When Does Determinacy Imply Expectational Stability?*

James Bullard†
Federal Reserve Bank of St. Louis

Stefano Eusepi‡
Federal Reserve Bank of New York

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Abstract

Since the introduction of rational expectations there have been issues with multiple equilibria and equilibrium selection. We study the connections between determinacy of rational expectations equilibrium and learnability of that equilibrium in a general class of purely forward-looking models. Our framework is sufficiently flexible to encompass lags in agents' information and either finite horizon or infinite horizon approaches to learning. We are able to isolate conditions under which determinacy does and does not imply learnability, and also conditions under which long-horizon forecasts make a clear difference for learnability. Finally, we apply our result to a relatively general New Keynesian model.

Keywords: Long-horizon expectations, multiple equilibria, learnability of equilibrium, E-stability.

JEL codes: E3, E4, D8.

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†Email: bullard@stls.frb.org. Any views expressed are those of the authors and do not necessarily reflect the views of the Federal Reserve Bank of St. Louis, the Federal Reserve Bank of New York, or the Federal Reserve System.

‡Email: stefano.eusepi@ny.frb.org

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1 Introduction

Since the introduction of rational expectations there have been issues with multiple equilibria and equilibrium selection. One generally accepted selection criterion appeals to the notion of equilibrium determinacy, advanced by Blanchard and Kahn (1980). This approach advocates selecting equilibria that are saddlepath stable and deliver a unique bounded rational expectations equilibrium. An alternative approach to equilibrium selection has been advanced by the adaptive learning literature, in particular by Marcet and Sargent (1989) and Evans and Honkapohja (2001). This approach replaces rational expectations with an econometric model whose reduced form is consistent with the equilibrium law of motion. An equilibrium is expectationally stable (or E-stable) if it is attainable as the limiting outcome of a learning process where the parameters of the model are updated in real time.[1]

In this paper we study the connection between determinacy and E-stability in a class of infinite-horizon dynamic stochastic general equilibrium (DSGE) models. A number of authors have studied the connection between determinacy and E-stability. For example, if there was an equivalence between E-stability and determinacy, then there would be a powerful argument for focusing attention on determinate models. Conversely, if for a large class of economic models determinacy does not imply E-stability, then "robust" rational expectations equilibria should satisfy both determinacy and E-stability conditions. A recent debate between Cochrane (2009) and McCallum (2009b) on equilibrium selection in New Keynesian models indicates that the relationship between determinacy and learnability is still an open issue. On the one hand, Cochrane (2009) argues that

[1] We use the terms *expectational stability*, *E-stability*, and *learnability* interchangably in this paper. The connections between the expectational stability condition and local convergence of systems under real time recursive learning are discussed extensively in Evans and Honkapohja (2001).

determinacy cannot rule out explosive equilibria, while learning cannot select bounded equilibria. On the other hand McCallum (2007, 2009a) argues that learning in the sense of Evans and Honkapohja (2001) can be used to rule out explosive equilibria and will generally select a determinate equilibrium.

In the literature there is no general presumption that determinacy implies learnability. An examination of the "general linear model" case in Evans and Honkapohja (2001) makes it plain that determinacy does not imply learnability. Nevertheless, the discussion in Woodford (2003a, 2003b) as well as in Bullard and Mitra (2002) highlights cases where the conditions for determinacy of equilibrium are the same as the conditions for expectational stability.[2] There appears to be a close relationship between the two criteria in many applications, and this is a puzzle we would like to help resolve. In particular, we would like to better understand the nature of the relationship between determinacy and learnability in economic terms.

Two approaches have emerged to the analysis of learning dynamics and E-stability in infinite-horizon DSGE models. The first approach, discussed in Evans and Honkapohja (2001), uses first order conditions from agents' dynamic optimization as decision rules[3] under learning. In this paper we refer to this approach as 'learning under *Finite-Horizon* (FH) decision rules', given that the models' first order conditions can often be expressed in terms of one-step-ahead forecasts. Preston (2005, 2006) notes, however, that such a model of learning dynamics does not share the same microfoundations as the model under rational expectations, where agents make plans with an infinite horizion.[4] We refer to this approach as 'learning under *Infinite-Horizon* (IH) decision rules'.

[2] Woodford (2003a, p. 1180) states, "Thus both criteria ... amount in this case to a property of the eigenvalues of [a matrix] A, and the conditions required for satisfaction of both criteria are related, though not identical."

[3] This is also known as the *Euler equation* approach.

[4] See also Sargent (1993) and Woodford (2003b) for further discussion.

These alternative approaches to the study of E-stability can, but do not always, change the conclusions one would draw concerning the learnability of a particular rational expectations equilibrium. We would like to understand more about the economics of the relationship between determinacy and learnability under both finite-horizon and infinite-horizon decision rules.

To meet these aims, we consider a general class of purely forward-looking models where, in equilibrium, current endogenous variables are determined by infinite horizon expectations (as in most microfounded macroeconomic models). We then investigate when the conditions for determinacy are the same as the conditions for stability under learning. We also investigate what aspects of the economic model might cause learnability and determinacy to be governed by differing conditions. To further understand these economic aspects we apply our general results to a fairly general version of the New Keynesian (NK) macroeconomic model.

Our work is closely related to both McCallum (2007) and Ellison and Pearlman (2011). These authors explore the connection between E-stability and determinacy in a general class of models and isolate conditions under which determinacy implies E-stability. However, they do not consider learning under infinite horizon decision rules. Another point of departure relative to this previous research is the treatment of information available to the model's agents when they learn. This is discussed in detail in section 3.1.

In the next section we illustrate our framework using a simple model of consumption determination. In section 3 we develop the general infinite-horizon model and review and characterize the E-stability conditions. We then discuss the propositions indicating when determinacy and learnability conditions will coincide, and when they will not. In section 4 we discuss the application to the microfounded New Keynesian model. We summarize our findings in the

concluding section, and we also include two appendices to the paper which contain details about the general model and proofs of the propositions.

2 A simple model of consumption

In this paper we discuss equilibrium uniqueness and stability under adaptive learning in a class of economic models where agents make decisions with *an infinite planning horizon*. We begin with a simple model of consumption in order to introduce the framework and to preview the results from the general model. There is a continuum of measure-zero households and each agent i maximizes her intertemporal utility function in consumption (C_t^i)

$$\mathbb{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \ln C_T^i$$

subject to

$$A_t^i + C_t^i = (1 + r_t)A_{t-1}^i + y$$

and the transversality condition $\lim_{T\to\infty} \mathbb{E}_t^i \beta^T \frac{C_T}{C_{T+1}} A_T = 0$. Each household shares the same discount factor $0 < \beta < 1$, r_t is the market risk-free real interest rate, A_t^i is the agent's holding of the risk-free asset (in zero net supply) and y is an exogenous income process, assumed to be constant and identical across households. The operator $\mathbb{E}_t$ defines expectations and is discussed in detail in the next section. The problem's first order conditions yield the familiar Euler equation for consumption

$$\mathbb{E}_t^i \frac{C_{t+1}^i}{C_t^i} = \beta(1 + r_t).$$

Under rational expectations the unique equilibrium implies that the interest rate is constant and equal to the inverse of households' discount factor. Assuming for

simplicity that initial asset holdings for each household i are zero, consumption in equilibrium is equal to the endowment y.

2.1 Decision rules under learning

When analyzing learning, we assume that agents do not have rational expectations at the outset but form expectations using adaptive learning rules. The methodology to analyze the convergence properties of the agent's learning process has been developed by Marcet and Sargent (1989) and Evans and Honkapohja (2001). As mentioned in the introduction Evans and Honkapohja (2001) and Preston (2005) present alternative approaches to study the model's dynamics under learning. The two approaches share the same *learning rule* (based on adaptive learning algorithms) and thus the same methodology to evaluate the convergence properties of the learning process. However, they differ in the assumed *decision rules* (or behavioral rules) used by the model's agents.

