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Abstract

We study a dynamic model of collateralized lending under adverse selection in which the quality of collateral assets is endogenously determined by hidden effort. Complementarities in incentives lead to non-ergodic dynamics: Asset quality and output grow when asset quality is high, but stagnate or deteriorate otherwise. Inefficiencies remain, even in the most efficient competitive equilibrium—investment and output are vulnerable to spells of lending market illiquidity, and these spells may persist because of suboptimal effort. Nevertheless, benevolent regulators without commitment can destroy welfare by prioritizing liquidity over incentives. Optimal interventions with commitment call for large, long-term subsidies in excess of what is required to restore liquidity.

Key words: liquidity, government intervention, adverse selection, collateral

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1 Introduction

When lenders are worried about default, they may ask borrowers to post collateral: mortgages are secured by houses, corporate bonds are secured by firm assets, and securitized debt instruments are backed by pools of loans. While collateralization typically increases capital market liquidity, negative shocks to collateral quality may lead to sharp increases in borrowing costs and market breakdowns. This has motivated regulators around the world to intervene in asset markets and lower policy rates to support lending activity.[1]

Existing theories study interventions in impaired markets under the assumption that collateral quality is exogenously determined. Yet in many instances, collateral quality is a choice: homeowners may shirk on maintenance, firms may pursue risky investments, and issuers of asset-backed securities may pledge bad loans. This suggests that the resilience of collateralized lending markets, as well as the effects of policy interventions, ultimately depend on private incentives to maintain high-quality collateral. We develop a dynamic model where lending is subject to adverse selection and collateral quality is persistent but determined by hidden effort. Our framework offers both a tractable model of long-run asset quality dynamics in collateralized lending markets, and a laboratory to study the dynamic effects of regulatory interventions.

We arrive at three main insights, all of which are driven by borrowers taking into account expected future market conditions when choosing asset quality. First, markets can be *resilient*: even without interventions, asset quality can recover from adverse shocks that lead to market breakdowns. Second, a sufficiently severe shock may require intervention, but interventions designed to restore market liquidity at minimal cost can impede the recovery of asset quality and lower welfare relative to laissez-faire. At the same time, such policies are *uniquely optimal* from the perspective of benevolent regulator without commitment, or if asset quality is assume to be fixed. Third, the optimal intervention with commitment is larger, longer-term, and *not* contingent on illiquidity, i.e. it may need to be offered even

[1] Interventions have come in various forms during times of financial stress, including asset purchases, credit facilities, and securities lending facilities. Examples include the Federal Reserve's Commercial Paper Funding Facility and Troubled Asset Relief Program; the Bank of England's Special Liquidity Scheme; the Bank of Japan's equity ETF purchases; the European Central Bank's Securities Lending Programme and Targeted Longer-term Refinancing Operations; and the U.S. Home Affordable Refinancing Program.

when markets would be liquid on their own. These results provide a new perspective on why certain markets recover from crises while others fail, and have implications for the design of regulatory interventions in practice.

The basic structure of our infinite-horizon model is as follows. There is a continuum of liquidity-constrained, long-lived borrowers who each own a durable productive asset that is either *good* or *bad*. Types are persistent but evolve stochastically, with transition probabilities determined by hidden effort. In every period, investment in the asset generates an additional return. To capture this return, borrowers raise funds from short-lived lenders subject to limited pledgeability and asymmetric information about asset quality. This creates scope for adverse selection. In equilibrium, lenders require collateral to secure loans, and only provide funds when the share of good borrowers is above an endogenous *liquidity threshold*.

Our primary focus is on the evolution of asset quality. Private incentives to improve asset quality depend on the relative value of good and bad assets. Because of the feedback from asset quality to interest rates, however, asset values are endogenously determined by the expected path of average asset quality, which is in turn determined by an aggregation of effort across borrowers and time. Two economic forces shape incentives in equilibrium. The first is a *strategic complementarity* that operates when markets are liquid. Borrowers pay a lower interest rate and can pledge less collateral when there are many good borrowers. This increases the relative value of good assets because good borrowers can retain more of their collateral. Hence, effort incentives are stronger when the share of good assets are high. The second is a *local strategic substitutability* in a neighborhood around the liquidity threshold. At the liquidity threshold, borrowers can obtain funds only if they post *all* pledgeable cash flows as collateral. Below the threshold, borrowers cannot obtain funding and retain all collateral. Small increases in average asset quality that lead to market liquidity can thus lead to a sharp decrease in the relative value of retained cash flows. This induces borrowers to free-ride on others' efforts.

The interaction between these forces gives rise to a dynamic coordination problem in privately optimal effort strategies. We provide a tractable characterization, and show that there exist multiple competitive equilibria, including cycles driven by self-fulfilling beliefs. However, there exists a *unique* Pareto-dominant equilibrium that delivers the highest welfare among all competitive equilibria. We focus primarily on this equilibrium, which entails selecting beliefs most conducive to effort, and show that dynamic coordination failures may persist even in this most efficient equilibrium.

We fully characterize asset quality dynamics in this equilibrium and show that they follow a cut-off rule: when the initial share of good assets is above a threshold, all borrowers exert effort, asset quality grows, and the competitive equilibrium is efficient. When the initial share of good assets is low, effort is less than efficient and asset quality either deteriorates or stagnates. Hence, markets are resilient to moderate shocks but vulnerable to large shocks. Moreover, there is positive co-movement between liquidity, asset quality, and output. The key externality is that individual borrowers do not internalize their effect on the path of future interest rates. As a result, liquidity can be fragile: small differences in initial conditions, or small shocks to the discount factor or the persistence of types, may generate discontinuous drops in long-run asset quality and welfare. Contrary to existing models of adverse selection, these results are not due to contemporaneous breakdowns in trade. Instead, they arise because declines in expected market conditions discourage effort today. Hence markets where asset quality is less persistent and borrowers have short horizons are particularly prone to long-run inefficiencies.

The failure of markets to recover from large asset quality shocks motivates our policy analysis. As in Tirole (2012) and Philippon and Skreta (2012), we introduce a benevolent regulator who can provide subsidies to boost market liquidity. We go beyond their work by studying how such interventions affect private incentives to maintain good collateral, and how optimal policy responds to this feedback. Because the effort decision is the solution to a dynamic problem, we distinguish between regulators with and without commitment.

The optimal policy under limited commitment is analogous to Tirole (2012) and Philippon and Skreta (2012): because effort is sunk once capital must be injected, the regulator intervenes only if the lending market inefficiently freezes, and does so using the smallest-possible subsidy that restores liquidity. The resulting market interest rate is *as if* average asset quality were at the liquidity threshold. This minimizes the total subsidy because it forces borrowers to pay the highest feasible interest rate. But precisely because it maximally expropriates good borrowers, it simultaneously weakens incentives to the greatest extent possible. In the model, this effect may lead to *intervention traps* in which borrowers' expectations of future interventions hamper private incentives, leading to sustained illiquidity and continued interventions at increasing costs. Moreover, the share of good assets may fall to zero even if it would have recovered otherwise, and the intervention may lower welfare relative to laissez-faire.

One might conclude from this that regulators should avoid intervening at all cost. This intuition is incomplete. The downside of not intervening is that markets remain illiquid until asset quality has recovered. We show that an intervention aimed at improving market liquidity can preserve incentives using a *large* per-capita subsidy that drives down interest rates, thereby preserving the relative value of good assets. Interestingly, the total costs of large per-capita subsidies may be lower than that of the minimal intervention under limited commitment. This is due to a composition effect: there are now more good borrowers who are less likely to default. We illustrate this result by showing that a regulator with commitment can discontinuously improve long-run asset quality at *infinitesimal cost* if asset quality stagnates in the competitive equilibrium. Additionally, such interventions can rejuvenate markets that would otherwise remain illiquid indefinitely. The optimal commitment policy also has a dynamic component: to ensure that asset quality recovers in the long-run, the intervention must be maintained until effort is self-sustaining. This may entail providing subsidies even if asset quality has recovered beyond the liquidity threshold.

While stylized, our model has important implications for the design and evaluation of regulatory interventions in practice. The European Central Bank has recently engaged in a number of long-term interventions designed to kickstart lending and boost liquidity in collateralized markets. These programs include the Long-term Refinancing Operations (LTRO) of 2011, the Outright Monetary Transactions (OMT) of 2012, and more conventional monetary policy measures. Empirical studies have found that, while these programs have indeed increased market liquidity, they have not boosted firm investment, and may have triggered risk-shifting by affected banks via domestic sovereign bond purchases or zombie lending (see Drechsler et al. (2016), Acharya et al. (2019), Carpinelli and Crosignani (2018), and Daetz et al. (2019)). Indeed, Drechsler et al. (2016) argue that their findings are inconsistent with classical lender-of-last resort theory that predicts large positive effects, while Daetz et al. (2019) point to concerns about future bank health as a plausible mechanism. Both effects are consistent with our model where banks endogenously adjust the quality of their portfolios in response to interventions. Indeed, our focus on the long-run response of market participants to regulation is particularly relevant given that the ECB's Targeted Longer-Term Refinancing Operations of 2014 were extended in 2016, with further extensions discussed in 2019. A possible conclusion from our model is that the intervention failed not because it was long-term, as commentators have suggested, but because it was too small.

Related mechanisms may be at play in other markets. For example, Melzer (2017) shows that homeowners at risk of default strategically invest less in the collateral value of their houses. In the aftermath of the recent housing crisis, the U.S. government introduced the Home Affordable Refinance Program (HARP) to lower debt burdens and lower spillovers among households. Agarwal et al. (2017) suggest that a re-design of this program could have substantial welfare benefits.

1.1 Related Literature

The notion that limited availability of collateral hampers investment and output has been studied extensively in previous work (e.g. Kiyotaki and Moore (1997)). Our paper focuses on markets where the stock of collateral itself is subject to asymmetric information and endogenously determined (see Stroebel (2016) for empirical evidence). Fluctuations in information about collateral quality has been cited as a key catalyst of the recent financial crisis (e.g. Gorton and Ordoñez (2014)). Our contribution lies in analyzing the endogenous dynamics of collateral quality and the feedback between market liquidity and private incentives. In this respect, our setting provides sharp predictions on how asset quality responds to adverse shocks. This also differentiates our work from Bigio (2015), Kurlat (2013), and Eisfeldt (2004), who study how exogenous shocks to asset quality can depress investment and output in dynamic models with asymmetric information.

Our policy analysis relates to the theoretical literature on interventions in adversely-selected asset markets, including work by Tirole (2012) and Philippon and Skreta (2012) discussed above. Fuchs and Skrzypacz (2015) consider a model with dynamic trading but fixed asset quality. Camargo and Lester (2014) study decentralized markets in which the dynamics of trading are decided by selective exit of seller types. Chiu and Koeppl (2016) provide a search framework that emphasizes the option value of waiting and arrives at different implications for the optimal timing of interventions. Philippon and Schnabl (2013) study how to best recapitalize a banking sector plagued by debt overhang. They do not consider the endogenous response of asset quality. Camargo et al. (2016) study the design of interventions in the presence of an information externality from trade. We do not consider information externalities or dynamic trading, but focus on the endogenous determination of effort incentives in a dynamic environment. Our result on the harmful effects of interventions by a benevolent planner under limited commitment is related to Netzer and Scheuer (2010) who find a similar result in a two-period model of moral hazard and insurance. We consider a fully dynamic model with market freezes and discontinuous drops in output and payoffs, and show how endogenous quality dynamics may lead to vicious cycles. To the best of our knowledge, we are among the first papers to characterize optimal dynamic policy in a long-run model of endogenous asset quality.

Previous work has argued that increasing asset prices and market liquidity can *decrease* incentives to produce high-quality assets. Examples include Chemla and Hennessy (2014), Vanasco (2017) Neuhann (2018), Caramp (2017) Fukui (2018), Daley et al. (forthcoming), and Asriyan et al. (2018). These papers consider settings where agents are more likely to sell their assets, or sell an increasing fraction, when asset prices rise. The resulting lack of exposure to asset returns then reduces incentives. We study collateralized lending markets where the opposite is true: when interest rates fall, agents can borrow the same amount using less collateral. This *increases* exposure to asset quality, and generates *positive* comovement between asset quality and output. This highlights a distinction between markets where cash flows are reallocated via asset sales, and those where collateral pledging allows borrowers to retain part of the asset.

Our framework has similarities with other dynamic models of asymmetric information. Zryumov (2015) and Hu (2018) study adversely-selected markets in which asset quality fluctuates due to the entry of bad types (the entry rate of good types is exogenous). In their work, asset quality dynamics are thus mainly determined by the extensive margin, while we study the intensive margin and consider policy implications. As in Asriyan et al. (2018), incentives in our setting model are determined by expectations over future market conditions. They focus on sunspot fluctuations in a model with transient types, while the dynamic considerations in our model arise from the persistence of types and endogenous fluctuations in fundamentals. To highlight this difference, we focus on the Pareto-dominant equilibrium with the most favorable beliefs.

Because improving asset quality generates positive spillovers, market liquidity in our model has features of a commons. This relates our framework to theories of collective reputations such as Tirole (1996) and Levin (2009) in which shared histories can contribute to persistent differences in effort across groups. In contrast to their work, we show that effort can be sustained even without reputational motives. This also distinguishes our work from Board and Meyer-ter Vehn (2013), who emphasize the role of learning in a single-firm model of quality reputation. We abstract from learning and instead focus on the role of expected liquidity and interest rates in determining incentives.

The remainder of the paper is organized as follows. In Section 2, we outline the model environment. Section 3 characterizes the competitive equilibrium and analyzes efficiency. In Section 4, we introduce a regulator and analyze the impact of policy interventions. We conclude in Section 5. Proofs not provided in the main text can be found in Appendix A.

2 Model

2.1 Basic Environment.

Time is discrete and infinite. Periods are indexed by $t = 1, 2, \ldots \infty$. There is a unit mass of risk-neutral long-lived *borrowers* with discount factor $\beta \in [0, 1)$. In addition, a mass $m > 1$ of risk-neutral *lenders* is born with a per-capita endowment of one unit of capital each period, and lives for one period. Lenders can invest their endowment in a risk-free storage technology with gross return $1 + r_f$ or offer funding to borrowers. Borrowers receive no endowments, but own and operate long-lived investment opportunities called *assets*. At the outset of each period, each asset has a type θ that is either *good* or *bad*, $\theta \in \{g, b\}$.

Assets generate two types of cash flows: a fixed cash flow L_θ that requires no additional investment, and an investment return R_θ that accrues only if the borrower invests one unit of capital. There is no saving: cash flows are consumed at the end of each period.

Assumption 1 (Payoffs)**.** *Asset cash flows satisfy the following inequalities:*

$$L_g \geq 1 + r_f > L_b \tag{1}$$

$$R_\theta > L_\theta \ \ \forall \theta \tag{2}$$

$$R_g > 1 + r_f. \tag{3}$$

Condition (1) ensures that the fixed cash flow L_θ alone is enough to pay back a loan of one unit of capital at interest rate $1 + r_f$ if the asset is good, but not if the asset is bad. Condition (2) implies that all borrowers would offer fixed cash flow L_θ to obtain private return R_θ. Condition (3) ensures that investing in good assets is socially efficient.

Asset types are persistent but not fixed, and transition probabilities are determined by unobservable borrower effort. Given current type θ and effort intensity $\mu \in [0, 1]$, the probability that the asset becomes a good type is

$$p_\theta(g|\mu) = (1 - \pi) \cdot \mathbb{1}(\theta = g) + \pi \cdot \mu, \tag{4}$$

where $p_\theta(\theta'|\mu)$ denotes the probability that an asset of type θ becomes type θ' given an effort intensity μ. Both the persistence and the marginal value of effort are determined by $\pi \in (0, 1]$, with types being more persistent when π is low. Effort is privately costly according to cost function $C(\mu)$. To transparently discuss the main economic mechanisms, we assume linear costs, $C(\mu) = c\mu$. We discuss general convex costs in Online Appendix B.6. We refer to $\mu = 1$ as "full effort", and $\mu \in (0, 1)$ as "partial effort".

The evolution of types occurs within a period. The fraction of good assets at the beginning of period t is λ_{t-1}, and λ_t is the updated fraction once borrowers have exerted effort. If all agents of a given type choose effort intensity $\mu_{t,\theta}$ in period t, then

$$\lambda_t = \mu_{t,g}\lambda_{t-1} + (1 - \mu_{t,g})(1 - \pi)\lambda_{t-1} + \mu_{t,b}\pi(1 - \lambda_{t-1}). \tag{5}$$

The share of good assets at time zero is exogenously determined and denoted by λ_0. We view this initial condition as the result of an aggregate shock outside of the model, and characterize how asset quality evolves as a function of λ_0.

2.2 Lending Market

In every period, each borrower offers a debt contract to lenders specifying a promised repayment $B \leq L_g$ in exchange for the required unit of capital. Each borrower can agree to a contract with at most one lender. Following Tirole (2012), we assume that borrowing and lending is hampered by financial frictions. While the precise form of friction is not essential, it is important that there is limited pledgeability and private benefits from investment.

Assumption 2 (Financial Friction)**.** *L_θ is pledgeable and contractible. R_θ is not pledgeable.*

Because of limited pledgeability, lenders understand that they will ultimately receive $\min\{B, L_\theta\}$ from a borrower of type θ. Since $L_b < 1 + r_f$, bad types do not have enough pledgeable cash flows to sustain borrowing. While good types may want to separate from bad types by offering different contracts, bad types thus always strictly prefer to pool with good types. Hence it is without loss of generality to restrict attention to a single contract indexed by the promised payment B, which is also the class of contract that minimizes the adverse selection discount faced by good borrowers.[2]

Lemma 1 (Pooling)**.** *There does not exist a separating equilibrium in which at least one borrower obtains funding with positive probability. Moreover, bad types default with probability 1 if they obtain funding.*

To isolate the interaction between market liquidity and incentives, we assume that do not learn from borrowers' repayment histories. Relaxing this assumption, while cumbersome to characterize incentives, would not alter the main feature that lower borrowing costs disproportionately benefit good borrowers, who are less likely to default. For this reason, we also opt for a simple formulation where the cash flows earned by the two types have distinct support. It is straightforward to extend the model to stochastic cash flows with common support but type-dependent distributions.

[2]This argument extends to a broader set of contracts, e.g. when a borrower can offer any contract $\{B_b, B_g\}$ that promises a payment $B_\theta \leq L_g$ conditional on the realization L_θ, which is ex-ante unknown but contractible. Since a borrower, regardless of type, strictly prefers a contract with smaller B_θ, single-crossing condition is generically violated for the set of contracts under which funding is obtained with probability 1. For any equilibrium in which multiple contracts co-exist, both types must be indifferent between either contract. However, a good type will necessarily prefer contracts with $B_g = B_b = B$ to any type-specific contract, since it maximally expropriates the bad type and thus lowers the required payment for good types.

2.3 Timing of Events

Figure 1 summarizes the timing of events from the perspective of a generic borrower. A new generation of lenders is born at the beginning of each period.

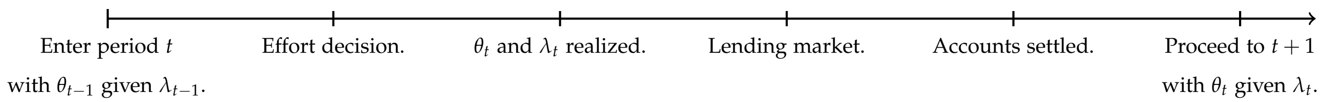

Figure 1: Timing of Events in period t.

The assumption that lending takes place after asset quality simplifies notation. If the order were reversed, lenders would form beliefs about realized asset quality that would be required to be correct in equilibrium. The equilibria we study would continue to exist, and the welfare ranking would be unchanged.

2.4 Decision Problems and Equilibrium Definition

Let $\lambda^t = (\lambda_0, \lambda_1, \ldots, \lambda_t)$ denote the public history of length $t+1$ summarizing the path of average asset quality. Analogously, let h^t to denote a borrower's private history of events from period 0 to t, summarizing private effort as well as the evolution of individual and average asset quality. We use the following definition of feasible histories.

Definition 1. *Public history λ^t is feasible given λ^{t-1} if it can be reached on the path of play given some effort strategy $\mu_t(\lambda^{t-1})$. Public history λ^t is feasible if it is feasible given λ^τ for all $\tau \leq t-1$.*

The definition of a feasible private history is analogous. We use $\mathcal{L}^t$ and $\mathcal{H}^t$ to denote the set of all feasible public and private histories of length $t+1$, respectively. The strategy of a borrower is a sequence of contract offers and effort decisions $\{B_t(h^t), \mu_t(h^{t-1})\}_{t=1,h^t \in \mathcal{H}^t}^{\infty}$ mapping private histories into the real line and the unit interval, respectively. A strategy for a lender in generation t is an acceptance probability $\phi_t : \mathcal{L}^t \to [0,1]$ mapping public histories into the unit interval. Aggregating over all generations yields the sequence $\{\phi_t(\lambda^t)\}_{t=1,\lambda^t \in \mathcal{L}^t}^{\infty}$.

