Technical Appendix to An Investigation of the Gains From Commitment in Monetary Policy

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Abstract

This appendix derives the solution to an optimal policy problem under quasi commitment in the context of a linear-quadratic economy. MATLAB code for the computation of this solution is available at http://nyfedeconomists.org/tambalotti/.

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A Quasi Commitment: The Linear Quadratic Case

This appendix derives the solution to an optimal policy problem under quasi commitment. It does so in the context of a fairly general linear-quadratic economy, of the kind considered for example by Svensson (1999a). Besides its analytical tractability, the main advantage of the LQ framework is that it provides a valid approximation to a very broad class of microfounded optimal fiscal and monetary policy problems, as shown in a series of recent papers by Benigno and Woodford (2004a, 2004b, 2004c).

The structure of the Appendix is as follows. In section A.1 we introduce the economic environment. In section A.2 we formulate a recursive version of the optimization problem under quasi commitment. Its main feature is to be recursive across policymakers' tenures, rather than across periods. In section A.3 we describe the implications of rationality and perfect information on private agents' expectations in a linear equilibrium. Finally, in section A.4, we derive the model's quasi commitment equilibrium by maximizing the Lagrangian associated with the constrained Bellman equation of section A.2 In this context, we also illustrate the three steps of the fixed point iterations used to solve for the parameters of the (quadratic) value function and of agents' (linear) expectations.

A.1 The Environment

Dynamics We consider a DSGE economy populated by a private sector and a policy authority. The behavior of the private sector is described by a system of linearized equilibrium conditions, which restrict the deviations of n_x predetermined (state) variables x_t and n_X non-predetermined (jump) variables X_t from a deterministic steady state. Their joint evolution is described by

$$\begin{bmatrix} x_{t+1} \\ GE_t X_{t+1} \end{bmatrix} = A \begin{bmatrix} x_t \\ X_t \end{bmatrix} + Bi_t + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}, \tag{1}$$

for a given sequence $\{i_t\}_{t\geq 0}$, where i_t is a q-dimensional vector of policy instruments. As usual, autocorrelated shocks are included in x_t, so that the structural shocks $\{\varepsilon_t\}_{t\geq 0}$ can be assumed to be i.i.d. with covariance matrix Σ. The matrices $\Sigma \in \mathbb{R}^{n_x \times n_x}$, $A \in \mathbb{R}^{(n_x+n_X)\times(n_x+n_X)}$, $G \in \mathbb{R}^{n_X \times n_X}$ and $B \in \mathbb{R}^{(n_x+n_X)\times q}$ contain functions of the structural parameters specific to the steady state, which are assumed to be known to all agents.

Policy Objective The policy authority maximizes the expected present discounted value of the stream of utility of the economy's representative agent, as approximated by (minus) a quadratic loss function of the form

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t L_t \right], \tag{2}$$

with

$$L_t \equiv \begin{bmatrix} x_t \\ X_t \\ i_t \end{bmatrix}' W \begin{bmatrix} x_t \\ X_t \\ i_t \end{bmatrix} \tag{3}$$

for some positive semidefinite symmetric matrix W. Note that constant target levels which differ from the system's steady state can be easily accommodated by augmenting the vector of predetermined variables x_t with an "intercept". This would result in a singular Σ and in a corresponding unit root of the system, to be considered stable when solving for the economy's rational expectations equilibrium. An example of a such a target level is the positive output gap x^* included in the central bank's loss function in the monetary model in the paper. That constant target is the source of the model's average inflation bias.

Quasi Commitment Equilibrium Recalling that changes in regime are triggered by positive realizations of the i.i.d. Bernoulli signals in the sequence $\{\eta_t\}_t$, with $E[\eta_t] = \alpha$, we can state the following