Infinite horizon decision rule. The infinite horizon (IH) decision rule, here expressed in log-deviations from its steady state,[5]

$$\hat{C}_t^i = -\mathbb{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t+1} \hat{r}_T. \tag{1}$$

is obtained by combining the Euler equation with the intertemporal budget constraint: the familiar permanent income theory of consumption determination. According to this decision rule consumption deviations from steady state depend on the expected evolution of the real interest rate[6]. Examples of models that incorporate IH decision rules under learning are Sargent (1993), which uses the same approach to study learning in a firm's investment problem with ad-

[5] That is, for a variable $X_t : \hat{X}_t = \ln X_t - \ln \bar{X}$, where $\bar{X}$ denotes its steady state value.

[6] Notice that current asset holdings and the income process do not enter the decision rule because current asset holding are assumed to be zero and the income process is a constant.

justment costs; Preston (2005) which studies the New Keynesian model, and Eusepi and Preston (2011) which focuses on a real business cycle (RBC) model.

Finite horizon decision rule. Under the *finite horizon* (FH) approach[7] the agents' Euler equation is assumed to be the behavioral rule for consumption decisions

$$\hat{C}_t^i = -\hat{r}_t + \mathbb{E}_t^i \hat{C}_{t+1}^i \tag{2}$$

so that current consumption is determined by both the current interest rate and the expected future consumption in the next period. More generally, the FH or Euler equation approach uses the first-order conditions of the agents' maximization[8] as behavioral rules. Examples include Evans and Honkapohja (2001), Chapter 10, which studies the standard RBC model, Bullard and Mitra (2002), documenting E-stability conditions in the New Keynesian model and Slobodyan and Wouters (2012), introducing learning in an estimated medium-scale DSGE model.[9]

Quasi-differencing. As discussed in Evans, Honkapohja and Mitra (2013) and Preston (2005), a useful way to link (1) and (2) is to "quasi differencing" (1). That is, leading (1) one period ahead and applying the expectation operator gives

$$\hat{E}_t \hat{C}_{t+1}^i = -\mathbb{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t+1} \hat{r}_{T+1} \tag{3}$$

which, combined with (1) and after re-arranging gives (2). We regard this link between the two decision rules as a useful mathematical formulation enabling us to compare their properties in the general class of models that we introduce

[7] See for example Evans, Honkapohja and Mitra (2013).

[8] In more complex models the Euler equations are combined with static equilibrium conditions, leading to a lower dimensional model. See for example the treatment of the RBC model in Evans and Honkapohja (2011).

[9] For a survey of the recent contributions in the literature, see Evans and Honkapohja (2013).

in next section. Under rational expectations, where consumption depends on the expected path of the real interest rate, the two decision rules yield the same optimal consumption decision. Hence, under rational expectations the model's properties are the same regardless of the FH or IH representation. However, this is generally not true under learning:[10] the two decision rules can deliver different convergence properties under learning dynamics. It is important to stress that neither of the two decision rules is fully optimal under learning. According to the IH decision rule agents revise their consumption plan every period as if they will never change the plan, even though that is exactly what they do. Under the FH decision rule, agents forecast their own decision variables and ignore their intertemporal budget constraint. Evans, Honkapohja and Mitra (2013), Preston (2005) and Eusepi and Preston (2011) discuss more broadly the relative merits and limits of these two approaches.

2.2 E-stability

Agents are endowed with a perceived law of motion (PLM) for Y_t^i, the variable they need to forecast, where $Y_t^{\underline{FH}} = \hat{C}_t^i$ (FH learning) and $Y_t^{\underline{IH}} = \hat{r}_t$ (IH learning). Throughout this paper we focus on a PLM which is consistent with the Minimum State Variable (MSV) solution of the model: here it takes the form

$$Y_t^j = a, \quad j = \underline{FH}, \underline{IH} \tag{4}$$

where a denotes a constant. Agents use (4) to form expectations. Substituting agents' forecasts in the decision rules, and using the market clearing condition that consumption needs to be equal to the endowment y (so that $\hat{C}_t^i = 0$) we obtain the actual law of motion (ALM) of the equilibrium interest rate, $\hat{r}_t$, and

[10] In particular, (3) only holds under rational expectations.

consumption under learning dynamics. This is defined as

$$Y_t^j = T(a)$$

where the form of $T(a)$ depends on the specific decision rule used. A mapping can be then defined between the PLM, defined as a, and the ALM, defined as $T(a)$. At the rational expectations equilibrium (REE), the PLM and ALM coincide, that is $T(a) = a = 0$. The notion of E-stability describes the interaction between PLM and ALM which arises during the learning process, where $T(a) \neq a$. The mapping captures the self-referential aspect of learning dynamics. Changes in beliefs (agents' PLM) affect consumption/saving decisions and thus the equilibrium interest rates (the ALM). This in turn has an impact on beliefs as agents update their PLM by observing recent consumption and interest rate outcomes. This interaction between PLM and ALM is captured by the following differential equation

$$\frac{d}{d\tau}a = T(a) - a \tag{5}$$

where τ denotes a measure of "artificial" time. The rational expectations equilibrium is defined to be E-stable if equation (5), evaluated at the REE, is locally stable –that is, if the PLM converges to the ALM. The ODE intuitively describes a stylized learning rule under which the parameter a is adjusted towards its REE value. In this simple model E-stability depends on the sign of $T'(a) - 1$, where $T'(a)$ denotes the first derivative of $T(a)$. Evans and Honkapohja (2001) discuss the tight connection between the convergence properties of real time learning algorithms, such as recursive least squares, and the concept of E-stability.

It is easy to verify that under the $\underline{\text{IH}}$ decision rule the E-stability condition

is

$$d\left(T\left(a\right)-a\right)/da=-\frac{1}{1-\beta}$$

so that the REE is always stable. The intuition is that if agents expect interest rates to be higher than in steady state (the REE), they would want to lower their consumption, thus increasing the supply of saving. This drives down the equilibrium real interest rate, inducing the desired correction in agents expectations about the real rate until convergence.

Under the FH decision rule the E-stability condition is simply

$$d\left(T\left(a\right)-a\right)/da=-1$$

so that, again, the REE is always stable. Actual consumption is always constant and equal to the endowment, independently of the interest rate: there are no feedbacks from the interest rate to expected consumption (i.e. $T(a)=0$). As a result, expected consumption and the real interest rate will revert back to their steady state equilibrium values.

The 'one-dimensional' model of consumption described above conveys the main intuition for the general proposition to follow. Finite and infinite horizon decisions rules yield the same stability conditions because the '*direction*' in the adjustment of expectations is the same. The only difference here is in the '*size*' of the adjustment, which depends on the agents' discount rate β. The IH model displays a $1/(1-\beta)$ times faster adjustment as changes in the PLM imply revisions in the whole forecast path in the infinite horizon. In the next section we discuss this finding in a general class of models.

3 The "general model"

In this section, we follow Evans and Honkapohja (2001) and think in terms of a general linear model.[11] We consider a class of infinite horizon purely forward-looking models. As in the simple model of consumption, we exploit the connection between FH and IH through quasi-differencing the IH decision rule. We first define a class of IH decision rules generalizing (1)

$$A_0 Y_t = A_1 \mathbb{E}_{t-\ell} Y_t + \sum_{d=1}^{n} A_{2,d} \mathbb{E}_{t-\ell} \sum_{T=t}^{\infty} \beta_d^{T-t} Y_{T+1} + \sum_{d=1}^{n} A_{3,d} \mathbb{E}_{t-\ell} \sum_{T=t}^{\infty} \beta_d^{T-t} X_T, \quad (6)$$

where Y_t denotes an $n-$dimensional vector of endogenous variables and X_t denotes a $k-$dimensional vector of shocks which evolve according to

$$X_t = H X_{t-1} + \epsilon_t. \quad (7)$$

The matrix H is assumed to be diagonal with elements $0 \leq h_{i,i} < 1$, for $i = 1...k$.

We assume that $A_{2,d}$ and $A_{3,d}$, (i) must have nonzero elements in their d^{th} row and, (ii) must have only zero elements in the remaining rows. This amounts to an assumption that each equation uses only one "discount factor" and that each equation must have at least one "discount factor" (possibly zero).[12] Any "discount factors" in (6) need not be discount factors in the traditional sense of economic theory, but may instead capture reduced-form discounting of future variables in the equilibrium dynamics of the model.[13]

[11] We relate the results from this section to a New Keynesian DSGE model in the next section.