Lender's Problem. Lemma 1 establishes that we can restrict attention to a single pooling contract in every period. Let B_t denote the contract offered in period t. Then the decision problem of a lender in generation t given λ^t is

$$\max_{\phi_t(\lambda^t)} \quad \phi_t(\lambda^t)\Big(\lambda B_t + (1-\lambda_t)L_b\Big) + (1-\phi_t(\lambda^t))\Big(1+r_f\Big) \tag{6}$$

Borrower's Problem. Let $\theta_t(h^t)$ and $\lambda^t(h^t)$ denote the borrowers type and average asset quality in period t associated with private history h^t, respectively. The cash flows accruing to the lender in period t given history h^t are then given by

$$u_{\theta_t(h^t)}(B_t(h^t), \lambda_t(h^t)) = \phi_t(\lambda^t)\Big(R_{\theta_t(h^t)} + L_{\theta_t(h^t)} - \min\{B_t(h^t), L_{\theta_t(h^t)}\}\Big) + (1-\phi_t)L_{\theta_t(h^t)}. \tag{7}$$

Transitions across types (and thus private histories) depend on effort. We denote the probability of feasible history h^t given effort strategy $\{\mu_t(h^{t-1})\}_{t=1,h^t\in\mathcal{H}^t}^{\infty}$ and initial condition $h_0 = \{\theta_0, \lambda_0\}$ by $f_0(h^t|\mu^t(h^{t-1}))$. Given the sequence of lenders' acceptance probabilities $\{\phi_t(\lambda^t(h^t))\}_{t=1,\lambda^t(h^t)\in\mathcal{L}^t(h^t)}^{\infty}$, the borrower's problem is

$$\max_{\{B_t(h^t),\mu_t(h^{t-1})\}_{t=1,h^t\in\mathcal{H}^t}^{\infty}} \quad \sum_{t=1}^{\infty}\sum_{h^t\in\mathcal{H}^t} f_0(h^t|\mu^t(h^{t-1}))\beta^{t-1}u_{\theta_t(h^t)}(B_t(h^t), \lambda_t(h^t)). \tag{8}$$

We are now ready to define a competitive equilibrium.

Definition 2 (Competitive Equilibrium)**.** *A* competitive equilibrium *is (i) a sequence of contract offers and effort strategies $\{B_t(h^t), \mu_t(h^{t-1})\}_{t=1,h^t\in\mathcal{H}^t}^{\infty}$ for each borrower and each feasible history, and (ii) a sequence $\{\phi_t(\lambda^t)\}_{t=1,\lambda^t\in\mathcal{L}^t}^{\infty}$ collecting the acceptance strategies of each lender generation and each feasible public history such that:*

1. *$\phi_t(\lambda^t)$ solves* (6) *given $B_t(\lambda^t)$ for all feasible public histories λ^t.*
2. *Each borrower's sequence of contract offers and effort strategies solves* (8) *given $\{\phi_t(\lambda^t)\}_{t=1,\lambda^t\in\mathcal{L}^t}^{\infty}$ and the effort strategies of all borrowers.*
3. *The set of feasible public and private histories is consistent with $\{\mu_t(h^{t-1})\}_{t=1,h^t\in\mathcal{H}^t}^{\infty}$ given λ_0.*
4. *Lenders' beliefs are consistent with Bayes' Rule wherever possible.*

3 Equilibrium

We solve for the equilibrium in two steps. First, we characterize the lending market equilibrium, and show that it satisfies a Markov property: λ_t is a sufficient statistic for all equilibrium outcomes in period t. Second, we use this result to analyze the optimal effort decision using a recursive representation of the borrower's problem with aggregate state variable λ.

3.1 Lending Market Equilibrium

By Equation (6), generation-t lenders are willing to accept contract B_t given λ_t only if

$$\lambda_t B_t + (1-\lambda_t) L_b \geq 1 + r_f \tag{9}$$

Without loss, restrict attention to contracts with $B_t \leq L_g$. Then (9) can be satisfied only if

$$\lambda_t \geq \bar{\lambda} \equiv \frac{1 + r_f - L_b}{L_g - L_b}. \tag{10}$$

Since $R_\theta > L_\theta$, all borrowers strictly prefer to borrow if doing so is feasible. We say that the market is *liquid* if (10) is satisfied, and call $\bar{\lambda}$ the *liquidity threshold*. Judiciously chosen off-equilibrium beliefs can sustain a number of equilibria in the lending market. We select the perfectly competitive outcome in which lenders' participation constraint (9) holds with equality.[3] This is the selection criterion that is most conducive to effort.

Proposition 1. *Define* $\bar{B}(\lambda_t) = \frac{1+r_f-(1-\lambda_t)L_b}{\lambda_t}$. *Then the equilibrium of the lending game conditional on public history* λ^t *is such that:*

1. *(Illiquid) If* $\lambda < \bar{\lambda}$, *then* $\phi^*(\lambda_t) = 0$.
2. *(Partially Liquid) If* $\lambda = \bar{\lambda}$ *then* $B_t^*(\lambda^t) = \bar{B}(\lambda_t)$ *and* $\phi^*(\lambda^t) \in [0,1]$.
3. *(Liquid) If* $\lambda > \bar{\lambda}$ *then* $B_t^*(\lambda^t) = \bar{B}(\lambda_t)$ *and* $\phi^*(\lambda_t) = 1$.

[3] For example, this equilibrium can be sustained if lenders believe that they face a good borrower with probability λ_t conditional on an off-equilibrium offer.

Since $\bar{B}(\lambda_t)$ is decreasing, it is cheaper to borrow when there are many good borrowers. Moreover, the lending market equilibrium is a function of λ_t only, and the evolution of asset quality is first-order Markov. Hence we can collapse borrowers' sequence problem to a recursive representation with state variable $s = (\theta, \lambda)$, where θ denotes the borrower's private type at the beginning of the period, and λ is average asset quality at the beginning of the period. The updated state is $s' = (\theta', \lambda')$, where λ' is the relevant share of good assets when the lending market opens. We use this recursive formulation to characterize the equilibrium going forward.

3.2 Optimal Effort

Given that all borrowers of type θ are symmetric, it is without loss for aggregate outcomes to focus on symmetric effort strategies by type. Denote these strategies by $\mu_\theta(\lambda)$, and let $\mu = \{\mu_g(\lambda), \mu_b(\lambda)\}$. Define the recursive version of law of motion (5) by

$$\lambda' = \Gamma(\lambda, \mu) \equiv \mu_g(\lambda)\lambda + (1 - \mu_g(\lambda))(1 - \pi)\lambda + \mu_b(\lambda)\pi(1 - \lambda). \tag{11}$$

We use $\Gamma^T(\lambda, \mu)$ to denote average asset quality after T iterations, and define $\Gamma^{-1}(\lambda, \mu)$ to be an *inverse law of motion* if $\Gamma\left(\Gamma^{-1}(\lambda, \mu), \mu\right) = \lambda$ for all λ. The smallest λ for which liquidity threshold $\bar{\lambda}$ is reached within a period given μ is

$$\bar{\lambda}^{-1}(\mu) = \min\{\lambda \in [0, 1] : \Gamma(\lambda, \mu) \geq \bar{\lambda}\}. \tag{12}$$

If average asset quality converges to a unique limit, we denote this limit by

$$\lambda^\infty(\mu) = \lim_{T \to \infty} \Gamma^T(\lambda_0, \mu). \tag{13}$$

We say that λ is *on the path of play given* μ if $\lambda = \Gamma^T(\lambda_0, \mu)$ for some positive integer T.

By Proposition 1, the cash flows earned by a type-θ borrower given λ are

$$u_\theta(\lambda) = \begin{cases} R_\theta + L_\theta - \min\{\bar{B}(\lambda), L_\theta\} & \text{if } \lambda > \bar{\lambda} \\ \phi^*(\lambda)R_\theta + (1-\phi^*(\lambda))L_\theta & \text{if } \lambda = \bar{\lambda} \\ L_\theta & \text{if } \lambda < \bar{\lambda}. \end{cases} \tag{14}$$

Hence the value function of a borrower of type θ at the beginning of a period satisfies

$$\begin{aligned} V_\theta(\lambda) = \max_{\mu_\theta} \quad & p_\theta(g|\mu_\theta)\Big(u_g(\lambda') + \beta V_g(\lambda')\Big) + p_\theta(b|\mu_\theta)\Big(u_b(\lambda') + \beta V_b(\lambda')\Big) \\ \text{s.t. } \; & \lambda' = \Gamma\left(\lambda, \mu^*(\lambda)\right), \end{aligned} \tag{15}$$

where $\mu^*(\lambda)$ is the equilibrium effort strategy that is taken as given by an individual borrower. We can then formally state the borrowers' incentive-compatibility constraint.

Lemma 2 (Incentive Compatibility)**.** *Let $\Delta u(\lambda) = u_g(\lambda) - u_b(\lambda)$ and $\Delta V(\lambda) = V_g(\lambda) - V_b(\lambda)$. Then the incentive-compatibility constraint for all borrowers given λ and λ' is*

$$\Delta(\lambda) \equiv \Delta u(\lambda') + \beta \Delta V(\lambda') \geq \frac{c}{\pi} \tag{16}$$

Hence the incentive-compatibility constraint depends only the relative value of good and bad assets. Going forward, we refer to $\Delta(\lambda)$ as *incentives* and to $\Delta u(\lambda)$ as the *cash flow difference*. Since (16) is independent of θ, it is without loss of generality with respect to aggregate outcomes to restrict attention to symmetric effort strategies *across types*, $\mu_g(\lambda) = \mu_b(\lambda) = \mu(\lambda)$. We use bold-face letters denote constant effort strategies: if $\mu(\lambda) = a$ for all λ, then $\mu = \mathbf{a}$. The next observation states that the cash flow difference need not be monotone in λ. We will later use this fact to demonstrate that incentives may also be non-monotonic.

Observation 1. *Let $\Delta L = L_g - L_b$ and $\Delta R = R_g - R_b$. Then*

$$\Delta u(\lambda) = \begin{cases} \Delta R + L_g - \bar{B}(\lambda) & \textit{if } \lambda > \bar{\lambda} \\ \phi^*(\lambda)\Delta R + (1-\phi^*(\lambda))\Delta L & \textit{if } \lambda = \bar{\lambda} \\ \Delta L & \textit{if } \lambda < \bar{\lambda}. \end{cases}$$

Hence $\Delta u(\lambda)$ is constant on $[0, \bar{\lambda})$ and strictly increasing on $(\bar{\lambda}, 1]$. If $\Delta R \geq \Delta L$, then $\Delta u(\lambda)$ is monotonically increasing. If $\Delta L > \Delta R$, then $\Delta u(\lambda)$ falls discontinuously at $\bar{\lambda}$.

The cash flow difference thus features a complementarity: if markets are liquid, borrowing costs decrease in the fraction of good types, making it relatively more valuable to own a good asset. (Bad types do not care about borrowing costs since they always default.) This complementarity is absent if the market is illiquid, in which case each borrower only consumes the asset's collateral value.

If $\Delta L > \Delta R$, the difference in collateral values is *higher* than the difference in investment returns, and so the cash flow difference drops at $\bar{\lambda}$. This is similar to the parameter specification in Tirole (2012). In this case, the cash flow difference is not globally monotonic in λ. The discontinuity occurs at $\bar{\lambda}$ because the adverse selection discount suffered by good types is maximal at $\bar{\lambda}$ and because borrowers retain all pledgeable cash flows by default if the market is illiquid. (Both types surrender all their collateral when borrowing at $\bar{\lambda}$ because $\bar{B}(\bar{\lambda}) = L_g$, eliminating ΔL as a source of relative value.)

Since average asset quality is endogenous, solving for the equilibrium requires finding a fixed point in privately optimal effort and the aggregate law of motion. The next result shows that incentives $\Delta(\lambda)$ may inherit the non-monotonicity of the cash flow difference at $\bar{\lambda}$. For this reason, effort may be a local strategic substitute in a neighborhood around $\bar{\lambda}$ and a strategic complement if λ is sufficiently high. This feature of the model will allow for equilibria with stagnant, increasing, and decreasing asset quality.

Lemma 3. *Let μ be such that $\Gamma(\lambda, \mu)$ is weakly increasing in λ on $[0, 1]$. Then:*

1. *V_θ is increasing on $[0, 1]$, and continuous and strictly increasing on $[\bar{\lambda}^{-1}(\mu), 1]$.*
2. *$\Delta(\lambda)$ is continuous and strictly increasing on $[\bar{\lambda}^{-1}(\mu), 1]$.*
3. *$\Delta(\lambda)$ is non-monotonic (i) only if $\Delta L > \Delta R$, and (ii) if $\pi < \min\left\{ \left(\frac{\bar{\lambda}}{1-\bar{\lambda}}\right)\left(\frac{\Delta L}{\Delta R} - 1\right), \bar{\lambda}\right\}$.*
4. *If $\mu = \mathbf{1}$, then $\Delta(\lambda)$ is non-monotonic if and only if $\Delta L > \Delta R$ and $\pi < \bar{\lambda}$. If these conditions are satisfied, then $\bar{\lambda}^{-1}(\mathbf{1}) = \arg\min_\lambda \Delta(\lambda)$.*

Because effort is a both a strategic complement and substitute in different regions of the state space, it is not surprising that the model admits a plethora of equilibrium paths for average asset quality driven by self-fulling beliefs regarding future asset quality. Online appendix B.1 provides the intuition for this result in a simplified model with $\beta = 0$, while online appendix B.2 provides an example of equilibrium multiplicity with deterministic cycles, stagnation, and growth. To eliminate avoidable coordination failures that reduce welfare, we select among equilibria by focusing on the equilibrium that delivers the highest possible welfare to all borrowers and show that it is unique.[4]

Definition 3. $\mu(\lambda)$ Pareto-dominates $\mu'(\lambda)$ *given* λ_0 *if* $V_\theta(\lambda_0|\mu) \geq V_\theta(\lambda_0|\mu')$ *for all* θ*.*

Definition 4. *A competitive equilibrium with effort strategy* μ^* *is a* Pareto-dominant competitive equilibrium (PCE) *given initial condition* λ_0 *if* μ^* *Pareto-dominates all other competitive equilibrium effort strategies* $\mu' \neq \mu^*$ *given* λ_0*.*

The next result establishes existence and uniqueness of the PCE for each λ_0 and shows that equilibrium dynamics are not ergodic: fixing parameters, long-run asset quality depends on initial condition λ_0.

Proposition 2. *For each* λ_0*, there exists a unique PCE in which* $\Gamma^T(\lambda_0, \mu^*)$ *is monotone in* T *and there exist thresholds* λ^s *and* $\lambda^e \geq \lambda^s$ *such that:*

1. *If* $\lambda_0 < \lambda^s$*, then* $\lambda^\infty(\mu^*) = 0$ *and* $\mu^*(\lambda) = 0$ *for all* $\lambda < \lambda^s$*.*

2. *If* $\lambda_0 \in [\lambda^s, \lambda^e]$*, then* $\lambda^\infty(\mu^*) \in \{\bar{\lambda}, 1\}$ *and* $\mu^*(\lambda) \in (0, 1]$ *for all* $\lambda \in [\lambda^s, \lambda^e]$*, with at least one inequality strict.*

3. *If* $\lambda_0 > \lambda^e$*, then* $\lambda^\infty(\mu^*) = 1$ *and* $\mu^*(\lambda) = 1$ *for all* $\lambda > \lambda^e$*.*

The proof is by construction and proceeds in three steps. First, we restrict attention to *monotone* equilibria, i.e. equilibria in which $\Gamma^T(\lambda_0, \mu^*)$ is monotone in T for $\forall T > 0$, and characterize the equilibrium with the highest equilibrium effort within this class. We are aided in this characterization by the result from Lemma 6 that incentives are strictly increasing in λ on $[\bar{\lambda}^{-1}, 1]$. This step eliminates all equilibria in which effort is low just because borrowers do not expect other borrowers to exert effort. Second, we show that this

[4] Eliminating fluctuations driven purely self-fulfilling beliefs also contrasts our work from other dynamic models of asymmetric information with coordination failures (e.g. Asriyan et al. (2018)).

maximum effort monotone equilibrium Pareto-dominates any other monotone equilibrium. Finally, we establish that it Pareto-improves on any other candidate effort profile μ' that satisfies incentive compatibility.

Because Pareto-optimality requires maximum incentive-compatible effort, the equilibrium follows a simple threshold structure: if λ_0 is sufficiently high, long-run asset quality converges to 1; if λ_0 is low, asset quality may never grow beyond the liquidity threshold, and may even deteriorate to 0. To whittle down the set of relevant parametric cases, we impose two minimal restrictions. The first ensures that effort is incentive compatible if $\lambda = 1$ indefinitely; the second implies that there exists an initial condition λ_0 such that markets are not liquid immediately even if all borrowers exert effort. The latter boils down to a restriction on π, the parameter that determines the persistence of types.

Assumption 3. $\Delta(1|\mathbf{1}) = \frac{\Delta R + L_g - (1+r_f)}{1-\beta(1-\pi)} > \frac{c}{\pi}$ *and* $\pi < \bar{\lambda}$.

Corollary 1 further characterizes the threshold structure of the PCE. In particular, we show that, conditional on the relative size of ΔL and ΔR, the dynamics of asset quality and effort can characterized by comparing incentives Δ under full effort ($\mu = \mathbf{1}$) and no effort ($\mu = \mathbf{0}$), where for a constant effort profile $\mu = \mathbf{a}$, Δ satisfies the recursion

$$\Delta(\lambda|\mathbf{a}) = \Delta u\Big(\lambda(1-\pi) + \mathbf{a}\pi\Big) + \beta(1-\pi)\Delta\Big(\lambda(1-\pi) + \mathbf{a}\pi\Big) \tag{17}$$

The reason we can focus on $\mu = \mathbf{1}$ and $\mu = \mathbf{0}$ in constructing the thresholds is that the PCE is the monotone equilibrium with maximum effort. Since incentives are increasing in λ on $[\bar{\lambda}^{-1}, 1]$, the first step in its construction is thus checking whether effort is incentive compatible if all other borrowers are expected to exert effort ($\mu = \mathbf{1}$). To construct the upper bound on the initial condition λ_0 for which effort is *not* incentive-compatible, we must likewise check if shirking is privately optimal if all other borrowers are expected to shirk ($\mu = \mathbf{0}$). In this latter case, because $\Delta u(\lambda) = \Delta L$ for all $\lambda < \bar{\lambda}$, we can characterize in closed form the threshold incentives $\Delta(\bar{\lambda}|\mathbf{0}) = \frac{\Delta L}{1-\beta(1-\pi)}$. The following result characterizes equilibrium dynamics as a function of the initial condition and the cost of effort.

Corollary 1. *Suppose that $\Delta R \geq \Delta L$, so that incentives are monotonic given $\mu = \mathbf{1}$. Then:*

(a) $\lambda^s = \lambda^e$, λ^e *satisfies* $\Delta(\lambda^e|\mathbf{1}) = \frac{c}{\pi}$, *and* $\lambda^e = 0$ *if and only if* $\Delta V(0|\mathbf{1}) \geq \frac{c}{\pi}$.

Next, suppose that $\Delta L > \Delta R$, so that incentives are non-monotonic given $\mu = \mathbf{1}$. Then:

(b) if $\frac{c}{\pi} \leq \min\left\{\Delta(\bar{\lambda}|\mathbf{0}), \Delta(\bar{\lambda}|\mathbf{1})\right\}$, then $\lambda^s = 0$ and $\lambda^\infty(\mu^*) = 1$ for all λ_0, and:

(i) if $\frac{c}{\pi} \leq \Delta(\bar{\lambda}^{-1}|\mathbf{1})$, then $\lambda^e = 0$ and there is full effort in every period.

(ii) if $\frac{c}{\pi} > \Delta(\bar{\lambda}^{-1}|\mathbf{1})$, then $\lambda^e > 0$ and there is partial effort for all $\lambda \in [\lambda^s, \lambda^e)$.

(c) if $\frac{c}{\pi} \in \left(\Delta(\bar{\lambda}|\mathbf{1}), \Delta(\bar{\lambda}|\mathbf{0})\right)$, then $\lambda^e > \lambda^s = 0$ and $\lambda^\infty(\mu^*) = \bar{\lambda}$ for all $\lambda_0 \leq \lambda^e$.

(d) if $\frac{c}{\pi} > \Delta(\bar{\lambda}|\mathbf{0})$, then $\lambda^s > 0$.