Definition 1 *(*Quasi Commitment Equilibrium*) A Quasi Commitment Equilibrium with parameter α is a sequence $\{x_t, X_t, i_t\}_{t\geq 0}$, such that:*

i) The policy plan $\{i_t\}_{t\geq 0}$ maximizes (2), given (1) and a quasi commitment technology with $E[\eta_t] = \alpha$

ii) The sequence $\{x_t, X_t\}_{t\geq 0}$ is the unique stable solution of (1), given the policy plan $\{i_t\}_{t\geq 0}$

iii) Expectations in (1) are rational, given $E[\eta_t] = \alpha$

A.2 The Policy Problem

The problem faced by the policy authority under quasi commitment is to choose a path for its instruments to minimize the discounted sum of expected period losses, subject to the constraints given by the equilibrium conditions (1) and by the limited commitment technology available. By redefining one-step-ahead expectations as independent variables, $X_t^e \equiv E_t X_{t+1}$,we can write this problem as

$$\begin{aligned} &\min_{\{i_t\}_{t\geq 0}} E_0 \left[\sum_{t=0}^{\infty} \beta^t L_t \right] \\ &\qquad x_{t+1} - A_{11}x_t - A_{12}X_t - B_1 i_t - \varepsilon_{t+1} = 0, \quad x_0 = x \\ &\text{s.t.} \qquad G X_t^e - A_{21}x_t - A_{22}X_t - B_2 i_t = 0 \\ &\qquad (1-\eta_t)\left(X_{t-1}^e - E_{t-1}X_t\right) = 0. \end{aligned} \tag{4}$$

The last equation simply states that, in selecting the optimal path for the economy, the policy authority can "choose" private agents' expectations, but under the constraint that they be rational.

Since commitment is only imperfect however, this constraint does not bind new policymakers. This formally represents the assumption that policymakers are allowed to (and will in equilibrium) formulate a new optimal plan at the beginning of each regime, namely whenever $\eta_t = 1$.

Since expectations of today's conditions, X^e_{t-1}, are formed before the realization of η_t is observed, new policymakers' reoptimizations come as a "surprise" to agents. However, this is true only in an ex-post sense, since we also assume that agents correctly perceive the probability of regime changes. In other words, agents have rational expectations both within and across regimes, so that their predictions are accurate on average, at least given a sample with enough reoptimizations. In this sense then, they are just as "surprised" when the current regime continues for another period.

To characterize its solution, it is useful to analyze the optimal policy problem by grouping losses by regime, rather than by period. To this end, define the dates of regime changes $\{\tau_j\}_{j\geq 0}$ and regime durations $\{\Delta\tau_j\}_{j\geq 0}$ beyond the initial period as

$$\begin{aligned} \tau_j &= \min\{t \,|\, t > \tau_{j-1},\, \eta_t = 1\},\ \tau_0 \equiv 0 \\ \Delta\tau_j &= \tau_{j+1} - \tau_j - 1. \end{aligned} \tag{5}$$

Thus the jth regime starts at date $t = \tau_j$ and is in effect for $t \in \{\tau_j, \ldots, \tau_j + \Delta\tau_j\}$, that is up until time $t = \tau_{j+1}$, when the $j+1^{\text{st}}$ regime starts.

With this notation, the policymaker's objective (4) can be rewritten in terms of a sum of losses over individual policy regimes

$$\min_{\{i_t\}_{t\geq 0}} E_0 \sum_{j=0}^{\infty} \beta^{\tau_j} \left[\sum_{k=0}^{\Delta\tau_j} \beta^k L_{\tau_j+k} \right]. \tag{6}$$

The recursive structure of the quasi commitment problem should now be clear. Positive realizations of the regime-change signal, $\eta_{\tau_j} = 1$, by dropping the expectational constraint from problem (4), sever the feedback from current policy choices to past private sector behavior. This feedback is precisely what makes the full commitment problem non recursive, and therefore time-inconsistent, as first pointed out by Kydland and Prescott (1977, Section II). Breaking that feedback at the inception of each regime then, makes the quasi commitment problem recursive *across regimes*, although not across periods.

We can then write the problem's Bellman equation as[1]

$$\begin{aligned} V(x_{\tau_j}) = &\max_{\{\varphi_{k+1}\}_{k\geq\tau_j}} \min_{\{x_{k+1}, X_k, i_k\}_{k\geq\tau_j}} E_{\tau_j} \left[\sum_{k=0}^{\Delta\tau_j} \beta^k \mathcal{L}_{\tau_j+k} + \beta^{\Delta\tau_j+1} V(x_{\tau_{j+1}}) \right] \\ \text{s.t.} \quad & x_{t+1} - A_{11}x_t - A_{12}X_t - B_1 i_t - \varepsilon_{t+1} = 0 \\ & \varphi_{\tau_j} = 0, \end{aligned} \tag{7}$$

where

$$\mathcal{L}_t \equiv L_t + 2\varphi'_{t+1} \left(G E_t X_{t+1} - A_{21}x_t - A_{22}X_t - B_2 i_t \right).$$

[1] See Marcet and Marimon (1999) for a formal treatment of "recursive saddle point" functional equations.

