilibria that arise from non-linearities in Braun et al. (2012) are not E-learnable. In addition, Christiano and Eichenbaum (2012) show that the linear approximation are accurate except on extreme cases, such as when output deviates by more than 20 percent from steady state.

While important, we do not seek to address these issues here and simply use the standard specification in the literature.

the New Keynesian Phillips Curve (NKPC), the log-linear version of firms' first-order conditions when they maximize profits by picking the price of consumption goods subject to consumers' demand and Calvo pricing. The constant $\rho > 0$ is the representative consumer's discount rate and $\kappa > 0$ is related to the amount of price stickiness in the economy. As $\kappa \to \infty$, the economy converges to a fully flexible price economy while prices are fully rigid when $\kappa = 0$.

The exogenous path for the natural rate is:

$$r_t = \begin{cases} \underline{r} < 0 & , \quad 0 \leq t < T \\ \overline{r} > 0 & , \quad T \leq t \end{cases} . \tag{4}$$

The constants $T > 0$, $\underline{r} < 0$ and $\overline{r} > 0$ are given. None of our results change if we pick a different path for r_t as long as $r_t < 0$ for $t < T$ and $r_t > 0$ for $t \geq T$.

Definition. *A* ***rational expectations equilibrium*** *consists of bounded paths for output, inflation and the nominal interest rate* $\{x_t, \pi_t, i_t\}_{t\geq 0}$ *that, given a path* $\{r_t\}_{t\geq 0}$ *for the natural rate, satisfy equations* (1)-(3).

There are three elements of the definition that are worth discussing in our context. First, the requirement that output and inflation remain bounded at all times is equivalent to the asymptotic conditions

$$\lim_{t\to\infty} |x_t| < \infty, \tag{5}$$

$$\lim_{t\to\infty} |\pi_t| < \infty. \tag{6}$$

The justification and role that (6) plays for determinacy of equilibria is controversial in the literature[7]. We do not attempt to contribute to that debate and instead adopt the

[7] Cochrane (2011) argues that there is no good economic reason to prevent nominal explosions. McCallum (2009) and Atkeson, Chari, and Kehoe (2009) agree and, among others in an active area of research, propose different criteria to eliminate or select equilibria. Woodford (2003), Thomas (2013) and others defend the approach. In our specific setup, inflation explodes if and only if the real output gap explodes, making it difficult to differentiate nominal from real explosions.

simplest, most conventional approach. Second, paths for x_t and π_t that satisfy equations (1) and (2) must be continuous.[8] If there were any jumps, the representative consumer's Euler equation would be violated due to the existence of arbitrage opportunities. Third, neither the definition of equilibrium nor the dynamics of the economy in equations (1) and (2) make any explicit reference to fiscal policy although, as stressed by Woodford (1995), Sims (1994), Benhabib et al. (2001), Cochrane (2011) and others, determinacy or the lack thereof is a result of the joint monetary-fiscal regime. In order to focus solely on monetary policy, we assume the fiscal authority always adjusts taxes or spending ex-post to validate any path of the endogenous variables that may arise, generating a "Ricardian" (in the nomenclature of Woodford (2001)) or "passive" (in the nomenclature of Leeper (1991)) fiscal regime.

3 The Socially Optimal Equilibrium

The social welfare loss function for the economy is

$$V = \frac{1}{2} \int_0^\infty e^{-\rho t} \left(x_t^2 + \lambda \pi_t^2 \right) dt. \tag{7}$$

The constant $\lambda > 0$ is a preference parameter that dictates the relative importance of deviations of output and inflation from their desired value of zero. This quadratic loss objective function can be obtained as a second order approximation around zero inflation to the economy's true social welfare function when the flexible price equilibrium is efficient (Woodford, 2003). A socially optimal equilibrium is an equilibrium that minimizes (7).

[8] Strictly speaking, differentiability of x_t and π_t would also be required for all t. We instead use a weaker solution concept —viscosity solutions— for the system of ODEs (1) and (2) that, in our context, allows for non-differentiability (in the classical "calculus 1" sense) in a set of measure zero. Without this modification, the assumed discontinuous process for the natural rate r_t in equation (4) would imply that (1) and (2) have no solution. More importantly, even if r_t were smooth, the central bank's control problem of Section 3 would have no solution since the solution always requires a jump in the nominal rate i_t.

Werning (2011) solves for the socially optimal equilibrium $\{x_t^*, \pi_t^*, i_t^*\}_{t\geq 0}$. He finds that it is unique and that the path of the policy rate satisfies

$$i_t^* = \begin{cases} 0 & , \quad 0 \leq t < t^* \\ (1 - \kappa\sigma\lambda)\,\pi_t^* + r_t & , \quad t \geq t^* \end{cases} \tag{8}$$

for some $t^* > T$ that can be found as a function of the parameters of the model. We refer to t^* as the liftoff date. The socially optimal policy is to commit to zero nominal interest rates for longer than the natural rate r_t is negative — one of the principal aspects of forward guidance. However, equation (8) is not a policy rule. Indeed, the optimal path $(1 - \kappa\sigma\lambda)\,\pi_t^* + r_t$ is a single fixed path, a function of time only. It is neither contingent on the actions of the central bank nor on whether realized inflation, output or their expectations happen to take one value or another. As such, it addresses neither the on nor the off-equilibrium behavior of the central bank and hence says nothing about implementability or indeterminacy.

Plugging in (8) into (1)-(2) gives the optimal paths for inflation and output for all $t > 0$. Inserting these paths into equation (7) makes the objective V a function of initial output x_0 and inflation π_0. The optimality conditions[9]

$$\frac{\partial V}{\partial x_0} = 0 \quad \text{and} \quad \frac{\partial V}{\partial \pi_0} = 0 \tag{9}$$

determine optimal initial output x_0^* and inflation π_0^*. The optimal initial level of output is always negative. The optimal initial level of inflation can be positive, negative or zero depending on parameters.[10]

Figure 1 shows the optimal path for three parameter configurations. It is most easily understood in three stages, starting from the last one and then working backwards in time. In the third and last stage, when $t \geq t^*$, the economy has positive natural and nominal rates. To ensure that inflation and output remain bounded so as to satisfy equations (5) and (6),

[9] Equivalently, we can use the Maximum Principle and set the initial value of the co-state variables to zero as in Werning (2011).

[10] For all these results, see Proposition 4 in Werning (2011).

the economy must move towards the unique steady state $(x^{ss}, \pi^{ss}) = (0, 0)$ along the saddle path given by $x = \phi\pi$ with $\phi = \frac{1}{2\kappa}\left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2}\right)$. Because the ZLB will never bind after t^*, the solution in this stage is independent of the process for the natural rate. Moreover, the paths for inflation and output are the same as if the central bank ignored the ZLB but, unlike what would happen in a fully unconstrained model, took initial inflation and output as given (by its own actions before t^*).

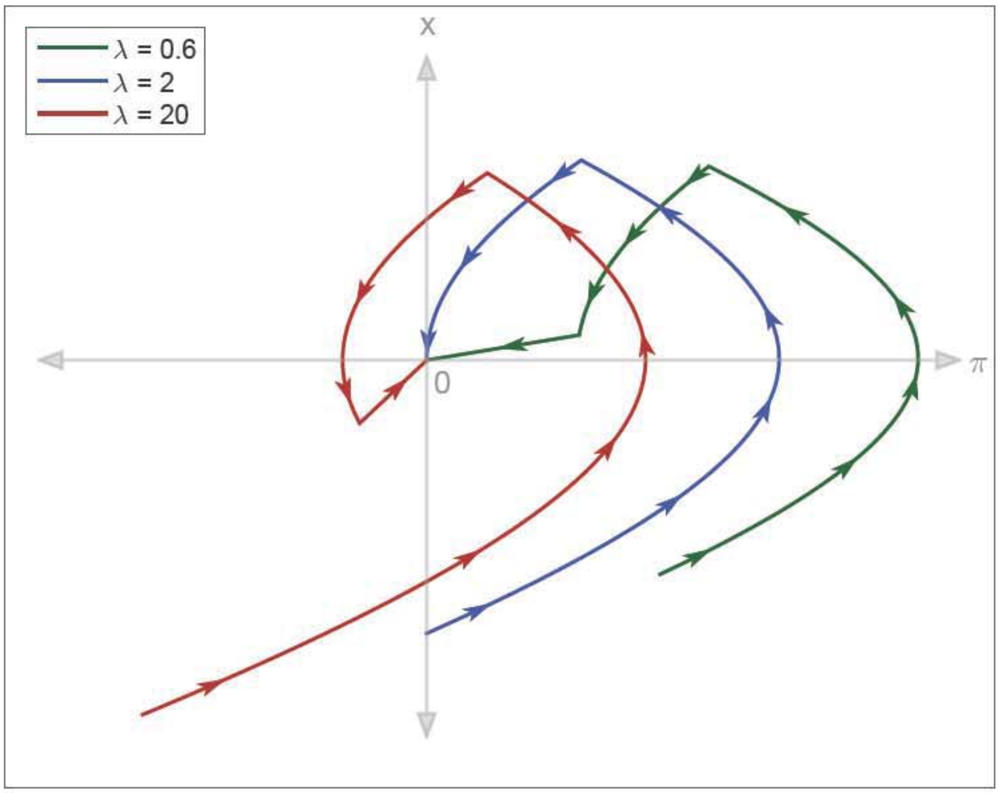

Figure 1: Optimal paths for three values of λ, the weight that the central bank places on inflation relative to output. The rest of the parameters used are taken from Werning (2011) ($T = 2$, $\sigma = 1$, $\kappa = 0.5$, $\rho = 0.01$, $r_h = 0.04$ and $r_l = -0.04$).