3.3 Graphical Illustration of Equilibrium

Figures 2 to 4 graphically illustrates three key cases from Corollary 1. The graphs are organized as follows. The left panel shows the value functions of good types (in blue) and bad types (in red). Hypothetical value functions conditional on $\mu = \mathbf{1}$ are depicted in thin lines, while actual PCE value functions are shown using thick lines. Value functions are discontinuous at the vertical dashed lines because there are discontinuous changes in the number of periods required to reach $\bar{\lambda}$. The green shaded region depicts the region in which the market is liquid. The dashed vertical lines depict the number of periods required to reach the liquidity region if all borrowers exert effort. The second panel shows equilibrium incentives in the PCE (in thick blue), under the assumption that $\mu = \mathbf{1}$ (in thin blue), and under the assumption that markets are illiquid and $\mu = \mathbf{0}$) (in cyan). The cost of effort is shown in red. The solid vertical line depicts λ^e. The third panel illustrates the equilibrium effort strategy μ^* in thick blue, the law of motion for asset quality given μ^* in thick green, and the hypothetical law of motion given $\mu = \mathbf{1}$ in thin green. The upward-sloping dashed line is the 45-degree line. The horizontal dashed line is $\bar{\lambda}$. The fourth panel shows simulated paths for asset quality for various initial conditions λ_0 (in dashed lines). The solid horizontal lines show the cutoffs λ^e, λ^s and $\bar{\lambda}$.

Figure 2 corresponds to case (a) in Corollary 1. Incentives are monotone conditional on $\mu = \mathbf{1}$, but c is such that full effort cannot be sustained for all λ. Hence $\lambda^s = \lambda^e > 0$, and the equilibrium is such that no farmer exerts effort if $\lambda < \lambda^e$ and all farmers exert effort if $\lambda \geq \lambda^e$. Accordingly, there is a discontinuous drop in the equilibrium value functions at λ^e.

Shirking is incentive compatible for all $\lambda < \lambda^e$ because $\frac{c}{\pi} > \Delta(\lambda^e|\mathbf{0})$. The simulated paths for effort in the fourth panel highlight this non-ergodicity: long-run asset quality differs sharply depending on the initial condition.

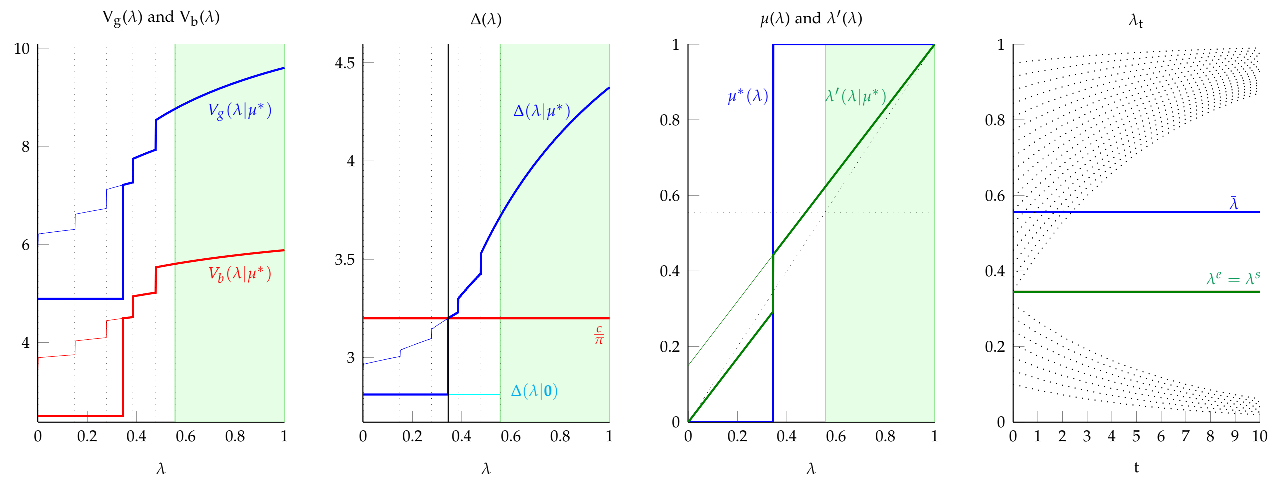

Figure 2: Case (a). Monotone incentives and effort. Parameters: $R_g = 2$, $R_b = 1$, $L_g = 1.4$, $L_b = 0.5$, $\beta = 0.8$, $\pi = 0.15$, $r_f = 0$, $\frac{c}{\pi} = 3.2$.

Figure 3 corresponds to Case (b,ii) in Corollary 1. The only parameters that differ from Figure 2 are R_g (1.5 instead of 2) and $\frac{c}{\pi}$ (1.975 instead of 3.2). Given that $\Delta L > \Delta R$, incentives are now non-monotonic conditional on $\mu = \mathbf{1}$. Since $\frac{c}{\pi} < \Delta(\bar{\lambda}|\mathbf{0})$, there does not exist an equilibrium in which no borrower exerts effort; hence asset quality must weakly improve for all $\lambda < \bar{\lambda}$. Since $\frac{c}{\pi} < \Delta(\bar{\lambda}|\mathbf{1})$, moreover, there are strict incentives to exert effort once $\bar{\lambda}$ is reached. Hence asset quality asymptotes to 1 for all initial conditions. However, not all borrowers exert effort in every period since $\frac{c}{\pi} > \Delta(\bar{\lambda}^{-1}|\mathbf{1})$, which implies that there is incentive to free-ride on others' effort close to the liquidity threshold. As a result, the unique PCE is such that borrowers choose partial effort when close to $\bar{\lambda}$, and full effort once $\bar{\lambda}$ is reached. The simulations in the fourth panel reflect this slowdown in effort close to $\bar{\lambda}$, and the convergence to 1 thereafter.

Figure 4 corresponds to Case (c) in Corollary 1. As in Figure 3, incentives are non-monotonic conditional on $\mu = \mathbf{1}$. The only change parameters is $\frac{c}{\pi}$ (2.3 instead of 1.975). As in the previous case, asset quality must weakly improve for all $\lambda < \bar{\lambda}$ because $\frac{c}{\pi} < \Delta(\bar{\lambda}|\mathbf{0})$. The difference now is that $\frac{c}{\pi} > \Delta(\bar{\lambda}|\mathbf{1})$, which implies that individual borrowers have no

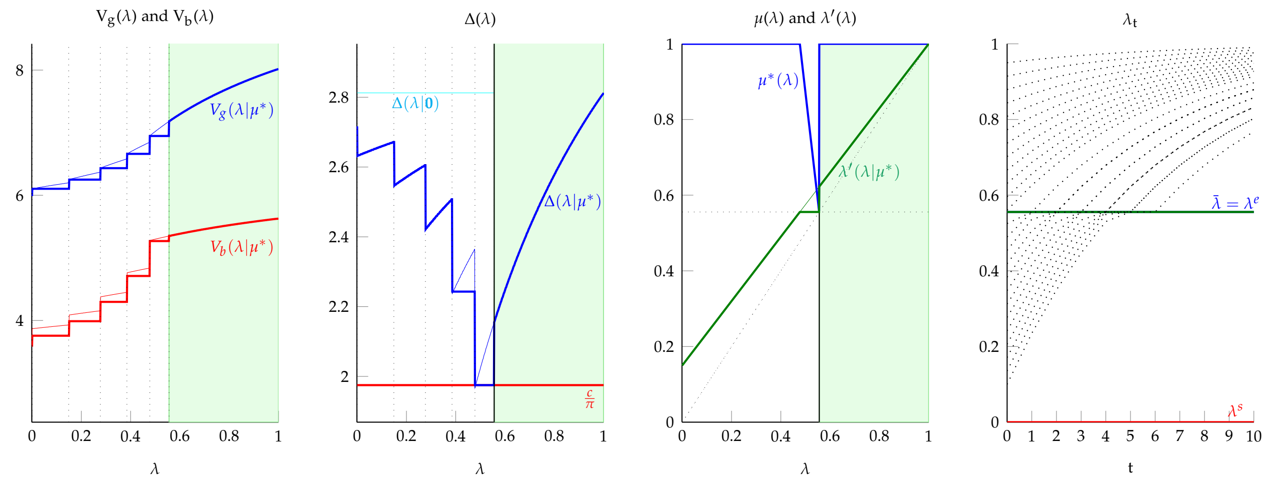

Figure 3: Case (b, ii). Non-monotonic incentives. is required on the equilibrium path, but asset quality asymptotes to 1 for all λ_0. Parameters: $R_g = 1.5$, $R_b = 1$, $L_g = 1.4$, $L_b = 0.5$, $\beta = 0.8$, $\pi = 0.15$, $r_f = 0$, $\frac{c}{\pi} = 1.975$.

incentives to exert effort once $\bar{\lambda}$ is reached. As a result, the PCE features partial effort and partial liquidity near $\bar{\lambda}$. Incentives are such that average asset quality is able to *reach*, but not "jump" beyond, the liquidity threshold $\bar{\lambda}$. As a result, asset quality remains stuck at the liquidity threshold forever for all $\lambda_0 \leq \bar{\lambda}$, and a fraction of borrowers fail to obtain funding in every period.

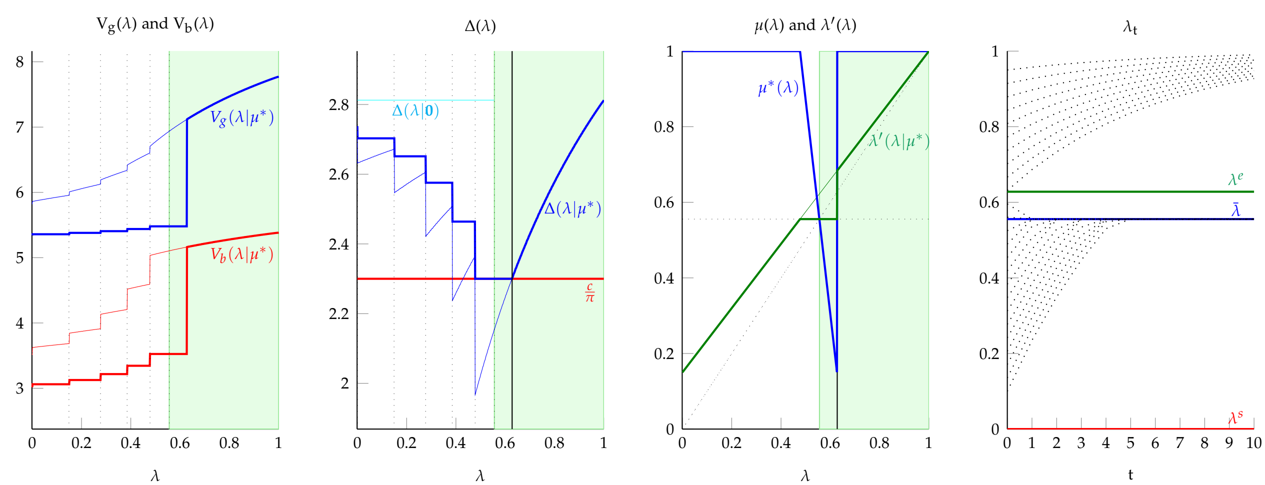

Figure 4: Case (c). Non-monotonic incentives, partial effort is required on the path of play, and $\lambda^\infty = \bar{\lambda}$ for all $\lambda_0 \leq lb$. Parameters: $R_g = 1.5$, $R_b = 1$, $L_g = 1.4$, $L_b = 0.5$, $\beta = 0.8$, $\pi = 0.15$, $r_f = 0$, $\frac{c}{\pi} = 2.3$.

3.4 Comparative Statics

In Figure 5, we consider the comparative statics of the incentive thresholds $\Delta(\bar{\lambda}|\mathbf{0})$, $\Delta(\bar{\lambda}|\mathbf{1})$, and $\Delta(\bar{\lambda}^{-1}|\mathbf{1})$ with respect to two key parameters, π (left panel) and β (right panel). Recall, π captures the transience of types (or equivalently, $1-\pi$ captures the persistence of types). If $\pi \to 1$, types are purely determined by effort within any given period; if $\pi \to 0$, effort is irrelevant and types are almost perfectly persistent. β measures borrower patience. To disentangle the effects of persistence from changes in the effective cost of effort, we set $c = \bar{c}\pi$ so that $\frac{c}{\pi}$ is a constant. The horizontal lines then plot two potential effective costs of effort.

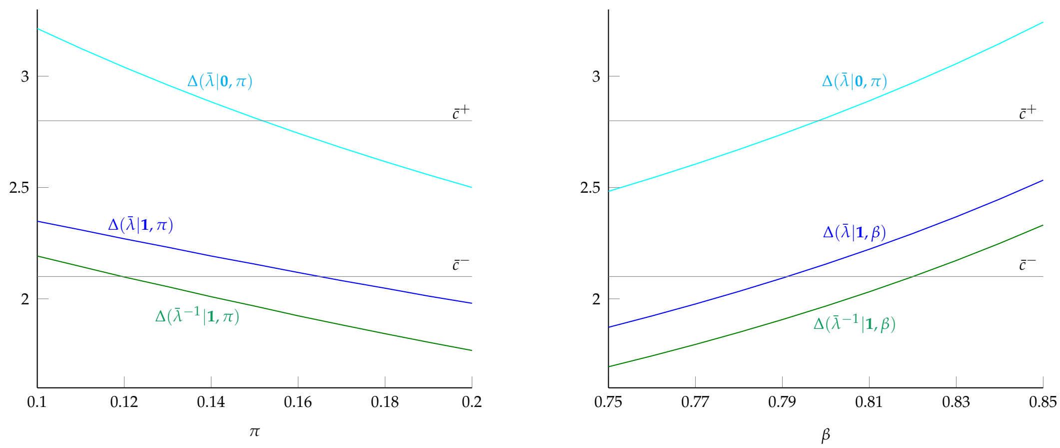

Figure 5: Comparative statics of $\Delta(\bar{\lambda}|\mathbf{0})$, $\Delta(\bar{\lambda}|\mathbf{1})$, and $\Delta(\bar{\lambda}^{-1}|\mathbf{1})$ w.r.t to π (left panel) and β (right panel).

Equation 17 shows that π affects incentives through two mechanisms. The first is that, conditional on $\mathbf{a} > 0$, average asset quality λ' is increasing in π. Since the per-period cash flow difference Δu is weakly increasing in λ, this effect increases the value of effort. The second is that lower persistence reduces the expected future value of effort, thereby lowering incentives to exert effort today. In our numerical example, the second mechanism dominates and all three incentive thresholds are strictly decreasing in π. Because the incentive-compatibility constraint requires that $\Delta \geq \bar{c}$, the figure shows that small shocks to π may lead to a discontinuous change in long-run average asset quality. If the effective cost of effort is $\bar{c}^-$, for example, then a small decrease in π may imply that $\Delta(\bar{\lambda}|\mathbf{1}) < \bar{c}^-$, so that

$\lambda^\infty = \bar{\lambda}$ rather than 1 for all $\lambda_0 \leq \bar{\lambda}$. If the effective cost of effort is $\bar{c}^+$, a small decrease in π may imply that $\Delta(\bar{\lambda}|\mathbf{0}) < \bar{c}^+$, so that $\lambda^\infty = 0$ rather than $\bar{\lambda}$ for all $\lambda_0 \leq \bar{\lambda}$.

The effects of the discount factor β can be understood similarly. Given that types are persistent, an increase in β raises the dynamic benefit of exerting effort today. As such, increases in β operate analogously to increases in type persistence $1 - \pi$. As a result, small negative shocks to β may reduce long-run average asset quality from 1 to $\bar{\lambda}$ or from $\bar{\lambda}$ to 0. In both cases, a small shock to parameters may thus discontinuously alter equilibrium asset quality dynamics. (It is clear that similar results apply with respect to ΔR and ΔL, also.)

3.5 Efficiency

To assess whether competitive equilibrium is efficient, we now consider a benevolent social planner who maximizes welfare by assigning effort profiles borrowers without having to respect the incentive compatibility constraint. We show that the social planner may want to assign higher effort than obtains under PCE. The difference between a competitive equilibrium outcome and the social planner's problem is that borrowers in a competitive equilibrium take the evolution of asset quality as given. The social planner internalizes this externality.

Proposition 3. *$\mu(\lambda) = \mathbf{1}$ Pareto-dominates any μ^* such that $\lambda^\infty(\mu^*) = \{\bar{\lambda}, 1\}$, and strictly if $\mu^*(\lambda) \in (0,1)$ for some λ on the equilibrium path. If $\lambda^\infty(\mu^*) = 0$ and Assumption 3 is satisfied, there exists $\bar{\beta} < 1$ such that $\mu(\lambda) = \mathbf{1}$ strictly Pareto-dominates μ^* if $\beta \geq \bar{\beta}$.*

The result can be verified using Figures 2-4. In all three cases, the hypothetical value functions under the assumption that all borrowers exert effort (plotted in thin lines) lie above the PCE value functions. Figure 2 shows a *dynamic coordination failure*: because the per-period payoff under illiquidity is too low, no borrower wants to exert effort for low values of λ. If borrowers could jointly commit to exerting effort for a number of periods, average asset quality would improve, and the cumulative benefits from liquidity and falling borrowing costs would be enough to outweigh the present value of increased effort. Notably, this coordination failure persists even though we have selected the Pareto-dominant competitive equilibrium, which circumvents any *static* coordination failures. An additional inefficiency is present in Figures 3 and 4. Here PCE is such that there is partial effort on the equilibrium path, which is sustained by liquidity rationing at $\bar{\lambda}$ (that is $\phi^*(\bar{\lambda}) < 1$). This leads to suboptimally low output and slow asset quality growth. The social planner eliminates these inefficiences by assigning full effort to all borrowers throughout. Importantly, inefficiencies obtain *only* if λ_0 is sufficiently low; for good initial conditions, the strategic complementarity under market liquidity is sufficient to induce efficient effort.

4 Policy Interventions

The previous section showed that effort and market liquidity may be inefficiently low even in the Pareto-dominant competitive equilibrium. Similar to Philippon and Skreta (2012) and Tirole (2012), we now introduce a benevolent regulator that can provide subsidies in order to restore liquidity or lower borrowing costs. Different from their work, we focus not just on alleviating market breakdowns, but also consider the impact on asset quality via private incentives. This is an important consideration because policy interventions have increasingly been used for extended periods of time, giving market participants time to adapt. (See online appendix B.3 for details.) Because the effort decision is a dynamic problem, we distinguish between limited and full commitment on the part of the regulator.

To establish a benchmark for liquidity interventions, we say that a market breakdown in competitive equilibrium is *inefficient* if not all borrowers are funded and λ' is such that total output is strictly increasing in the fraction of funded borrowers.

Lemma 4. *Market breakdowns are inefficient if and only if $\phi^*(\bar{\lambda}) < 1$ and $\lambda \in (\underline{\lambda}, \bar{\lambda}]$, where*

$$\underline{\lambda} = \frac{1 + r_f - R_b}{\Delta R}.$$

Proof. In competitive equilibrium, $\phi^*(\lambda) = 1$ for all $\lambda > \bar{\lambda}$. If $\phi^* < 1$, aggregate output strictly increases in the fraction of funded borrowers if $\lambda' R_g + (1 - \lambda') R_b > 1 + r_f$. □

Assumption 4. $\underline{\lambda} < \bar{\lambda}$.

The regulator has no informational advantage or disadvantage over lenders: he observes λ but not θ. Based on this information, he decides on a *subsidy rule* $s(\lambda')$ determining a payment made by the regulators to any lender whose borrower defaults when the realized asset quality within a given period is λ'. Such a subsidy makes lending less risky, and thereby fosters liquidity and reduces borrowing costs for borrowers. The subsidy can be chosen to deliver any feasible equilibrium interest rate. Since Philippon and Skreta (2012) show that the equilibrium interest rate is the sufficient statistic for assessing the real effects of any policy in environments such as this one, it is without loss of generality to study this policy only. Alternative policy implementations with the same aggregate consequences include changes to the risk-free rate r_f, outright asset purchases, or collateral exchanges.

Borrowers may react to the policy by changing their effort strategy. We let $\tilde{\mu}(s)$ denote the effort strategy given the policy, and let $\tilde{V}(\lambda)$ denote the associated borrower value functions, defined analogously to the case without interventions. We use $D(\lambda)$ to denote the *deficit* associated with the subsidy (that is, the total discounted sum of funds injected by the regulator). As in Tirole (2012) we assume that deficits have a social dead weight cost $\delta > 0$. Letting $\Pi(\lambda)$ denote the expected utility (or equivalently, gross returns) earned by lenders, utilitarian welfare is

$$W(\lambda) = \lambda \tilde{V}_g(\lambda) + (1-\lambda)\tilde{V}_b(\lambda) + \Pi(\lambda') - (1+\delta)D(\lambda) \tag{18}$$

We focus mainly on the limit $\delta \to 0$, where the deadweight loss acts as a selection device that chooses the "smallest possible" intervention among those that deliver a particular market allocation.

The regulator's objective is to choose a subsidy rule $s(\lambda)$ to maximize time-zero welfare $W(\lambda_0)$. The rule is announced at the date zero. We consider two scenarios. Under *limited commitment*, $s(\cdot)$ has to be such that the regulator finds it *ex-post* optimal to deliver the promised subsidy given realized asset quality λ'. That is, it must be ex-post welfare-maximizing to deliver the promised subsidy *given* borrowers' sunk effort decisions. Under *full commitment*, the regulator can credibly commit to any subsidy rule chosen at date zero, and s need not satisfy an ex-post optimality requirement.