In this expression, the state variables x_{τ_j} are predetermined as of the last period of the $j-1^{\text{st}}$ regime. The n_X predetermined Lagrange multipliers φ_{t+1}, attached to the constraints involving expectations, must instead satisfy the initial condition $\varphi_{\tau_j} = 0$. This condition corresponds to $1 - \eta_{\tau_j} = 0$ in (4). Its effect is to free new policymakers from their predecessors' outstanding promises regarding the continuation of their optimal policy plan. This also explains why the value function in (7) depends only on x_{τ_j}, and not on the predetermined Lagrange multipliers as well.

By the Bellman principle, this value function is the minimum achievable value of the objective (4), given the relevant constrains. Moreover, the state contingent policy plan optimally chosen by the jth policymaker will be optimal for all subsequent policymakers to follow. Therefore, the solution to problem (7) completely characterizes the economy's equilibrium under quasi commitment.

Given its particular recursive structure, solving this problem requires blending the solution strategies which are usually applied to the discretion and commitment cases separately (see for example Söderlind, 1999). On the one hand, the fact that optimal policy plans are systematically reformulated at the inception of each regime makes the problem similar to one with discretion, at least across regimes. This feature is reflected in the "regime-by-regime" formulation of the Bellman equation. As under discretion, it also requires to guess and iterate on agents' across regime expectations, as illustrated in the next section.

Unlike under discretion however, under quasi commitment policymakers are in fact able to credibly commit to a policy rule, if only for a random number of periods. Moreover, since that number is drawn from a geometric distribution, that rule has a positive probability of being in place for an arbitrarily large number of periods. This implies that policymakers must look infinitely into the future when formulating that plan. Within each regime then, the problem is formally identical to one with full commitment, except for a modified discount factor. For this reason, it is most convenient to characterize its solution with a Lagrangian method, as we will do in section A.4

A.3 Private Agents' Expectations

In our environment, at any given point in time, the behavior of private agents depends on expectations about the entire infinite future. In this respect, they differ significantly from policymakers, who need to worry only about their own term in office. The private sector must instead form expectations about the non-predetermined variables, $E_t X_{t+1}$, taking into account that every period a regime change may occur with probability α. This results in

$$E_t[X_{t+1}] = (1-\alpha)E_t[X_{t+1}|\ \underbrace{\eta_{t+1} = 0}_{\text{within regime}}\ ] + \alpha E_t[X_{t+1}|\ \underbrace{\eta_{t+1} = 1}_{\text{across regime}}\ ].$$

Note that the exogeneity of the regime-change shock is crucial in maintaining the linearity of the expectation. As a consequence, it is a fundamental ingredient in the relative simplicity of the solution.

As pointed out above, the recursive structure of equation (7) implies that each policymaker solves her optimization problem by focusing exclusively on the evolution of the economy within

her regime. Therefore, within-regime expectations are determined as part of the optimal plan. On the contrary, across-regime expectations are taken as given, or rather as a given function of the current state, just as in the case of discretion. Given the LQ structure of the problem, we guess this function to be linear, with coefficients $H \in \mathbb{R}^{n_X \times n_x}$ and $\tilde{H} \in \mathbb{R}^{n_X \times n_X}$

$$\begin{aligned} E_t\left[X_{t+1}|\eta_{t+1}=1\right] &= HE_t\left[x_{t+1}|\eta_{t+1}=1\right] + \widetilde{H}E_t\left[\varphi_{t+1}|\eta_{t+1}=1\right] \\ &= HE_t[x_{t+1}]. \end{aligned} \tag{8}$$

The last equality follows from the fact that, conditional on the current regime lasting exactly t periods, i.e. $\eta_{t+1}=1$, the intervening reoptimization implies $\varphi_{t+1}=0$. Furthermore, the fact that x_{t+1} is predetermined implies that knowledge of η_{t+1} does not help to predict its value, so that the expectation in (8) is just conditioned on information available at time k.