The second stage is $t \in [T, t^*)$, when the natural rate is positive but the nominal rate is zero. Starting at a given (x_T, π_T), inflation and output move so as to minimize the time it takes to reach the saddle path $x = \phi\pi$. This is accomplished by pegging nominal rates to zero. To be able to reach the saddle path at all, it must be the case that $x_T^* > \phi\pi_T^*$. The inequality implies that inflation and output are eventually non-increasing (when t is close enough to t^*), although they may be increasing during earlier parts of their trajectory. When

x_t and π_t hit the saddle path, the third stage begins.

Finally, in the first stage given by $t \in [0, T)$, the natural rate is negative and nominal rate is at the ZLB. The low contemporaneous nominal rates and the zero nominal rates past time T from the second stage decrease today's savings and lower the real interest rate. Inflation and output eventually become positive during this stage as a result. The initial point (x_0^*, π_0^*) is determined by the optimality conditions (9).

4 Central Bank Policy, Rules and Indeterminacy

In this section, we study a central bank that attempts to implement the socially optimal path (8) as the unique equilibrium of the economy by means of an interest rate rule. An initial natural canditate rule is based on (8):

$$i_t = \begin{cases} 0 & , \quad 0 \leq t < t^* \\ (1 - \kappa\sigma\lambda)\,\pi_t^* + r_t & , \quad t \geq t^* \end{cases} . \tag{10}$$

Although equations (8) and (10) look very similar, they are conceptually different. While equation (8) describes the single path i_t^*, equation (10) is a rule —a policy response function— by which the central bank commits to set interest rates in all possible states of the world. It therefore provides both the on and off-equilibrium behavior of the central bank. For this particular rule, the behavior of the central bank is the same for all states of the world; the rule states that interest rates will follow the optimal path (which is the same in all states of the world) come what may. The results from the previous section imlpy that if the central bank follows rule (10), then it can implement the socially optimal equilibrium whenever $(x_0, \pi_0) = (x_0^*, \pi_0^*)$. However, Cochrane (2013) shows that many other equilibria are also consistent with this rule, leading to an indeterminate outcome. In fact, he goes further and shows that if the central bank commits to *any* given non-explosive path of nominal rates, irrespective of its optimality or the central bank's commitment ability, the economy suffers from indeterminacy. More formally, rules of the form $i_t = f(t)$ produce indeterminacy for any bounded choice of f.

The intuition for this result is as follows. By virtue of equations (1) and (2), the choice of i_t directly affects —we may even say control— inflation and output for all $t > 0$, but not for $t = 0$. The path of nominal interest rates only affects *changes* in x and π starting at $t = 0$, but cannot directly influence x_0 or π_0. The inability to affect current inflation and output with current interest rates, economically speaking, stems from the forward-looking nature of the IS equation. Initial inflation and output, instead of being control variables like in the last section, are now non-predetermined or "jump" variables. Can the central bank nevertheless influence x_0 and π_0 in some way? For the rules we examine in this section, the answer is no. In the next section, we propose rules that can indeed uniquely select the desired equilibrium by guaranteeing that $(x_0, \pi_0) = (x_0^*, \pi_0^*)$.

To understand the inability of the central bank to influence x_0 or π_0 with a rule like (10), interpret $i_t = f(t)$ as a Taylor rule with a time-varying intercept and coefficients of zero on inflation and output. Such a rule does not satisfy the Taylor principle and always produces dynamics that are saddle path stable for $t \geq t_1$. The existence of a saddle path breeds indeterminacy. We can construct one equilibrium for each point in the saddle path. Pick a point $(\tilde{x}, \tilde{\pi})$ on the saddle path and consider a candidate equilibrium with $(x_{t_1}, \pi_{t_1}) = (\tilde{x}, \tilde{\pi})$. For $t \geq t_1$, the economy follows the dynamics (1) and (2) by moving along the saddle path towards the stady state. Then trace the dynamics (1) and (2) backwards in time, from $t = t_1$ to $t = T$, starting at $(x_{t_1}, \pi_{t_1}) = (\tilde{x}, \tilde{\pi})$ and ending at (x_T, π_T). Again follow (1) and (2) backwards in time from T to $t = 0$, now with initial conditions (x_T, π_T) inherited from the previous step. The resulting path is bounded, continuous, obeys the IS equation, the Phillips Curve and the ZLB: It is an equilibrium.

Because of the linearity of the system, the set of points x_0 and π_0 that put the system on the saddle path at t_1, and the saddle path itself, are both lines in the x-π plane. The set of rational expectations equilibria is thus indexed by points in a line, which we can take to be the saddle path or the x_0-π_0 line that gets the economy on the saddle path at t_1. Figure 2 shows these two lines together with inflation and output from equilibria that start with different x_0 and π_0. The blue line is the socially optimal equilibrium. The red line is an equilibrium that reaches steady state $(x^{ss}, \pi^{ss}) = (0, 0)$ at t_1, just when the ZLB period ends. The yellow line is one of the "local-to-frictionless" equilibria from Cochrane (2013) in which inflation and output do not explode backwards in time. The remaining paths have two arbitrary values for π_0 and illustrate the kinds of behavior that the equilibria can exhibit. All of these equilibria are consistent with the central bank following (10).

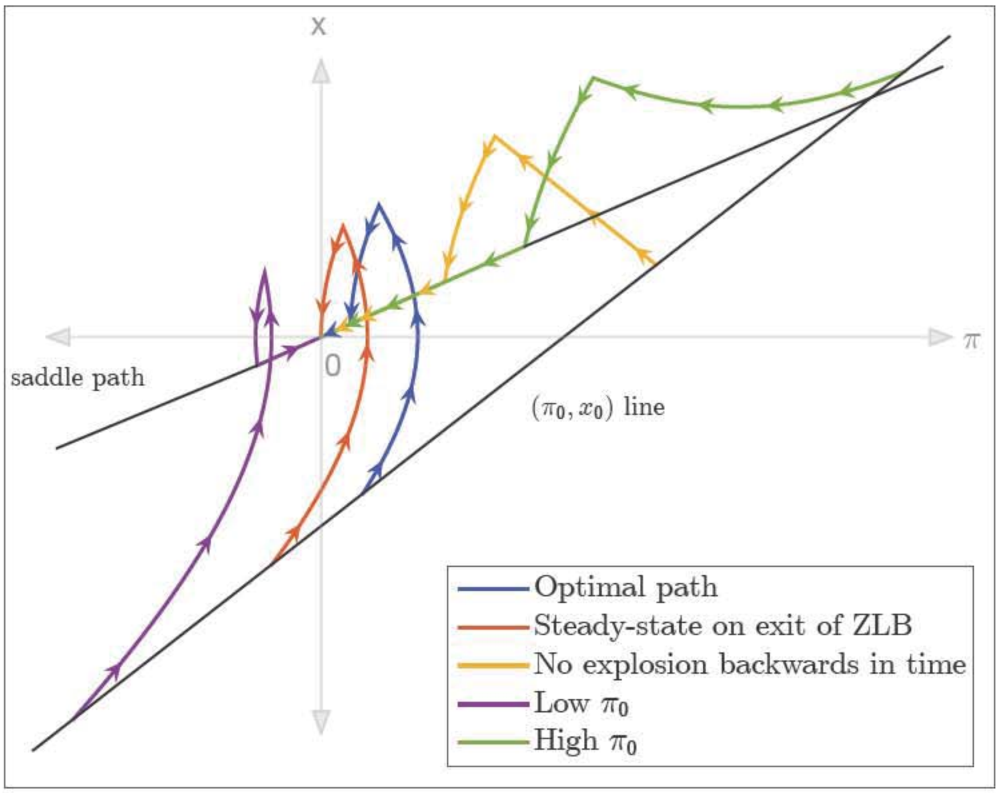

Figure 2: Multiple equilibria when the central bank follows the interest rate rule in equation (10). The parameters used are $T = 1.5$, $\sigma = 1$, $\kappa = 0.5$, $\lambda = 0.6, \rho = 0.01$, $r_h = 0.04$ and $r_l = -0.04$.