[12] We stress that even with a discount factor of zero, say $\beta_d = 0$, the dth row in $A_{2,d}$ would still have nonzero elements.

[13] We thank the editor, Jesus Fernandez-Villaverde, for pointing this out.

3.1 Timing of expectations and informational delays

A key aspect of our approach is that we allow for differing information sets to be available to agents at the time expectations are formed and decisions are made. To accomplish this, we use the operator $\mathbb{E}_{t-\ell}$, where $\ell \geq 0$, which we think of as an information lag or a "delay" when $\ell > 0$, and which is just the standard $\mathbb{E}_t$ when $\ell = 0$.

Many models in macroeconomics envision date t quantities and prices being determined simultaneously with date t expectations. For many purposes under the rational expectations assumption it may not be too important, although it is rarely analyzed in the literature. In an environment with learning, the dating of the expectations operator may be more critical, and Evans and Honkapohja (2001) have suggested that the $t-1$ dating of the expectations operator may be more natural when learning is explicitly considered. In fact, for a given dating structure under rational expectations, learning is generally implemented using a variety of dating assumptions.[14] In this paper, we assume the *same dating of expectations* under both learning and rational expectations. Accordingly, in models with $\ell > 0$, expectations of the current state, $\mathbb{E}_{t-\ell}Y_t$, determine Y_t; that is, quantities and prices at time t are determined using $t - \ell$ periods old information and thus without knowledge of Y_t, independently of how expectations are formed. Conversely, under $\ell = 0$, expectations are formed using the current values of Y_t and X_t.[15]

[14] See Evans and Honakpohja (2001).

[15] In the case of real time learning we would need to assume that agents use $t-1$ information to update *model coefficients* while using time t variables to *form expectations*. This avoids simultaneity problems in using real time learning algorithms. See for example Eusepi and Preston (2011).

3.2 An equivalence result

Before illustrating the general results we provide an example that reveals, at a more general level than the one-dimensional consumption models presented above, the connection between E-stability conditions under $\underline{\text{FH}}$ and $\underline{\text{IH}}$ decision rules. We thus consider a two-dimensional version of the 'general model', where $A_0 = \mathbf{I}_2$, $A_1 = A_3 = \mathbf{0}_2$ so that agents learn with time t information and only learn about the steady state of the economy. By following the same steps as in the consumption decision model[16] the E-stability conditions under the two decision rules are related to the following matrices

$$M^{\underline{FH}} = A_{21} + \beta_1 \cdot \mathbf{I}_2 + A_{22} + \beta_2 \cdot \mathbf{I}_2 - \mathbf{I}_2$$

and

$$M^{\underline{IH}} = A_{21} \cdot (1 - \beta_1)^{-1} + A_{22} \cdot (1 - \beta_2)^{-1} - \mathbf{I}_2.$$

E-stability implies that the real part of eigenvalues of M^i, $i = \underline{FH}, \underline{IH}$, should be negative. Below we present two examples that compare E-stability conditions under the two decision rules.

Case 1: $\beta_1 = \beta_2 = \beta$. It is easy to verify that

$$M^{\underline{FH}} = (1 - \beta) \cdot M^{\underline{IH}}$$

so the eigenvalues of the two matrices have the same sign and are in fact proportional. As in the one dimensional consumption model, the *direction* of adjustment in expectations is the same while the *size* of the adjustment is proportional to the discount factor. Hence the two decision rules imply the same stability

[16] See Appendix A.

conditions.

Case 2: $\beta_1 \neq \beta_2$. Here the E-stability conditions for the two rules need not coincide. An immediate way to see this is to look at the trace of the two matrices (that is the sum of the two eigenvalues). When the discounts rates are different the trace (Tr) of the matrices is

$$Tr(M^{\underline{FH}}) = \beta_1 - \beta_2 + Tr(A_{21}) + TrA_{22} - 2(1-\beta_2)$$

and[17]

$$Tr(M^{\underline{IH}}) = (1-\beta_2)^{-1}\left[Tr(A_{22}) + \frac{(1-\beta_2)}{(1-\beta_1)}Tr(A_{21}) - 2(1-\beta_2)\right].$$

For β_2 'small' and β_1 'large' it is easy to verify that a set of parameters exists such that the trace of $M^{\underline{FH}}$ can be negative while the trace of $M^{\underline{IH}}$ is positive. For example, let us assume[18] $\beta_1 = .98$; $\beta_2 = 0.2$, $A_{21}(1,1) = 0.1$, $A_{21}(1,2) = 0.1$, $A_{22}(2,1) = -10$, $A_{22}(2,2) = 0.01$. It is straightforward to verify that under the $\underline{\text{FH}}$ decision rule the REE is E-stable, while it is E-*un*stable under the $\underline{\text{IH}}$ decision rule. Conversely, by setting $\beta_1 = 0.2$; $\beta_2 = 0.98$, $A_{21}(1,1) = 10$, $A_{21}(1,2) = 1.2$, $A_{22}(2,1) = -10$, $A_{22}(2,2) = -1$ the stability properties of the equilibrium are reversed. E-stability is obtained under the $\underline{\text{FH}}$ horizon rule while the equilibrium is unstable under the $\underline{\text{IH}}$ rule. Here different discount rates imply that *both the size and the direction of adjustment in expectations are different under the two decision rules.* This is because the adjustment in expectations about one variable depends not only on the discount rate related to that variable but possibly on all (different) discount rates in the model. In other words, the *direction* of adjustment in expectations depends on the

[17] Recall that $Tr(A_{21}) = A_{21}(1,1)$ and $Tr(A_{22}) = A_{22}(2,2)$.

[18] Recall that the other entries of the two matrices are zero by assumption.

relative size of the adjustment of each variable which, in turn, is connected to the different discount rates. Of note, the two examples suggest that neither of the two decision rules has a more stabilizing effect on the learning dynamics. Which rule delivers (in)-stability under learning will depend on the specific model and parameter values. The following proposition generalizes this insight to the general model (6).

Proposition 1 *Consider the model (6) and assume that $\beta_d = \beta \ \forall \ d$. Then finite horizon (FH) and infinite horizon (IH) learning models deliver the same E-stability conditions.*

Proof. See Appendix B. ■

The next proposition explores the links between determinacy and E-stability.

Proposition 2 *Consider the model (6) with $\ell = 0$.*

1. *Under FH decision rules, determinacy implies learnability (McCallum, 2007).*
2. *Assume $\beta_d = \beta \ \forall \ d$. Then determinacy implies E-stability under both FH and IH decision rules.*

Proof. See Appendix B. ■

These results for the general model are summarized in Table 1. From the two propositions it is immediate that determinacy implies E-stability under both IH and FH decision rules only in a subset of cases, namely when economic agents have: (i) information about contemporaneous variables and (ii) the same "discount factors." The first result in Proposition 2 is the specific case considered in McCallum (2007). In contrast with McCallum (2007), the proposition makes

Table 1.
Does determinacy imply E-stability?

	$\ell = 0$	$\ell > 0$
$\beta_d = \beta$	yes	no
$\beta_d \neq \beta_{d'}$	no	no

Table 1: Conditions under which determinacy does and does not imply E-stability.

clear that, even under contemporaneous information, determinacy need not imply E-stability under IH learning.

Concerning informational delays, it is immediate to recognize that, in the class of models considered here, the key to our results is the matrix A_1. If $A_1 = \mathbf{0}$, E-stability conditions are independent of ℓ.[19] However, as discussed in Section 2.1, in a model with delays agents do not observe contemporaneous variables when taking date t decisions. Hence, we generally expect the model to deliver $A_1 \neq \mathbf{0}$. The example in the next section clarifies this issue further.

Related to this point, McCallum (2007) and Ellison and Pearlman (2011) provide learning environments where lagged information does not matter for E-stability conditions. Here we describe briefly how we can reconcile our findings with theirs. To use one example,[20] McCallum (2007) re-writes the finite horizon representation[21] of (6), with $\ell = 1$, as

$$z_t = M\mathbb{E}_t z_{t+1} + N z_{t-1} + e_t, \tag{8}$$

where $z_t = (Y_t, \mathbb{E}_t Y_{t+1}, \mathbb{E}_t Y_{t+2})$ and M and N are defined appropriately, and concludes that this model is E-stable as its variables depend on current informa-

[19] This case has been proved in McCallum (2009a). However this equivalence is only valid for purely forward-looking models. For a counter-example in a model with lagged endogenous variables, see Bullard and Mitra (2002).