A.4 Quasi Commitment Equilibrium

With a solution for agents' forecasting problem in hand, we are now ready to solve the saddle-point problem on the right-hand side of Bellman equation (7). The only remaining challenge is that the running cost function involves a sum with a random number of terms, given the uncertainty on the length of the current regime. Also in this instance however, the assumed exogeneity of the regime-change signals represents a major simplification.

In fact, with $\{\eta_t\}_t$ a sequence of i.i.d. Bernoulli draws, $\Delta\tau_j \sim$Geometric(α), so that $\Pr\{\Delta\tau_j = m\} = (1-\alpha)^m\alpha$. This implies, for example

$$\begin{aligned} E_{\tau_j}\left\{\sum_{k=0}^{\Delta\tau_j}\beta^k L_{\tau_j+k}\right\} &= \sum_{m=0}^{\infty}\alpha(1-\alpha)^m E_{\tau_j}\left[\sum_{k=0}^{\Delta\tau_j}\beta^k L_{\tau_j+k}\middle|\,\Delta\tau_j=m\right] \\ &= \sum_{k=0}^{\infty}\left[\beta(1-\alpha)\right]^k\sum_{m=0}^{\infty}\alpha(1-\alpha)^m E_{\tau_j}\left[L_{\tau_j+k}|\Delta\tau_j=k+m\right] \\ &= \sum_{k=0}^{\infty}\left[\beta(1-\alpha)\right]^k E_{\tau_j}\left[L_{\tau_j+k}\right], \end{aligned}$$

where the last equality follows from the fact that L_{τ_j+k} must be measurable with respect to $\mathcal{I}_{\tau_j+k}$, the information set available at time τ_j+k. It cannot therefore be a function of the length of the regime after that time. Identical reasoning can be applied to all the terms in the square bracket in (7), since they are all τ_j+k-measurable, including of course the one-step-ahead expectation.

Given a guess for the form of the value function

$$V(x) = x'Px + \rho, \tag{9}$$

with $P \in \mathbb{R}^{n_x \times n_x}$ and $\rho \in \mathbb{R}$, we can then write the Lagrangian associated with the extremum

problem in (7) as

$$\begin{aligned}\mathcal{L} = E_{\tau_j} \Bigg\{ \sum_{k=\tau_j}^{\infty} \left[\beta(1-\alpha)\right]^{k-\tau_j} \big[& L_k + \alpha\beta x'_{k+1} P x_{k+1} \\ & +2\varphi'_{k+1} \left((1-\alpha) G E_k \left[X_{k+1} | \eta_{k+1} = 0\right] + \alpha G H x_{k+1} - A_{21} x_k - A_{22} X_k - B_2 i_k\right) \\ & +2\phi'_{k+1} \left(x_{k+1} - A_{11} x_k - A_{12} X_k - B_1 i_k\right)\big]\Bigg\},\end{aligned} \tag{10}$$

where we have introduced the n_x non-predetermined Lagrange multipliers ϕ_{k+1}. Note that the probability of a regime change has three effects. First, it modifies the discount rate to take into account the survival probability of the regime. Second, the continuation value is multiplied by $\alpha(1-\alpha)^{k-\tau_j}$, to account for the probability of the jth regime ending at time $k+1$. Third, the expectational constraint is modified to take into account that the non-predetermined variables would jump in case of a regime change at time $k+1$.

This expression also makes clear the connection between quasi commitment, discretion and full commitment. Consider first $\alpha \to 1$. In this case, only the first term in the infinite sum remains. Since a new regime starts every period with probability one, $\varphi_t = 0\ \forall t$ and (10) reduces to the familiar expression for the optimal discretionary policy. Next, consider the opposite extreme, $\alpha \to 0$. In this case, the running cost function is identical to the objective function under commitment, the terminal value drops out of the sum and the term $\alpha G H x_{k+1}$ disappears from the constraint. This is the standard Lagrangian formulation of the commitment problem.

A.4.1 Solution Procedure

Just as in the case of discretion, it is not possible to solve simultaneously for the value function, P and ρ, private agents' expectations, H, and the state space representation of the optimal equilibrium dynamics. However, it is not difficult to solve for any one of these objects given the other two. This motivates the following iterative solution procedure.