We now extend the results from Cochrane (2013) to Taylor rules. The next proposition establishes that simple linear rules with a constant liftoff date are unable to solve the problem of implementing a determinate optimal equilibrium. We allow the Taylor rule coefficients to depend on x_0 and π_0 to highlight that the key obstacle for determinacy is a constant liftoff date t_1 and not a path-dependent choice of coefficients[11]. In the next section, we will allow

[11]Cochrane (2013) shows that allowing for a time-varying intercept that is a function of the optimal path or

t_1 to be a function of x_0 and π_0 and implement the optimal equilibrium uniquely.

Proposition 1. *Let t_1 be a constant with $t_1 \geq T$. Let $\xi_\pi(x_0, \pi_0)$ and $\xi_x(x_0, \pi_0)$ be arbitrary functions. If $\kappa\sigma\lambda \neq 1$, the rule*

$$i_t = \begin{cases} 0 & , \quad 0 \leq t < t_1 \\ \max\left(0, \xi_\pi(x_0, \pi_0)\pi_t + \xi_x(x_0, \pi_0)x_t + r_t\right) & , \quad t_1 \leq t < \infty \end{cases} \tag{11}$$

can never implement the socially optimal path (8) as the unique equilibrium of the economy.

Proof: Assume that rule (11) implements the optimal path. We then have that $t_1 = t^*$. Consider the point (x_0, π_0) that reaches $(x_t, \pi_t) = (0, 0)$ at $t = t^*$ when following (1) and (2). If $\kappa\sigma\lambda \neq 1$, this point is different from (x_0^*, π_0^*) and is an equilibrium for any choice of functions $\xi_\pi(x_0, \pi_0)$ and $\xi_x(x_0, \pi_0)$. Converesely, if the equilibrium is unique, the dynamics of the economy for $t \geq t_1$ must be explosive unless the steady state is reached at liftoff, i.e. unless $(x_{t_1}, \pi_{t_1}) = (x^{ss}, \pi^{ss}) = (0, 0)$. But optimality conditions imply that $\left(x_{t_1}^*, \pi_{t_1}^*\right) \neq (0, 0)$ whenever $\kappa\sigma\lambda \neq 1$, showing that the resulting unique equilibrium is suboptimal. When $\kappa\sigma\lambda = 1$, the optimal path happens to have $(x_t^*, \pi_t^*) = (0, 0)$ for all $t \geq t_1$ and can thus be uniquely implemented by a choice of coefficients in rule (11) that obey the Taylor principle.

The central bank faces a trade-off. It can either implement a determinate yet suboptimal equilibrium or pick a rule that can support the optimal equilibrium but also many other equilibira, resulting in indeterminacy. One direct implication of the proposition is that following the Taylor principle outside the ZLB does guarantee a unique equilibrium, as is the case in models without the ZLB, but at the cost of being suboptimal (except for the case $\kappa\sigma\lambda = 1$). The equilibrium in which the Taylor principle holds requires the economy to be at its steady state $(x^{ss}, \pi^{ss}) = (0, 0)$ at $t = t_1$, right after exiting the ZLB, since otherwise (5) and (6) would be violated. This equilibrium is similar to the no-commitment equilibrium

for time-varying inflation and output targets can also overcome the problem of indeterminacy. As explained in the introduction, we want to restrict ourselves to simpler rules that should be easier to implement and explain to the public.

described in Werning (2011), in which the economy must be at steady state at $t = T$ (and identical when $t_1 = T$). In sharp contrast to models without a binding ZLB in which the Taylor principle is meant to reap the benefits of commitment, the ZLB makes the Taylor principle produce suboptimality akin to that emerging from *lack* of commitment. Figure 3 displays the optimal equilibrium, the Taylor principle equilibrium with $t_1 = t^*$ and the no-commitment equilibrium for two parameter configurations. Inflation and output can deviate substantially from their ideal levels and approach the no-commitment equilibrium, as the left panel of Figure 3 illustrates. On the other hand, the right panel shows that when $\kappa\sigma\lambda$ is close to 1, the Taylor principle equilibrium implies small welfare losses compared with the optimal path.

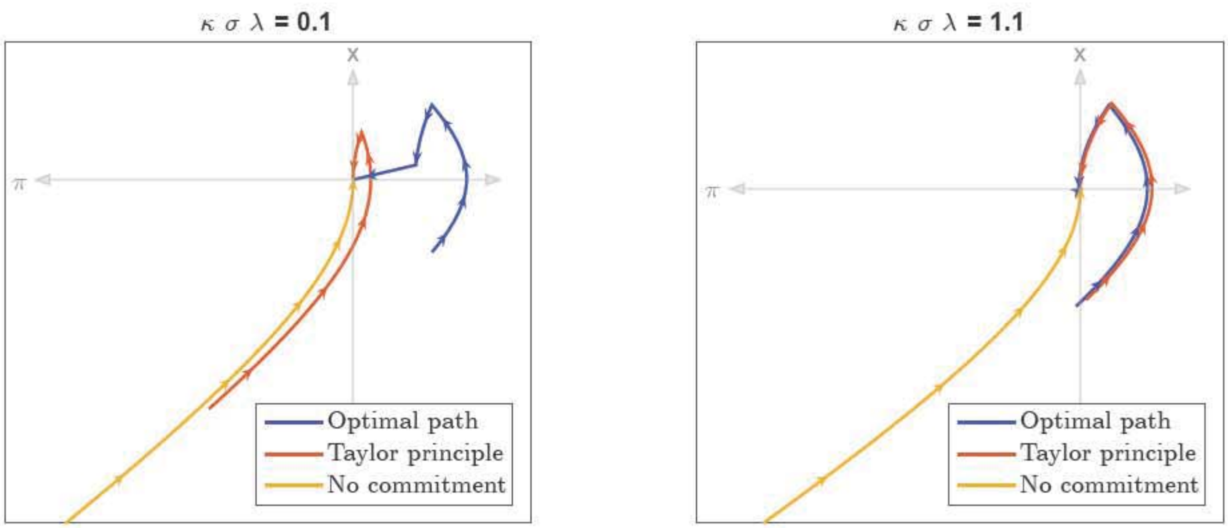

Figure 3: Comparison between the optimal path, the path without central bank commitment and the path when the central bank satisfies the Taylor principle after t^*. We use $T = 2$, $\sigma = 1$, $\kappa = 0.5$, $\rho = 0.01$, $r_h = 0.04$ and $r_l = -0.04$ for both panels. For the left and right panels, we choose $\lambda = 0.2$ and $\lambda = 2.2$, respectively.

5 A Rule to Implement the Socially Optimal Equilibrium without Indeterminacy

In this section, we show the main result of our paper: Allowing the time t_1 of exit from the ZLB to depend on initial inflation and output makes it possible to implement the optimal equilibrium without indeterminacy. We focus our analysis on implementing the socially optimal equilibrium because it is what a central bank in our model would prefer to do. However, our techniques can be used to implement any other equilibrium uniquely, which may be of use if the researcher or central banker believes, for whatever reason, that the optimal path is unreasonable or unattainable.

Instead of conditioning t_1 on π_0 and x_0, we could have as easily decided to condition on other points from the paths for inflation and output and obtained identical results. Both due to theoretical and practical reasons, we find it more compelling to focus on variables at $t = 0$. Theoretically, the idea of conditioning actions on π_0 and x_0 goes to the heart of the indeterminacy problem, since these are the values that cannot be directly controlled by changes in the nominal rate, as argued in the last section. For a real-world central bank, there are several practical benefits that are admittedly outside the model but nevertheless relevant. Conditioning the liftoff date on time-zero variables has the advantage that its specific value does not need to be known until at least $t = T$, sidestepping the need to produce forecasts and minimizing the need for surveys of private sector expectations. Private sector agents should show less disagreement with each other and with the central bank when thinking about a past value instead of a forecast. The lag between the realization of π_0 and x_0 and when their values need to be used may also reduce policy mistakes arising from statistical revisions, which are fairly common and sometimes quantitatively significant. Allowing t_1 to depend on x_{t_1} in addition to x_0 and π_0 reduces some of the benefits just discussed but can further simplify the communication of the rule, as we show in the last part of this section.