Given deadweight cost δ, the regulator will never find it optimal to offer a subsidy so generous that good borrowers will prefer defaulting on their loans.[5] Hence we can take as

[5]To see why, observe that the lending market equilibrium is *as if* all borrowers were high quality if the

given that only bad borrowers default, so that the deficit satisfies the recursion

$$D(\lambda) = (1-\lambda')s(\lambda') + \beta D(\lambda') \qquad \text{s.t.} \qquad \lambda' = \Gamma(\lambda, \tilde{\mu}(s)) \tag{19}$$

Lenders' break-even condition under the subsidy is $\lambda' B + (1-\lambda')(L_b + s(\lambda')) \geq 1 + r_f$. Borrowers continue to offer a contract such that this constraint holds with equality. Hence the face value is

$$\tilde{B}(\lambda', s) = \frac{1 + r_f - (1-\lambda')(L_b + s(\lambda'))}{\lambda'}$$

Observation 2. *Borrowers obtain funding if and only if $\tilde{B}(\lambda', s) \leq L_g$. Hence the minimum subsidy required to ensure that borrowers can obtain funding is*

$$\underline{s}(\lambda') = \frac{1 + r_f - \lambda' L_g - (1-\lambda')L_b}{1-\lambda'}. \tag{20}$$

4.1 Limited Commitment

Under limited commitment, the subsidy rule $s(\lambda')$ must be statically welfare-maximizing for all λ'. Since effort is sunk at the time of the intervention, moreover, continuation values are independent of the current policy. Because it is efficient for all borrowers to obtain funding if $\lambda = \bar{\lambda}$ (and this can be achieved by the regulator at zero cost), the per-period utility earned by borrower θ given $s(\lambda')$ is

$$\tilde{u}_\theta(\lambda', s) = \begin{cases} R_\theta + L_\theta - \min\{\tilde{B}(\lambda', s), L_\theta\} & \text{if } \tilde{B}(\lambda', s) \leq L_g \\ L_\theta & \text{if } \tilde{B}(\lambda', s) > L_g \end{cases} \tag{21}$$

Since lenders earn $1 + r_f$ in every period, and the within-period deficit is $(1-\lambda')s(\lambda')$, ex-post optimality can be defined as follows.

Definition 5. *$s(\lambda')$ is ex-post optimal if and only if*

$$s(\lambda') = \arg\max_{s'} \qquad \lambda' \tilde{u}_g(\lambda', s') + (1-\lambda')\tilde{u}_b(\lambda', s') + (1 + r_f) - (1+\delta)(1-\lambda')s'.$$

regulator offers subsidy ΔL. Under such a subsidy, effort incentives are maximal, all borrowers obtain funding, and yet there are no gains from good borrowers defaulting.

Proposition 4. *There exists a unique ex-post optimal subsidy rule* $\tilde{s}(\lambda')$ *satisfying*

$$\tilde{s}(\lambda') = \begin{cases} \underline{s}(\lambda') & \text{if } \lambda \in (\tilde{\lambda}(\delta), \bar{\lambda}) \\ 0 & \text{otherwise} \end{cases} \qquad \text{where } \frac{\partial \tilde{\lambda}(\delta)}{\partial \delta} > 0 \text{ and } \lim_{\delta \to 0} \tilde{\lambda}(\delta) = \underline{\lambda}$$

and all borrowers are financed if $\lambda = \bar{\lambda}$.

The ex-post optimal subsidy is such that the regulator intervenes minimally only if there are inefficient market breakdowns under laissez-faire. In the limit $\delta \to 0$, the subsidy is offered *if and only if* there is an inefficient market breakdown. The regulator does not need to intervene at $\bar{\lambda}$, but the promise of a vanishingly small intervention is enough to eliminate all statically inefficient equilibria of the lending game with $\phi^*(\bar{\lambda}) < 1$.

Since λ' is taken as given, the ex-post optimal subsidy rule focuses *only* on restoring liquidity, and ignores effort incentives. While this unambiguously increases welfare conditional given λ', it may reduce welfare once the feedback to effort is taken into account.

Proposition 5. *Let PCE be such that* $\lambda^\infty(\mu^*) \in \{\bar{\lambda}, 1\}$ *and* $\mu^*(\lambda) \in (0,1)$ *for some* λ *on the equilibrium path. Then average asset quality converges to* $\max\{\underline{\lambda}, 0\} < \mu^*(\lambda) \in (0,1)$ *under the ex-post optimal subsidy. There exist parameters such that the ex-post optimal subsidy strictly lowers welfare relative to PCE.*

The intuition is that the intervention creates a region of the state space where incentives are "as if" $\lambda = \bar{\lambda}$. Since the adverse selection discount is highest at $\bar{\lambda}$, this hampers per-period incentives relative to the laissez-faire equilibrium. In online appendix B.4, we provide a parametric example where the ex-post optimal intervention strictly reduces output and welfare relative to PCE *in every period*.

Figures 6 and 7 illustrate the mechanism more broadly by comparing outcomes in PCE (in dotted **black**) and under the ex-post optimal subsidy in the limit $\delta \to 0$ (in dashed red) using the parameters from Figures 3 and Figures 4. The first panel plots the evolution of average asset quality for various initial conditions $\lambda_0 \leq \bar{\lambda}$, while the second panel plots the associated *total net cash flows* in PCE.[6] The third panel plots the per-period deficit $d(\lambda') =$

[6] Net cash flows in PCE and under the subsidy are $CF^{\text{PCE}}_{t-1} = \phi^*(\lambda^*_t)\Big(\lambda^*_t R_g + (1-\lambda^*_t)R_b - 1\Big) + \lambda^*_t L_g +$

$(1-\lambda')\underline{s}(\lambda')$, while the fourth panel plots welfare as a function of the initial condition λ_0. In both cases, parameters are such that $\underline{\lambda} = 0$.

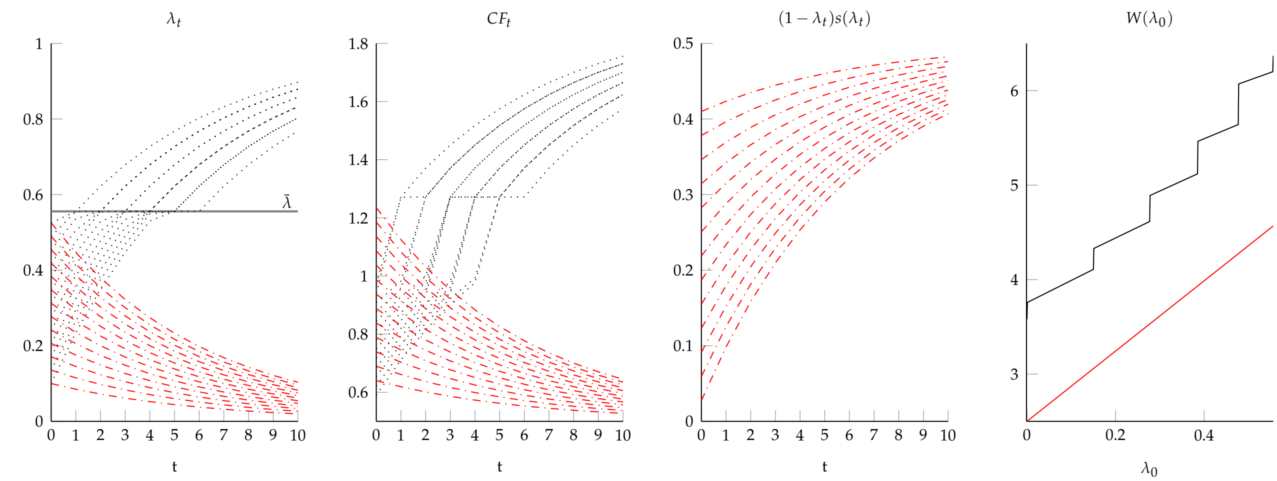

Figure 6: Welfare comparison between PCE and the ex-post optimal subsidy. Case (b,ii) from Corollary 1. Parameters: $R_g = 1.5$, $R_b = 1$, $L_g = 1.4$, $L_b = 0.5$, $\beta = 0.8$, $\pi = 0.15$, $r_f = 0$, $\frac{c}{\pi} = 1.975$. Parameters are such that $\underline{\lambda} = 0$.

Figure 6 corresponds to Case (b,ii) in Corollary 1, where the PCE features partial effort along the equilibrium path and $\lambda^{\infty} = 1$. By Proposition 5, asset quality declines under the subsidy (first panel). If λ_0 is low, this concern is mitigated by the fact that the policy allows all borrowers to obtain funding, thereby boosting net cash flows. Eventually, however, asset quality improves and markets become liquid in PCE, and cash flows are *higher* without the subsidy. Moreover, the fact that asset quality declines under the subsidy implies that the per-period subsidy rises, further lowering welfare. Since the number of periods in which markets are liquid is falling in λ_0, the welfare difference is increasing in λ_0.

Figure 7 corresponds to Case (c) in Corollary 1, where the PCE is such that there is partial effort and asset quality remains stuck at the liquidity threshold, $\lambda^{\infty} = \bar{\lambda}$. The equilibrium under the subsidy is unchanged, and the basic mechanisms are the same as in the previous case. The key difference is that asset quality never improves beyond $\bar{\lambda}$ in PCE, and that only a fraction of borrowers obtain funding at the the liquidity threshold

---

$(1-\lambda_t^*)L_b$ and $CF_{t-1}^{\text{subsidy}} = \tilde{\lambda}_t(R_g + Lg) + (1-\tilde{\lambda}_t)(R_b + L_b) - 1$, respectively, where λ_t^* and $\tilde{\lambda}_t$ denote the respective asset quality. Note that all borrowers get funded in every period under the subsidy because $\underline{\lambda} = 0$.

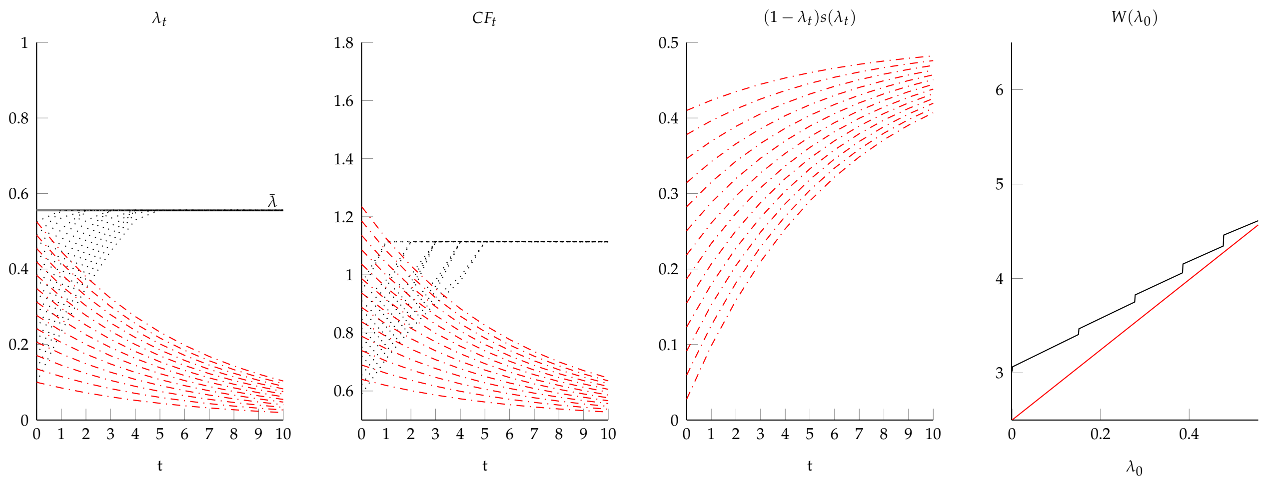

Figure 7: Welfare comparison between PCE and the ex-post optimal subsidy. Case (c) from Corollary 1. Parameters: $R_g = 1.5$, $R_b = 1$, $L_g = 1.4$, $L_b = 0.5$, $\beta = 0.8$, $\pi = 0.15$, $r_f = 0$, $\frac{c}{\pi} = 2.3$. Parameters are such that $\underline{\lambda} = 0$.

($\phi^*(\bar{\lambda}) < 1$). This long-run inefficiency of the PCE increases the relative value of the subsidy. As a result, the welfare difference is smaller than in Case (b,ii). The discounted deficit is decreasing in λ_0 because good borrowers do not default. As a result, the welfare difference decreases in the initial condition λ_0.

Remark 1. *The ex-post optimal subsidy need not lower welfare. For example, it is easy to see that there exists parameters such that subsidy is welfare-improving for β sufficiently small, since the adverse effects on future asset quality are then irrelevant.*

4.2 Full Commitment

Under full commitment, the regulator chooses a subsidy rule $s(\lambda')$ at time zero and remains committed to it for all realizations of λ'. Our goal is to demonstrate how the policy under commitment differs qualitatively from the ex-post optimal rule, and how this difference manifests in welfare. For simplicity, we focus on the case where full effort is efficient but not attained in PCE, and $\underline{\lambda} \leq 0$ (that is, it is efficient to fund all borrowers). Consider a subsidy rule $\hat{s}$ under which effort is privately optimal for all λ. Given that only bad borrowers default, the difference in per-period cash flows under $\hat{s}$ is

$$\Delta u(\lambda', \hat{s}) = \Delta R + \Delta L + \hat{s}(\lambda') - \frac{1 + r_f - L_b - \hat{s}(\lambda')}{\lambda'},$$

and is strictly increasing in $\hat{s}(\lambda')$. Let $\hat{\mu}$ and $\hat{V}_\theta(\lambda)$ denote the value functions associated with $\hat{s}$, and define $\Delta\hat{V}(\lambda)$ and $\hat{\Delta}(\lambda)$ accordingly. The IC constraint given λ and $\hat{s}$ is

$$\Delta u(\lambda', \hat{s}) + \beta \Delta \hat{V}(\lambda') \geq \frac{c}{\pi}. \tag{22}$$

Proposition 6. *There exists a unique optimal subsidy rule $\hat{s}$ such that all borrowers exert effort in all periods and all borrowers are able to obtain funding under $\hat{s}$ whenever they obtain funding under the ex-post optimal subsidy rule $\tilde{s}$. $\hat{s}$ is such that*

1. $\hat{s}(\lambda') = 0$ *for all* $\lambda' \geq \bar{\lambda}$ *such that* $\mu^*(\Gamma^{-1}(\lambda', \mu^*)) = 1$.

2. *Constraint* (22) *holds with equality for any* λ *such that* $\hat{s}(\Gamma(\lambda, \mathbf{1})) > 0$.

3. $\hat{s}(\lambda') \geq \tilde{s}(\lambda')$ *for all* λ'.

Hence the regulator does not intervene if the market is liquid and the PCE already features full effort, and provides the smallest effort-inducing subsidy otherwise. The next result shows that the optimal effort-inducing policy differs qualitatively the optimal policies in Philippon and Skreta (2012) and Tirole (2012). In particular, the regulator may need to intervene even if asset quality is high enough to sustain liquidity in the absence of interventions. While this is ex-post inefficient, it is ex-ante efficient because borrowers respond by exerting more effort. It may also lead to lower total costs. (Online Appendix B.5 provides an example in which the regulator can induce effort with an infinitesimal subsidy.) The value of commitment thus is the ability to intervene in excess of what is required to restore liquidity. We also show that, while $\hat{s}$ must be weakly larger than $\underline{s}$ to induce effort, the *total* deficit $(1-\lambda)\tilde{s}$ may be lower than under the minimal subsidy. This is because an increase in good borrowers leads to lower total subsidy payout even if the per-capita subsidy is higher.

Proposition 7. $\hat{s}(\Gamma(\lambda, \mathbf{1})) > 0$ *for all* λ *such that* $\Gamma(\lambda, \mathbf{1}) \geq \bar{\lambda}$ *and* $\mu^*(\lambda) \in (0, 1)$. *There exist parameters such that $\hat{s}$ generates smaller per-period deficits than $\tilde{s}$ in every period.*

5 Conclusion

We propose a tractable dynamic framework to study liquidity and efficiency in collateralized lending markets with uninformed lenders. Our key innovation is that collateral quality is persistent but determined by hidden effort. Private effort imposes an externality on other borrowers because loans are priced based on average asset quality. We prove existence and uniqueness of a Pareto-dominant equilibrium that delivers strictly higher utility to all borrowers than any other competitive equilibrium but may remain inefficient. The dynamics of collateral quality and liquidity are non-ergodic: the fraction of good collateral asymptotes to one if it is high initially, but may deteriorate to zero or the liquidity threshold otherwise. Liquidity is fragile: small shocks to the persistence of collateral quality or borrower patience may trigger discontinuous drops in long-run collateral quality. These comparative statics may help to assess which markets are particularly prone to freezes: these are asset classes whose payoffs are easy to alter by originators, or whose market participants are relatively short-termist. Competitive equilibria are inefficient given poor initial conditions, but regulators without commitment may further reduce welfare by prioritizing ex-post liquidity provision over effort incentives. Regulators with commitment offer large per-capita subsidies, and may optimally intervene even if the market is liquid. These results contrasts sharply with existing policy prescriptions. The tractability of our framework may permit further applications.

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A Proofs

A.1 Proof of Lemma 1

Since $L_b < 1 + r_f$, bad borrowers cannot obtain funding in any separating equilibrium. Hence lenders are willing to accept the good type's contract only if $B \geq 1 + r_f$. A bad borrower who offers the same contract thus defaults on the promise with probability one and repays L_b rather than B. Since $R_g > L_b$, doing so yields a strictly higher return than offering any other contract.

A.2 Proof of Lemma 2

Use (4) in (15), and difference the value functions given $\mu_\theta = 1$ and $\mu_\theta = 0$.

A.3 Proof of Lemma 3

1. Observe that $R_\theta > L_\theta$ for all θ. Moreover, $u_b(\lambda)$ is a constant for $\lambda \geq \bar{\lambda}$, while $u_g(\lambda)$ is strictly increasing and continuous. The result then follows from Theorem 4.7 in Stokey et al. (1989).

2. Observe that $\Delta V(\lambda)$ and $\Delta(\lambda)$ satisfy the recursions $\Delta V(\lambda) = (1-\pi)\left[\Delta u(\lambda') + \beta \Delta V(\lambda')\right]$ and $\Delta(\lambda) = \Delta u(\lambda') + \beta(1-\pi)\Delta(\lambda')$. Since $\Delta u(\lambda')$ is strictly increasing, the result again follows from Theorem 4.7 in Stokey et al. (1989).

3. Observe that $\Delta u(\lambda)$ is non-monotonic only if $\Delta L > \Delta R$, and that $\bar{\lambda}^{-1}(\mu) > 0$ for all μ if and only if $\pi < \bar{\lambda}$. Given this, establishing sufficiency is equivalent to showing that there does not exist μ such that $\Delta u(\Gamma(\lambda, \mu)) \geq \Delta L$ for all λ such that $\lambda = \Gamma^T(\lambda_0, \mu)$ for some $T \geq 1$ and λ_0. Conditional on $\lambda \geq \bar{\lambda}$, $\Delta u(\lambda) \geq \Delta L$ if and only if $\lambda \geq \lambda^L = \bar{\lambda}\frac{\Delta L}{\Delta R}$. Hence we require that λ^L cannot be reached from $\bar{\lambda}$ under any μ such that $\Gamma(\lambda, \mu)$ is weakly increasing in λ. This is the case if $\bar{\lambda} + \pi(1-\bar{\lambda}) < \lambda^L$, which is equivalent to the stated condition. To prove the fourth statement, observe that $\pi < \bar{\lambda}$ implies that there exists a unique $\lambda_0 > 0$ such that $\Gamma(\lambda_0, \mu) = \bar{\lambda}$. By continuity of the law of motion on $[0, 1]$, and the discontinuity in $\Delta u(\lambda)$ at $\bar{\lambda}$, there exists a unique $\lambda_0' < \lambda_0$ arbitrarily close to λ^0 such that $\Gamma(\lambda_0', \mu) < \bar{\lambda}$ and $\lim_{\lambda_0' \to \lambda^0} \Gamma^T(\lambda_0', \mu) = \Gamma^T(\lambda_0, \mu)$ for all $T \geq 2$. By the discontinuity in $\Delta u(\lambda)$ at $\bar{\lambda}$ and the continuity of $\Delta(\lambda)$ on $[\bar{\lambda}^{-1}(\mu), 1]$, $\exists \lambda_0'$ such that $\Delta(\lambda_0') > \Delta(\lambda_0)$.