[20] The comparison with Ellison and Pearlman (2011) yields similar arguments.

[21] See Appendix A, equation (35).

tion only. The model (8) relies on two specific assumptions. First, despite the assumption that decisions are taken with date $t-1$ information, according to (8) agents are assumed to form date t expectations after observing z_t.[22] Second, agents are able to observe *aggregate expectations* $\mathbb{E}_t Y_{t+1}$ and $\mathbb{E}_t Y_{t+2}$.[23] This assumption is rather unusual in the learning literature where agents learn using VARs which include only realized endogenous and exogenous variables (see Evans and Honkapohja, 2001). Regardless of the validity of such an assumption, in (8) agents have a *larger* information set than what is assumed in this paper. Therefore, we should not expect the stability conditions to be the same.

Summing up, the above propositions and the atheoretical examples indicate that, for the class of models and equilibria considered here, determinacy and E-stability do not generally select the same equilibria. In the next section we apply our results to a simple monetary model that has been widely studied in the New Keynesian literature. We find this a compelling case to examine, given that the literature has devoted much attention to issues about determinacy and learnability.

4 A simple monetary model

4.1 Model environment

In the recent literature, problems of equilibrium selection have often been connected to monetary policy design. Below we describe a simple New Keynesian (NK) model that has been widely explored in the literature. It is simple enough to fit the general class of models described above and it provides a useful starting point to discuss the connection between determinacy and E-stability in mone-

[22] That is, agents form expectations according to $\mathbb{E}_t z_{t+1} = \Omega z_t$ where z_t clearly contains time t variables.

[23] That is, the agents' PLM is $z_t = \Omega z_{t-1} + e_t$ where z_t contains aggregate expectations.

tary policy design.

4.1.1 Households

There is a continuum of households that consume, save, and supply labor in a homogeneous labor market. A household i maximizes utility over an infinite horizon

$$\mathbb{E}_{t-\ell}^{i} \sum_{T=t}^{\infty} \beta^{T-t} \left[\ln \left(C_T^i \right) - \psi h_T^i \right], \tag{9}$$

where the parameter $\beta \in (0,1)$ is the discount factor, ψ is the parameter controlling the disutility of labor and we assume time separable preferences with log utility of consumption (C_t^i), and linear disutility in labor (h_t^i). For tractability we assume that in the case of informational delays, $\ell > 0$, *only consumption decisions are predetermined.*[24] This means that labor supply decisions and saving decisions are taken with date t information, independently of the value of ℓ. Consumption C_t and the price level P_t are given by standard Dixit-Stiglitz aggregators,

$$C_t^i \equiv \left[\int_0^1 c_t^i(j)^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}}, \; P_t \equiv \left[\int_0^1 p_t(j)^{1-\theta} dj \right]^{\frac{1}{1-\theta}}.$$

The budget constraint of household i is given by

$$M_{t+1}^i + P_t C_t^i \leq M_t^i - D_t^i + (1+i_t) D_t^i + W_t h_t^i + P_t \int \phi_t^f(j) dj + P_t \phi_t^M, \tag{10}$$

where M_t^i denotes money holdings[25] at the beginning of the period and D_t denotes the amount on deposit at the financial intermediary, which pays the

[24] The assumption allows hours to adjust in the current period so that equilibrium output (partially determined by the current shock to productivity) equals consumption (which is predetermined). The evolution of hours (the labor supply) has no implications for the stability conditions discussed below.

[25] We assume that money bears no interest.

gross nominal interest rate $(1 + i_t)$. The variable W_t is the economy-wide nominal wage determined in a perfectly competitive labor market, and $\phi_t(j)$ and ϕ_t^M denote real profits from firms and the financial intermediary. Each agent i is assumed to have an equal share of each firm j and the financial intermediary. These assumptions guarantee that the households' income profiles are identical, even in the case of incomplete markets. The household also faces the cash-in-advance constraint

$$P_t C_t^i \leq M_t^i + W_t h_t^i - D_t, \tag{11}$$

which takes this form because households receive their wages at the beginning of the period.[26] Solving the household problem gives the optimal consumption decision given current beliefs[27] in log-linear deviations from the nonstochastic steady state

$$\hat{C}_t^i = \mathbb{E}_{t-\ell}^i \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)\hat{Y}_T^i - \beta(i_T - \pi_{T+1}) \right], \tag{12}$$

where $\hat{Y}_t$ denotes real income.[28] The labor supply, decided with period t information is

$$\psi C_t^i = \frac{w_t}{P_t}. \tag{13}$$

[26] In an equilibrium with a positive nominal interest rate, the cash in advance constraint is binding in every period.

[27] This is solved by combining households' intertemporal budget constraint and consumption Euler equation.

[28] Following Preston (2005), for simplicity we assume that each agent forecasts total income and not its single components. This has no implication for our conclusions.

4.1.2 Firms

Each firm j produces a differentiated good and has market power. They face a demand for their output given by

$$Y_t^D(j) = A_t h_t(j) = \left(\frac{p_t(j)}{P_t}\right)^{-\theta} C_t \tag{14}$$

where C_t denotes aggregate consumption and where labor is the only input in the (linear) production function. The term A_t denotes an AR(1) technology exogenous process. The labor market is viewed as economy-wide and perfectly competitive. Firms are subject to a cash constraint, which gives rise to a *cost channel* for monetary policy.[29] In particular, the firms must anticipate their wage bill to the workers and therefore have to borrow funds from the financial intermediary in the amount corresponding to a fraction γ of the wage bill $h_t(j) W_t$. In the sequel we only consider a model with cost channel ($\gamma = 1$) and without cost channel ($\gamma = 0$).

We study a standard model of nominal pricing rigidities a là Calvo. In order to simplify the analysis we assume that, independently of informational delays, firms can observe the current aggregate price level when deciding their optimal price.[30] With informational delays the optimal relative price depends on the expected current and future marginal cost. Similarly to households, firms choose their labor input (and the amount of funds to borrow) using current information—delays can occur only at the pricing stage. The (log-linearized) pricing decision that maximizes discounted expected profits subject to demand

[29] Empirical evidence in favor of a cost channel is shown in Ravenna and Walsh (2006).

[30] This assumption is made to simplify the exposition and has no important implications for the stability conditions discussed below.

and Calvo constraints is

$$\hat{p}_t^*(j) = \mathbb{E}_{t-\ell} \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[(1-\alpha\beta)\, \hat{s}_T + \alpha\beta\pi_{T+1} \right] \tag{15}$$

where $\hat{p}_t^* = \ln\left(P_t^*/P_t\right)$ and

$$s_t = \frac{w_t}{P_t A_t} \left(1 + \gamma i_t\right), \tag{16}$$

is the real marginal cost of production, which is a function of the real wage and the opportunity cost of holding cash.[31] Given homogeneous factor markets implies that each firm that chooses the optimal price will choose the same price $\hat{p}_t^*(j) = \hat{p}_t^*$. While Calvo pricing implies a distribution of prices across firms, nevertheless the model specification leads to a simple log-linear relation between the optimal price and inflation

$$\pi_t = \frac{1-\alpha}{\alpha} \hat{p}_t^*. \tag{17}$$

Finally, financial intermediaries make a profit of

$$\begin{aligned} P_t \phi_t^M &= (1+i_t)\, \gamma \int W_t h(j) dj - (1+i_t) \int D_t^i di && (18) \\ &= (1+i_t)\, T_t && (19) \end{aligned}$$

where T_t is a cash injection from the government to be defined below.