Step 1: Solve the optimal policy problem. For given values of H, P and ρ, solve the policymaker's problem by taking first order conditions in (10). The solution of the resulting linear rational expectations system within the jth regime can be written, $\forall t \in \{\tau_j, \ldots, \tau_j + \Delta\tau_j\}$, as

$$\begin{bmatrix} x_{t+1} \\ \varphi_{t+1} \end{bmatrix} = M \begin{bmatrix} x_t \\ \varphi_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix} \tag{11a}$$

$$\begin{bmatrix} i_t \\ X_t \end{bmatrix} = \begin{bmatrix} \xi' \\ C \end{bmatrix} \begin{bmatrix} x_t \\ \varphi_t \end{bmatrix} \tag{11b}$$

$$\varphi_{\tau_j} = 0, \tag{11c}$$

with x_{τ_j} predetermined as of the last period of the $j-1^{\text{st}}$ regime.

Step 2: Update H. Use the state space form (11) to compute the one-step-ahead expectation

$$E_t \left[X_{t+1} | \eta_{t+1} = 1\right] = C \begin{pmatrix} I \\ 0 \end{pmatrix} E_t x_{t+1}. \tag{12}$$

Matching this expression with (8), we get an equation for H

$$H = C\begin{pmatrix} I_{n_x \times n_x} \\ 0_{n_X \times n_x} \end{pmatrix},$$

which must be satisfied for expectations to be consistent with the proposed equilibrium.

Step 3: Update P and ρ. Substitute the state space form (11) into the Bellman equation (7) and match terms to find an expression for P and ρ

$$\begin{aligned} P &= \begin{pmatrix} I_{n_x \times n_x} \\ 0_{n_X \times n_x} \end{pmatrix}' V_1 \begin{pmatrix} I_{n_x \times n_x} \\ 0_{n_X \times n_x} \end{pmatrix} + \alpha\beta \begin{pmatrix} I_{n_x \times n_x} \\ 0_{n_X \times n_x} \end{pmatrix}' M' V_2 M \begin{pmatrix} I_{n_x \times n_x} \\ 0_{n_X \times n_x} \end{pmatrix} \\ \rho &= \frac{\beta}{1-\beta} \begin{pmatrix} (1-\alpha) tr\{\left(\begin{smallmatrix} I_{n_x \times n_x} \\ 0_{n_X \times n_x} \end{smallmatrix}\right)' V_1 \left(\begin{smallmatrix} I_{n_x \times n_x} \\ 0_{n_X \times n_x} \end{smallmatrix}\right) \Sigma\} \\ +\alpha tr\{\left(\begin{smallmatrix} I_{n_x \times n_x} \\ 0_{n_X \times n_x} \end{smallmatrix}\right)' V_2 \left(\begin{smallmatrix} I_{n_x \times n_x} \\ 0_{n_X \times n_x} \end{smallmatrix}\right) \Sigma\} \end{pmatrix} \end{aligned}$$

where the matrices V_1, V_2 solve the Sylvester equations

$$\begin{aligned} V_1 &= \begin{pmatrix} I_{n_x \times n_x} & 0_{n_x \times n_X} \\ C & \\ \xi' & \end{pmatrix}' W \begin{pmatrix} I_{n_x \times n_x} & 0_{n_x \times n_X} \\ C & \\ \xi' & \end{pmatrix} + \beta(1-\alpha) M' V_1 M \\ V_2 &= \begin{pmatrix} I_{n_x \times n_x} \\ 0_{n_X \times n_x} \end{pmatrix} P \begin{pmatrix} I_{n_x \times n_x} \\ 0_{n_X \times n_x} \end{pmatrix}' + \beta(1-\alpha) M' V_2 M. \end{aligned}$$

Starting with an initial guess for H, P and ρ, a fixed point of this procedure will result in: *(i)* a state contingent policy plan, $\{i_t\}_t$, which is optimal from the perspective of each policymaker, taking as given agents' expectations; *(ii)* a corresponding path for the endogenous variables, $\{x_t, X_t\}_t$; *(iii)* rational expectations, given the optimal sequence $\{i_t, x_t, X_t\}_t$. This is our definition of a quasi commitment equilibrium.

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