4. If $\mu(\lambda) = 1$ for all λ, then there exists a unique inverse law of motion. Hence there exists a unique $\bar{\lambda}^{-1}$ such that $\Gamma(\bar{\lambda}^{-1}, 1) = \bar{\lambda}$, and $\Gamma(\lambda, 1) < \bar{\lambda}$ for all $\lambda < \bar{\lambda}^{-1}$. $\pi < \bar{\lambda}$ ensures that $\bar{\lambda}^{-1} > 0$. By Part 3 of this Lemma, moreover, $\Delta(\bar{\lambda}^{-n}) < \Delta(\bar{\lambda}^{-n} - \epsilon)$ for $\epsilon \approx 0$. This proves the first part of the claim. Next, we show that $\bar{\lambda}^{-1} = \arg\min_{\lambda} \Delta(\lambda)$ given $\Delta L > \Delta R$ and $\pi < \bar{\lambda}$. First, observe that there exist discontinuities $\Delta(\lambda)$ at a sequence of points $\{\bar{\lambda}^{-n}, \ldots, \bar{\lambda}\}_{n=1}^{N}$, where n is an integer, $\bar{\lambda}^{-n}$ is defined to be such that $\Gamma^n(\lambda^{-n}, \mu = \mathbf{1}) = \bar{\lambda}$, and N denotes the maximum number of steps required to reach $\bar{\lambda}$ given that all borrowers exert effort. Since $u_\theta = L_\theta$ for all θ and all $\lambda \leq \bar{\lambda}$, the fact that $\Delta(\lambda)$ is strictly increasing on $[\bar{\lambda}^{-1}, 1]$ immediately implies $\Delta(\lambda)$ is strictly increasing on $[\bar{\lambda}^{-(n-1)}, \bar{\lambda}^{-n})$ for all $n \geq 1$. Hence the minimum over $\Delta(\lambda)$ is attained at a point of discontinuity. By an abuse of notation, let $\Delta^{-n} = \Delta(\bar{\lambda}^{-n})$. Then

$$
\begin{aligned}
\Delta^{-n} &= \Delta u(\bar{\lambda}^{-(n-1)}) + \beta(1-\pi)\Delta^{-(n-1)} \\
&= \Delta u(\bar{\lambda}^{-(n-1)}) + \beta(1-\pi)\Delta u(\bar{\lambda}^{-(n-2)}) + \Big(\beta(1-\pi)\Big)^2 \Delta^{-(n-2)}
\end{aligned}
$$

Since the market is illiquid for all $n \geq 1$, continuing to iterate gives

$$
\Delta^{-n} = \Delta L \left(\sum_{i=0}^{n-1} (1-\pi)^i \beta^i \right) + (1-\pi)^n \beta^n \Big[\Delta R + \beta \Delta V(\bar{\lambda})\Big] \tag{23}
$$

We proceed by induction. Suppose $\Delta^{-2} > \Delta^{-1}$. Then

$$\Delta L + (1-\pi)\beta\Big[\Delta R + \beta \Delta V(\bar{\lambda})\Big] > \Delta R + \beta \Delta V(\bar{\lambda}) \tag{24}$$

For all $n > 2$, the difference in incentives at the points of discontinuity is $\Delta^{-n} - \Delta^{-(n-1)} = \Delta L\left((1-\pi)\beta\right)^{n-1} + \left((1-\pi)\beta\right)^{n}\left[\Delta R + \beta \Delta V(\bar{\lambda})\right] - \left((1-\pi)\beta\right)^{n-1}\left[\Delta R + \beta \Delta V(\bar{\lambda})\right]$. Hence $\Delta^{-n} - \Delta^{-(n-1)} > 0$ for any $n > 2$ if and only if $\Delta L + (1-\pi)\beta\left[\Delta R + \beta \Delta V(\bar{\lambda})\right] > \Delta R + \beta \Delta V(\bar{\lambda})$, which is equivalent to the first induction Step (see Equation 24). Moreover, it follows immediately that we cannot have $\Delta^{-2} \leq \Delta^{-1}$, since the above arguments would then imply the contradiction that ΔV is globally monotone.

A.4 Proof of Proposition 2

Proof. Our proof is organized as follows. Let us refer to an equilibrium as a *monotone* equilibrium if $\Gamma^{T}(\lambda_0, \mu^*)$ is monotone in T for $\forall T > 0$. First, we show the existence of a monotone equilibrium with $\lambda^{\infty} \in \{1, \bar{\lambda}, 0\}$ given λ_0 for the entire parameter space. For shorthand, we refer to a monotone equilibrium with $\lambda^{\infty} = x$, as an *x-monotone equilibrium*. Second, we identify the Pareto monotone equilibrium given λ_0, i.e. the monotone equilibrium that Pareto-dominates any other monotone equilibrium that may co-exist. Third, we show that Pareto monotone equilibrium Pareto-dominates any competitive equilibrium, which establishes that it is the Pareto competitive equilibrium. In addition, we show that for the Pareto monotone equilibrium, there exists cutoffs λ^{s} and λ^{e}, where

- $\mu^*(\lambda) = 0$ for $\lambda < \lambda^{s}$;
- $\mu^*(\lambda) \leq 1$ for $\lambda \in [\lambda^{s}, \lambda^{e}]$ with the inequality strict for at least one $\lambda \in [\lambda^{s}, \lambda^{e}]$;
- $\mu^*(\lambda) = 1$ for $\lambda \geq \lambda^{e}$.

Step 1. We first show existence of a monotone equilibrium for the entire parameter space. We show by characterizing the conditions under which a 1-monotone equilibrium exists, followed by $\bar{\lambda}$, and 0. Then, we establish that the superset of the three spans the parameter space.

First, note that under Assumption 3, there exists a 1-monotone equilibrium if $\lambda_0 = 1$. This is because given $\frac{\Delta u(1)}{1-\beta(1-\pi)} > \frac{c}{\pi}$, for a sequence $\lambda_t = 1$ for $t = 0, 1, 2, ...$, individual borrowers find it optimal to exert effort, since $\Gamma(1,1) = 1$. Similarly, if Assumption 3 does not hold, there exists an equilibrium with $\lambda^{\infty} = 0$, since for all $\lambda \in [0,1]$, borrower's incentive compatibility condition does not hold, i.e. a necessary condition for a 1- or $\bar{\lambda}$-monotone equilibrium to exist is $\frac{\Delta u(1)}{1-\beta(1-\pi)} > \frac{c}{\pi}$. For the remainder of the proof, we assume so.

Following a similar argument, we can show that more generally, there exists some cutoff $\lambda^l < 1$ such that for any $\lambda_0 > \lambda^l$, there exists a 1-monotone equilibrium. Since a 1-monotone equilibrium exists for $\lambda_0 = 1$, for some small ϵ:

$$\frac{\Delta u(1)}{1-\beta(1-\pi)} - \left(\Delta u(1-\epsilon) + \sum_t \beta^t \Delta u(\Gamma^t(1-\epsilon, \mu = \mathbf{1})\right) = \left[1 + r_f - \frac{1 + r_f - \epsilon L_b}{1-\epsilon}\right] + \beta\left[1 + r_f - \frac{1 + r_f - \epsilon(1-\pi)L_b +}{1 - \epsilon + \epsilon\pi}\right] + ...$$

which approaches 0 as $\epsilon \to 0$. Hence, for some $\bar{\epsilon} > 0$, $\frac{1-\bar{\epsilon}-\pi}{1-\pi}$ is such that $\lim_{T\to\infty} \Gamma^T\left(\frac{1-\bar{\epsilon}-\pi}{1-\pi}, \mu = \mathbf{1}\right) = 1$ and where $\mu = \mathbf{1}$ is incentive compatible, i.e. there exists a 1-monotone equilibrium. Going forward, let λ^l be the minimum value in $[\bar{\lambda}, 1]$ for which given $\lambda_0 \geq \frac{\lambda^l - \pi}{1-\pi}$, there exists a 1-monotone equilibrium with $\mu^* = \mathbf{1}$.

So far, we established that there exists some nonempty set of $\lambda_0 \in [0, 1]$ such that there exists a 1-monotone equilibrium. This implies that there exists some λ^l such that for $\lambda_0 \in [\frac{\lambda^l-\pi}{1-\pi}, 1]$, there exists a 1-monotone equilibrium in which $\mu^*(\lambda) = \mathbf{1}$. We now proceed by identifying the necessary and sufficient conditions under which it exists for the remaining set of λ_0. We do so by splitting it into two cases, when (1) $\bar{\lambda} + (1-\bar{\lambda})\pi \geq \lambda^l$, and when (2) $\bar{\lambda} + (1-\bar{\lambda})\pi < \lambda^l$. Consider Case (1), and suppose that $\bar{\lambda} + (1-\bar{\lambda})\pi \geq \lambda^l$. This means that $\bar{\lambda} > \frac{\lambda^l-\pi}{1-\pi}$, i.e. there exists values of $\lambda_0 < \bar{\lambda}$ for which there exists a 1-monotone equilibrium. To pin down the set of λ_0 for which there exists a 1-monotone equilibrium, we consider two subcases, where given $\mu(\lambda) = \mathbf{1}$, $\Delta(\lambda)$ is (a) monotonic or (b) non-monotonic. This corresponds to when (a) $\Delta L < \Delta R$ or $\pi > \bar{\lambda}$ and (b) $\Delta L > \Delta R$ and $\pi < \bar{\lambda}$, as given by Lemma 3.

First consider Subcase (a), and suppose that $\Delta(\lambda)$ is monotonic. Given some $\lambda \leq \bar{\lambda}$, let τ be the smallest integer such that $\Gamma^\tau(\lambda, \mu = \mathbf{1}) \geq \lambda^l$. We can express borrowers' gains from effort at λ conditional on $\mu = \mathbf{1}$ as:

$$\Delta L + \beta(1-\pi)\Delta L + ... + \beta^{\tau-1}(1-\pi)^{\tau-2}\left((1-\pi)\Delta u(\Gamma^\tau(\lambda, \mu = \mathbf{1})) + \beta \Delta V(\Gamma^\tau(\lambda, \mu = \mathbf{1}))\right) \tag{25}$$

Effort is incentive compatible at λ if the above expression is greater than $\frac{c}{\pi}$. Since $\Delta(\lambda)$ is monotonic, $\Delta u(\Gamma^\tau(\lambda, \mu = \mathbf{1})) \geq \Delta L$. This implies that the above expression increases in λ.

Hence, there exists some $\lambda^e < \bar{\lambda}$ (and consequently $\lambda^e < \lambda^l$) such that $\Delta(\lambda, \mu = \mathbf{1}) \geq \frac{c}{\pi}$ if and only if $\lambda \geq \lambda^e$. If, in addition, $\frac{c}{\pi} < \frac{\Delta L}{1-\beta(1-\pi)}$, then $\Delta(0, \mu = \mathbf{0}) > \frac{c}{\pi}$. Since $\Delta(\lambda)$ is monotonic, this implies that $\lambda^e = 0$. Instead, if $\frac{c}{\pi} > \frac{\Delta L}{1-\beta(1-\pi)}$, then $\lambda^e = 0$ if $\Delta(0, \mu = \mathbf{1}) \geq \frac{c}{\pi}$, and $\lambda^e \in (0, \bar{\lambda})$ otherwise. Finally, note that there exists a monotone equilibrium with $\lambda^\infty(\mu^*) = 0$ for $\lambda_0 < \lambda^e$, since $\frac{c}{\pi} > \frac{\Delta L}{1-\beta(1-\pi)}$ implies that a decreasing $\Gamma(\lambda, \mu)$ is consistent with borrowers' incentives not to exert effort.

Next, consider Subcase (2) where $\Delta(\lambda)$ is non-monotonic, i.e. there exists a discontinuity in $\Delta u(\lambda)$ at $\lambda = \bar{\lambda}$. This implies that for $\mu = \mathbf{1}$, $\Delta(\bar{\lambda} - \epsilon) > \Delta(\bar{\lambda})$ for some $\epsilon > 0$. Consider the set $\in [\Gamma^{-1}(\bar{\lambda}, \mu = \mathbf{1}), \Gamma^{-1}(\lambda^l, \mu = \mathbf{1})]$. Since given $\mu = \mathbf{1}$, for some $x \in [\Gamma^{-1}(\bar{\lambda}, \mu = \mathbf{1}), \Gamma^{-1}(\lambda^l, \mu = \mathbf{1})]$, $\Delta(\lambda)$ is increasing, effort is not incentive compatible since $\Delta(\Gamma(x, \mu = \mathbf{1})) < \frac{c}{\pi} = \Delta(\lambda^l)$. Given $\mu(x)$ such that $\Gamma(x, \mu) = \bar{\lambda}$, i.e. $\mu(x) = \frac{\bar{\lambda} - x(1-\pi)}{\pi}$ and $\mu(\lambda) = \mathbf{1}$ for $\lambda \geq \Gamma^{-1}(\lambda^l, \mu = \mathbf{1})$, the incentive constraint is given by

$$\Delta u(\bar{\lambda}) + \beta(1-\pi)\Delta\Gamma(\bar{\lambda}, \mu = \mathbf{1}) \geq \frac{c}{\pi}. \tag{26}$$

Since $\Delta\Gamma(\bar{\lambda}, \mu = \mathbf{1}) > \frac{c}{\pi}$, the condition holds as long as $\Delta u(\bar{\lambda}) \geq (1 - \beta(1-\pi)\frac{c}{\pi}$. If $\frac{c}{\pi} < \frac{\Delta L}{1-\beta(1-\pi)}$, then there exists some $\phi < 1$ such that:

$$(1-\phi)\Delta L + \phi \Delta R = (1 - \beta(1-\pi))\frac{c}{\pi}. \tag{27}$$

Hence, borrowers' indifference holds between effort and no effort. This implies that for $\lambda_0 < \frac{\lambda^l - \pi}{1-\pi}$, there exists a 1-monotone equilibrium where $\mu^*(\lambda) = \mathbf{1}$ for $\lambda \notin \left[\frac{\bar{\lambda}-\pi}{1-\pi}, \frac{\lambda^l - \pi}{1-\pi}\right]$ and $\mu^*(\lambda) = \frac{\bar{\lambda} - \lambda(1-\pi)}{\pi}$ for $\lambda \in \left[\frac{\bar{\lambda}-\pi}{1-\pi}, \frac{\lambda^l-\pi}{1-\pi}\right]$. Let λ^s be defined as the threshold value where $\mu^*(\lambda) < 1$ for some λ along the equilibrium path. Then, here $\lambda^s = 0$. In contrast, if $\frac{c}{\pi} > \frac{\Delta L}{1-\beta(1-\pi)}$, then given that $\Gamma(\lambda, \mu)$ weakly increases, it follows that $\Delta u(\bar{\lambda}) < (1 - \beta(1-\pi))\frac{c}{\pi}$ for any $\phi \in [0, 1]$. Hence, a 1-monotone equilibrium exists if only if $\lambda_0 \geq \Gamma^{-1}(\lambda^l, \mu = \mathbf{1})$. Furthermore, since there does not exist any ϕ such that $\frac{\Delta u(\bar{\lambda})}{1-\beta(1-\pi)} = \frac{c}{\pi}$, there can not exist a $\bar{\lambda}$-monotone equilibrium. Finally, conditional on $\Gamma(\lambda, \mu)$ weakly decreasing, there exists a 0-monotone equilibrium for $\lambda_0 < \Gamma^{-1}(\lambda^l, \mu = \mathbf{1})$, since $\frac{c}{\pi} > \frac{\Delta L}{1-\beta(1-\pi)}$ implies that a decreasing $\Gamma(\lambda, \mu)$ is consistent with borrowers' incentives not to exert effort.

Consider Case (2), where $\bar{\lambda} + (1-\bar{\lambda})\pi < \lambda^l$. Since it has been shown that a 1-monotone equilibrium exists for $\lambda_0 \geq \Gamma^{-1}(\lambda^l, \mu = \mathbf{1})$, we focus on the remaining interval where $\lambda_0 < \Gamma^{-1}(\lambda^l, \mu = \mathbf{1})$. First, we establish that there does not exist a 1-monotone equilibrium for $\lambda_0 < \Gamma^{-1}(\lambda^l, \mu = \mathbf{1})$. Given $\bar{\lambda} + (1-\bar{\lambda})\pi < \lambda^l$ we have that there exists some $\lambda^e \in (\bar{\lambda}, \lambda^l)$

such that some $\lambda \in [\bar{\lambda}, \lambda^e)$, individual borrowers' effort decision is such that no effort is strictly dominating strategy at conditional on beliefs that all other borrowers exert effort in the current and all future periods. From this, we can infer that there does not exist feasible transition path to 1: any candidate path with $\lambda_0 < \frac{\lambda^l - \pi}{1-\pi}$, must have some T at which $\Gamma^T(\lambda_0, \mu) \in [\bar{\lambda}, \lambda^e)$, as there does not exist a $\lambda \leq \bar{\lambda}$ where $\lambda + (1-\lambda)\pi \geq \lambda^e$. However, effort is not incentive compatible given any $\lambda \in [\bar{\lambda}, \lambda^e)$, so we are done. Given this, we proceed by identifying the conditions under which there exists a $\bar{\lambda}$-monotone equilibrium. We show existence requires that $\frac{c}{\pi} \geq \frac{\Delta R}{1-\beta(1-\pi)}$ by contradiction. To see this, note that $\frac{\Delta R}{1-\beta(1-\pi)}$, which corresponds to the marginal value of effort conditional on a sequence of repeated $\bar{\lambda}$ and $\phi = 1$. Since given $\phi = 1$, $\frac{\Delta R}{1-\beta(1-\pi)} < \Delta(\bar{\lambda}, \mu = \mathbf{1})$ under Lemma 3, if $c < \frac{\Delta R}{1-\beta(1-\pi)}$, then a 1-monotone equilibrium exists. Given this, suppose that $\frac{\Delta L}{1-\beta(1-\pi)} > \frac{c}{\pi} > \frac{\Delta R}{1-\beta(1-\pi)}$. We maintain our attention to $\lambda_0 < \Gamma^{-1}(\lambda^l, \mu = \mathbf{1})$. Consider the following μ:

$$\mu(\lambda) = \begin{cases} \begin{cases} \frac{\bar{\lambda} - \lambda(1-\pi)}{\pi} & \text{for } \lambda \in (\frac{\bar{\lambda}-\pi}{1-\pi}, \frac{\lambda^l - \pi}{1-\pi}) \\ 1 & \text{for } \lambda < \frac{\bar{\lambda}-\pi}{1-\pi} \end{cases} & \text{if } \lambda_0 < \bar{\lambda} \\ \begin{cases} \frac{\bar{\lambda} - \lambda(1-\pi)}{\pi} & \text{for } \lambda \in (\bar{\lambda}, \frac{\bar{\lambda}}{1-\pi}) \\ 0 & \text{for } \lambda > \frac{\bar{\lambda}}{1-\pi} \end{cases} & \text{if } \lambda_0 \in (\bar{\lambda}, \frac{\lambda^l - \pi}{1-\pi}) \end{cases} \tag{28}$$

Given μ, there exists some T such that for $t > T$, $\Gamma^t(\lambda_0, \mu) = \bar{\lambda}$, i.e. $\lambda^\infty = \bar{\lambda}$. As such, it suffices to show that μ is incentive compatible. Consider a borrower's incentive constraint given $\Gamma(\lambda, \mu) = \bar{\lambda}$. Note, there exists some $\phi < 1$ such that:

$$(1-\phi)\Delta L + \phi \Delta R = (1 - \beta(1-\pi))\frac{c}{\pi} \tag{29}$$

This implies that borrowers are indifferent between effort and no effort, since $\frac{(1-\phi)\Delta L + \phi\Delta R}{1-\beta(1-\pi)} = \frac{c}{\pi}$. Hence, we show incentive compatibility for the cases where $\mu < 1$. Next, consider when $\Gamma(\lambda, \mu) < \bar{\lambda}$. It follows from $\frac{c}{\pi} \leq \frac{\Delta L}{1-\beta(1-\pi)}$ that $\Delta L + \beta V(\Gamma(\lambda, \mu)) > \frac{c}{\pi}$. Finally, for any $\lambda \in (\bar{\lambda}, \frac{\lambda^l - \pi}{1-\pi})$, since $\frac{\Delta R}{1-\beta(1-\pi)}$ and $\Gamma(\lambda) > \bar{\lambda}$, no effort is consistent since $\Delta R + \beta V(\Gamma(\lambda, \mu)) < \frac{c}{\pi}$. Hence, a $\bar{\lambda}$-monotone equilibrium for $\lambda_0 < \frac{\lambda^l - \pi}{1-\pi}$ if $\frac{\Delta L}{1-\beta(1-\pi)} > \frac{c}{\pi}$.

Within Case (2), it remains to consider when $\frac{\Delta L}{1-\beta(1-\pi)} < \frac{c}{\pi}$. We show that a $\bar{\lambda}$-monotone equilibrium does not exist. Recall that we showed that a $\bar{\lambda}$-monotone equilibrium does not exist if $\frac{\Delta R}{1-\beta(1-\pi)} \geq \frac{c}{\pi}$. Let $\frac{\Delta R}{1-\beta(1-\pi)} < \frac{c}{\pi}$. Then, it holds that $\frac{\max\{\Delta L, \Delta R\}}{1-\beta(1-\pi)} < \frac{c}{\pi}$. Since $(1-\phi)\Delta L + \phi\Delta R < (1-\beta(1-\pi))\frac{c}{\pi}$, for any $\phi \in [0,1]$, there does not exist an incentive compatible μ such that $\Gamma(\bar{\lambda}, \mu) = \bar{\lambda}$. This rules out a $\bar{\lambda}$-monotone equilibrium.