4.1.3 Monetary and fiscal policies

We assume that the fiscal authority operates a zero debt, zero spending fiscal policy. The fiscal variable T_t denotes a cash injection from the government

[31] Financial intermediaries operate in a perfectly competitive market for funds. Therefore the cost of borrowing for each firm is $\gamma i_t \frac{W_t}{P_t} h_t(j)$.

which is equal to

$$T_t = M_{t+1} - M_t. \tag{20}$$

Monetary policy is described by a simple Taylor-type rule of the (log-linear) form

$$\hat{\imath}_t = \phi_\pi \mathbb{E}_{t-\ell}\pi_{t+j} + \phi_y \mathbb{E}_{t-\ell}\hat{Y}_{t+j} \tag{21}$$

where the monetary authority reacts to *private sector* expectations about inflation and output, where $j = 0$ indicates a contemporaneous policy rule and $j = 1$ is a forward-looking policy rule.

4.1.4 Equilibrium: IH and FH representations

Imposing equilibrium in the labor and goods markets and aggregating implies that aggregate demand can be expressed as

$$\hat{Y}_t = \mathbb{E}_{t-\ell} \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)\hat{Y}_T - \beta(\hat{\imath}_T - \pi_{T+1}) \right], \tag{22}$$

where $\hat{Y}_t = \hat{C}_t$ denotes aggregate spending. Quasi-differencing this expression we obtain the familiar forward-looking IS curve, including one extra term for information delays

$$\hat{Y}_t = (1-\beta)\mathbb{E}_{t-\ell}\hat{Y}_t - \beta\hat{E}_{t-\ell}\left(i_t - \pi_{t+1}\right) + \beta\hat{E}_{t-\ell}\hat{Y}_{t+1}. \tag{23}$$

Inflation dynamics, obtained by combining (13) , (16) and (17) with (15), are described by

$$\pi_t = \mathbb{E}_{t-\ell} \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\xi \left(\hat{Y}_T + \tilde{\gamma}\hat{\imath}_T - \hat{A}_T \right) + \beta(1-\alpha)\pi_{T+1} \right], \tag{24}$$

where $\xi = \frac{(1-\alpha\beta)(1-\alpha)}{\alpha} > 0$, and where $\tilde{\gamma} = \frac{\gamma(1+\bar{\imath})}{1+\gamma\bar{\imath}} = 1$ ($\gamma = 0$) denotes the

model with (without) a cost channel. Quasi-differencing we obtain

$$\pi_t = \beta \mathbb{E}_{t-\ell} \pi_{t+1} + \xi \mathbb{E}_{t-\ell} \left(\hat{Y}_t + \tilde{\gamma} \hat{\imath}_t - \hat{A}_t \right), \tag{25}$$

which, again, depends only on one-period-ahead forecasts. Assuming no information delays ($\ell = 0$), under the IH decision rule households forecast income, the nominal interest rate and inflation, while under FH decision rule they forecast income and inflation only. Conversely, in presence of information delays ($\ell = 1$) the interest rate is forecasted under both FH and IH decision rules.

4.1.5 Knowledge about the monetary policy rule

Preston (2006), Eusepi (2005) and Eusepi and Preston (2010) show that knowledge about the monetary policy rule has important consequences on E-stability conditions in models with both FH and IH decision rules. If agents know the policy rule (21) then they do not need to forecast the interest rate independently: they can use the restriction implied by the policy rule to forecast the interest rate as a function of their inflation and output forecasts. That is (22) becomes

$$\hat{Y}_t = \mathbb{E}_{t-\ell} \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)\hat{Y}_T - \beta(\phi_\pi \pi_{T+j} + \phi_y Y_{T+j} - \pi_{T+1}) \right]$$

and the Phillips curve is

$$\pi_t = \mathbb{E}_{t-\ell} \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\xi \left(\hat{Y}_T + \tilde{\gamma}\phi_\pi \pi_{T+j} + \tilde{\gamma}\phi_y Y_{T+j} - \hat{A}_T \right) + \beta(1-\alpha)\pi_{T+1} \right],$$

Notice that the interest rate forecast affects inflation only in the presence of a cost channel. Under the FH decision rules agents need to forecast the interest rate *only* in the presence of information delays. In this case agents may use their knowledge of the policy rule when forming expectations about the cur-

rent interest rate. Conversely, if the monetary policy is unknown, agents will forecast the interest rate independently. In the sequel we evaluate E-stability conditions under both assumptions and show that knowledge about the policy rule is an important but not the unique factor breaking the equivalence between determinacy and E-stability conditions under alternative decision rules.

4.2 Analysis of determinacy and learnability

Most papers in the adaptive learning literature use versions of equations (23) and (25) to evaluate the E-stability properties of different policy rules.[32] We compare conditions for determinacy with the E-stability conditions obtained using (23) and (25) and the E-stability conditions that arise when using (22) and (24). In order to tease out the different factors breaking determinacy and E-stability conditions we evaluate the model along four dimensions: 1) the existence of a cost channel, 2) the monetary policy rule, 3) knowledge about the policy rule and 4) the existence of information delays.

Table 2, which summarizes the results, complements Table 1 in showing that determinacy implies E-stability only in a subset of cases. As stated in the general Propositions, models with multiple "discount rates," such as the one discussed above,[33] can display different E-stability conditions under IH and FH decision rules. In turn, the disconnect between determinacy and FH E-stability conditions depend only on the existence of information delays.

Model without information delays ($\ell = 0$). In the 'baseline' version of the model, with a contemporaneous monetary policy rule ($j = 0$) determinacy

[32] See for example Bullard and Mitra (2002), among others.

[33] In particular, this model can be re-expressed in general form (6) with $n = 3$, $\beta_1 = \beta$ (for aggregate demand), $\beta_2 = \alpha\beta$ (for the Phillips curve) and $\beta_3 = 0$ (for the policy rule).

Table 2.
Determinacy and E-stability in the New Keynesian model

	$\gamma = 0$		$\gamma = 1$	
	$j = 0$	$j = 1$	$j = 0$	$j = 1$
MP rule is unknown				
$\ell = 0$	$\underline{FH}/\underline{IH}$	$\underline{FH}$	$\underline{FH}/\underline{IH}$	$\underline{FH}$
$\ell = 1$	*NO*	*NO*	*NO*	*NO*
MP rule is known				
$\ell = 0$	$\underline{FH}/\underline{IH}$	$\underline{FH}/\underline{IH}$	$\underline{FH}/\underline{IH}$	$\underline{FH}/\underline{IH}$
$\ell = 1$	$\underline{FH}/\underline{IH}$	$\underline{FH}/\underline{IH}$	*NO*	*NO*

Table 2: This table shows the relation between determinacy and E-stability conditions for alternative model specifications. Entry FH/IH denotes that determinacy implies E-stability for both IH and FH learning rules; entries FH and IH correspond to the case where determinacy implies E-stability for FH or IH only. Finally, NO refers to the case where determinacy does not imply E-stability under either decision rules.

implies E-stability under both FH and IH decision rules.[34] However, this is no longer true with monetary policy rules that respond to expectations ($j = 1$). In particular, while determinacy implies E-stability under FH decision rules (consistently with the general results in McCallum, 2007), determinacy does not imply E-stability under IH decision rules.[35] As apparent from inspecting Table 2, this result depends on agents' knowledge of the policy rule: that is, whether or not they need to forecast the interest rate independently of inflation and output. Numerical experiments suggest that, under the IH decision rule, forward-looking policy rules are both determinate and E-*un*stable for a wide range of parameter values. In particular, a positive monetary policy response to output together with a substantial degree of nominal rigidities are required for determinate equilibria to be E-stable.[36]

[34] Bullard and Mitra (2002) and Preston (2005) show for FH and IH decision rules respectively that determinacy implies E-stability.

[35] See Preston (2006).

[36] For example, assume $\beta = 0.99$, $\gamma = 0$ and $\phi_\pi = 1.5$. The equilibrium is both determinate

Model with information delays ($\ell = 1$). In the presence of information delays, the connection between E-stability under FH decision rules and determinacy is broken; determinate equilibria can be E-*un*stable under either of the two decision rules. As shown in Preston (2006), Eusepi (2005) and Eusepi and Preston (2010), only monetary policy rules that, in addition to satisfying the Taylor principle, display a sufficiently strong response to output induce E-stability. As shown in the bottom section of the table, agents' full knowledge about the monetary policy rule is not enough to guarantee E-stable equilibria when the cost channel of monetary policy is active ($\gamma = 1$). Under this model specification, for a wide range of parameter values determinate equilibria are E-*un*stable unless monetary policy responds with enough strength to output. Numerical experiments are discussed below.