We start with the smallest necessary deviation from rule (10) that produces a determinate optimal equilibrium. The new feature is to make the liftoff time be equal to t^*, $t^* + 1$ or

$t^* + 2$ depending on the realized values of x_0 and π_0. This example shows that a rule for the liftoff time is a powerful tool to fight indeterminacy at the ZLB.

Proposition 2. *There exist a function $f(x_0, \pi_0)$ such that the interest rate rule*

$$\begin{aligned} i_t &= \begin{cases} 0 & , \quad 0 \le t < t_1 \\ (1 - \kappa\sigma\lambda)\,\pi_t^* + r_t & , \quad t \ge t_1 \end{cases} \\ t_1 &= t^* + f(x_0, \pi_0) \end{aligned} \tag{12}$$

implements the optimal equilibrium path without indeterminacy.

Proof. See Appendix.

The proof is constructive: An example of the desired function is

$$f(x_0, \pi_0) = \begin{cases} 0 & , \quad \textit{if } (\pi_0 = \pi_0^* \textit{ and } x_0 = x_0^*) \textit{ or } Ax_0 + B\pi_0 \neq C \\ 1 & , \quad \textit{if } Ax_0 + B\pi_0 = C \textit{ and } Dx_0 + E\pi_0 \neq C \\ 2 & , \quad \textit{otherwise} \end{cases} \tag{13}$$

where A, B, C and D are appropriately selected constants given in the Appendix.

When $\pi_0 = \pi_0^*$ and $x_0 = x_0^*$, the rule gives $t_1 = t^*$, $i_t = i_t^*$ and therefore the optimal path is an equilibrium. In all other cases, we pick a t_1 that ensures no equilibrium is possible. To do so, we proceed as follows. Consider a candidate equilibrium with initial conditions π_0 and x_0 different from (x_0^*, π_0^*) and let the economy flow over time using the IS and the NKPC. We consider three cases. First, if the economy is not on the saddle path at t^*, set $f(x_0, \pi_0) = 0$. The equation $Ax_0 + B\pi_0 \neq C$ in rule (13) describes the set of points (x_0, π_0) for which this happens. Because the economy is not on its saddle path at the time of liftoff, either inflation and output instantaneously jump a discrete amount to reach the saddle path, or else inflation and output become unbounded. In either case, we have precluded an equilibrium. For the second case, consider the points that are on the saddle path at t^* but not on the saddle path at t^*+1. In rule (13), this corresponds to the case $Ax_0 + B\pi_0 = C$ and $Dx_0 + E\pi_0 \neq C$. For those points, we assign $f(x_0, \pi_0) = 1$. As explained in the last section, the set of initial conditions that reach the saddle path at t^* constitute a line in the x-π plane. For points in that line that do not reach the saddle path at t^*+1, we have precluded an equilibrium from forming by the same argument as in the first case. Those that do reach the saddle path at t^*+1 constitute the third case, for which we set $f(x_0, \pi_0) = 2$. There is at most one point in this category, since it is given by the intersection of two distinct lines: The line of initial conditions that reaches the saddle path at t^* and the one that reaches it at t^*+1. This point, if it exists, cannot reach the saddle path at t^*+2, the time of liftoff, and is therefore not an equilibrium. Since we have covered all possible π_0 and x_0 in the plane, the proof is complete.

The amount of time spent at the ZLB takes one of three values: zero, one or two years. We have picked these concrete values for simplicity. The argument in the last paragraph makes clear that, other than requiring $f(x_0^*, \pi_0^*) = 0$, any three or more distinct values (no smaller than $-t^*$, of course) for the other cases would also work.

The way equilibria are eliminated can be alternatively cast in terms of expectations. Suppose agents have an expectation $\tilde{\pi}_0$ and $\tilde{x}_0$ for initial inflation and output. If the central bank is credibly committed to rule (13), liftoff time will be rationally expected to be at $t_1 = t^* + f(\tilde{x}_0, \tilde{\pi}_0)$. On the other hand, rational expectations also require non-explosive paths for inflation and output. Therefore, agents have two rational ways to form expectations for x_{t_1} and π_{t_1}. The first is to trace the evolution of x_t and π_t from $t = 0$ until $t_1 = t^* + f(\tilde{x}_0, \tilde{\pi}_0)$ starting at $(\tilde{\pi}_0, \tilde{x}_0)$, giving an expected outcome of $(\tilde{\pi}_{t_1}, \tilde{x}_{t_1})$. The second is to realize that non-explosive paths are expected to be on the saddle path at t_1. If $(\tilde{\pi}_{t_1}, \tilde{x}_{t_1})$ is not on the saddle path, we cannot have an equilibrium, since the two computations give contradictory expectations. By appropriately picking the value of $f(\tilde{\pi}_0, \tilde{x}_0)$, the central bank can always create the expectation that liftoff will occur at a time when the economy is not on the saddle path, thus eliminating any undesired equilibria.

The same rule for t_1 given by equation (13), with different constants A, B, C and D, can also implement the optimal equilibrium without indeterminacy for Taylor rules of the form

$$\begin{aligned} i_t &= \begin{cases} 0 & , \quad 0 \leq t < t_1 \\ \xi_\pi (x_0, \pi_0) \pi_t + \xi_x (x_0, \pi_0) x_t + r_t & , \quad t_1 \leq t < \infty \end{cases}, \qquad (14) \\ t_1 &= t^* + f(x_0, \pi_0) \qquad (15) \end{aligned}$$

For example, setting $\xi_\pi (x_0, \pi_0) = 1 - \kappa\sigma\lambda$, $\xi_x (x_0, \pi_0) = 0$ and $f(x_0, \pi_0)$ as in equation (13) implements the optimal equilibrium uniquely with constant Taylor rule coefficients. This means that once the economy exits the ZLB, all policy is memoryless, which is one of the key distinctions from price-targeting regimes. Just as rule (12)-(13) was a generalization of (10), the rule in equations (14)-(15) can be thought of as an improvement over rule (11). Compared with equation (12), we now allow interest rates outside the ZLB to respond contemporaneously to inflation and output, with Taylor rule coefficients that are constant after $t = 0$ but in general path-dependent since they can be functions of time-zero inflation and output.

Before presenting other concrete choices for ξ_π, ξ_x and f, we state a straightforward necessary and sufficient condition for the optimal path to be a determinate equilibrium: Expectations that initial inflation and output are (x_0^*, π_0^*) must lead to the optimal path while all other expectations must lead to paths that are either discontinuous or explosive. Under rule (14)- (15), the initial optimal point (x_0^*, π_0^*) leads to the optimal path if and only if

$$f(x_0^*, \pi_0^*) = 0 \qquad (16)$$

and

$$\xi_\pi (x_0^*, \pi_0^*) \pi_{t_1}^* + \xi_x (x_0^*, \pi_0^*) x_{t_1}^* = (1 - \kappa\sigma\lambda) \pi_{t_1}^* \qquad (17)$$

Equations (16) and (17) put the economy on the optimal saddle path $x = \phi\pi$ at t^*. We show explicit necessary and sufficient conditions for paths to be continuous and non-explosive in the Appendix, which must be violated for all $(x_0, \pi_0) \neq (x_0^*, \pi_0^*)$ if a determinate optimal equilibrium is to arise. The condition for continous paths follows from making the right and left limits of x_t and π_t equal to each other at $t = T$ and $t = t_1$. They are "continuous pasting" conditions that connect the paths of x_t and π_t when either r_t or i_t jump. The conditions for bounded paths are identical to those in models without the ZLB —the economy must be on a saddle path or steady-state at t_1— since after liftoff we assume the ZLB is not binding.

Although instructive and simple in many dimensions, a rule like (13) for t_1, whether coupled with (12) or (14), may not be the easiest to communicate to the public, as it references somewhat complicated conditions on initial inflation and output like $Ax_0 + B\pi_0 \neq C$. In addition, the degrees of freedom in the choice of f, both in the number of distinct values it can take and in the numeric values themselves, may call into question the central bank's economic rationale behind any particular choice. We thus now propose rules for t_1 that attempt to make the function $f(x_0, \pi_0)$ a continuous function of (x_0, π_0). We seek to model the intuitive real-world phenomenon that, at least during non-crisis times, small changes in economic conditions generally lead to changes in monetary policy of commensurate size. This feature is present in all interest rate and targetting rules in the literature and, in the present context, reveals additional intuition about how a central bank should tackle the ZLB. Whether a continuous rule is easier to explain and communicate to the public than something like rule (13) is left for the reader to decide.