To complete the existence of a monotone equilibrium, consider instead a 0-monotone equilibrium. Let $\mu = \mathbf{0}$ for $\lambda < \frac{\lambda^l - \pi}{1-\pi}$, which means $\Gamma(\lambda, \mu = \mathbf{0}) = (1-\pi)\lambda$. Since $\frac{c}{\pi} > \frac{\Delta L}{1-\beta(1-\pi)} = \frac{\Delta u(0)}{1-\beta(1-\pi)}$, the law of motion is consistent with borrower's individual decision not to exert effort for the path of λ. Hence, for $\frac{\Delta L}{1-\beta(1-\pi)} < \frac{c}{\pi}$, a 0-monotone equilibrium exists given $\lambda_0 < \frac{\lambda^l - \pi}{1-\pi}$.

We have shown general existence of a monotone equilibrium. We summarize equilibrium characteristics over the parameter space. Formally, let λ^e be a threshold at which for $\lambda > \lambda^e$, $\mu = \mathbf{1}$ is incentive compatible, and let λ^s be a threshold at which for $\lambda > \lambda^s$, $\mu \in (\mathbf{0}, \mathbf{1}]$. For $\bar{\lambda} + (1-\bar{\lambda})\pi \geq \lambda^l$:

- If $\Delta(\lambda)$ monotonic, then a 1-monotone equilibrium with $\mu^* = \mathbf{1}$ exists for $\lambda_0 > \lambda^e$, where $\lambda^e \in [0, \bar{\lambda})$. Otherwise, a 0-monotone equilibrium with $\mu^* = \mathbf{0}$ exists.

- if $\Delta(\lambda)$ non-monotonic, then a 1-monotone equilibrium with $\mu^* = \mathbf{1}$ exists for $\lambda_0 > \lambda^e$; a 1-monotone equilibrium with $\mu^* \leq \mathbf{1}$ exists for $\lambda_0 \in (\lambda^s, \lambda^e)$, where $\lambda^s = 0$ if $\frac{\Delta L}{1-\beta(1-\pi)} > \frac{c}{\pi}$ and $\lambda^s = \lambda^e$ if $\frac{\Delta L}{1-\beta(1-\pi)} \leq \frac{c}{\pi}$; a 0-monotone equilibrium with $\mu^* = \mathbf{0}$ exists if $\lambda_0 < \lambda^s$.

For $\bar{\lambda} + (1-\bar{\lambda})\pi < \lambda^l$:

- A 1-monotone equilibrium with $\mu^* = \mathbf{1}$ exists for $\lambda_0 \geq \lambda^e$, where $\lambda^e \in (\bar{\lambda}, 1]$.

- A $\bar{\lambda}$-monotone equilibrium exists for $\lambda_0 \in [\lambda^s, \lambda^e)$ if $\frac{\Delta L}{1-\beta(1-\pi)} > \frac{c}{\pi}$, where $\lambda^s = 0$.

- A 0-monotone equilibrium with $\mu^* = \mathbf{0}$ for $\lambda_0 < \lambda^s$ exists if $\frac{\Delta L}{1-\beta(1-\pi)} > \frac{c}{\pi}$, where $\lambda^s = \lambda^e$.

Step 2. We proceed to show that the above set of monotone equilibria Pareto-dominates any other monotone equilibrium whenever they co-exist. To do so, first we recognize a sufficient condition under which one competitive equilibrium dominates another. Since $V_\theta(\lambda)$ strictly increases in λ, an equilibrium with greater $\Gamma^T(\lambda_0, \mu)$ for all T Pareto-dominates:

Lemma 5. *Given initial condition λ_0, a competitive equilibrium with effort profile μ Pareto-dominates a competitive equilibrium with effort profile μ' if $\Gamma^T(\lambda_0, \mu) \geq \Gamma^T(\lambda_0, \mu')$ for $\forall T$.*

We apply Lemma 5 to the set of monotone equilibria. First, note that under the conditions for which a 1-monotone equilibrium with $\mu^* = \mathbf{1}$ exists, it is trivially a Pareto monotone equilibrium. Next, consider the conditions under which a 1-monotone equilibrium with $\mu^*(\lambda) < \mathbf{1}$ for some λ exists. It is straightforward to see that the equilibrium explicitly characterized in the existence proof dominates any $\bar{\lambda}$- or 0-monotone equilibrium if it exists. Hence, is suffices to show whether it is dominated by a different 1-monotone equilibrium. This equilibrium is dominated by some equilibrium with effort profile μ' only if $\mu'(\lambda) > \mu^*(\lambda)$ for $\lambda \in \left[\frac{\bar{\lambda}-\pi}{1-\pi}, \frac{\lambda^l-\pi}{1-\pi}\right]$. This requires that for some $\lambda \in \left[\bar{\lambda}, \lambda^l\right]$, it must hold that $\Delta(\lambda) > \frac{c}{\pi}$. However, no such λ exists given the definition of λ^l. Hence, it is a Pareto monotone equilibrium. Consider the remaining cases, when a 1-monotone equilibrium does not exist, but when a $\bar{\lambda}$-monotone equilibrium exists as outlined above. It follows from the proof that the $\bar{\lambda}$-monotone equilibrium shown to exist above also Pareto dominates any other $\bar{\lambda}$- or 0-monotone equilibrium given Lemma 5. For the remaining cases when only a 0-monotone equilibrium exists, it is trivially the Pareto monotone equilibrium.

Step 3. What remains is to compare the Pareto monotone equilibrium to any other potential competitive equilibrium. We do so by showing that there cannot exist any other incentive compatible effort profile that improves on the Pareto monotone equilibrium. Suppose that for some λ_0, the Pareto monotone equilibrium is a 1-monotone equilibrium. Earlier, we showed that there exists some threshold λ^e, for which if $\lambda_0 \geq \Gamma^{-1}(\lambda^e, \mu = \mathbf{1})$, then there exists a 1-monotone equilibrium. Note the following:

Lemma 6. *Suppose that an equilibrium effort profile μ is such that $\mu(\lambda) = \mathbf{1}$ for $\lambda \geq \lambda_0$. Then, given initial condition λ_0, it is a Pareto competitive equilibrium.*

This follows directly from Lemma 5. Since a full effort profile obtains the period-by-period largest feasible λ, it is Pareto dominant. In addition, since $\lim_t \Gamma^t(\lambda, \mu = \mathbf{1}) = 1$, this equilibrium asymptotes to 1.

The remaining set of Pareto 1-monotone equilibrium is one in which $\mu^*(\lambda) < \mathbf{1}$ for some λ. We show that there does not exist any incentive compatible effort profile μ' that Pareto dominates μ^*. Note that for $\lambda_0 < \Gamma^{-1}(\bar{\lambda}, \mu = \mathbf{1})$, the sequence of $\Gamma(\lambda_0, \mu^*)$ to $\Gamma^{-1}(\bar{\lambda}, \mu^*)$ satisfy the condition under Lemma 5 since $\mu^*(\lambda) = \mathbf{1}$. Now, consider when $\lambda = \Gamma^{-1}(\bar{\lambda}, \mu^*)$. It suffices to show that given some initial value $\Gamma^{-1}(\bar{\lambda}, \mu^*)$, there does not exist an alternative incentive compatible effort profile μ' such that $\Gamma^{-1}(\bar{\lambda}, \mu^*) < \Gamma^{-1}(\bar{\lambda}, \mu')$. Recall, $\mu(\lambda) = \mathbf{1}$ for $\lambda \geq \bar{\lambda}$, i.e. μ is such that full effort is exerted for $\lambda > \bar{\lambda}$ along the path. By Lemma 3, $\Delta(\lambda)$ increases in this region. Hence, μ induces the largest incentives. This implies that there does not exist any μ' that can improve on μ.

Next, suppose that given λ_0, the Pareto monotone equilibrium is a $\bar{\lambda}$-monotone equilibrium. Note that in this equilibrium, along the equilibrium path, $\mu(\lambda) = \mathbf{1}$ for $\lambda < \frac{\bar{\lambda}-\pi}{1-\pi}$ and $\mu(\lambda) < 1$ for $\lambda \in \frac{\bar{\lambda}-\pi}{1-\pi}$. Existence of the equilibrium requires $\frac{c}{\pi} < \frac{\Delta L}{1-\beta(1-\pi)}$. As before, note that for the sequence of $\Gamma(\lambda_0, \mu^*)$ to $\Gamma^{-1}(\bar{\lambda}, \mu^*)$ satisfy the condition under Lemma 5 when $\mu^* = \mathbf{1}$. Consider when $\lambda > \bar{\lambda}$. From earlier, we showed that effort is not incentive compatible conditional on full effort. Since $\Delta u(\lambda)$ increases in λ for $\lambda > \bar{\lambda}$, this implies that there does not exist any incentive compatible $\mu'(\lambda) > \mu(\lambda)$.

Hence, it suffices to show that there does not exist some incentive compatible effort profile μ' for which given $\Gamma^{-1}(\bar{\lambda}, \mu^*)$, it holds that $\Gamma(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu') > \bar{\lambda}$. Consider incentives when $\lambda = \bar{\lambda}^{-1}(\mu)$. Without loss of generality, we focus on when $\lambda_0 < \bar{\lambda}$. Suppose by contradiction there does exist such μ'. This requires that under effort profile μ',

$$\Delta u(\Gamma(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')) + \beta \Delta V(\Gamma(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')) \geq \frac{c}{\pi} \tag{30}$$

In a $\bar{\lambda}$-monotone equilibrium we require $\Delta u(\bar{\lambda}) + \beta \Delta V(\bar{\lambda}) = \frac{c}{\pi}$. We can further explicitly express this as $\Delta u(\bar{\lambda}) + \beta \frac{\Delta u(\bar{\lambda})}{1-\beta(1-\pi)} = \frac{c}{\pi}$. We claim that $\Delta u(\Gamma(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')) \leq (1 - \beta(1-\pi)\frac{c}{\pi}$. Suppose by contradiction, that $\Delta u(\Gamma(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')) > (1 - \beta(1-\pi))\frac{c}{\pi}$. Under Lemma 3, this implies that $\Delta(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu'), \mu = \mathbf{1}) > \frac{c}{\pi}$, since $\Delta(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu'), \mu = \mathbf{1}) > \frac{\Delta u(\Gamma(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu'))}{1-\beta(1-\pi)}$. However, this implies that there exists a 1-monotone equilibrium.

Since $\Delta u(\Gamma(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')) \leq (1 - \beta(1-\pi))\frac{c}{\pi}$, combined with Inequality 30 we get the following:

$$\Delta u(\Gamma(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')) \leq \Delta u(\bar{\lambda}) \tag{31}$$

$$\Delta V(\Gamma(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')) > \Delta V(\bar{\lambda}) \tag{32}$$

In the Pareto monotone equilibrium, a constant partial effort strategy is used by borrowers at $\bar{\lambda}$, which implies that $\Delta V(\bar{\lambda}) = (1-\pi)\frac{c}{\pi}$. Using this in the above inequality, we get:

$$\Delta V(\Gamma(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')) > (1-\pi)\frac{c}{\pi} \tag{33}$$

This implies that given $\Gamma^2(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')$, individual borrowers must strictly prefer to exert effort. It follows that $\Gamma^2(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu') > \Gamma(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')$. Writing out Inequality 33:

$$\Delta u(\Gamma^2(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')) + \beta \Delta V(\Gamma^2(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')) > \frac{c}{\pi} \tag{34}$$

Suppose that $\Delta V(\Gamma^2(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')) > (1-\pi)\frac{c}{\pi}$. Then, by the same argument, $\Gamma^3(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu') > \Gamma^2(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')$. If $\Delta V(\Gamma^T(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')) > (1-\pi)\frac{c}{\pi}$ for $T > 2$, then $\lim_{t\to\infty} \Gamma^t(\Gamma^{-1}(\bar{\lambda}, \mu), \mu')) = 1$, i.e. $\lambda^\infty = 1$. We reach a contradiction. Suppose instead that T is the minimum integer at which $\Delta V(\Gamma^T(\Gamma^{-1}(\bar{\lambda}, \mu), \mu')) < (1-\pi)\frac{c}{\pi}$. This implies that:

$$\Delta u(\Gamma^T(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')) > (1 - \beta(1-\pi))\frac{c}{\pi} \tag{35}$$

However, given Lemma 3, $\Delta u(\lambda) > \Delta u(\Gamma^T(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu'))$ for $\lambda > \Gamma^T(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')$, which means:

$$\Delta u(\Gamma^T(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu = \mathbf{1})) > \Delta u(\Gamma^T(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')) \tag{36}$$

$$> (1 - \beta(1 - \pi))\frac{c}{\pi}. \tag{37}$$

The above condition, combined with the fact that $\Delta V(\Gamma^t(\Gamma^{-1}(\bar{\lambda}, \mu^*), \mu')) \geq (1 - \pi)\frac{c}{\pi}$ for $t \in (1, T)$ satisfies the conditions for there to exist a 1-monotone equilibrium. We reach a contradiction. Hence, we show that there does not exist any such μ' that dominates the $\bar{\lambda}$-monotone equilibrium outlined in the existence proof.

Finally, consider when the Pareto monotone equilibrium is a 0-monotone equilibrium. This holds when $\frac{c}{\pi} < \frac{\Delta L}{1-\beta(1-\pi)}$ and $\lambda_0 < \lambda^s$. Suppose that $\frac{c}{\pi} < \frac{\Delta L}{1-\beta(1-\pi)}$ and $\lambda_0 < \lambda^s$. First, note that $\frac{c}{\pi} < \frac{\Delta L}{1-\beta(1-\pi)}$ implies that for any μ' such that $\Gamma^t(\lambda_0, \mu') < \bar{\lambda}$ for all t, effort is not incentive compatible in any period. Hence, a necessary (but not sufficient) condition for effort to be incentive compatible, is if given λ_0, effort profile μ' induces an equilibrium path whereby $\Gamma^t(\lambda_0, \mu') > \bar{\lambda}$ for some t. Suppose that such incentive compatible effort profile μ' exists. This implies that

$$\Delta u(\Gamma^{-1}(\lambda, \mu')) + \beta \Delta V(\Gamma^{-1}(\lambda, \mu')) > \frac{c}{\pi} \tag{38}$$

for some $\lambda \geq \bar{\lambda}$. Note that under Lemma 3,

$$\Delta u(\Gamma(\lambda_0, \mu = \mathbf{1})) + \beta \Delta V(\Gamma(\lambda_0, \mu = \mathbf{1})) > \Delta u(\Gamma(\lambda_0, \mu')) + \beta \Delta V(\Gamma(\lambda_0, \mu')) \tag{39}$$

and $\lambda_0 < \lambda^s$ implies that effort is not incentive compatible conditional on a full effort profile, i.e. $\Delta u(\Gamma(\lambda_0, \mu = \mathbf{1})) + \beta \Delta V(\Gamma(\lambda_0, \mu = \mathbf{1})) < \frac{c}{\pi}$. Hence, no such μ' can exist.

□

A.5 Proof of Proposition 3

Proof. We first show that a full-effort profile $\mu = \mathbf{1}$ (weakly) Pareto-dominates PCE if $\lambda^\infty \in \{1, \bar{\lambda}\}$.

First, consider the conditions under which the PCE where $\lambda^\infty = 1$. It suffices to consider the case where $\mu^*(\lambda) < 1$ for some λ along the equilibrium path. (Otherwise $\mu^* = \mathbf{1}$). Consider when $\lambda \in \left(\frac{\bar{\lambda}-\pi}{1-\pi}, \lambda^e\right)$. Given $\mu(\lambda) = \mathbf{1}$, the payoffs of a good and bad type of borrower is given by:

$$u_g(\Gamma(\lambda,\mu(\lambda)=\mathbf{1}))+\beta V_g(\Gamma(\lambda,\mu(\lambda)=\mathbf{1}))-\frac{c}{1-\beta}\text{ and} \tag{40}$$

$$\pi\left[u_g(\Gamma(\lambda,\mu(\lambda)=\mathbf{1}))+\beta V_g(\Gamma(\lambda,\mu(\lambda)=\mathbf{1}))\right]+(1-\pi)\left[u_b(\Gamma(\lambda,\mu(\lambda)=\mathbf{1}))+\beta V_b(\Gamma(\lambda,\mu(\lambda)=\mathbf{1}))\right]-\frac{c}{1-\beta} \tag{41}$$

The payoff of a good and bad type under μ^* can be written as:

$$u_g(\Gamma(\lambda,\mu^*))+\beta V_g(\Gamma(\lambda,\mu^*))-\frac{c}{1-\beta}\text{ and} \tag{42}$$

$$\pi\left[u_g(\Gamma(\lambda,\mu^*))+\beta V_g(\Gamma(\lambda,\mu^*))\right]+(1-\pi)\left[u_b(\Gamma(\lambda,\mu^*))+\beta V_b(\Gamma(\lambda,\mu^*))\right]-\frac{c}{1-\beta} \tag{43}$$

By Lemma 3, both type's payoff under $\mu(\lambda)=\mathbf{1}$ is strictly larger than μ^*.

Next, consider when the PCE where $\lambda^\infty=\bar{\lambda}$. If the PCE is a $\bar{\lambda}$-monotone equilibrium, this means $\frac{\Delta L}{1-\beta(1-\pi)}>\frac{c}{\pi}$. For $\Gamma(\lambda,\mu^*)=\bar{\lambda}$, the payoff of a good and bad type under μ^* is:

$$\frac{u_g(\bar{\lambda})}{1-\beta}-\frac{c}{1-\beta}\text{ and} \tag{44}$$

$$\pi\left[u_g(\bar{\lambda})+\beta V_g(\bar{\lambda})\right]+(1-\pi)\left[u_b(\bar{\lambda})+\beta V_b(\bar{\lambda})\right]-\frac{c}{1-\beta} \tag{45}$$

Note that $R_\theta>L_\theta$ for $\theta=g,b$. This implies that

$$\frac{u_g(\bar{\lambda}+\epsilon)}{1-\beta}>\frac{u_g(\bar{\lambda})}{1-\beta}\text{ and} \tag{46}$$

$$\pi\left[u_g(\bar{\lambda}+\epsilon)+\beta V_g(\bar{\lambda}+\epsilon)\right]+(1-\pi)\left[u_b(\bar{\lambda}+\epsilon)+\beta V_b(\bar{\lambda}+\epsilon)\right] \tag{47}$$

$$>\pi\left[u_g(\bar{\lambda})+\beta V_g(\bar{\lambda})\right]+(1-\pi)\left[u_b(\bar{\lambda})+\beta V_b(\bar{\lambda})\right] \tag{48}$$

For some arbitrarily small $\epsilon>0$. It follows that given $\mu=\mathbf{1}$, the payoffs of a good and bad type of borrower is given by:

$$u_g(\Gamma(\lambda,\mu=\mathbf{1}))+\beta V_g(\Gamma(\lambda,\mu=\mathbf{1}))-\frac{c}{1-\beta}\text{ and} \tag{49}$$

$$\pi\left[u_g(\Gamma(\lambda,\mu=\mathbf{1}))+\beta V_g(\Gamma(\lambda,\mu=\mathbf{1}))\right]+(1-\pi)\left[u_b(\Gamma(\lambda,\mu=\mathbf{1}))+\beta V_b(\Gamma(\lambda,\mu=\mathbf{1}))\right]-\frac{c}{1-\beta} \tag{50}$$

Under Lemma 3, note that both type's payoff under $\mu = \mathbf{1}$ is strictly larger than μ^*. In addition, note that since V_θ increases on $[0,1]$, if $\mu = \mathbf{1}$ dominates for some λ, then it also holds for $\lambda' > \lambda$.

Second, we show under Assumption 3 that there exists some threshold $\bar{\beta} < 1$ such that a full-effort profile $\mu = \mathbf{1}$ Pareto-dominates the PCE as long as $\beta < \bar{\beta}$. It suffices to show that as β approaches 1, full effort is Pareto dominant, and the PCE does not always have full effort.