E-stability under alternative decision rules. Comparing the E-stability conditions under the two decision rules, we find that; a) FH and IH decision rules generally imply different E-stability conditions and, b) neither decision rule has more stabilizing effects on learning dynamics. We focus the discussion on the model with information delays ($\ell = 1$). First, when agents ignore the monetary policy rule, numerical experiments suggest that the FH decision rule implies E-stability for a larger set of parameter values than the IH rule. In particular, under the FH decision rule a lower response to output in the monetary policy rule is required in order to guarantee E-stability.[37] The result holds in-

and E-stable for $\alpha = 0.85$ and $\phi_y = 1.8$ and it becomes E-unstable for lower values of ϕ_y or α.

[37] Assume $\beta = 0.99; \alpha = 0.6; \phi_\pi = 1.5$, $\gamma = 0$. Under the FH decision rule the equilibrium is E-stable for $\phi_y > 0.3$ while a $\phi_y > 1.1$ is required for E-stability under the IH decision rule. The same condition holds under both conteporaneous and forward-looking policy rules ($j = 0, 1$).

dependently of the form of the monetary policy rule and the existence of a cost channel of monetary policy. Second, consider the model where the monetary policy rule is known to the public and the cost channel of monetary policy is active ($\gamma = 1$). Numerical experiments suggest that, in this case, the IH decision rule implies a larger set of parameters for which E-stability occurs relative to the FH decision rule.[38]

Summing up, determinacy does not imply E-stability in a simple New Keynesian model. The disconnect in this model can be sourced to the agents' information set (the timing of information and their knowledge about the monetary policy rule) and to the monetary transmission mechanism (the presence of a cost channel of monetary policy). Moreover, FH and IH decision rules induce different stability conditions and neither decision rule appears to exert a more stabilizing effect on the learning process.

5 Conclusions

We have studied a general class of models and investigated the connection between determinacy and E-stability. As an example we have applied our findings to a New Keynesian model generalized on certain dimensions in an attempt to delineate the differences between conditions for equilibrium determinacy and conditions for equilibrium learnability in terms of meaningful economic assumptions.

In general, we conclude that determinacy does not imply E-stability and

[38] Set $\beta = 0.99; \alpha = 0.6; \phi_\pi = 1.5$ and $\gamma = 1$. Recall that the REE equlibrium is determinate for values of ϕ_y which are positive and below 0.5. Under the IH decision rule the equilibrium is E-stable for $\phi_y > 0.01$ while $\phi_y > 0.4$ is required to obtain E-stability under the FH decision rule.

therefore that both conditions need to be verified if the goal is to study robust rational expectations equilibria. While there are important classes of models in which determinacy does indeed imply learnability, we have shown how such a result can be sensitive to the specific approach used to study learning dynamics and to the information set available to the model's agents.

James Bullard, Federal Reserve Bank of St Louis

Stefano Eusepi, Federal Reserve Bank of New York

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6 Appendix

A Framework

A.1 E-stability in the General Model

The Perceived Law of Motion (PLM) which is consistent with the Minimum State Variable (MSV) solution of (6) takes the form

$$Y_t = a + bX_{t-\ell}, \tag{26}$$

where a denotes the intercept vector, and b denotes a $n \times k$ matrix of coefficients. Substituting the agent forecasts obtained from (26) for aggregate expectations we obtain the actual law of motion (ALM) of Y_t under learning dynamics, defined as

$$Y_t = T(a, b) \cdot \begin{bmatrix} 1 \\ X_{t-\ell} \end{bmatrix},$$

which depends on exogenous variables and the coefficients in the PLM. The notion of E-stability describes the interaction between PLM and ALM which arises during the learning process, where $T(a, b) \neq (a, b)$. It is determined by the following matrix differential equations

$$\frac{d}{d\tau}(a, b) = T(a, b) - (a, b) \tag{27}$$

where τ denotes "artificial" time. The local asymptotic stability of (27) depends on the eigenvalues of the Jacobian of (27).

A.2 Infinite horizon (IH) learning

To study the infinite horizon approach to learning, we assume agents use the PLM (26) to form expectations: the discounted infinite sum in equation (6) can then be expressed as

$$\mathbb{E}_{t-\ell}\sum_{T=t}^{\infty}\beta_d^{T-t}Y_{T+1}=\frac{1}{1-\beta_d}a+b\left(1-\beta_d H\right)^{-1}HX_{t-\ell}. \tag{28}$$

Also, the discounted infinite sum of exogenous variables is

$$\mathbb{E}_{t-\ell}\sum_{T=t}^{\infty}\beta_d^{T-t}X_T=\left(1-\beta_d H\right)^{-1}X_{t-\ell}. \tag{29}$$

Substituting (28) and (29) in (6) we obtain the actual law of motion (ALM) of Y_t,

$$\begin{aligned} Y_t &= \tilde{A}_1\left(a+bX_{t-\ell}\right) \\ &\quad +\sum_{d=1}^{n}\tilde{A}_{2,d}\left(\frac{1}{1-\beta_d}a+b\left(1-\beta_d H\right)^{-1}HX_{t-\ell}\right)+A_{\Sigma X}X_{t-\ell}, \end{aligned} \tag{30}$$

where $\tilde{A}_1=A_0^{-1}A_1$, and $\tilde{A}_{2,d}=A_0^{-1}\ A_{2,d}$, and where

$$A_{\Sigma X}=\sum_{d=1}^{n}\tilde{A}_{3,d}\left(1-\beta_d H\right)^{-1}. \tag{31}$$

Equation (30) defines the following mapping between the PLM and the ALM

$$T^{\underline{IH}}(a,b)=\left(\tilde{A}_1 a+\sum_{d=1}^{n}\tilde{A}_{2,d}\frac{1}{1-\beta_d}a,\quad \tilde{A}_1 b+\sum_{d=1}^{n}\tilde{A}_{2,d}b\left(1-\beta_d H\right)^{-1}H+A_{\Sigma X}\right). \tag{32}$$

According to (27), the matrix governing expectational stability of the intercept is thus

$$M^{\underline{IH}}(a)=\tilde{A}_1+\sum_{d=1}^{n}\tilde{A}_{2,d}\frac{1}{1-\beta_d}-\mathbf{I}_n. \tag{33}$$

For the matrix of coefficients b, after vectorizing the second argument of $T^{\underline{IH}}(a,b)$ we obtain

$$M^{\underline{IH}}(b) = \left(\mathbf{I}_k \otimes \tilde{A}_1\right) + \left\{\sum_{d=1}^{n}\left[\left(\mathbf{I}_k - \beta_d H\right)^{-1} H\right]' \otimes \tilde{A}_{2,d}\right\} - \mathbf{I}_k \otimes \mathbf{I}_n. \quad (34)$$

E-stability obtains if and only if the real parts of the eigenvalues of $M^{\underline{IH}}(a)$ and $M^{\underline{IH}}(b)$ are negative. We stress that to obtain the case of contemporaneous expectations ($\ell = 0$) all that is needed is to re-define the matrix A_0 and set $\tilde{A}_1 = 0$.