The first concrete rule we propose has (16) and (17) together with

$$\xi_\pi > 1 \tag{18}$$

$$\xi_x = 0 \tag{19}$$

$$f(x_0, \pi_0) = \bar{t} + A\left|\pi_0 - \pi_0^*\right| \tag{20}$$

whenever $(x_0, \pi_0) \neq (x_0^*, \pi_0^*)$, where $\bar{t}$ and A are any large enough positive constants. This rule prescribes zero nominal interest rates for a period of time that depends on deviations of initial inflation from optimal and a Taylor rule with $\xi_x = 0$ and any constant $\xi_\pi > 1$ thereafter. More generally, we can replace the choice of ξ_x and ξ_π by any combination that satisfies the Taylor principle, given by

$$\kappa\left(\xi_\pi - 1\right) + \rho\xi_x > 0 \text{ and } \xi_x + \sigma\rho > 0. \tag{21}$$

The intuition for why this rule eliminates indeterminacy is as follows. Because $\xi_\pi > 1$, the Taylor principle holds for $t \geq t_1$ so any value of (x_{t_1}, π_{t_1}) different from the steady state $(0, 0)$ leads to explosive paths. In other words, the central bank is "tough" on inflation after the ZLB, inducing the economy to have $\pi_t = 0$ for all $t \geq t_1$. On the other hand, the central bank provides additional stimulus by extending the amount of time spent at the ZLB as π_0 moves away from π_0^*. If the stimulus from this extension is large enough —meaning $\bar{t}$ and A are large enough— the expectations of the ensuing high inflation are inconsistent with the expectations of zero inflation that arise from following the Taylor principle. Hence, no equilibrium of this type can exist.

One way to think about the inconsistent expectations of a large stimulus during the ZLB and of zero inflation after the ZLB is that the central bank generates such high inflation during the ZLB that inflation expectations become unanchored. Another way, equally compatible with the equations, is to think that the non-existence of equilibrium does not arise because the central bank is unable to rein in the self-inflicted inflation. On the contrary, the central bank makes a credible commitment to stimulate the economy during the ZLB and can completely revert back to zero inflation by using the Taylor principle outside the ZLB. The key inconsistency in this version of the story is that there is no π_0 so dire —so far away from π_0^*— that requires the central bank to put in action the large stimulus and large reversal specified by the rule. More concretely, consider initial inflation expectations of $\tilde{\pi}_0 \neq \pi_0^*$. Rule (20) implies that expectations of t_1 are $\bar{t} + A\left|\tilde{\pi}_0 - \pi_0^*\right|$. Using the IS and the NKPC to trace the path of the economy backwads in time from $(0, 0)$ at $t = t_1$ to (x_0, π_0)

always reveals that $\pi_0 \neq \tilde{\pi}_0$. While inflation expectations are finite and the central bank keeps its commitment throughout, the equilibrium cannot exist because no initial level of inflation is rational.

Agents form rational expectations about several objects: π_0, π_{t_1}, t_1, etc. Non-existence of equilibria only requires that one of these expectations is not validated. In our first interpretation, we assumed expectations of π_0 were correct and concluded inflation expectations must be unbounded. In the second case, we started by asserting inflation is bounded and could not find a rational expectation for π_0. There are, of course, other ways to tell the non-existence story based on the expectations that we choose to assume are unquestionably rational and the ones we subsequently check for consistency.

We now show another variation of the rule that also implements the optimal equilibrium uniquely but without satisfying the Taylor principle. Define the functions

$$m(u) = \frac{r_h}{\kappa}\frac{\phi_1^2 e^{-\phi_2 u} - \phi_2^2 e^{-\phi_1 u}}{\phi_1 - \phi_2} + \left(\frac{r_h - r_l}{\kappa}\right)\frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\phi_1 - \phi_2} - \frac{r_l \rho}{\kappa} \tag{22}$$

$$h(u) = r_h \frac{\phi_1 e^{-\phi_2 u} - \phi_2 e^{-\phi_1 u}}{\phi_1 - \phi_2} + (r_h - r_l)\frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\phi_1 - \phi_2} - r_l \tag{23}$$

where

$$\phi_1 = \frac{1}{2}\rho + \frac{1}{2}\sqrt{\rho^2 + 4\frac{\kappa}{\sigma}} \tag{24}$$

$$\phi_2 = \frac{1}{2}\rho - \frac{1}{2}\sqrt{\rho^2 + 4\frac{\kappa}{\sigma}} \tag{25}$$

The rule is

$$\xi_\pi < 1 \tag{26}$$

$$\xi_x = \sigma\kappa\frac{m'(t_1)}{h'(t_1)}(\xi_\pi - 1) + \frac{h'(t_1)}{m'(t_1)} + \sigma\rho \tag{27}$$

$$f(x_0, \pi_0) = \bar{t} + A\left|\pi_0 - \pi_0^*\right| \tag{28}$$

where $\bar{t}$ and A are any large enough positive constants and ξ_π is any constant smaller than

1. Both $\frac{m'(t_1)}{h'(t_1)}$ and $\frac{h'(t_1)}{m'(t_1)}$ are positive; $\frac{m'(t_1)}{h'(t_1)}$ is decreasing in $|\pi_0 - \pi_0^*|$ and $\frac{h'(t_1)}{m'(t_1)}$ is increasing in $|\pi_0 - \pi_0^*|$. Picking any other ξ_π (including $\xi_\pi > 1$) will give identical results as long as

$$\kappa\left(\xi_\pi - 1\right) + \rho\xi_x < 0 \tag{29}$$

since the economy is then guaranteed to have a saddle path, which is the essential distinction from rules that satisfy the Taylor principle given in equation (21).

Compared to the previous rule, the form of stimulus during the ZLB remains unchanged, requiring that t_1 increases fast enough as a function of $|\pi_0 - \pi_0^*|$. However, instead of envisioning the economy to be in steady-state immediately after liftoff by being aggressive on inflation, the central bank pursues a more gradual adjustment strategy. The economy is anticipated to be on its saddle path at t_1 and then travel on it towards the steady-state as time passes.

The intuition for why suboptimal equilibria cannot be sustained is similar to before. In the previous rule, we picked t_1 so that the economy could not reach steady-state for any $(x_0, \pi_0) \neq (x_0^*, \pi_0^*)$ under rational expectations. In this rule, we choose policy so that the economy cannot reach its saddle path. Avoiding the saddle path (a line) requires more fine-tuning than avoiding the steady state (a point). Given a coefficient ξ_π for inflation, the output coefficient ξ_x is engineered so that expectations for π_0 remain the same when expectations for x_{t_1} and π_{t_1} change. Under the last rule described in equations (18)-(20), expectations of x_{t_1} and π_{t_1} do matter for π_0, but they are anchored at $x_{t_1} = \pi_{t_1} = 0$ because of the Taylor principle. Effectively, then, π_0 depends only on the time spent at the ZLB. Under the new rule described in equations (26)-(28), the central bank contends with the more intricate case in which expectations of π_0 depend on expectations of π_{t_1} and x_{t_1} that are not anchored by the Taylor principle. The relative influence that expectations of π_{t_1} and x_{t_1} have on π_0 is influenced by the relative weight that the central bank places on inflation and output in its Taylor rule. The choice of ξ_x given by equation (27) ensures that the central bank places a relative weight of inflation and output that makes any changes in the expectation of π_0 stemming from changes in expectations of inflation at t_1 be exactly offset by changes in expectations of output at t_1, and vice-versa. As a result, the saddle path of the economy is the same for any $\pi_0 \neq \pi_0^*$. Because the central bank can achieve this strategy by picking the relative weight it places on inflation and output, the absolute level of its reaction to inflation, ξ_π, is free (as long as $\xi_\pi < 1$ so that we are not in the realm of the Taylor principle, analyzed in the last rule).

While the last two rules have liftoff times t_1 that are continuous for all $(x_0, \pi_0) \neq (x_0^*, \pi_0^*)$, t_1 is discontinuous at (x_0^*, π_0^*) since $t^* + \bar{t}$ is not in a small neighborhood of t^*. In addition, the rule $t_1 = t^* + \bar{t} + A\,|\pi_0 - \pi_0^*|$ with large enough positive $\bar{t}$ and A has a decidedly dovish flavor, as not only lower, but also *higher* initial inflation prolong the time spent at the ZLB, which (eventually) makes inflation and output larger than when $t_1 = t^*$. In the next propositions, we show that these two features are common to all rules that implement the optimal path as a unique equilibrium and have a t_1 that is continuous in (x_0, π_0) for all off-equilibrium paths. In addition, Proposition 4 shows that the Taylor rules we considered in equations (18)-(19) and (26)-(27) are the only two possible types of rules that can be followed outside the ZLB to implement a unique optimal equilibrium, providing a complete characterization of the class of rules with a state-dependent t_1.