Under full effort, $\lambda^\infty = 1$. Recall, we've already shown that $\mu = \mathbf{1}$ dominates any PCE with $\lambda^\infty = 1, \bar{\lambda}$. Suppose that the PCE is such that $\lambda^\infty = 0$. Note that the long-run per-period payoff difference between $\mu = \mathbf{1}$ and μ^* is given by $u(1) - c - L_g > 0$. Let $\tilde{u}_\theta^t(\mu)$ be the expected one-period payoff in period t of a borrower that is type θ at time 0 conditional on μ. Note that for any λ_0, there exists some t sufficiently large such that $\Gamma^t(\lambda_0, \mu = \mathbf{1}) > \bar{\lambda} > \Gamma^t(\lambda_0, \mu = \mathbf{0})$. For such t, the expected one-period payoffs of each type is under $\mu = \mathbf{1}$:

$$\tilde{u}_g^t(\mu = \mathbf{1}) = u_g(\Gamma^t(\lambda_0, \mu = \mathbf{1})) + \beta V_g(\Gamma^t(\lambda_0, \mu = \mathbf{1})) \tag{51}$$

$$\tilde{u}_b^t(\mu = \mathbf{1}) = (1 - (1-\pi)^t)\left[u_g(\Gamma^t(\lambda_0, \mu = \mathbf{1})) + \beta V_g(\Gamma^t(\lambda_0, \mu = \mathbf{1}))\right] \tag{52}$$

$$+ (1-\pi)^t \left[u_b(\Gamma^t(\lambda_0, \mu = \mathbf{1})) + \beta V_b(\Gamma^t(\lambda_0, \mu = \mathbf{1}))\right] \tag{53}$$

As $t \to \infty$, $\tilde{u}_\theta^t(\mu = \mathbf{1}) \to \frac{R_g + L_g - (1+r_f) - c}{\pi}$ for both $\theta = g, b$. Under $\mu = \mathbf{0}$, it is given by

$$\tilde{u}_g^t(\mu = \mathbf{0}) = (1-\pi)^t \left[u_g(\Gamma^t(\lambda_0, \mu = \mathbf{0})) + \beta V_g(\Gamma^t(\lambda_0, \mu = \mathbf{0}))\right] \tag{54}$$

$$+ (1 - (1-\pi)^t)\left[u_b(\Gamma^t(\lambda_0, \mu = \mathbf{0})) + \beta V_b(\Gamma^t(\lambda_0, \mu = \mathbf{0}))\right] \tag{55}$$

$$\tilde{u}_b^t(\mu = \mathbf{0}) = u_b(\Gamma^t(\lambda_0, \mu = \mathbf{0})) + \beta V_b(\Gamma^t(\lambda_0, \mu = \mathbf{0})) \tag{56}$$

As $t \to \infty$, $\tilde{u}_\theta^t(\mu = \mathbf{0}) \to \frac{L_b}{\pi}$ for both $\theta = g, b$. Given Assumption 3, $R_g + L_g - (1 + r_f) - c = \Delta R + L_g - (1 + r_f) - c + R_b > L_b$. Given that $\tilde{u}_\theta^t(\mu = \mathbf{1})$ increases in t, and $\tilde{u}_\theta^t(\mu = \mathbf{0})$ (weakly) deceases in t, there exists some τ period at which $\tilde{u}_\theta^t(\Gamma^\tau(\lambda_0, \mu = \mathbf{1})) > \tilde{u}_\theta^t(\Gamma^\tau(\lambda_0, \mu = \mathbf{0}))$. Furthermore, this implies that $\tilde{u}_\theta^t(\mu = \mathbf{1}) < \tilde{u}_\theta^t(\mu = \mathbf{0})$ for $\theta = g, b$ and all $t > \tau$. Since the differential payoffs between $\mu = \mathbf{1}$ and $\mu^* = 0$ for $t > \tau$ is an increasing sequence, the sum is infinite. Hence, full effort is Pareto dominant for sufficiently large β.

What remains is to show that full effort is not generally obtained in a PCE even as $\beta \to 1$. We prove by example. Set $\beta = 1$. It suffices to show that there exists a set of parameters under which for some λ_0, the PCE does not admit full effort. Consider some

$\lambda_0 = \frac{\bar{\lambda}-\pi}{1-\pi}$. Note that under $\mu = \mathbf{1}$,

$$\Delta u(\Gamma(\lambda, \mu = \mathbf{1})) = \Delta R + L_g - \frac{1 + r_f - (1 - \Gamma(\lambda, \mu = \mathbf{1}))L_b}{\Gamma(\lambda, \mu = \mathbf{1})} \tag{57}$$

$$= \Delta R + \Delta L - \frac{1 + r_f - L_b}{\Gamma(\lambda, \mu = \mathbf{1})} \tag{58}$$

Let us use $f(\lambda) = \frac{1+r_f-L_b}{\lambda}$. Effort at λ_0 is incentive compatible conditional on full effort if:

$$\sum_{a=1}^{\infty} (1-\pi)^{a-1}\pi(\Delta R + \Delta L - f(\Gamma^a(\lambda_0, \mu = \mathbf{1})) - c) > 0 \tag{59}$$

Given Lemma 3, note that the LHS of Inequality 59 is strictly less than the incentives associated with a sequence of $\lambda_t = \bar{\lambda}, 1, 1, 1, 1, ...$ Hence, a sufficient condition is to show that effort is not incentive compatible given a sequence $\lambda_t = \bar{\lambda}, 1, 1, 1, 1, ...$:

$$\pi(\Delta R + \Delta L - f(\lambda_0(1-\pi) + \pi) - c) + (1-\pi)(\Delta R + \Delta L - f(1) - c) \leq 0 \tag{60}$$

Given that we require $u(1) > c$ for $\beta = 1$, i.e. $\Delta R + \Delta L - f(1) - c > 0$, c must be such that

$$\Delta R + L_g - (1 + r_f) > c > \Delta R + (1-\pi)(L_g - (1 + r_f)) \tag{61}$$

Since $\pi < \bar{\lambda}$, parameters exist such that the relevant interval is non-empty. □

A.6 Proof of Proposition 4

Given that cash flows are constant in the region of illiquidity, the regulator either does not offer a subsidy or offers a subsidy that induces liquidity. If markets are illiquid, static welfare

$$\underline{w} = \lambda' L_g + (1 - \lambda')L_b + (1 + r_f).$$

If the regulator intervenes and markets are liquid, static welfare is

$$\begin{aligned}\bar{w}(s(\lambda')) &= \lambda'(R_g + L_g - \tilde{B}(\lambda', s)) + (1-\lambda')R_b + (1 + r_f) - (1+\delta)(1-\lambda')s(\lambda') \\ &= \lambda'(R_g + L_g) + (1-\lambda')(R_b + L_b) - \delta(1-\lambda')s(\lambda').\end{aligned}$$

Since $\bar{w}(s(\lambda'))$ is strictly decreasing in $s(\lambda')$, the regulator never intervenes if markets are liquid in competitive equilibrium, and chooses a subsidy no higher than $\underline{s}(\lambda')$. Intervening minimally is optimal if and only if $\bar{w}(\underline{s}(\lambda')) > \underline{w}$ or, equivalently, if

$$\lambda' R_g + (1-\lambda')R_b - (1+r_f) > \delta\Big(1 + r_f - \lambda' L_g - (1-\lambda')L_b\Big).$$

In the limit as $\delta \to 0$, this condition is satisfied if and only if $\lambda' \geq \underline{\lambda}$. To see why there cannot exist a competitive equilibrium with $\phi^*(\bar{\lambda}) < 1$, observe that the regulator would find it strictly optimal to intervene, and could induce funding for all borrowers by offering a subsidy of size ϵ.

A.7 Proof of Proposition 5

Let λ_m be such that $\mu^*(\lambda_m) \in (0,1)$. For partial effort to be consistent with PCE, it must be the case that (i) $\Delta L > \Delta R$, (ii) $\Gamma(\lambda_m, \mu^*) = \bar{\lambda}$, and (iii) that the incentive constraint is violated at $\mu(\lambda_m) = 1$. (If the incentive constraint were weakly satisfied at given $\mu(\lambda_m) = 1$, full effort would strictly Pareto-dominate $\mu^*(\lambda_m)$). It then follows that the IC constraint at λ_m is

$$\phi^*(\bar{\lambda})\Delta R + (1-\phi^*(\bar{\lambda}))\Delta L + \beta \Delta V(\bar{\lambda}) = \frac{c}{\pi}$$

for some $\phi^*(\bar{\lambda}) < 1$. $\Delta u(\bar{\lambda}) = \Delta R < \phi^*(\bar{\lambda})\Delta R + (1-\phi^*(\bar{\lambda}))\Delta L$ under the optimal subsidy, while continuation values are unaffected. Hence the IC constraint is violated at λ_m, and $\Gamma(\lambda_m, \tilde{\mu}) < \bar{\lambda}$. Since payoffs are constant under the subsidy on $(\underline{\lambda}, \bar{\lambda})$, the IC constraint is violated for all $\lambda : \Gamma(\lambda, 0) \geq \bar{\lambda}$, and asset quality asymptotes to $\max\{\underline{\lambda}, 0\}$.

We prove the second part of the statement by construction. Let $\underline{\lambda} \leq 0$. Consider a PCE such that $\lambda^\infty(\mu^*) = \bar{\lambda}$ and $\Gamma(\lambda_0, \mu^*) = \bar{\lambda}$ where $\mu^*(\lambda_0) < 1$. (That is, the steady state is reached within the initial period, and borrowers choose a partial effort strategy.) In any such PCE, we must have $c > \pi\Delta(\bar{\lambda}, \mu = \mathbf{1})$, $\phi^*(\bar{\lambda}) = 1$, and $\mu^*(\bar{\lambda}) = \bar{\lambda}$ by Corollary 1. Since the IC constraint holds with equality, borrowers are indifferent between effort and shirking. Hence value functions can be equivalently written *as if* borrowers always shirk. Then value functions in competitive equilibrium satisfy

$$V_b(\lambda_0) = V_b(\bar{\lambda}) = \frac{\phi^*(\bar{\lambda})R_b + (1-\phi^*(\bar{\lambda}))L_b}{1-\beta}$$

$$V_g(\lambda_0) = V_g(\bar{\lambda}) = \frac{(1-\pi)\Big[\phi^*(\bar{\lambda})R_g + (1-\phi^*(\bar{\lambda})L_g)\Big] + \pi V_b(\lambda_0)}{1-\beta(1-\pi)}.$$

The value difference and incentives in the PCE with partial effort are given by

$$\Delta V(\bar{\lambda}) = \frac{(1-\pi)\left(\phi \Delta R + (1-\phi)\Delta L\right)}{1-\beta(1-\pi)} \qquad \Rightarrow \qquad \Delta(\bar{\lambda}) = \frac{\phi \Delta R + (1-\phi)\Delta L}{1-\beta(1-\pi)}$$

Lender profits are $\Pi(\lambda_0) = \frac{1+r_f}{1-\beta}$, and the government runs no deficit. Hence welfare is

$$W^{\mathrm{PCE}}(\lambda_0) = \lambda_0 V_g(\bar{\lambda}) + (1-\lambda_0)V_b(\bar{\lambda}) + \frac{1+r_f}{1-\beta} = V_b(\bar{\lambda}) + \lambda_0 \Delta V(\bar{\lambda}) + \frac{1+r_f}{1-\beta}$$

Now consider the equilibrium given the subsidy. We have already shown that the incentive constraint is violated given $\underline{s}$. Since $\underline{\lambda} \leq 0$, the government intervenes indefinitely, and the per-period cash flow is $u_\theta = R_\theta$ for all future periods. Hence value functions under subsidy are constants and satisfy $\tilde{V}_b(\lambda_0) = \tilde{V}_b(\bar{\lambda}) = \frac{R_b}{1-\beta}$ and $\tilde{V}_g(\lambda_0) = \tilde{V}_g(\bar{\lambda}) = \frac{(1-\pi)R_g + \pi \tilde{V}_b(\lambda_0)}{1-\beta(1-\pi)}$. The difference in values and incentives under the intervention are given by $\Delta\tilde{V} = \frac{(1-\pi)\Delta R}{1-\beta(1-\pi)}$ and $\tilde{\Delta} = \frac{\Delta R}{1-\beta(1-\pi)}$. The per-period deficit is $d(\lambda') = (1-\lambda')\underline{s}(\lambda') = 1 + r_f - L_b - \lambda' \Delta L$, and the law of motion satisfies $\lambda' = (1-\pi)\lambda$. Hence the deficit satisfies the recursion

$$D(\lambda) = 1 + r_f - L_b - (1-\pi)\lambda \Delta L + \beta D((1-\pi)\lambda)$$

Since this expression is linear in λ, it can be solved in closed-form to give

$$D(\lambda_0) = \frac{1+r_f-L_b}{1-\beta} - \lambda_0 \frac{(1-\pi)\Delta L}{1-\beta(1-\pi)} > 0.$$

Since lender profits are unchanged, welfare under the optimal subsidy given $\delta \to 0$ is

$$\begin{aligned} W(\lambda_0) &= \lambda_0 \tilde{V}_g + (1-\lambda_0)\tilde{V}_b + \frac{1+r_f}{1-\beta} - D(\lambda_0) \\ &= \tilde{V}_b + \lambda_0 \Delta\tilde{V} + \frac{1+r_f}{1-\beta} - D(\lambda_0) \end{aligned}$$

Hence the welfare difference is

$$W^{\mathrm{PCE}}(\lambda_0) - W(\lambda_0) = -\frac{1-\phi}{1-\beta}(R_b - L_b) + \lambda_0 \frac{(1-\pi)(1-\phi)}{1-\beta(1-\pi)}(\Delta L - \Delta R) + D(\lambda_0)$$

In PCE, $\phi = \frac{\Delta L - \frac{c}{\pi}(1-\beta(1-\pi))}{\Delta L - \Delta R}$ and so $1-\phi = \frac{\frac{c}{\pi}(1-\beta(1-\pi)) - \Delta R}{\Delta L - \Delta R}$. Using this gives

$$W^{\text{PCE}}(\lambda_0) - W(\lambda_0) = \frac{1 + r_f - R_b + \phi(R_b - L_b)}{1-\beta} + (1-\pi)\lambda_0\left(\frac{c}{\pi} - \frac{\Delta R + \Delta L}{1-\beta(1-\pi)}\right)$$

The first term may be positive, while the second term is unambigously negative. Consider the limit as $\beta \to 1$. The limit of the second term is $\frac{1-\pi}{\pi}(c - (\Delta L + \Delta R))$, which is finite. For $R_b \leq 1$, the first term is strictly positive, and asymptotes to $+\infty$. Hence the welfare difference is positive in the limit. The conjectured PCE is feasible starting at $\lambda_0 = \bar{\lambda}$ if $c \in (\Delta u(1), \Delta L)$.

A.8 Proof of Proposition 6

Proof. Parts (i) and (ii)Part (ii) follow directly from the presence of deadweight costs δ and the fact that incentives are monotonically increasing in the subsidy for each λ'. Part (iii) follows from noting that the minimal subsidy provides the smallest incentives conditional on liquidity. Uniqueness follows from Part (ii) and because incentives are strictly increasing in the s. □

A.9 Proof of Proposition 7

Proof. $\mu^*(\lambda) \in (0,1)$ only if Δ is non-monotonic conditional on $\mu = \mathbf{1}$, and $\Gamma(\lambda, \mu^*) = \bar{\lambda}$ if $\mu^*(\lambda) \in (0,1)$. Hence $\lambda' = \Gamma(\lambda, \mathbf{1}) > \bar{\lambda}$ for any such λ, and $\Delta(\lambda') < \frac{c}{\pi}$. Hence the IC constraint is satisfied at λ' only if $\hat{s}(\lambda') > 0$. $\mu = \mathbf{1}$. The second statement is proven by construction. See Example 4 in the main text. □

B Online Appendix – Not For Publication

B.1 Example with $\beta = 0$.

We begin by studying a simple example that highlights the basic mechanisms.

Example 1 ($\beta = 0$)**.** *Let $\beta = 0$. Then borrowers' decision problem is static, and incentives are $\Delta(\lambda) = \Delta u(\lambda')$. Observe that λ' and thus $\Delta(\lambda)$, are endogenously determined by borrowers' effort. If borrowers exert effort ($\mu^* = 1$), then $\lambda' = \pi + \lambda(1-\pi)$ and*

$$\Delta(\lambda|\mu^* = 1) = \begin{cases} \Delta R + L_g - \bar{B}(\pi + \lambda(1-\pi)) & \text{if } \pi + \lambda(1-\pi) \geq \bar{\lambda} \\ \Delta L & \text{if } \pi + \lambda(1-\pi) < \bar{\lambda}. \end{cases}$$

If no borrower exerts effort ($\mu^* = 0$), then $\lambda' = \lambda(1-\pi)$ and

$$\Delta(\lambda|\mu^* = 0) = \begin{cases} \Delta R + L_g - \bar{B}(\lambda(1-\pi)) & \text{if } \lambda(1-\pi) \geq \bar{\lambda} \\ \Delta L & \text{if } \lambda(1-\pi) < \bar{\lambda}. \end{cases}$$

Since $\bar{B}(\lambda)$ is strictly increasing, incentives are weakly higher if other borrowers are believed to exert effort. This mechanism is shown in Figure 8 for the case where Δu is monotone, and in Figure 9 for the case where Δu is non-monotonic. The third panel plots incentives under the conjectured effort, with $\mu = 1$ shown in blue, and $\mu = 0$ shown in cyan. The horizontal red line depicts the right-hand side of the incentive constraint, $\frac{c}{\pi}$. A conjectured effort strategy is an equilibrium if the incentive

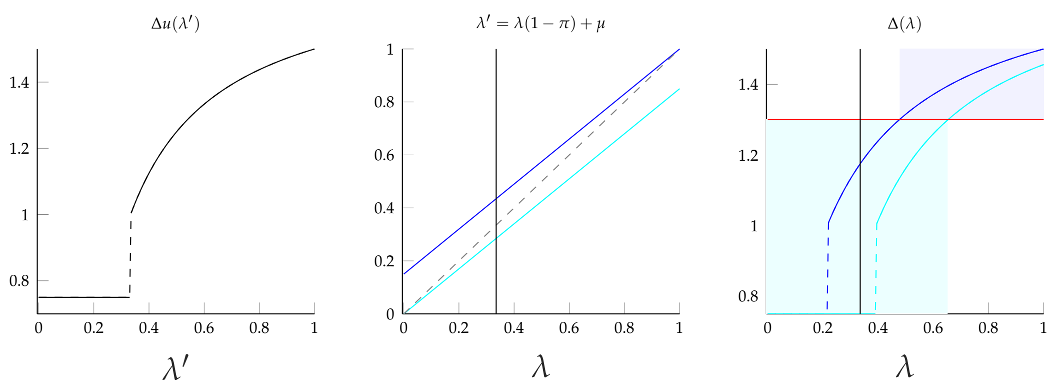

Figure 8: Monotone cash flow differences and incentives: $\Delta R \geq \Delta L$.

is satisfied (if $\mu = 1$) or violated (if $\mu = 0$). Blue and cyan shadings represent values of λ for which $\mu = 1$ and $\mu = 0$ are consistent with equilibrium. There are regions in which the effort externality

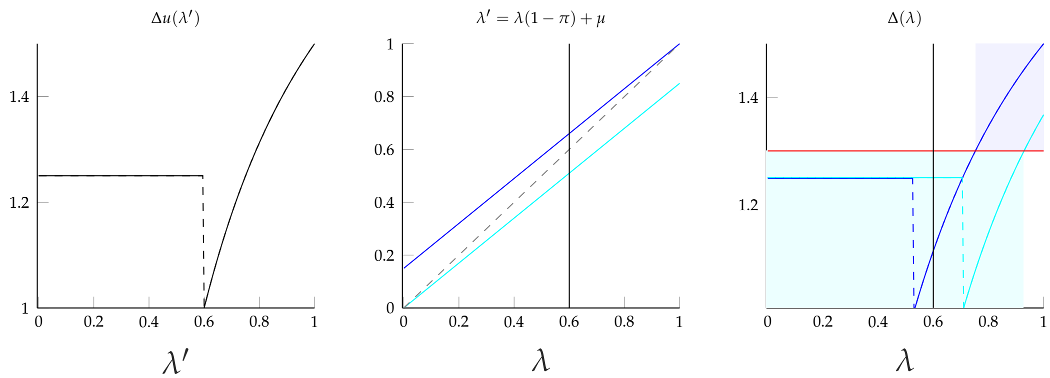

Figure 9: Non-monotone cash flow differences and incentives: $\Delta L > \Delta R$.

leads to self-fulfilling incentives: if all agents exert effort, asset quality rises sufficiently to make effort worthwhile; if asset quality is expected to fall, shirking is privately optimal. Effort equilibria may fail to exist if λ_0 is too small.