A.3 Finite horizon (FH) learning

We define FH learning as the outcome of quasi-differencing (6). Forwarding (6) one period we get

$$E_{t-\ell}A_0Y_{t+1} = A_1\mathbb{E}_{t-\ell}Y_{t+1} + \sum_{d=1}^{n} A_{2,d}\mathbb{E}_{t-\ell}\sum_{T=t+1}^{\infty}\beta_d^{T-t-1}Y_{T+1} + \mathbb{E}_{t-\ell}A_{\Sigma X}X_{t-\ell+1}.$$

Next, we premultiply the above $D = diag\left([\beta_1 \ldots \beta_n]\right)$ to get

$$DE_{t-\ell}A_0Y_{t+1} = DA_1\mathbb{E}_{t-\ell}Y_{t+1} + \sum_{d=1}^{n} A_{2,d}\mathbb{E}_{t-\ell}\sum_{T=t+1}^{\infty}\beta_d^{T-t}Y_{T+1} + D\hat{E}_{t-\ell}A_{\Sigma X}X_{t-\ell+1}.$$

using $DA_{2,d} = A_{2,d}\beta_d$. Second, we rewrite (6) as

$$A_0Y_t = A_1\mathbb{E}_{t-\ell}Y_t + \sum_{d=1}^{n} A_{2,d}\mathbb{E}_{t-\ell}Y_{t+1} + \sum_{d=1}^{n} A_{2,d}\mathbb{E}_{t-\ell}\sum_{T=t+1}^{\infty}\beta_d^{T-t}Y_{T+1} + A_{\Sigma X}X_{t-\ell}$$

and combining the two expressions we get

$$Y_t = \tilde{A}_1\mathbb{E}_{t-\ell}Y_t + \left[\sum_{d=1}^{n}\tilde{A}_{2,d} + A_0^{-1}DA_0\left(\mathbf{I}_n - \tilde{A}_1\right)\right]\mathbb{E}_{t-\ell}Y_{t+1} + \tilde{A}_{\Sigma X}X_{t-\ell}, \quad (35)$$

where $\tilde{A}_{\Sigma X} = A_0^{-1}\left(A_{\Sigma X} - DA_{\Sigma X}H\right)$. The ALM (35) gives the following mappings

$$T^{\underline{FH}}(a) = \tilde{A}_1 a + \left[\sum_{d=1}^{n} \tilde{A}_{2,d} + A_0^{-1} DA_0 \left(\mathbf{I}_n - \tilde{A}_1\right)\right] a$$

and

$$T^{\underline{FH}}(b) = \tilde{A}_1 b - A_0^{-1} DA_0 \tilde{A}_1 bH + \sum_{d=1}^{n} \tilde{A}_{2,d} bH + A_0^{-1} DA_0 bH + A_0^{-1}\left(A_{\Sigma X} - DA_{\Sigma X}H\right).$$

The mapping between the PLM and the ALM yields the following Jacobian matrix for the intercept

$$M^{\underline{FH}}(a) = \tilde{A}_1 + \left[\sum_{d=1}^{n} \tilde{A}_{2,d} + A_0^{-1} DA_0 \left(\mathbf{I}_n - \tilde{A}_1\right)\right] - \mathbf{I}_n \tag{36}$$

which is the counterpart to (33). For the regression coefficients we obtain (after vectorization) the following Jacobian

$$M^{\underline{FH}}(b) = \mathbf{I}_k \otimes \tilde{A}_1 - H \otimes A_0^{-1} DA_0 \tilde{A}_1 + H \otimes \sum_{d=1}^{n} \tilde{A}_{2,d} + H \otimes A_0^{-1} DA_0 - \mathbf{I}_k \otimes \mathbf{I}_n \tag{37}$$

which, in turn, is the counterpart to (34).

A.4 Determinacy

Finally, determinacy can be evaluated by inspecting the following matrix

$$M^D = \left(\mathbf{I}_n - \tilde{A}_1\right)^{-1} \left[\sum_{d=1}^{n} \tilde{A}_{2,d} + A_0^{-1} DA_0 \left(\mathbf{I}_n - \tilde{A}_1\right)\right], \tag{38}$$

obtained from (35), after imposing rational expectations on expected current variables. Determinacy is obtained if and only if the eigenvalues of M^D lie within the unit circle.

B Proofs

B.1 Proof of Proposition 1

Consider first the matrix $M^{\underline{FH}}(a)$ in (36). Imposing $\beta_d = \beta$, $\forall\ d$, we obtain

$$\begin{aligned} M^{\underline{FH}}(a) &= \tilde{A}_1 + \left[\sum_{d=1}^{n} \tilde{A}_{2,d} + A_0^{-1}\beta I_n A_0 \left(\mathbf{I}_n - \tilde{A}_1\right)\right] - \mathbf{I}_n \\ &= (1-\beta)\tilde{A}_1 + \sum_{d=1}^{n} \tilde{A}_{2,d} - (1-\beta)\,\mathbf{I}_n = (1-\beta)M^{\underline{IH}}(a) \end{aligned}$$

where the last equality is immediate from (33) and the assumption $\beta_d = \beta$, $\forall$ d. Concerning the regression coefficients, from (37) we obtain

$$\begin{aligned} M^{\underline{FH}}(b) &= \mathbf{I}_k \otimes \tilde{A}_1 - H \otimes \beta\tilde{A}_1 + H \otimes \sum_{d=1}^{n} \tilde{A}_{2,d} + H \otimes \beta\mathbf{I}_n - \mathbf{I}_k \otimes \mathbf{I}_n \\ &= \mathbf{I}_k \otimes \tilde{A}_1 - \beta H \otimes \tilde{A}_1 + H \otimes \sum_{d=1}^{n} \tilde{A}_{2,d} + \beta H \otimes \mathbf{I}_n - \mathbf{I}_k \otimes \mathbf{I}_n \\ &= (\mathbf{I}_k - \beta H) \otimes \tilde{A}_1 + H \otimes \sum_{d=1}^{n} \tilde{A}_{2,d} - (\mathbf{I}_k - \beta H) \otimes \mathbf{I}_n \end{aligned}$$

where the two equalities follow from the properties of the Kroneker product . Setting $\beta_d = \beta\ \forall\ d$ in (34), which gives

$$\begin{aligned} M^{\underline{IH}}(b) &= \left(\mathbf{I}_k \otimes \tilde{A}_1\right) + \left\{\sum_{d=1}^{n} \left[(\mathbf{I}_k - \beta H)^{-1} H\right]' \otimes \tilde{A}_{2,d}\right\} - \mathbf{I}_k \otimes \mathbf{I}_n \\ &= \left(\mathbf{I}_k \otimes \tilde{A}_1\right) + \left[(\mathbf{I}_k - \beta H)^{-1} H\right]' \otimes \sum_{d=1}^{n} \tilde{A}_2 - \mathbf{I}_k \otimes \mathbf{I}_n. \end{aligned}$$

Post-multiplying by $(\mathbf{I}_k - \beta H) \otimes \mathbf{I}_n$ we get

$$M^{\underline{FH}}(b) = M^{\underline{IH}}(b) \cdot \left[(\mathbf{I}_k - \beta H) \otimes \mathbf{I}_n\right], \tag{39}$$

where we use the following equalities[39]

$$\left(\mathbf{I}_k \otimes \tilde{A}_1\right)\left[\left(\mathbf{I}_k - \beta H\right) \otimes \mathbf{I}_n\right] = \mathbf{I}_k \left(\mathbf{I}_k - \beta H\right) \otimes \tilde{A}_1 \mathbf{I}_n,$$

$$\left(I_k \otimes I_n\right)\left[\left(\mathbf{I}_k - \beta H\right) \otimes \mathbf{I}_n\right] = \left(I_k - \beta H\right) \mathbf{I}_k \otimes \mathbf{I}_n,$$

and

$$\begin{aligned}\left(\left[\left(\mathbf{I}_k - \beta H\right)^{-1} H\right]' \otimes \sum_{d=1}^{n} \tilde{A}_2\right)\left[\left(\mathbf{I}_k - \beta H\right) \otimes \mathbf{I}_n\right] &= \left[\left(\mathbf{I}_k - \beta H\right)^{-1} H \left(\mathbf{I}_k - \beta H\right) \otimes \sum_{d=1}^{n} \tilde{A}_2\right] \\ &= H \otimes \sum_{d=1}^{n} \tilde{A}_2.\end{aligned}$$

Back to (39), the matrix $\left(\mathbf{I}_k - \beta H\right) \otimes \mathbf{I}_n$ is a $nk \times nk$ block diagonal matrix

$$\left(\mathbf{I}_k - \beta H\right) \otimes \mathbf{I}_n = \begin{bmatrix} \left(1 - \beta h_{11}\right) \mathbf{I}_n & ... & 0 \\ 0 & ... & 0 \\ 0 & ... & \left(1 - \beta h_{kk}\right) \mathbf{I}_n \end{bmatrix},$$

so that

$$M^{\underline{IH}}(b) \cdot \left[\left(\mathbf{I}_k - \beta H\right) \otimes \mathbf{I}_n\right] = \begin{bmatrix} \bar{A}_{11} & ... & 0 \\ 0 & ... & 0 \\ 0 & ... & \bar{A}_{kk} \end{bmatrix}. \tag{40}$$