Proposition 3. *If $f(x_0, \pi_0)$ is continuous at (x_0^*, π_0^*) and $\kappa\sigma\lambda \neq 1$, then the rule given in equations (14)-(15) cannot implement the optimal equilibrium without indeterminacy.*

Proof. Appendix

Proposition 4. *Let $g(x_0, \pi_0)$ be a continuous function with $g(x_0^*, \pi_0^*) \neq 0$ and set*

$$f(x_0, \pi_0) = \begin{cases} 0 & , \ \text{if } (x_0, \pi_0) = (x_0^*, \pi_0^*) \\ g(x_0, \pi_0) & , \ \text{if } (x_0, \pi_0) \neq (x_0^*, \pi_0^*) \end{cases}$$

The interest rate rule given by equations (14)-(15) implements the optimal path as the unique equilibrium of the economy if and only if the following two conditions are satisfied:

1. *Either (21) holds, or (27) and (29) hold*

2. For all $x_0 \geq m(T)$ and $\pi_0 \geq h(T)$, the liftoff time $t_1 = t^* + f(x_0, \pi_0)$ satisfies

$$t_1 \geq \max\left\{m^{-1}(x_0), h^{-1}(\pi_0)\right\}, \tag{30}$$

where the functions m and h are given by equations (22) and (23).

Proof. Appendix

The first condition in the proposition asserts that expectations of x_0 and π_0 are invariant with respect to expectations of x_{t_1} and π_{t_1}. When the Taylor principle holds, expectations of x_{t_1} and π_{t_1} are fixed at zero lest we have unbouded outcomes, so there cannot be changes in expectations of x_0 and π_0 stemming from x_{t_1} or π_{t_1}. When the Taylor principle does not hold, the choice of ξ_x given by (27) makes the influences of π_{t_1} and x_{t_1} on x_0 and π_0 exactly cancel each other. Irrespective of whether they Taylor principle holds, once x_0 and π_0 are decoupled from x_{t_1} and π_{t_1}, they are solely controlled by t_1. The second condition in the proposition then makes the choice of t_1 inconsistent with any rational expectations agents may form for x_0 and π_0 whenever $(x_0, \pi_0) \neq (x_0^*, \pi_0^*)$. For $x_0 < m(T)$ and $\pi_0 < h(T)$, no equilibrium is possible independent of the choice of t_1, as initial inflation and output are too far from the steady-state for the economy to ever get there by following the IS and NKPC. For $x_0 \geq m(T)$ and $\pi_0 \geq h(T)$, we eliminate undesired equilibria by following the same strategy as in the examples we have already examined. The second condition in Proposition 4 makes the stimulus "strong enough", where the meaning of strong enough is made precise by equation (30). The functions $m^{-1}(x_0)$ and $h^{-1}(\pi_0)$, being the inverses of m and h, are continuous, increasing, concave, bounded below by T and tend to infinity as their arguments grow to infinity. As initial inflation and output increase, so must the time spent at the ZLB, confirming the dovish nature of the class of rules we study in this paper.

One immediate consequence of the proposition is that the two examples we proposed in (18)-(19) and (26)-(27) indeed implement the optimal path as the unique equilibrium. The proposition is also useful to construct additional rules. The bound (30) on t_1 is independent of ξ_π and ξ_x, so we can always first choose a rule for liftoff without thinking about what to do after t_1 and then pick appropriate ξ_π and ξ_x that satisfy the first condition in Proposition 4. For example, a particularly simple alternative to (20) is to set $t_1 = \bar{t} + A \max(0, \pi_0)$.

To make the results more concrete, we use the parameter values in Werning (2011) ($T = 2$, $\sigma = 1$, $\kappa = 0.5$, $\lambda = 1/\kappa$, $\rho = 0.01$, $r_h = 0.04$ and $r_l = -0.04$) to numerically examine the rules. Based on (30), the tightest bounds on $\bar{t}$ and A for rule (18)-(20) are[12]

$$\begin{aligned} \bar{t} &\geq t^* = 2.6 \\ A &\geq 11.2 \end{aligned}$$

These magnitudes mean, for example, that a deviation of π_0 from π_0^* of two percentage points (in either direction) must lead to an extension of the ZLB period from 2.6 to at least 2.8 years in order to eliminate indeterminacy. In this case, fairly modest extensions in the length of the ZLB can be enough to preclude indeterminacy. For the rule (26)-(28), the functions $a(\pi_0)$ and $b(\pi_0)$ at π_0^* are

$$a(\pi_0^*) = 0.07 \tag{31}$$

$$b(\pi_0^*) = 0.68 \tag{32}$$

If we set $\xi_\pi = 0.5$, the Taylor rule is then given by

$$i_t = 0.5\pi_t + 0.41x_t + r_t$$

If $|\pi_0 - \pi_0^*| = 2\%$, then

$$a(\pi_0) = 0.04 \tag{33}$$

$$b(\pi_0) = 0.69 \tag{34}$$

[12]Note that in this example $\kappa\sigma\lambda = 1$ so we are able to make f continuous at (x_0^*, π_0^*).

and the Taylor rule is given by

$$i_t = 0.5\pi_t + 0.39x_t + r_t.$$

Finally, we present a variation that simplifies the communication of the rule for liftoff time t_1, at the cost of allowing ξ_π and ξ_x to depend not only on expectations of initial inflation, but also on expectations of output at t_1.

Proposition 5. *If we allow ξ_π and ξ_x to depend on x_{t_1} then we can implement the optimal path as the unique equilibrium with $f(x_0, \pi_0)$ continuous at (x_0^*, π_0^*).*

Proof. See Appendix.

The intuition for Proposition 5 is that the central bank can now directly affect expectations of x_0 and π_0 not just by using t_1 but also by influencing expectations of x_{t_1} by changing the coefficients of the Taylor rule. This gives the central bank more flexibility to pick t_1, as some of the burden of a "high enough" t_1 can be eased by regulating x_{t_1}. The need to make f discontinuous in the previous cases was required to create expectations of a large enough boom when expectations were very close to optimal. Now that the central bank can directly condition its actions on expectations of output after the ZLB, there is no need to increase output expectations through the choice of f around (x_0^*, π_0^*). A corollary of this proposition is that we can set $\bar{t} = 0$ in equation (20). The advantage in this case is that the central bank need not concern itself with the choice of $\bar{t}$ and can now use the same formula $t_1 = t^* + A|\pi_0 - \pi_0^*|$ for all on and off equilibrium paths alike. One disadvantage is that it requires a more complicated choice of ξ_π and ξ_x that uses contemporaneous information at t_1, in contrast to the previous rules that only required knowledge of π_0 at t_1.

6 Conclusion

We have presented a new way to manage a liquidity trap. The core of our proposal is to make the time of liftoff from the ZLB contingent on inflation (or the output gap) at the beginning of the liquidity trap episode. If liftoff is sufficiently delayed as initial inflation (or the output gap) increase, then the class of rules we examine can eliminate indeterminacy in the context of a rational expectations New Keynesian model and uniquely implement the socially optimal (or any other feasible) path for the economy. Another virtue of the rules we propose is their relative simplicity: they are easy to communicate, can be made independent of the deep parameters of the model, can be made memoryless once the ZLB is over, do not hinge on active or non-Ricardian fiscal policy and, perhaps more importantly, do not require a change in regime as would be the case for price-level targets, money-supply rules, exchange rate management or time-varying inflation targets.

Although our rules have desirable properties in the workhorse New Keynesian model — which we believe is a useful first step towards understanding them— we have not studied their broader applicability. Assessing the robustness and efficiency of the rule in different models is an important next step towards real-world use. A central bank may benefit from understanding the properties of our rule in the presence of informational frictions, learning, more comlpex investment dynamics, imports and exports, coordination with fiscal policy, limited commitment, imperfect credibility, heterogeneous agents, financial intermediation and financial stability concerns, etc.

Maybe it's difficult to increase t1 when inflation is high, but because of spiral nature of dynamics, inflation will decrease going forward, perhaps making the CB appear more prescient.