If $\Delta(\lambda)$ is non-monotonic and $c < \Delta L$, no pure strategy equilibrium may exist near the liquidity threshold. In particular, it may be privately optimal to exert effort if markets are expected to be illiquid, and optimal to shirk if asset quality is expected to be just above $\bar{\lambda}$ and borrowing costs are expected to be high. In this case, the equilibrium requires partial effort effort μ and lenders' acceptance probability such that $\lambda' = \bar{\lambda}$ and $\Delta(\lambda) = \frac{c}{\pi}$. That is, $\mu^* = \bar{\lambda}$ and $\phi^*\Delta R + (1-\phi^*)\Delta L = \frac{c}{\pi}$. We depict this equilibrium in **thick green** *in Figure 10.*

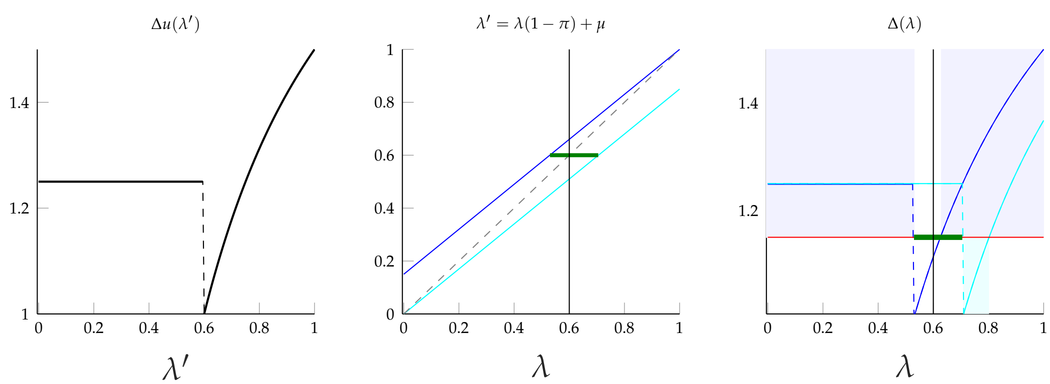

Figure 10: Non-existence of pure strategy equilibrium

B.2 Equilibrium Multiplicity with $\beta > 0$

The following example shows how expectations about future asset quality shape incentives by constructing three different equilibria for the same fundamentals. In the first equilibrium, stagnation is self-fulfilling because borrowers cannot be sure they will obtain funding, lowering the value of owning a good asset (and consequently the private value of effort). In the second equilibrium, all borrowers exert effort initially, but the belief that asset quality will fall back to $\bar{\lambda}$ (leading to an increase in borrowing costs) is self-fulfilling – as a consequence, the economy cycles between a lower and higher average asset quality forever. In the third equilibrium, the belief that all borrowers will always exert effort provides incentives to exert effort – as a result, asset quality continually improves. In all three cases, the mechanism that sustains the belief is that the equilibrium interest rate is decreasing in the fraction of good borrowers so that the incentive to become a good borrower is increasing in average asset quality.

Example 2 (Multiplicity with $\beta > 0$**).** *Let $L_b = \pi = \beta = \frac{1}{2}$, and $L_g = \frac{3}{2}$, so that $\bar{\lambda} = \frac{1}{2}$. Let $R_b = 1$, $R_g = \frac{5}{3}$, and $c = \frac{22}{45}$. If $\lambda_0 = \frac{1}{2}$, the following three equilibria exist:*

1. *An equilibrium in which average asset quality remains at $\bar{\lambda} = \frac{1}{2}$ forever, $\mu\left(\frac{1}{2}\right) = \frac{1}{2}$, and only a fraction $\phi(\frac{1}{2}) = \frac{4}{5}$ of farmers obtains funding in every period.*
2. *A cyclical equilibrium in which average asset quality jumps back and forth between $\lambda_0 = \frac{1}{2}$ and $\lambda_1 = \frac{3}{4}$ forever, all farmers always obtain funding, and $\mu\left(\frac{1}{2}\right) = 1$ and $\mu\left(\frac{3}{4}\right) = \frac{1}{4}$.*
3. *An equilibrium in which $\mu(\lambda) = \phi(\lambda) = 1$ for all $\lambda \geq \lambda_0$ and asset quality converges to* 1.

Since equilibrium effort strategies are privately optimal and value functions are increasing in θ by Lemma 3, it is easy to verify that the third equilibrium generates strictly higher values for all borrowers than the first two.

Construction. We first construct a cyclical equilibrium with duration two where asset quality moves from some $\lambda_0 \geq \frac{1}{2} \geq \bar{\lambda}$ to λ_1 and back to λ_0 indefinitely. Conjecture and verify that $\mu(\lambda_0) = 1$ and $\mu(\lambda_1) = \mu^* \in (0,1)$. Then $\lambda_1 = \pi + \lambda_0(1-\pi)$. To return to λ_0 in the second period, μ^* must solve $\lambda_0 = \lambda_1(1-\pi) + \mu^*\pi = (\pi + \lambda_0(1-\pi))(1-\pi) + \mu^*\pi$. Hence $\mu^* = \lambda_0(2-\pi) - (1-\pi)$ and $\mu^* > 0$ for all $\pi \in (0,1)$ if and only if $\lambda \geq 0.5$. Define $B^0 = \frac{1+r_f-(1-\lambda_0)L_b}{\lambda_0}$ and $B^1 = \frac{1+r_f-(1-\lambda_1)L_b}{\lambda_1}$. Let superscript t denote dependence on λ_t.

Then value functions satisfy

$$V_g^0 = R_g + L_g - B^1 + \beta V_g^1 - c \tag{62}$$

$$V_g^1 = \Big(1 - \pi(1-\mu^*)\Big)\Big[R_g + L_g - B^0 + \beta V_g^0\Big] + \pi(1-\mu^*)\Big[R_b + \beta V_b^0\Big] - \mu^* c \tag{63}$$

$$V_b^0 = \pi\Big[R_g + L_g - B^1 + \beta V_g^1\Big] + (1-\pi)\Big[R_b + \beta V_b^1\Big] - c \tag{64}$$

$$V_b^1 = \mu^*\pi\Big[R_g + L_g - B^0 + \beta V_g^0\Big] + (1-\mu^*\pi)\Big[R_b + \beta V_b^0\Big] - \mu^* c \tag{65}$$

Define the following four constants denoting expected cash flows net of costs: $\tilde{u}_g^0 = R_g + L_g - B^1 - c$, $\tilde{u}_g^1 = \Big(1 - \pi(1-\mu^*)\Big)\Big[R_g + L_g - B^0\Big] + \pi(1-\mu^*)R_b - \mu^* c$, $\tilde{u}_b^0 = \pi\Big[R_g + L_g - B^1\Big] + (1-\pi)R_b - c$, and $\tilde{u}_b^1 = \mu^*\pi\Big[R_g + L_g - B^0\Big] + (1-\mu^*\pi)R_b - \mu^* c$. Then $V_g^0 = \tilde{u}_g^0 + \beta V_g^1$, $V_g^1 = \tilde{u}_g^1 + \beta V_g^0 - \beta\pi(1-\mu^*)\Delta V^0$, $V_b^0 = \tilde{u}_b^0 + \beta V_b^1 + \beta\pi\Delta V^1$, and $V_b^1 = \tilde{u}_b^1 + \beta V_b^0 + \beta\mu^*\pi\Delta V^0$. Differencing $V_g^0 - V_b^0$ and $V_g^1 - V_b^1$ and substituting gives

$$\Delta V^0 = \frac{\tilde{u}_g^0 - \tilde{u}_b^0 + \beta(1-\pi)\left(\tilde{u}_g^1 - \tilde{u}_b^1\right)}{1-\beta^2(1-\pi)^2} \qquad \text{and} \qquad \Delta V^1 = \frac{\tilde{u}_g^1 - \tilde{u}_b^1 + \beta(1-\pi)\left(\tilde{u}_g^0 - \tilde{u}_b^0\right)}{1-\beta^2(1-\pi)^2}$$

By definition of the constants, $\tilde{u}_g^0 - \tilde{u}_b^0 = (1-\pi)\Big(\Delta R + L_g - B^1\Big)$ and $\tilde{u}_g^1 - \tilde{u}_b^1 = (1-\pi)\Big(\Delta R + L_g - B^0\Big)$. Hence $\tilde{u}_g^0 - \tilde{u}_b^0 > \tilde{u}_g^1 - \tilde{u}_b^1$. Next verify the incentive constraint. By Lemma 2 we have

$$\Delta^0 = \Big(\Delta R + L_g - B^1\Big) + \beta\Delta V^1 \qquad \text{and} \qquad \Delta^1 = \Big(\Delta R + L_g - B^0\Big) + \beta\Delta V^0.$$

For our construction to be valid, we need $\frac{c}{\pi} = \Delta^1 < \Delta^0$. Rearranging gives

$$\Delta^0 = \frac{\Big(\Delta R + L_g - B^1\Big) + (1-\pi)\beta\Big(\Delta R + L_g - B^0\Big)}{1-\beta^2(1-\pi)^2}$$

and

$$\Delta^1 = \frac{\Big(\Delta R + L_g - B^0\Big) + \beta(1-\pi)\Big(\Delta R + L_g - B^1\Big)}{1-\beta^2(1-\pi)^2}$$

Since $B^1 < B^0$, we have that $\Delta^0 > \Delta^1$ if $\beta, \pi \in (0,1)$, as is necessary. Hence the construction is valid if c is such that $\frac{c}{\pi} = \Delta^1$. Plugging in the stated parameters then gives the desired result.

To construct the equilibrium in which asset quality remains at $\bar{\lambda}$ indefinitely, observe that incentives with constant asset quality satisfy

$$\Delta(\bar{\lambda}) = \frac{\phi \Delta R + (1-\phi)\Delta L}{1-\beta(1-\pi)}$$

For asset quality to remain constant, we must have $\mu(\bar{\lambda}) = \bar{\lambda}$. Partial effort is determined such that the IC constraint to hold with equality. Given c, we can then solve for ϕ such that $\Delta(\bar{\lambda}) = \frac{c}{\pi}$. The stated result then follows.

The construction of the equilibrium with $\mu(\lambda) = 1$ follows directly from Lemma 3 and observing that incentives must be strictly higher than in the cyclical equilibrium at $\lambda = \bar{\lambda}$.

B.3 Policy Interventions in Practice

In recent years, regulatory authorities have increasingly relied on asset purchase programs and liquidity injections to stimulate economic activity and rejuvenate lending markets in times of stress. These programs, which initially focused on short-term purchases of government debt, have recently been expanded to include long-running purchases of broader set of privately-produced financial assets, such as corporate bonds, mortgage-backed securities, or bank-originated asset-backed securities. The European Central Bank began exchanging investment grade corporate bonds for government debt under the Eurosystem's Corporate Sector Purchase Programme starting in 2016. Other recent policies, including the Feds Term Securities Lending Facility (stopped in 2010) and the ECBs Securities Lending Programme (continued), were explicitly aimed at upgrading the collateral available to private borrowers by providing downside insurance to lenders or by engaging in swaps for low-quality collateral prone to adverse selection.

B.4 Example: Interventions Strictly Lower Output and Welfare

The following example illustrates Proposition 5 by providing parameters under which a regulator with limited commitment strictly reduces output and welfare relative to the PCE *in every period.* This results from the fact that the ex-post optimal subsidy destroys effort incentives, lowering asset quality period-by-period and forcing the regulator to intervene time and again. As a result, asset quality continues to fall and the deficit increases over time.

Example 3. *Let* $\lambda_0 \in \left(\bar{\lambda}^{-1}(\mathbf{1}), \bar{\lambda}\right)$ *and assume that* $\frac{c}{\pi} \in \left(\Delta(\bar{\lambda}^{-1}(\mathbf{1})), \Delta(\bar{\lambda})\right)$. *By Corollary 1, the unique PCE is then such that* $\mu^*(\lambda_0) \in (0,1)$, $\Gamma(\lambda_0, \mu^*) = \bar{\lambda}$, $\phi^*(\bar{\lambda}) = \frac{\Delta L - \Delta R + \Delta(\bar{\lambda}^{-1}(\mathbf{1})) - \frac{c}{\pi}}{\Delta L - \Delta R}$, *and* $\mu^*(\lambda) = 1$ *for all* $\lambda \geq \bar{\lambda}$. *That is, borrowers exert partial effort so that asset quality reaches* $\bar{\lambda}$ *in the initial period, supported by a lending market equilibrium in which only a fraction of borrowers receive funding given* $\bar{\lambda}$. *From then on, all borrowers exert effort and receive funding in every period, and asset quality converges to 1. Under the ex-post optimal subsidy, the cash flow is* $\tilde{u}_\theta = R_\theta$ *forever. Hence, incentives under the subsidy are* $\tilde{\Delta} = \frac{\Delta R}{1-\beta(1-\pi)} \leq \Delta(\bar{\lambda}^{-1}(\mathbf{1}))$, *borrowers do not exert effort in any period, asset quality asymptotes to zero, and the government provides a strictly positive subsidy in every period. Now consider the limit as* $c \to \pi\Delta(\bar{\lambda}^{-1}(\mathbf{1}))$. *Then* $\phi^*(\bar{\lambda}) \to 1$, *and so* $u_\theta \to R_\theta$ *in the initial period of the PCE. It follows that the PCE generates approximately the same cash flows as under the subsidy in the initial period, strictly higher cash flows in every period thereafter, and does not require the regulator to run a deficit in any period. Consequently, the PCE generates strictly higher welfare than under the ex-post optimal subsidy for any* β.

B.5 Example: Subsidy with Infinitesimal Cost

Example 4 (Example 3 continued). *Consider the same equilibrium as in Example 3, and let the initial condition be as small as possible,* $\lambda_0 \to \frac{\bar{\lambda}-\pi}{1-\pi} = \Gamma^1(\bar{\lambda}, \mathbf{1})$. *If all borrowers were to exert effort, then* $\lambda_1 = \bar{\lambda} + \epsilon$ *for* ϵ *small and positive. Since* $\frac{c}{\pi} > \Delta(\bar{\lambda}^{-1}(\mathbf{1}))$, *not all borrowers exert effort at* λ_0 *in the unique PCE. Guess and verify that* $\hat{s}(\lambda_1) > 0$ *and* $\hat{s}(\lambda) = 0$ *for all* $\lambda \geq \lambda_1$. *By Proposition* **??**, *the subsidy rule* $\hat{s}(\lambda_1)$ *must then satisfy* $\Delta(\lambda_0) + \frac{\lambda_1}{1-\lambda_1}\hat{s}(\lambda_1) = \frac{c}{\pi}$, *where* $\Delta(\lambda_0)$ *denotes incentives at* λ_0 *in PCE. As in Example 3, now consider the limit* $c \to \pi\Delta(\bar{\lambda}^{-1}(\mathbf{1}))$. *Then* $\Delta(\lambda_0) \to \frac{c}{\pi}$, *which implies that* $\hat{s}(\lambda_1) \to 0$. *Since* $\frac{c}{\pi} \leq \Delta(\bar{\lambda})$, *it follows that* $\frac{c}{\pi} < \Delta(\lambda_1)$, *and all borrowers exert effort in every period. No subsidies are required in any period but the initial one.*

B.6 Extension to General Cost Functions

In this extension, we consider the robustness of our results to general convex cost functions. Since the effort problem is symmetric, take as given that optimal effort is also symmetric across types. The first-order condition for an interior optimum is

$$\pi\Big[\Delta u(\lambda') + \beta\Delta V(\lambda')\Big] = C'(\mu) \tag{66}$$

where $\lambda' = \lambda(1-\pi) + \pi \cdot \mu$. Relative to the benchmark model with linear costs, convex costs introduce two additional channels. The first is technical: since both the left-hand side and right-hand side of (66) depend on μ, there is an additional fixed point that needs to be solved for in every period in addition to the value function iteration. While this introduces no substantial computational complexity, it obscures the basic patterns of substitutability and complementarity that is the focus of our paper. In particular, in the baseline model we could guess and verify that $\mu = 1$, $\mu = 0$, or that the IC constraint holds with equality. With convex costs, we would have to guess and verify whether asset quality is increasing or decreasing, where both the costs and benefits are now a function of the current state.

The second is more substantive: since asset quality grows if only if $\mu \geq \lambda$, sustained growth requires increasing effort over time. Sufficiently convex costs of effort may therefore reduce the scope for high long-run asset quality. We provide two examples to argue that the basic mechanisms remain even with convex costs. First, we consider the $\beta = 0$ with simple quadratic costs. This a useful benchmark because the shape of the general incentive function Δ is inherited from the per-period cash flow difference Δu (see Lemma 3 and Example 1 in Online Appendix B.1). We show that the same non-monotonicity around the liquidity threshold obtains as in the baseline model; hence the fundamental model dynamics are unchanged. They key difference is that increasing marginal costs make it more difficult to sustain high levels of effort. Hence asset quality may not converge to 1, but rather to an interior steady state $\lambda^{ss} \in (\bar{\lambda}, 1]$. However, this does not change the fundamental policy implication that the feedback of interventions to asset quality induces larger subsidies that must be offered even if the market is liquid on its own.

Example 5 ($\beta = 0$ and $C(\mu) = \frac{c}{2}\mu^2$)**.** *If the market is expected to be illiquid, the interior solution is $\mu^* = \frac{\Delta L}{\gamma}$. If the market is expected to be liquid, an interior optimum satisfies $\gamma\mu = \Delta L + \Delta R - \frac{1+r_f-L_b}{\lambda(1-\pi)+\pi\mu}$. The solution is*

$$\mu^* = \frac{\Delta L + \Delta R - c\lambda(1-\pi) + \sqrt{\Big(\Delta L + \Delta R - c\lambda(1-\pi)\Big)^2 + 4c\Big(\lambda(1-\pi)(\Delta L + \Delta R) - (1+r_f) + L_b\Big)}}{2c}$$

Figure 11 plots the implied evolution of asset quality as a function of the λ for parameters that deliver the non-monotonicity of incentives at $\bar{\lambda}$. The left panel plots optimal effort conditional on guessing that the market is illiquid and liquid, respectively. The implied law of motion is on the right-hand side. An equilibrium requires that the conjectured degree of market liquidity is verified. The upper steady state is given by the intersection of the green line and the 45 degree line. In this example, the model would therefore again feature inefficient convergence to $\bar{\lambda}$ from poor initial conditions.

In the second example, we introduce *aggregate* returns to scale in effort by considering cost functions of the form $C(\mu, \lambda)$ with C convex in μ and decreasing in λ. Such a cost structure could be motivated by information spillovers among borrowers in identifying high-quality collateral, or by a complementarity among real assets. In the context of mort-

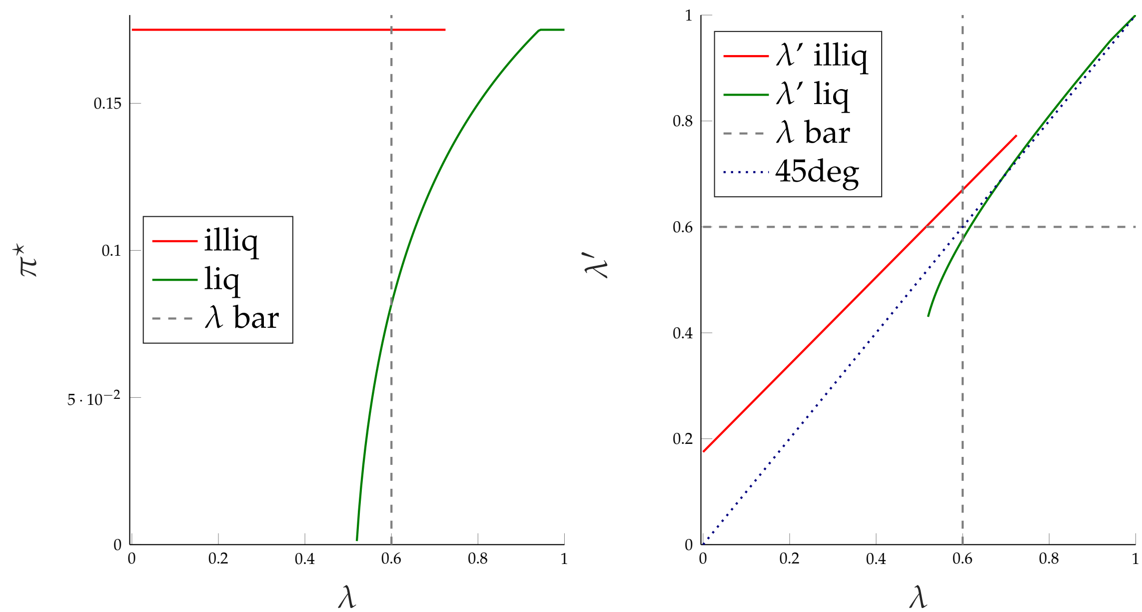

Figure 11: Non-monotonicity of incentives and the equilibrium law of motion around $\bar{\lambda}$. Parameters: $R_g = 2$, $R_b = 1$, $L_g = 1.5$, $L_b = 0.25$, $\beta = 0.5$, $\pi = 0.15$, $r_f = 0$, $c = 1.15$.

gages, for example, it is well-known that defaults are contagious. A higher percentage of good borrowers may thus make it easier to originate additional mortgages with low default risk.

Example 6 (Returns to Scale)**.** *Assume that the cost function is of the form* $C(\mu, \lambda) = \frac{c\mu^2}{2\gamma(1+\lambda)}$. *Then the first-order condition for an interior optimum is*

$$\pi\left[\Delta u(\lambda') + \beta \Delta V(\lambda')\right] = \frac{c\mu}{\gamma(1+\lambda)}$$

Then μ^* *is strictly increasing in* λ*, and* $\mu^* \geq \lambda$ *for all* $\lambda \geq 1$ *if* γ *is sufficiently large. Hence aggregate economies of scale circumvent the result that it becomes increasingly difficult to increase growth as* λ *grows large, and permits convergence of* λ *to 1.*