The eigenvalues of (40) are given by the eigenvalues of the sub-matrices

$$\bar{A}_{ii} = \left(1 - \beta h_{ii}\right)\left[\tilde{A}_1 + \left(1 - \beta h_{ii}\right)^{-1} h_{ii} \cdot \sum_{d=1}^{n} \tilde{A}_2 - \mathbf{I}_n\right] \text{ for } i = 1...k.$$

[39] In particular, the mixed-product property of the Kroneker product implies

$$\left(A \otimes B\right)\left(C \otimes D\right) = AC \otimes BD.$$

We then obtain the eigenvalues as

$$\begin{aligned} eig\left((1-\beta h_{ii})\left[\tilde{A}_1+(1-\beta h_{ii})^{-1}h_{ii}\cdot\sum_{d=1}^{n}\tilde{A}_2-\mathbf{I}_n\right]\right) &= \\ (1-\beta h_{ii})\,eig\left(\left[\tilde{A}_1+(1-\beta h_{ii})^{-1}h_{ii}\cdot\sum_{d=1}^{n}\tilde{A}_2-\mathbf{I}_n\right]\right) \text{ for } i &= 1...k, \end{aligned}$$

which proves the claim. Similarly, the reverse also holds true, given that

$$M^{\underline{IH}}(b)=M^{\underline{FH}}(b)\cdot[(\mathbf{I}_k-\beta H)\otimes\mathbf{I}_n]^{-1}$$

is well defined under the maintained assumptions about H.

B.2 Proof of Proposition 2

1. Set $A_1=0$ in (6) so that, quasi-differencing yields

$$\begin{aligned} Y_t &= \left[\sum_{d=1}^{n}\tilde{A}_{2,d}+A_0^{-1}DA_0\right]\mathbb{E}_tY_{t+1}+A_0^{-1}\left(A_{\Sigma X}-DA_{\Sigma X}H\right)X_t \\ &= M^D\mathbb{E}_tY_{t+1}+NX_t \end{aligned}$$

   and thus belongs to the class of models considered in McCallum (2007).[40] Hence determinacy implies learnability under <u>FH</u> learning.

2. This is straightforward from Proposition 2 and point 1 in Proposition 3.

C Results in Table 2

Most of the results (determinacy imply E-stability) shown in Table 2 can be found in the existing literature or are implications of Proposition 2. The tables

[40] However, we also have a proof available in the working paper version of this paper.

TABLE 3.
Determinacy and E-stability in the NK model ($\gamma = 0$)

	Finite Horizon	
	$j = 0$	$j = 1$
MP rule is unknown		
$\ell = 0$	Proposition 2	Proposition 2
$\ell = 1$	*NO*	*NO*
MP rule is known		
$\ell = 0$	Proposition 2	Proposition 2
$\ell = 1$	Eusepi (2005)	(1)
	Infinite Horizon	
MP rule is unknown		
$\ell = 0$	Preston (2005)	*NO*
$\ell = 1$	*NO*	*NO*
MP rule is known		
$\ell = 0$	Eusepi and Preston (2012)	Preston (2006)
$\ell = 1$	Eusepi and Preston (2010)	(2)

Table 3: This table refers to the model without cost-channel. NO refers to the case where determinacy does not imply E-stability.

below details the sources. The proofs below fill the gaps: they are numbered from (1) to (5). To simply the analysis we assume no exogenous processes and we focus only on the dynamics of the intercept.

Proof of (1-2). Consider first the model with the FH decision rule. Using $M^{FH}(a)$, in equation (36) (appendix A.3) gives a 2×2 matrix: necessary and sufficient conditions for E-stability are that the trace should be negative and the

TABLE 4.
Determinacy and E-stability in the NK model ($\gamma = 1$)

	Finite Horizon	
	$j = 0$	$j = 1$
MP rule is unknown		
$\ell = 0$	Proposition 2	Proposition 2
$\ell = 1$	*NO*	*NO*
MP rule is known		
$\ell = 0$	Proposition 2	Proposition 2
$\ell = 1$	*NO*	*NO*
	Infinite Horizon	
MP rule is unknown		
$\ell = 0$	(3)	*NO*
$\ell = 1$	*NO*	*NO*
MP rule is known		
$\ell = 0$	(4)	(5)
$\ell = 1$	*NO*	*NO*

Table 4: This table refers to the model with cost-channel. NO refers to the case where determinacy does not imply E-stability.

determinant positive. The trace and determinant can be simplified[41] to yield

$$\text{trace} = -(1-\beta) - \phi_y \beta < 0$$

and

$$\text{determinant} = \beta \cdot \big(\xi(\phi_\pi - 1) + (1-\beta)\phi_y\big)$$

so that the usual restriction $\xi(\phi_\pi - 1) + (1-\beta)\phi_y > 0$ holds. This corresponds to the condition for determinacy also known as the Taylor Principle. Hence, determinacy implies E-stability. Moving to the model with the IH decision rule, Using $M^{IH}(a)$, in equation (33) (appendix A.2) gives the following trace and determinant

$$\text{trace} = -\frac{1-\beta}{1-\alpha\beta} - \frac{\beta\phi_y}{1-\beta} < 0$$

and

$$\text{determinant} = \frac{\beta \cdot \big(\xi(\phi_\pi - 1) + (1-\beta)\phi_y\big)}{(1-\alpha\beta)(1-\beta)}$$

so that, again determinacy implies E-stability.

Proof of (3-4). As shown in Preston (2005) and Eusepi and Preston (2012) knowledge of the monetary policy rule does not matter for stability in the case of a contemporaneous Taylor rule, as no feed-back effects from learning are present. We then focus on the case where agents know the policy rule, which simplifies the analytical calculations. Again, using equation (33) in appendix A2 gives a 2×2 matrix. The trace and determinant can be simplified[42] to yield

$$\begin{aligned} \text{trace} \quad = \quad & -\frac{\left[\xi\left(\phi_\pi - 1\right) + \phi_y(1-\beta-\xi)\right]}{(1-\beta)} + \\ & -\frac{\left(1+\phi_y\right)(1-\beta)}{(1-\alpha\beta)} \end{aligned}$$

[41] These conditions are obtained after simplifying the output of the matlab file EstabilityI-Hgamma0j1.m. The file is available on request.

[42] These conditions are obtained after simplifying the output of the matlab file EstabilityI-Hgamma1j0.m. The file is available on request.

$$\text{determinant} = \xi\left(\phi_\pi - 1\right) + \phi_y(1 - \beta - \xi).$$

Necessary and sufficient condition for E-stability is then: $\xi\left(\phi_\pi - 1\right) + \phi_y(1 - \beta - \xi) > 0$. As shown in Llosa and Tuesta (2009, Proposition 1) this is also a necessary condition for determinacy.

Proof of (5). Following the same steps as above, the trace and determinant associated to (36) (FH decision rule) give the following conditions[43]

$$\text{trace} = \phi_y + \xi < 1 - \beta$$

$$\text{determinant} = \xi\left(\phi_\pi - 1\right) + \phi_y(1 - \beta - \xi) > 0.$$

Using Proposition 2, these are also necessary conditions for determinacy. The trace and determinant associated with (33) (IH decision rule) are

$$\text{trace} = \frac{(\beta + \phi_\pi\xi - 1)}{1 - \alpha\beta} - \frac{\phi_y - \xi + \phi_\pi\xi}{(1 - \beta)}$$

$$\text{determinant} = \xi\left(\phi_\pi - 1\right) + \phi_y(1 - \beta - \xi).$$

By substituting the condition $\phi_\pi\xi > \xi - \phi_y(1 - \beta - \xi)$ in the expression for the trace and after algebraic manipulation we get the following condition for the trace

$$\left(\xi - \phi_y\right) - \psi < 1 - \beta$$

where

$$\psi = \phi_y\left(\beta + \xi\right)\frac{\beta\left(1 - \alpha\right)}{1 - \beta}$$

so that determinacy implies E-stability.

[43] These conditions are obtained after simplifying the output of the matlab file EstabilityI-Hgamma1j1.m. The file is available on request.