7 Bibliography

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8 Appendix

8.1 Preliminaries

Consider an interest rate rule of the form:

$$
\begin{aligned}
i_t &= \begin{cases} 0 & , \quad 0 \le t < t_1 \\ \xi_\pi(\pi_0, x_0)\,\pi_t + \xi_x(\pi_0, x_0)\,x_t + r_t & , \quad t_1 \le t < \infty \end{cases} && (35) \\
t_1(\pi_0, x_0) &= t^* + f(\pi_0, x_0) && (36) \\
g(\pi_0, x_0) &\ge T - t^* && (37)
\end{aligned}
$$

where $\xi_\pi(\pi_0, x_0)$, $\xi_x(\pi_0, x_0)$ and $f(\pi_0, x_0)$ are functions of initial inflation and output. Using this interest rate rule, the system of ODEs (1)-(2) in matrix form is

$$
\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sigma} r_l \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{\sigma} \\ -\kappa & \rho \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} \text{ for } t \in [0, T) \qquad (38)
$$

$$
\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sigma} r_h \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{\sigma} \\ -\kappa & \rho \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} \text{ for } t \in [T, t_1) \qquad (39)
$$

$$
\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma}\xi_x & \frac{1}{\sigma}(\xi_\pi - 1) \\ -\kappa & \rho \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} \text{ for } t \in [t_1, \infty) \qquad (40)
$$

Because the solution must be continuous, we require that the solutions to (38) and (39) "paste continuously" at $t = T$ and that the solutions to (39) and (40) paste continuously at time $t = t_1$:

$$
\lim_{t \to T\text{-}} x_t = x_T,\ \lim_{t \to T\text{-}} \pi_t = \pi_T,\ \lim_{t \to t_1^-} x_t = x_{t_1},\ \lim_{t \to t_1^-} \pi_t = x_{t_1}. \qquad (41)
$$

Let $\phi_1 > 0$, $\phi_2 < 0$ be the eigenvalues of

$$A_{zlb} = \begin{bmatrix} 0 & -\frac{1}{\sigma} \\ -\kappa & \rho \end{bmatrix}$$

given in equations (24) and (25). Let α_1, α_2 be the eigenvalues of the steady-state dynamics matrix

$$A_{ss} = \begin{bmatrix} \frac{1}{\sigma}\xi_x & \frac{1}{\sigma}\left(\xi_\pi - 1\right) \\ -\kappa & \rho \end{bmatrix}$$

given by

$$\begin{aligned} \alpha_1 &= \frac{1}{2\sigma}\left(\xi_x + \sigma\rho + \sqrt{\xi_x^2 - 2\sigma\rho\xi_x + \sigma^2\rho^2 - 4\kappa\sigma\left(\xi_\pi - 1\right)}\right) \\ \alpha_2 &= \frac{1}{2\sigma}\left(\xi_x + \sigma\rho - \sqrt{\xi_x^2 - 2\sigma\rho\xi_x + \sigma^2\rho^2 - 4\kappa\sigma\left(\xi_\pi - 1\right)}\right) \end{aligned}$$

Explicitly solving (38)-(40), the continuous pasting condition (41) in terms of t_1, x_0, π_0, x_{t_1} and π_{t_1} is

$$\begin{aligned} x_0 &= \frac{\left(\phi_1 e^{-\phi_2 t_1} - \phi_2 e^{-\phi_1 t_1}\right)}{\left(\phi_1 - \phi_2\right)} x_{t_1} - \frac{1}{\sigma}\frac{\left(e^{-\phi_1 t_1} - e^{-\phi_2 t_1}\right)}{\left(\phi_1 - \phi_2\right)}\pi_{t_1} + r_h \frac{\phi_1^2 e^{-\phi_2 t_1} - \phi_2^2 e^{-\phi_1 t_1}}{\kappa\left(\phi_1 - \phi_2\right)} \quad (42) \\ &\quad + \left(r_h - r_l\right)\frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\kappa\left(\phi_1 - \phi_2\right)} - \frac{r_l\rho}{\kappa}, \\ \pi_0 &= -\kappa\frac{\left(e^{-\phi_1 t_1} - e^{-\phi_2 t_1}\right)}{\left(\phi_1 - \phi_2\right)} x_{t_1} + \frac{\left(\phi_1 e^{-\phi_1 t_1} - \phi_2 e^{-\phi_2 t_1}\right)}{\left(\phi_1 - \phi_2\right)}\pi_{t_1} + r_h \frac{\phi_1 e^{-\phi_2 t_1} - \phi_2 e^{-\phi_1 t_1}}{\left(\phi_1 - \phi_2\right)} \quad (43) \\ &\quad + \left(r_h - r_l\right)\frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\left(\phi_1 - \phi_2\right)} - r_l. \end{aligned}$$

By analyzing the asymptotic behavior of the solution to (40), the transversality conditions (5) and (6) can be equivalently written as

$$
\begin{aligned}
(1-\xi_\pi)\,\pi_{t_1} &= (\xi_x-\sigma\alpha_2)\,x_{t_1}, \text{ if } \det A_{ss}<0 && (44)\\
\pi_{t_1} &= x_{t_1}=0, \text{ if } \det A_{ss}>0 \text{ and } \operatorname{trace} A_{ss}>0 && (45)\\
\rho\pi_1 &= \kappa x_1, \text{ if } \det A_{ss}=0 \text{ and } \xi_x+\sigma\rho\geq 0 && (46)\\
\text{any } \pi_{t_1}, x_{t_1} &\in \mathbb{R}, \text{ otherwise}^{13} && (47)
\end{aligned}
$$

The first case corresponds to saddle-path stability (system explodes unless it starts on the saddle path), the second to instability (system explodes unless it starts at the steady-state), the third to a knife-edge case (system either explodes or remains constant at its starting point) and the last to stability (system never explodes).

8.2 Constants in Proposition 2

$$
\begin{aligned}
A &= \frac{\phi_1}{\phi_2}-\frac{\phi_2}{\phi_1}\\
B &= \kappa\left(\frac{1}{\phi_1}-\frac{1}{\phi_2}\right)\\
C &= e^{f\phi_2}\left(r_l+e^{-T\phi_2}\left(r_h-r_l\right)\right)-e^{f\phi_1}\left(r_l+e^{-T\phi_1}\left(r_h-r_l\right)\right)\\
D &= \frac{\phi_1}{\phi_2}e^{\phi_1}-\frac{\phi_2}{\phi_1}e^{\phi_2}\\
E &= \kappa\left(\frac{e^{\phi_2}}{\phi_1}-\frac{e^{\phi_1}}{\phi_2}\right)
\end{aligned}
$$

8.3 Proof of Proposition 3

If $f(x_0,\pi_0)$ is continuous at (x_0^*,π_0^*) and $\kappa\sigma\lambda\neq 1$, then the rule given in equations (14)-(15) cannot implement the optimal equilibrium without indeterminacy.

Proof. Let $f(x_0,\pi_0)$ be a continuous function and $\kappa\sigma\lambda\neq 1$. Assume, for the sake of contradiction, that the rule in equations (14)-(15) implements the optimal equilibrium uniquely.

This means there the only solution to the system of equations (42)-(47) is $\left(\pi_0^*, \pi_{t_1}^*, x_0^*, x_{t_1}^*\right)$. The case (47) cannot occur since, without any restrictions on π_{t_1} and x_{t_1}, (42)-(43) clearly have an infite number of solutions. For the other three cases (44)-(46), the (non-linear, because of t_1) system of two equations (42)-(43) in the four uknowns $(\pi_0, \pi_{t_1}, x_0, x_{t_1})$ always has more than one solution unless

$$\frac{\left(\phi_1 e^{-\phi_2 t_1} - \phi_2 e^{-\phi_1 t_1}\right)}{(\phi_1 - \phi_2)} x_{t_1} - \frac{1}{\sigma} \frac{\left(e^{-\phi_1 t_1} - e^{-\phi_2 t_1}\right)}{(\phi_1 - \phi_2)} \pi_{t_1} = 0 \tag{48}$$

and

$$-\kappa \frac{\left(e^{-\phi_1 t_1} - e^{-\phi_2 t_1}\right)}{(\phi_1 - \phi_2)} x_{t_1} + \frac{\left(\phi_1 e^{-\phi_1 t_1} - \phi_2 e^{-\phi_2 t_1}\right)}{(\phi_1 - \phi_2)} \pi_{t_1} = 0 \tag{49}$$

Using the last two equations in (42) and (43), we get

$$\begin{aligned} x_0 &= r_h \frac{\phi_1^2 e^{-\phi_2 t_1} - \phi_2^2 e^{-\phi_1 t_1}}{\kappa(\phi_1 - \phi_2)} + (r_h - r_l) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\kappa(\phi_1 - \phi_2)} - \frac{r_l \rho}{\kappa}, & (50) \\ \pi_0 &= r_h \frac{\phi_1 e^{-\phi_2 t_1} - \phi_2 e^{-\phi_1 t_1}}{(\phi_1 - \phi_2)} + (r_h - r_l) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{(\phi_1 - \phi_2)} - r_l. & (51) \end{aligned}$$

Evaluating (42) and (43) at $\left(\pi_0^*, \pi_{t_1}^*, x_0^*, x_{t_1}^*\right)$ and substracting from (50) and (51), we get

$$\begin{aligned} x_0 - x_0^* &= r_h \frac{\phi_1^2 e^{-\phi_2 t_1} - \phi_2^2 e^{-\phi_1 t_1}}{\kappa(\phi_1 - \phi_2)} - r_h \frac{\phi_1^2 e^{-\phi_2 t^*} - \phi_2^2 e^{-\phi_1 t^*}}{\kappa(\phi_1 - \phi_2)} \\ &\quad - \frac{\left(\phi_1 e^{-\phi_2 t^*} - \phi_2 e^{-\phi_1 t^*}\right)}{(\phi_1 - \phi_2)} x_{t_1}^* & (52) \\ &\quad + \frac{2\kappa}{\sigma} \frac{\left(e^{-\phi_1 t^*} - e^{-\phi_2 t^*}\right)}{(\phi_1 - \phi_2)} \left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2}\right)^{-1} x_{t_1}^* & (53) \\ \pi_0 - \pi_0^* &= r_h \frac{\phi_1 e^{-\phi_2 t_1} - \phi_2 e^{-\phi_1 t_1}}{(\phi_1 - \phi_2)} - r_h \frac{\phi_1 e^{-\phi_2 t^*} - \phi_2 e^{-\phi_1 t^*}}{(\phi_1 - \phi_2)} \\ &\quad + \kappa \frac{\left(e^{-\phi_1 t^*} - e^{-\phi_2 t^*}\right)}{(\phi_1 - \phi_2)} x_{t_1}^* & (54) \\ &\quad - 2\kappa \frac{\left(\phi_1 e^{-\phi_1 t^*} - \phi_2 e^{-\phi_2 t^*}\right)}{(\phi_1 - \phi_2)} \left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2}\right)^{-1} x_{t_1}^* & (55) \end{aligned}$$

where we have used that $\pi_{t_1}^* = 2\kappa \left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2}\right)^{-1} x_{t_1}^*$, as show in Werning (2011).

Letting $(x_0, \pi_0) \to (x_0^*, \pi_0^*)$, continuity of $f(x_0, \pi_0)$ at (x_0^*, π_0^*) implies $t_1 \to t^*$ and thus

$$
\begin{aligned}
0 &= \left(\phi_1 e^{-\phi_2 t^*} - \phi_2 e^{-\phi_1 t^*} - \frac{2\kappa}{\sigma}\left(e^{-\phi_1 t^*} - e^{-\phi_2 t^*}\right)\left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2}\right)^{-1}\right) x_{t_1}^* \\
0 &= \left(-\kappa\left(e^{-\phi_1 t^*} - e^{-\phi_2 t^*}\right) + 2\kappa\left(\phi_1 e^{-\phi_1 t^*} - \phi_2 e^{-\phi_2 t^*}\right)\left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2}\right)^{-1}\right) x_{t_1}^*
\end{aligned}
$$

If $x_{t_1}^* = 0$, then we're in the case $\kappa\sigma\lambda = 1$. Otherwise,

$$
\begin{aligned}
\left(\phi_1 e^{-\phi_2 t^*} - \phi_2 e^{-\phi_1 t^*}\right) - \frac{2\kappa}{\sigma}\left(e^{-\phi_1 t^*} - e^{-\phi_2 t^*}\right)\left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2}\right)^{-1} &= 0 \\
-\kappa\left(e^{-\phi_1 t^*} - e^{-\phi_2 t^*}\right) + 2\kappa\left(\phi_1 e^{-\phi_1 t^*} - \phi_2 e^{-\phi_2 t^*}\right)\left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2}\right)^{-1} &= 0
\end{aligned}
$$

lead to a contradiction: Solving for t^* in each of the equations yields different solutions unless $\kappa^2\lambda = 0$, which is impossible since $\kappa > 0$ and $\lambda > 0$.

8.4 Proof of Proposition 4

Proposition 6. *Assume $f(x_0, \pi_0)$ is continuous for all $(x_0, \pi_0) \neq (x_0^*, \pi_0^*)$. The interest rate rule given by equations (14)-(15) implements the optimal path as the unique equilibrium of the economy if and only if*

1. *Either (21) holds, or (27) and (29) hold*

2. *For all $x_0 \geq m(T)$ and $\pi_0 \geq h(T)$, the liftoff time $t_1 = t^* + f(x_0, \pi_0)$ satisfies*

$$
t_1 \geq \max\left\{m^{-1}(x_0), h^{-1}(\pi_0)\right\}, \tag{56}
$$

where the functions m and h are given by equations (22) and (23).

Proof. Let $f(x_0, \pi_0)$ be continuous for all $(x_0, \pi_0) \neq (x_0^*, \pi_0^*)$. We first assume that conditions 1 and 2 hold and prove the optimal equilibrium is the unique equilibrium. The optimal equilibrium is an equilibrium by equations (14)-(15). If condition 1 holds, then for all $(x_0, \pi_0) \neq (x_0^*, \pi_0^*)$

$$\begin{aligned}
\frac{\left(\phi_1 e^{-\phi_2 t_1} - \phi_2 e^{-\phi_1 t_1}\right)}{(\phi_1 - \phi_2)} x_{t_1} - \frac{1}{\sigma} \frac{\left(e^{-\phi_1 t_1} - e^{-\phi_2 t_1}\right)}{(\phi_1 - \phi_2)} \pi_{t_1} &= 0 \\
-\kappa \frac{\left(e^{-\phi_1 t_1} - e^{-\phi_2 t_1}\right)}{(\phi_1 - \phi_2)} x_{t_1} + \frac{\left(\phi_1 e^{-\phi_1 t_1} - \phi_2 e^{-\phi_2 t_1}\right)}{(\phi_1 - \phi_2)} \pi_{t_1} &= 0
\end{aligned}$$

Equations (42)-(43) then give

$$\begin{aligned}
x_0 &= m(t_1), \\
\pi_0 &= h(t_1).
\end{aligned}$$

Condition 2 ensures that at least one of the last two equations is violated, precluding all non-optimal equilibrium.

Conversely, assume that the optimal path is the unique equilibrium. From the proof of the last proposition, we know that the case (47) cannot occur. The case (46) cannot occur either, since the solution to the ODE (40) would be constant, allowing for equilibria other than the optimal one. Indeed, any path for x_t and π_t that is in the line $\rho\pi = \kappa x$ at t_1 is an equilibrium. From the proof of the last proposition, we also know that (48) and (49) must hold. If equation (45) holds, (48) and (49) are immediately satisfied. If equation (44) holds, (48) and (49) are satisfied if and only if (27). It follows that condition 1 is true.

Because the optimal equilibrium is unique and (48) and (49) hold, we must have that, for all $(x_0, \pi_0) \neq (x_0^*, \pi_0^*)$, either

$$x_0 \neq m(t_1) \tag{57}$$

or

$$\pi_0 \neq h(t_1) \tag{58}$$

(or both). For $x_0 < m(T)$, (57) is immediately satisfied since $t_1 \geq T$ bounds $m(t_1)$ below by $m(T)$. For the same reason, (58) is immediately satisfied for $\pi_0 < h(T)$. Now assume

$x_0 \geq m(T)$ and $\pi_0 \geq h(T)$. If there exist $\bar{x}_0$ and $\bar{\pi}_0$ such that

$$t_1 < \max\left\{m^{-1}(\bar{x}_0), h^{-1}(\bar{\pi}_0)\right\} \tag{59}$$

then there is more than one equilibrium. To see this, note that t_1 is bounded below by T and thus there exist constants $\bar{m}$ and $\bar{h}$ such that

$$m(t_1) \geq \bar{m} \tag{60}$$

$$h(t_1) \geq \bar{h} \tag{61}$$

On the other hand, we can always find $\underline{x}_0$ and $\underline{\pi}_0$ such that

$$\underline{x}_0 < \bar{m} \leq m(t_1)$$

$$\underline{\pi}_0 < \bar{h} \leq h(t_1)$$

Because m and h is monotonic, the last two equations imply

$$m^{-1}(\underline{x}_0) < t_1 \tag{62}$$

$$h^{-1}(\underline{\pi}_0) < t_1 \tag{63}$$

By continuity of $f(x_0, \pi_0)$, (59), (62) and (63) together imply that there exist $(x_0^{**}, \pi_0^{**}) \neq (x_0^*, \pi_0^*)$ such that $x_0^{**} = m(t_1)$ and $\pi_0^{**} = h(t_1)$. The paths for inflation and output starting at (x_0^{**}, π_0^{**}) are an equilibrium, and therefore we must have, for all $(x_0, \pi_0) \neq (x_0^*, \pi_0^*)$ that

$$t_1 \geq \max\left\{m^{-1}(x_0), h^{-1}(\pi_0)\right\} \tag{64}$$

which proves that condition 2 is satisfied.